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Purity- and Gaussianity-bounded uncertainty relations

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Abstract
Bounded uncertainty relations provide the minimum value of the uncertainty assuming some additional information on the state. We derive analytically an uncertainty relation bounded by a pair of constraints, those of purity and Gaussianity. In a limiting case this uncertainty relation reproduces the purity-bounded derived by *Man'ko and Dodonov* and the Gaussianity-bounded one (Mandilara and Cerf 2012 *Phys. Rev.* A **86** 030102R).

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(Some figures may appear in colour only in the online journal)

1. Introduction

The only states that saturate the Schrödinger–Robertson (SR) [1] uncertainty relation for the canonically conjugated coordinates of position and momenta, are the pure Gaussian states. However, if some additional information about the state is available then the set of states which minimize the uncertainty, or else of *minimizing states* (MSs), is modified while a tighter lower bound on the uncertainty can be derived.

A basic characteristic of a state is its degree of mixedeness. The minimizing set obtained by imposing the constraint of fixed degree of mixedeness depends on the measure that one chooses to quantify this degree, i.e. purity, purities of higher order or various entropies (see [2] for a review). For instance, in the case of the *purity -bounded uncertainty relation* suggested by Dodonov and Man'ko [3] the minimizing set is composed by mixed states, or more precise, mixtures of number states. On the other hand, in the case of the von-Neumann entropy, the set

of the MSs is composed by the 'thermal states' whose temperature is increasing as the fixed entropy tends to infinity [4]. In both cases, the lower bound on the uncertainty is increasing with the degree of mixedeness since the mixedeness adds extra 'amount' of classical (statistical) uncertainty.

In a recent work [5] we have suggested an uncertainty relation bounded by the degree of Gaussianity, a quantity which we introduced in that same work. There the non-Gaussian MSs are identified for a fixed degree of Gaussianity and among them one finds all the eigenstates of the harmonic oscillator. Along with the Gaussianity- bounded uncertainty relation, we have presented a general method for deriving bounded uncertainty relations that reduces the problem to an eigenvalue problem.

In this work we employ the method exhibited in [5] to derive an uncertainty relation where the bound depends on two characteristics of the state, namely its purity and Gaussianity. To our knowledge this is the first two-dimensional bounded uncertainty relation that has been derived so far. The uncertainty relation is represented via parametric relations which connect three quantities, namely, the purity, Gaussianity and uncertainty. This exact expression is difficult to handle analytically and for this reason we also provide an approximate expression. The derived relation provides the boundaries for three basic characteristics of a state and can be used as a tool for visualizing and partitioning the space of non-Gaussian mixed states.

The structure of the paper is the following. In first step, in section 2 we derive the set of MSs of the purity and Gaussianity-bounded uncertainty relation and then in section 3 we provide exact and approximate expressions for this uncertainty relation. We discuss our and conclude in section 4.

2. Minimizing states

Let us start with the position $\hat{x}$ and momentum $\hat{p}$ of a quantum particle in one dimension, which could also be the quadratures of a single mode of the electromagnetic field, in a state defined by the density operator $\hat{\rho}$. In its most general form, the SR uncertainty relation [1] for the position and momentum of this particle reads

$$(\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2)(\langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2) - \tfrac{1}{4}(\langle\hat{x}\hat{p} + \hat{p}\hat{x}\rangle - 2\langle\hat{x}\rangle\langle\hat{p}\rangle)^2 \geqslant \hbar^2/4. \tag{1}$$

The left-hand side is invariant under *linear canonical transformations* (LCT), i.e. the direct sum of the symplectic transformations SL(2, R) and translations T(2). In quantum optics language, LCT correspond to squeezing, rotations and displacements, which form the set of Gaussian operations. The invariance of the uncertainty with respect to LCT becomes directly evident if we express the left hand side of equation (1) in terms of the *covariance matrix* γ of the state $\hat{\rho}$, defined through its matrix elements as

$$\gamma_{ij} \equiv \tfrac{1}{2}\,\mathrm{Tr}(\{(\hat{r}_i - d_i), (\hat{r}_j - d_j)\}\hat{\rho}) \tag{2}$$

where $\hat{\mathbf{r}} = (\hat{x}, \hat{p})^T$, $\mathbf{d} = \mathrm{Tr}(\hat{\mathbf{r}}\hat{\rho})$ is the displacement vector, and $\{\cdot, \cdot\}$ is the anticommutator. The left-hand side of equation (1) is simply $\det\gamma$ and therefore is invariant under LCT. For simplicity in the presentation, we define here the dimensionless quantity

$$\alpha(\hat{\rho}) \equiv \sqrt{\det\gamma}/(\hbar/2)$$

which we call *uncertainty*. With this definition the SR uncertainty relation simply reads $\alpha \geqslant 1$.

The alternative method of derivation of the SR uncertainty relation presented in [5], exploits the invariance of the uncertainty α under LCT. Due to this invariance, it becomes possible to confine our search of MSs into a specific class of states into which all states can be reduced under the action of LCT. By constraining the MSs to belong to this class, we are

led to solve an optimization problem for α under constraints, which we tackle with Lagrange multipliers' method. Apart from the constraints of the class, one may impose additional constraints and thus derive bounded uncertainty relations depending on other characteristics of the state such as the purity [3] or the Gaussianity [5].

Before we proceed with the derivation of the purity- and Gaussianity-bounded uncertainty relation let us first introduce these two quantities. The *purity* μ of a state $\hat{\rho}$ is defined as

$$\mu(\hat{\rho}) \equiv \mathrm{Tr}(\hat{\rho}^2)$$

while the degree of *Gaussianity* is defined as [5]

$$g(\hat{\rho}) \equiv \mathrm{Tr}(\hat{\rho}\hat{\rho}_G)/\mathrm{Tr}(\hat{\rho}_G^2) \tag{3}$$

where $\hat{\rho}_G$ is a reference Gaussian state uniquely defined by the mean vector $\mathbf{d}$ and covariance matrix γ of the state $\hat{\rho}$. The Gaussianity exhibits the following properties (see [5] for the proofs):

(i) g is invariant under LCT
(ii) g is a bounded quantity, that is, $2/e \leqslant g \leqslant 2$, while
$g = 1$ for Gaussian states (but the converse is not true)
(iii) g together with α confines the set of mixed states with strictly positive Wigner function.

The aim is to find the states that minimize α under the constraints of fixed μ and g. All three quantities are invariant under LCT and therefore, as in [5], without loss of generality we can confine our search among a specific class of states with covariance matrix proportional to the unity (in Williamson form) and $\mathbf{d} = \mathbf{0}$. We should note here that every state can be reduced in this form under LCT and in this specific class the reference Gaussian state is just a thermal state $\hat{\rho}_G = \mathrm{e}^{-\beta\hat{n}}/A$ where $\hat{n}$ is the number operator and $A = (\alpha + 1)/2$ the normalization factor. Our choice to work within this specific class of states can be translated as constraints on the state $\hat{\rho}$

$$\mathrm{Tr}(\hat{\rho}\hat{x}) = \mathrm{Tr}(\hat{\rho}\hat{p}) = 0 \tag{4}$$

$$\mathrm{Tr}(\hat{\rho}(\hat{x}\hat{p} + \hat{p}\hat{x})) = \mathrm{Tr}(\hat{\rho}(\hat{x}^2 - \hat{p}^2)) = 0. \tag{5}$$

In addition we require that the states $\hat{\rho}$ which minimize the uncertainty

$$\alpha = \mathrm{Tr}(\hat{\rho}\ (2\hat{n} + 1)) \tag{6}$$

are of fixed purity and Gaussianity degree,

$$\mu = \mathrm{Tr}(\hat{\rho}^2) \tag{7}$$

$$g = \frac{1}{N}\mathrm{Tr}(\hat{\rho}\,\mathrm{e}^{-\beta\hat{n}}) \tag{8}$$

where $\mathrm{e}^{-\beta} = \frac{\alpha-1}{\alpha+1}$ and $N = (\alpha + 1)/2\alpha$.

We proceed now with the optimization procedure for finding states $\hat{\rho}$ which satisfy equations (4)–(5), (7)–(8) and extremize α. For each state $\hat{\rho}$ an eigenbasis exists such that $\hat{\rho} = \sum c_n|\Psi_n\rangle\langle\Psi_n|$ with $0 \leqslant c_n \leqslant 1$ and $\sum c_n = 1$. We can also rewrite the state as $\hat{\rho} = \sum |\psi_n\rangle\langle\psi_n|$, by using the unnormalized eigenvectors $|\psi_n\rangle = \sqrt{c_n}|\Psi_n\rangle$, while additionally imposing the normalization constraint

$$\mathrm{Tr}(\hat{\rho}) = 1. \tag{9}$$

In this way the positivity of $\hat{\rho}$ is ensured since the mixing coefficients c_n are just the squared norms $c_n = \langle\psi_n|\psi_n\rangle$.

The next step is to choose an orthonormal basis $\{|i\rangle\}$ and decompose the vectors $|\psi_n\rangle = \sum \psi_n^i |i\rangle$. We can re-express accordingly the uncertainty (6) and constraints (4)–(5), (7)–(8), and (9) as functions of the complex amplitudes ψ_n^is. This gives

$$\alpha = \sum_{n,i,j} \psi_n^{i*} \psi_n^j \langle i|(2\hat{n}+1)|j\rangle \tag{10}$$

and

$$\sum_{n,i,j} \psi_n^{i*} \psi_n^j \langle i|\hat{x}|j\rangle = 0 \tag{11}$$

$$\sum_{n,i,j} \psi_n^{i*} \psi_n^j \langle i|\hat{p}|j\rangle = 0 \tag{12}$$

$$\sum_{n,i,j} \psi_n^{i*} \psi_n^j \langle i|(\hat{x}\hat{p}+\hat{p}\hat{x})|j\rangle = 0 \tag{13}$$

$$\sum_{n,i,j} \psi_n^{i*} \psi_n^j \langle i|(\hat{x}^2-\hat{p}^2)|j\rangle = 0 \tag{14}$$

$$\sum_{n,i,j} \psi_n^{i*} \psi_n^j \langle i|\,\mathrm{e}^{-\beta\hat{n}}|j\rangle/N = g \tag{15}$$

$$\sum_{n,m,i,j} \psi_n^{i*} \psi_n^j \psi_m^{j*} \psi_m^i = \mu \tag{16}$$

$$\sum_{n,i} \psi_n^i \psi_n^{i*} = 1. \tag{17}$$

The Lagrange multipliers method is well suited as an optimization procedure for this problem. This method provides necessary conditions on the solution, which remains invariant under the exchange of any of the constraints with the quantity to be optimized. Since it is more convenient for us to optimize over the purity while setting the uncertainty as a constraint, we proceed accordingly. After differentiating over the amplitudes ψ_n^i we obtain the following necessary condition on the eigenvectors $|\psi_n\rangle$

$$(\lambda_0 I + \lambda_1 \hat{n} + \lambda_2\,\mathrm{e}^{-\beta\hat{n}}/N + \lambda_3\hat{x} + \lambda_4\hat{p} + \lambda_5(\hat{x}\hat{p}+\hat{p}\hat{x}) + \lambda_6(\hat{x}^2-\hat{p}^2) - \hat{\rho})|\psi_n\rangle = 0. \tag{18}$$

The term $\hat{\rho}$ here appears as a consequence of the purity term $\mathrm{Tr}(\hat{\rho}^2)$. This condition can be re-written as $(\hat{H}-\hat{\rho})|\psi_n\rangle = 0$ where $\hat{H}$ is a Hermitian operator defined as

$$\hat{H} = \lambda_0 I + \lambda_1 \hat{n} + \lambda_2\,\mathrm{e}^{-\beta\hat{n}}/N + \lambda_3\hat{x} + \lambda_4\hat{p} + \lambda_5(\hat{x}\hat{p}+\hat{p}\hat{x}) + \lambda_6(\hat{x}^2-\hat{p}^2). \tag{19}$$

We can employ the fact that $c_n = \langle\psi_n|\psi_n\rangle$ to express this condition in the form

$$\hat{H}|\psi_n\rangle - c_n|\psi_n\rangle = 0 \tag{20}$$

or equivalently as

$$\hat{H}|\Psi_n\rangle - c_n|\Psi_n\rangle = 0. \tag{21}$$

One can conclude that the eigenvectors $|\Psi_n\rangle$ of the solution $\hat{\rho}$ are the eigenvectors $|\phi_n\rangle$ of the Hermitian operator $\hat{H}$ while the mixing coefficients c_n are the corresponding*positive* eigenvalues ε_n^+ of $\hat{H}$. In other words, the Lagrange multipliers method provides a necessary condition on the expression of the solution $\hat{\rho}$. It is written as

$$\hat{\rho} = \sum_n \varepsilon_n^+ |\Psi_n\rangle\langle\Psi_n|.$$

We should note here an important difference between the condition that we obtain here, (21), and the necessary condition obtained in [5] where all constraints are linear, i.e. can be expressed in the form $\mathrm{Tr}(\hat{\rho}B)$ with $\hat{B}$ a Hermitian operator. In that case the condition dictates that the all the eigenvectors of the solution $\hat{\rho}$ should correspond to the same eigenvalue of a Hermitian operator $\hat{H}$. The degeneracy constraint is lifted here due to the presence of the nonlinear constraint of the purity $\mathrm{Tr}(\hat{\rho}^2)$. With this more general example, we complete the description of the method for the derivation of bounded uncertainty relation originally described in [5].

One should now proceed with the identification of the eigenvectors of the Hermitian operator $\hat{H}$, a task that is not that simple because of the presence of the term $\mathrm{e}^{-\beta\hat{n}}/N$ in equation (19). The problem can be simplified, as shown in the appendix. There it is proven that the states that the purity possess a phase-independent Wigner function and therefore are confined to be mixtures of number states

$$\hat{\rho} = \sum_n \varepsilon_n^+ |n\rangle\langle n|. \tag{22}$$

Obviously, the solution to the optimization problem consists of states which either maximize or minimize the purity μ for fixed uncertainty α and Gaussianity degree g. As we are going to show at the end, the states which maximize the purity are not relevant for our purposes here, and thus we proceed by identifying the states of minimum purity which can be expressed as in equation (22).

Having restricted ourselves to states of the form equation (22), we ensure that the constraints equations (11)–(14) are automatically satisfied and the restriction of the Hermitian operator $\hat{H}$ on this class of states becomes

$$\hat{H}_0 = \lambda_0 \hat{I} + \lambda_1 \hat{n} + \lambda_2 \,\mathrm{e}^{-\beta\hat{n}}/N. \tag{23}$$

It is important to mention here that the symmetrized Hamiltonian, equation (23), is not a result of the application of the principle of symmetric criticality [6] but it follows as a result of the independent proof presented in the appendix. The eigenstates of $\hat{H}_0$ are the number states $|n\rangle$ (for the non-degenerate case) and the corresponding spectrum is

$$\varepsilon_n = \lambda_0 + \lambda_1 n + \lambda_2 \frac{2\alpha(\alpha-1)^n}{(\alpha+1)^{n+1}}, \tag{24}$$

where λs are to be identified by the constraints equations (6), (8) and (9). The difficult part is to identify among the eigenstates of $\hat{H}_0$ those with positive eigenvalues ε_n^+. To do so one should first identify the possible structures of positive spectrum that correspond to different possible values of λ_0, λ_1 and λ_2 (or equivalently to different values of α and g). In figure 1 we present four different representative 'shapes' of the spectrum corresponding to a fixed α and varying g. By inserting the positive spectrum ε_n^+ into equation (22) one gets straightforwardly the value of the purity $\mu = \sum_n (\varepsilon_n^+)^2$.

One can see that for all cases the *positive* spectrum corresponds to *successive* number states, so we can conclude that the MSs have the form

$$\hat{\rho} = \sum_{n_{\min}}^{n_{\max}} \varepsilon_n^+ |n\rangle\langle n|. \tag{25}$$

where $n_{\min}$ and $n_{\max}$ are parameters which also depend on the constraints α and g in a complicated fashion.

3. Bounded uncertainty relation

Having identified the form of the MSs we can proceed with the identification of λ_0, λ_1 and λ_2 in equation (24) by imposing the constraints of normalization, uncertainty α and Gaussianity

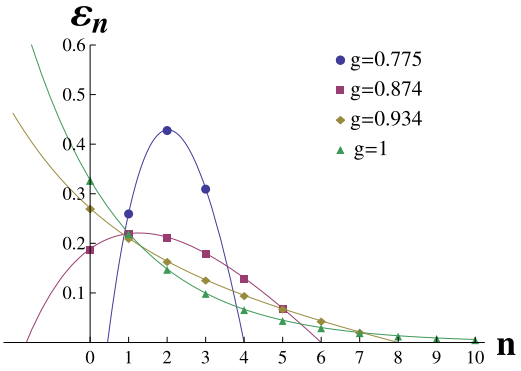

Figure 1. Different representative shapes of the positive spectrum ε_n of the Hermitian operator $\hat{H}_0$ for $\alpha = 5.1$. The corresponding values of the Gaussianity and purity are $(g, \mu) = \{(0.775, 0.348), (0.874, 0.183), (0.934, 0.176), (1, 0.196)\}$.

g. This should be done for all pairs of $n_{\min}$ and $n_{\max}$, but then one should go back and keep only the pairs for which the eigenvalues $\{\varepsilon_{n_{\min}}, \varepsilon_{n_{\min}}, \ldots \varepsilon_{n_{\max}}\}$ are positive while all the rest of eigenvalues are negative. Obviously, for some choices of values of uncertainty and Gaussianity no pair of $n_{\min}$ and $n_{\max}$ satisfying these conditions, exists. Otherwise, we can in principle deduce the values of $n_{\min}$ and $n_{\max}$ which are consistent with the constraints of normalization, uncertainty α and Gaussianity g. This finally yields the extremal purity μ.

In what follows, we expose one possible way for simplifying this complicated procedure by fixing $n_{\max}$ instead of the Gaussianity g. Then we still have to check all values of $n_{\min}$ and keep those that satisfy the positive spectrum condition. The key observation is that if equation (24) is re-written substituting the discrete index n by a continuous variable x,

$$\lambda_0 + \lambda_1 x + \lambda_2 \frac{2\alpha(\alpha-1)^x}{(\alpha+1)^{x+1}} \tag{26}$$

then the zeros of this equation (which are maximum two in number) define $n_{\min}$ and $n_{\max}$. More precisely, if the equation has two positive roots, x_1 and x_2 ($>x_1$), then $n_{\min} = \lceil x_1 \rceil$, $n_{\max} = \lfloor x_2 \rfloor$ (where $\lceil x \rceil$ is ceiling function and $\lfloor x \rfloor$ the floor function). In the case where $x_1 < 0$ or we have only one root $x_2 > 0$ (see yellow curve in 1) then $n_{\min} = 0$. In view of these results, we are able to propose a protocol to obtain in a systematic way the whole set of MSs where the constraint on the Gaussianity is replaced by a constraint on the second root x_2 of equation (26) which indirectly fixes $n_{\max}$.

(i) Fix a value for $\alpha > 1$, a non-negative integer value for $n_{\min}$ and a real positive value for x_2 such that $x_2 - n_{\min} > 1$.

(ii) Solve the system of equations

$$\sum_{n=n_{\min}}^{\lfloor x_2 \rfloor} \left(\lambda_0 + \lambda_1 n + \lambda_2 \frac{2\alpha(\alpha-1)^n}{(\alpha+1)^{n+1}}\right)(2n+1) = \alpha \tag{27}$$

$$\sum_{n=n_{\min}}^{\lfloor x_2 \rfloor} \left(\lambda_0 + \lambda_1 n + \lambda_2 \frac{2\alpha(\alpha-1)^n}{(\alpha+1)^{n+1}}\right) = 1 \tag{28}$$

$$\lambda_0 + \lambda_1 x_2 + \lambda_2 \frac{2\alpha(\alpha-1)^{x_2}}{(\alpha+1)^{x_2+1}} = 0 \tag{29}$$

for λ_0, λ_1, and λ_2 and obtain ε_n from equation (24) as a function of $n_{\min}$, α and x_2. The equations (27)–(28) express the constraints equations (6) and (9) respectively while equation (29) ensures that x_2 is the root of equation (26).

(iii) Verify using the spectrum provided by equation (24) that the lowest index of positive part of the spectrum is indeed $n_{\min}$. In other words we check that $\varepsilon_{n_{\min}} \geqslant 0$ and also that $\varepsilon_{n_{\min}-1} < 0$ if $n_{\min} > 0$.

This procedure gives the values of the Gaussianity g and the minimum purity μ corresponding to the chosen parameters α and x_2, namely

$$g = \sum_{n_{\min}}^{\lfloor x_2 \rfloor} \varepsilon_n \frac{2\alpha(\alpha-1)^n}{(\alpha+1)^{n+1}} \tag{30}$$

$$\mu = \sum_{n_{\min}}^{\lfloor x_2 \rfloor} \varepsilon_n^2. \tag{31}$$

This yields one MS. To obtain the whole set of MSs this procedure should be repeated by varying the parameters α and x_2.

According to our studies the above procedure always yields $g < 1$, meaning that the derived MSs cannot cover values of g greater than 1. For $g = 1$ we have the limiting case where $\lambda_0 = \lambda_1 = 0$, there is no roots for equation (29) and the MSs are the so called 'thermal' states (see green curve in figure 1). For $g > 1$ there is no combination of λs which gives positive spectrum solution but by extrapolating the results in [5] one can construct a bound of minimum purity with the following states

$$\hat{\rho} = r\hat{\rho}_G + (1-r)|n\rangle\langle n| \tag{32}$$

where $narrow\infty$, $rarrow1$ while $\mu(\hat{\rho})$ is kept constant, and $\hat{\rho}_G$ a thermal state

$$\hat{\rho}_G = \sum_{m=0}^{\infty} \frac{2(\beta-1)^m}{(\beta+1)^{m+1}} |m\rangle\langle m|$$

of purity $\mu(\hat{\rho}_G) = \frac{1}{\beta}$. The uncertainty α for the MSs in equation (32) can be easily calculated

$$\alpha = \frac{g}{(2-g)\mu}, \quad g < 1 \tag{33}$$

where g is the Gaussianity and μ the purity of the state $\hat{\rho}$.

One may employ the parametric relations equations (30)–(31) for $g < 1$, to represent graphically the surface that stands for the Gaussianity- and purity-bounded uncertainty relation. For $g > 1$ one should employ the much simpler relation given by equation (33). In figure 2 we represent the purity and Gaussianity uncertainty relation projected on three, mutually orthogonal planes. One can also see on the same figure the lines which represent the purity-bounded uncertainty relation and the Gaussianity-bounded one. These one-dimensional uncertainty relations appear as outer boundaries (see red and blue line in figure 2) of the surface standing for the purity and Gaussian uncertainty relation. In figure 3 we give a three-dimensional view of the uncertainty relation.

One can observe that the surface representing the uncertainty relation is convex for $g > 1$ and its boundaries are laying on the plane of pure states. For $g < 1$ a part of the convex surface is somehow 'etched' by concave grooves delimited by (blue) loops so that the projections of the loops on $\mu = 1$ plane coincide with the intersection of the uncertainty surface with the plane. This reflects the fact that for any given value of α the points of the boundary of the grooves have the same value of g for all available values of μ. Therefore, the curves on the uncertainty surface which correspond to any given α are convex. This convexity makes sufficient our analysis of the states which minimize the purity because we perform it independently for all given values of α and therefore, there is no need to search for the states which maximize the purity.

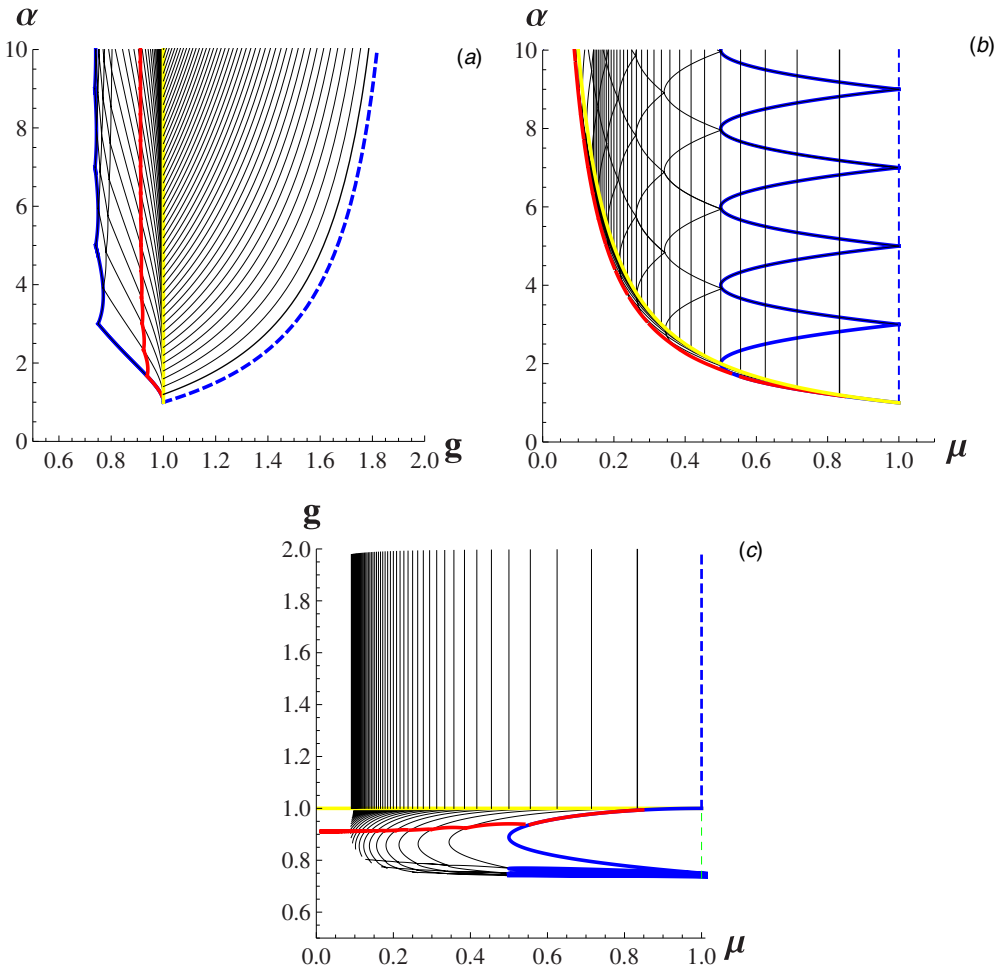

Figure 2. The surface that represents the purity- and Gaussianity-bounded relation projected on three planes: (*a*) α–g , (*b*) α–μ, and (*c*) g–μ. The red line stands for the purity-bounded uncertainty relation, the blue (dashed and solid) line for the Gaussianity-bounded uncertainty relation and the yellow line for the Gaussian states. For $g > 1$ the solid lines are of constant purity (see equation (32)). For $g < 1$ the surface is separated into segments of MSs with fixed $n_{\min}$ and $n_{\max}$ (see equation (25)). The lowest and highest value of uncertainty α for each segment define respectively $n_{\min}$ and $n_{\max}$ according to the relation $\alpha = 2n + 1$. For the plots we have worked in the domain of uncertainty $1 \leqslant \alpha \leqslant 10$ and for this reason (c) appears incomplete.

Note that, the parametric relation for $g < 1$ is not very convenient since it is not expressed in the desired form $\alpha \geqslant f(g, \mu)$ where for a given value of g and μ one may conclude on the smallest possible value on the uncertainty α. For this reason we have derived the following approximate relation for $f(g, \mu)$ when $g < 1$,

$$f_{\text{app}}(g, \mu) = \frac{2 - g}{g^{7/2}\mu^{g^2}} \left(1 + \frac{0.2\sqrt{\mu}}{\sqrt{0.01\pi}}\, \mathrm{e}^{-100(g-0.87)^2}\right)^{-1}. \tag{34}$$

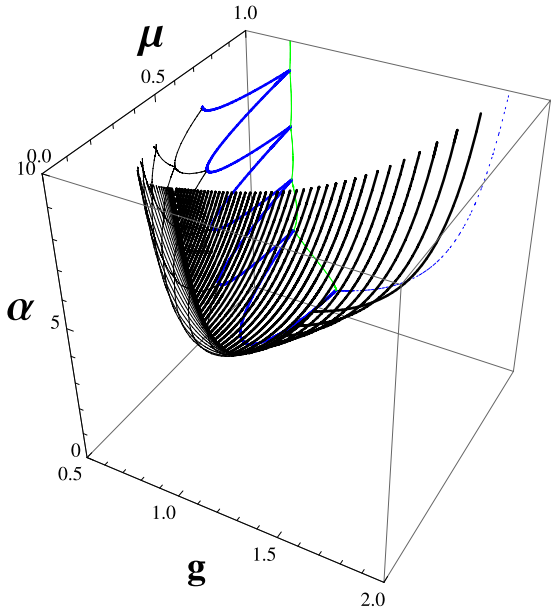

Figure 3. The surface that represents the purity- and Gaussianity-bounded relation. For a given value of purity μ and Gaussianity g, this surface provides the lowest possible value of the uncertainty α for all quantum states (pure or mixed). The blue line represents the Gaussianity-bounded uncertainty relation [5].

For every pair of values (g, μ) it holds that $f(g, \mu) > f_{\text{app}}(g, \mu)$ thus the approximate formula provides a lower estimation on the real bound of the uncertainty $f(g, \mu)$. In view of these results, we summarize the Gaussianity- and purity-bounded uncertainty relation as

$$\alpha \geqslant \left\{ \begin{array}{ll} \dfrac{g}{(2-g)\mu}, & 2 > g \geqslant 1 \\ f_{\text{app}}(g, \mu), & 1 > g \geqslant 2/e \end{array} \right\}. \tag{35}$$

In a previous work [7] we have derived following a similar but less mathematically consistent method, an uncertainty relation that is bounded by the degree of von-Neumann entropy of a state $\hat{\rho}$ and the quantity of overlap $\text{Tr}(\hat{\rho}\hat{\rho}_G)$ between the state and the reference Gaussian state $\hat{\rho}_G$. The approach which we follow here, permits us to assert the positivity of the density matrix of the solution in the Lagrange multipliers method while in [7] we 'impose' the positivity on the solution provided by the optimization method. On the other hand, the relation obtained in this work in three-dimensional representation, strongly resembles the one in [7], with the main difference being on the set of MSs. In [7] all MSs are of infinite rank while here the rank of the solution (the number of eigenvectors of the solution density matrix) remains finite in the general case ($g \neq 1$).

4. Conclusions

We have introduced and studied an uncertainty relation for the quantum variables of position and momentum, which is tighter than both the Schrödinger–Robertson [1] and the purity-bounded uncertainty relation by Dodonov and Man'ko [3]. Our new relation makes the minimum on the uncertainty α a function of the purity μ of quantum states and their degree

of Gaussianity g. Thus the whole set of quantum states of one-dimensional moving particle (or one optical mode) becomes bounded below in terms of α by a non-trivial surface in three-dimensional parametric space of μ, g, and α. Being projected on the plane $\mu - \alpha$ our bound recovers the purity-bounded uncertainty relation by Dodonov and Man'ko [3] while its projection on the $g - \alpha$ plane recovers the Gaussianity-bounded uncertainty relation [5]. Whereas for $g > 1$ our surface is given by an explicit function $\alpha = f(g, \mu)$ the part of the surface for $g < 1$ is obtained only as a parametric function. In order to express our result in the desired form for $g < 1$ we have constructed an approximation by function $f_{\text{app}}(g, \mu)$. This function determines a surface which for any (g, μ) lays slightly below the actual surface: $f(g, \mu) > f_{\text{app}}(g, \mu)$ and thus provides a less tight, but still valid, bound. Finally, our results allow us to visualize the whole set of quantum states in three-dimensional parametric space and accurately bound the uncertainty of x and p taking into account the purity and Gaussianity the states for which the uncertainty is evaluated.

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Appendix. Rotation invariance of minimizing states

Proposition. The states of minimum purity for given uncertainty and Gaussianity can be expressed as mixtures of number states

Let us consider a general density matrix $\hat{\rho}$ of purity μ and Gaussianity g, whose covariance matrix has been set via LCT proportional to the unity (in Williamson form) and its displacement vector $\mathbf{d}$ to zero. In this case the uncertainty α of the state completely characterizes the reference Gaussian state $\hat{\rho}_G$ which is just a thermal state. In the Wigner representation the reference state, $\hat{\rho}_G$,

$$W_G(r) = \frac{1}{\pi\alpha}\, \mathrm{e}^{-r^2/\alpha}, \quad r = \sqrt{x^2 + p^2} \tag{A.1}$$

has no dependence on the angular degree of freedom φ. In contrast, in the general case the state $\hat{\rho}$ itself possess an angular-dependent Wigner function $W(r, \varphi)$ and its purity can be expressed via $W(r, \varphi)$ as

$$\mu = 2\pi \iint W(r, \varphi)^2 r \, \mathrm{d}r \, \mathrm{d}\varphi \tag{A.2}$$

while its Gaussianity as

$$g = 2\pi\alpha \iint W(r, \varphi) W_G(r) r \, \mathrm{d}r \, \mathrm{d}\varphi. \tag{A.3}$$

The next step is to prove that for any given state $\hat{\rho}$, another state $\hat{\rho}_s$ exists of the same α and g and smaller or equal purity, which possess a phase-independent Wigner function. Let us define this new state $\hat{\rho}_s$ by phase-averaging the Wigner function $W(r, \varphi)$ of the initial state $\hat{\rho}$

$$W_s(r) = \frac{1}{2\pi} \int W(r, \varphi) \, \mathrm{d}\varphi. \tag{A.4}$$

Here $W_s(r)$ is the Wigner function of the new state $\hat{\rho}_s$. The reference Gaussian state of $\hat{\rho}_s$ (and consequently the uncertainty α) is the same as for $\hat{\rho}$, since phase-averaging cannot affect the angular-independent Wigner function equation (A.1). The Gaussianity degree equation (A.3) remains the same, as well. This is a straightforward result of substitution of the phase-independent Wigner function given by equation (A.4) into equation (A.3). On the other hand, the purity μ_s of the symmetrized state $\hat{\rho}_s$ is constrained to be smaller than, or equal to, that of $\hat{\rho}$. Indeed, by applying the Cauchy–Schwarz inequality we have

$$\begin{aligned}\mu_s &= 2\pi \iint W_s(r)^2 r\,\mathrm{d}r\,\mathrm{d}\varphi \\ &= \iiint W(r,\varphi)W(r,\Phi) r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\Phi \\ &\leqslant \sqrt{\iiint W(r,\varphi)^2 r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\Phi}\sqrt{\iiint W(r,\Phi)^2 r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\Phi} \\ &= 2\pi \iint W(r,\varphi)^2 r\,\mathrm{d}r\,\mathrm{d}\varphi = \mu.\end{aligned}$$

This concludes the proof of this proposition.

From this proposition it is straightforward to deduce that the minimizing states we are looking for, are states of angular-independent Wigner function and therefore can be expressed [8] as a convex combination of the *number* (Fock) states

$$\hat{\rho} = \sum \rho_n |n\rangle\langle n|. \tag{A.5}$$

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