

Quantum water-filling solution for the capacity of Gaussian information channels

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ABSTRACT

We study the transmission of classical information via optical Gaussian channels with a classical additive noise under the physical assumption of a finite input energy including the energy of classical signal (modulation) and the energy spent on squeezing the quantum states carrying information. Multiple uses of a certain class of memory channels with correlated noise is equivalent to one use of parallel independent channels generally with a phase-dependent noise. The calculation of the channels capacity implies finding the optimal distribution of the input energy between the channels. Above a certain input energy threshold, the optimal energy distribution is given by a solution known in the case of classical channels as *water-filling*. Below the threshold, the optimal distribution of the input energy depends on the noise spectrum and on the input energy level, so that the channels fall into three different classes: the first class corresponds to very noisy channels excluded from information transmission, the second class is composed of channels in which only one quadrature (q or p) is modulated and the third class corresponds to the *water-filling* solution. Although the non-modulated quadrature in the channels of the second class is not used for information transmission, a part of the input energy is used for the squeezing the quantum state which is a purely quantum effect. We present a complete solution to this problem for one mode and analyze the influence of the noise phase dependence on the capacity. Contrary to our intuition, in the highly phase-dependent noise limit, there exists a universal value of the capacity which neither depends on the input energy nor on the value of noise temperature. In addition, similarly to the case of lossy channels for weak thermal contribution of the noise, there exists an optimal squeezing of the noise, which maximizes the capacity.

Keywords: Gaussian channels, classical capacity, quantum channels, memory channels, correlated noise

1. INTRODUCTION

One of the central problems of quantum communication is to determine the classical capacity of a quantum channel, which is defined as the supremum of the rate of classical information that can be transmitted via the channel.

The key question in finding the capacity is its additivity meaning that the maximum transmission rate achieved when using several channels together does not exceed the sum of maximum transmission rates for each of the channels used separately. The capacity of additive channels is achieved by encoding classical information into product states. Intuitively, this conjecture is natural at least for channels, for which temporally separated uses of the channel are independent. Although the additivity conjecture recently was shown to be not true in general,¹ for many memoryless channels additivity was established.^{2,3} On the other hand, memory effects or correlations may lead to the optimality of entangled input states achieving the classical capacity. This effect was shown first for channels with discrete alphabets: a depolarizing channel⁴ and a quasiclassical depolarizing channel.⁵ These results demonstrate the existence of a threshold value of the memory parameter above which entangled states improve the transmission rate with respect to product states. Bounds on the classical and quantum capacities of a qubit channel with finite memory were derived.^{6,7} Classical and quantum capacities of memory channels were discussed in a general framework.⁸ For general Pauli channels with memory it was also shown⁹ that the optimal states are either product states or Bell states separated by a memory threshold. In the case of higher dimensions, it has been shown that the capacity of qudit channels exhibits the same threshold phenomenon as Pauli qubit channels.^{10,11}

For quantum memory channels with continuous alphabet, the first studies considered two uses correlated of an additive optical channel and a lossy optical channel,^{12,13} the classical capacity was achieved for input states with some degree of entanglement. Later, lower and upper bounds on the classical capacity were derived for a lossy optical memory channel.^{14–16} In a recent paper¹⁷ we have evaluated the classical capacity of certain Gaussian channels with additive noise, in particular, Markov correlated noise, which introduces memory in the channel. It was shown that for an input energy above a certain threshold, the optimal input states are entangled states where the squeezing of the input has to match the anisotropy of the environment and the energy is uniformly distributed between the channels which corresponds to the classical *water-filling* solution.¹⁸

In the present paper we study the optical Gaussian channel with additive phase-dependent noise, i.e. a noise with different variance in the q and p -quadrature. For certain class of Gaussian memory channels with correlated noise the capacity is equivalent to the capacity of a Gaussian channel with the noise, which is independent between the modes but may be phase dependent within each mode.^{16,17,19} Additivity of Gaussian channels under Gaussian inputs make this problem equivalent to the one of finding the “one-shot” capacity of one mode. The solution of the one-mode problem may eventually help us to find the solution of the problem for arbitrary number of modes and the limit of infinite number of modes, which is necessary to determine the capacity of memory channels with correlated noise. However, in this paper we do not go beyond one-mode and consider in addition another problem, which may have an applied interest. Usually the problem of capacity is posed for some given channel, however, when designing optical connections one can think of the choice of the properties of the channel itself. Recently, in Ref.¹⁶ such a question was posed for a lossy channel and it was shown that the higher the asymmetry of the noise the higher is the channel capacity for a wide range of channel losses (modeled by a beamsplitter). They have found that in the limit of infinite noise squeezing the capacity tends to some universal value no matter which is the channel loss. In addition for channels with a high transmittivity there exists a finite value of the noise squeezing which maximizes the capacity for given transmittivity. We study the influence of the noise squeezing on the capacity of a Gaussian channel with lossless transmission and additive noise and start our analysis for the one-mode case.

In section 2 we shortly sketch the formulation of the problem of finding classical capacity of quantum channels and its application to Gaussian channels with correlated noise. In section 3 we discuss the optimization problem for the n -mode parallel channel which arises when one calculates channel capacity. Section 4 is devoted to the detailed study for one-mode problem in different regimes depending on the asymmetry of the noise and the available amount of the input energy. In section 5 we consider the influence of the properties of the channel noise on the classical capacity. In section 6 we discuss the results and highlight the problems which are still open.

2. CLASSICAL CAPACITY OF QUANTUM GAUSSIAN CHANNELS

In order to use a channel to transmit classical information one has to choose an alphabet that contains letters labeled by an index i . The use of a quantum channel requires furthermore, to associate each letter i to a quantum state ρ_i^{in} . The input states ρ_i^{in} are sent through the channel, interact with the environment and thus are modified at the output. The quantum channel T is a completely positive, trace-preserving map acting on the input “letter” states:

$$\rho_i^{\text{out}} = T[\rho_i^{\text{in}}]. \quad (1)$$

In addition the letters i that form the messages appear with a certain probability p_i so that the overall modulated input state is given by a mixture $\rho^{\text{in}} = \sum_i p_i \rho_i^{\text{in}}$. By linearity of T , Eq. (1) determines as well the action of the channel on the overall modulated input $\bar{\rho} \equiv \sum_i p_i \rho_i^{\text{out}} = T[\rho^{\text{in}}]$, where $\bar{\rho}$ will be referred to as the overall modulated output. In order for $\bar{\rho}$ be physical it has to obey the energy constraint

$$\sum_i p_i \text{Tr}(\rho_i^{\text{in}} a^\dagger a) \leq \bar{n}, \quad (2)$$

where \bar{n} is the maximum mean photon number per use of the channel and will be referred to as “input energy” in the following.

The classical capacity $C(T)$ of the channel T represents the supremum on the amount of classical bits which can be transmitted per invocation of the channel via quantum states in the limit of an infinite number of channel

uses. This quantity can be calculated with the help of the so-called *one-shot* capacity, defined as²⁰

$$C_1(T) = \sup_{\{\rho_i^{\text{in}}, p_i\}} \chi, \quad (3)$$

where the Holevo χ -quantity reads

$$\chi = S\left(\sum_i p_i T[\rho_i^{\text{in}}]\right) - \sum_i p_i S(T[\rho_i^{\text{in}}]), \quad (4)$$

with the von Neumann entropy $S(\rho) = -\text{Tr}(\rho \log \rho)$ where \log denotes the logarithm to base 2. The supremum in (3) is taken over all ensembles of $\{p_i, \rho_i^{\text{in}}\}$ of probability distributions p_i and pure input “letter” states ρ_i^{in} .²⁰

The term “one-shot” means that only one invocation of T is needed to calculate Eq. (3). Using this quantity the capacity $C(T)$ defined as above may be evaluated in the following way. A number n of consecutive uses of the channel T can be equivalently considered as one parallel n -mode channel $T^{(n)}$, which is used only one time. Then the capacity $C(T)$ is evaluated in the limit:

$$C(T) = \lim_{n \rightarrow \infty} \frac{1}{n} C_1(T^{(n)}). \quad (5)$$

Let us now assume $T^{(n)}$ to be a n -mode optical additive channel with memory. In the following, the number of modes of this channel corresponds to the number of mono-modal channel uses. Each mode j is associated with the annihilation and creation operators a_j, a_j^\dagger , respectively, or equivalently to the quadrature operators $q_j = (a_j + a_j^\dagger)/\sqrt{2}, p_j = i(a_j - a_j^\dagger)/\sqrt{2}$ which obey the canonical commutation relation $[q_i, p_j] = i\delta_{ij}$, where δ_{ij} denotes the Kronecker-delta. By ordering the quadratures in a column vector $R = (q_1, \dots, q_n; p_1, \dots, p_n)^\top$, we can define the displacement vector m and covariance matrix γ of an n -mode state ρ as

$$\begin{aligned} m &= \text{Tr}[\rho R] \\ \gamma &= \text{Tr}[(R - m)\rho(R - m)^\top] - \frac{1}{2}J, \quad J = i \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \end{aligned} \quad (6)$$

where J is the symplectic or commutation matrix with the $n \times n$ identity matrix I . In this paper we focus on Gaussian states, which are fully characterized by m and γ . Furthermore, without loss of generality, we set the displacement of overall modulated states and the means of classical Gaussian distributions to zero, because displacements do not change the entropy.

For the optical channel, the encoding of classical information is made according to a continuous alphabet, where the previous discrete letter with index i is replaced by the real and imaginary part of a complex number α . A message of length n is therefore encoded in a $2n$ real column vector $\alpha = (\Re\{\alpha_1\}, \Re\{\alpha_2\}, \dots, \Re\{\alpha_n\}, \Im\{\alpha_1\}, \dots, \Im\{\alpha_n\})^\top$. Physically, this encoding corresponds to a displacement of the n -partite Gaussian input state in the phase space by α and is denoted by ρ_α^{in} . The Wigner function of ρ_α^{in} reads

$$W_\alpha^{\text{in}}(R) = \frac{\exp[-\frac{1}{2}(R - \sqrt{2}\alpha)^\dagger \gamma_{\text{in}}^{-1} (R - \sqrt{2}\alpha)]}{(2\pi)^n \sqrt{\det(\gamma_{\text{in}})}}. \quad (7)$$

Throughout this work, we do not question the conjecture that a coherent state minimizes the entropy of a mono-modal Gaussian thermal channel and use the result that squeezed, pure states are optimal in the case of a mono-modal Gaussian channel with anisotropic noise.¹⁷ Therefore, we only consider Gaussian distributions of the letters in the messages so that the overall modulated input state sent through the channel is a Gaussian mixture $\rho^{\text{in}} = \int d^{2n}\alpha f(\alpha)\rho_\alpha^{\text{in}}$, where $d^{2n}\alpha = d\Re\{\alpha_1\}d\Im\{\alpha_1\}\dots d\Re\{\alpha_n\}d\Im\{\alpha_n\}$ with (classical) Gaussian distribution $f(\alpha)$ with zero mean and covariance matrix γ_{mod} .

As we are no longer dealing with probability distributions p_i but with probability densities $f(\alpha)$, the summations in the formulae above are replaced by proper integrations. The action of $T^{(n)}$ on an input state carrying a message α reads as in¹²

$$T^{(n)}[\rho_\alpha^{\text{in}}] = \rho_\alpha^{\text{out}} = \int d^{2n}\beta f_{\text{env}}(\beta) D(\beta_n) \otimes \dots \otimes D(\beta_1) \rho_\alpha^{\text{in}} D^\dagger(\beta_1) \otimes \dots \otimes D^\dagger(\beta_n), \quad (8)$$

with $d^{2n}\beta = d\Re\{\beta_1\}d\Im\{\beta_1\}\dots d\Re\{\beta_n\}d\Im\{\beta_n\}$, $\beta = (\Re\{\beta_1\}, \dots, \Re\{\beta_n\}, \Im\{\beta_1\}, \dots, \Im\{\beta_n\})^\top$ and the displacement operator $D(\beta_j) = e^{\beta_j\hat{a}_j^\dagger - \beta_j^*\hat{a}_j}$. The displacement is applied according to the (classical) Gaussian distribution of the environment $f_{\text{env}}(\beta)$ with zero mean and covariance matrix γ_{env} . If this matrix is not diagonal, then the environment introduces correlations between the successive uses of the channel. These correlations model the memory of the channel.

Since we centered the distributions of the environment and modulation as well as the Wigner function of the input state around zero, the covariance matrices of the state carrying the message α at the output of the channel ρ_α^{out} and of the overall modulated output state $\bar{\rho}$, read, respectively,

$$\begin{aligned}\gamma_{\text{out}} &= \gamma_{\text{in}} + \gamma_{\text{env}} \\ \bar{\gamma} &= \gamma_{\text{out}} + \gamma_{\text{mod}}.\end{aligned}\tag{9}$$

The one-shot capacity of such a system is

$$C_1(T^{(n)}) = \sup_{\gamma_{\text{in}}, \gamma_{\text{mod}}} \chi_n, \quad \chi_n = S(\bar{\rho}) - S(\rho_\alpha^{\text{out}}).\tag{10}$$

In the case of a Gaussian state ρ , the von Neumann entropy can be expressed in terms of the symplectic eigenvalues ν_j of its covariance matrix:

$$\begin{aligned}S(\rho) &= \sum_j g\left(|\nu_j| - \frac{1}{2}\right) \\ g(x) &= \begin{cases} (x+1)\log(x+1) - x\log x & , x > 0 \\ 0 & , x = 0. \end{cases}\end{aligned}\tag{11}$$

In the following we analyze the case when all covariance matrixes (9) are diagonalized in the same basis. For one mode channel the one-shot capacity is achieved in such case and we conjecture that this will hold for n modes. Care should be taken to the diagonalization which has to be a symplectic transformation.¹⁷ In the absence of $q-p$ correlations the commutativity of transformation diagonalizing q and p blocks is sufficient for the diagonalization transformation to be symplectic. Therefore, our consideration is restricted to the channels for which the q and p blocks of the noise covariance matrix are diagonalized by the transformations which mutually commute. Then the symplectic eigenvalues in this case are functions of the eigenvalues of the corresponding covariance matrices:

$$\begin{aligned}\bar{\nu}_i &= \sqrt{\overline{\gamma_i^q \gamma_i^p}}, \quad \overline{\gamma_i^{q,p}} = \gamma_{\text{in},i}^{q,p} + \gamma_{\text{mod},i}^{q,p} + \gamma_{\text{env},i}^{q,p}, \\ \nu_{\text{out}} &= \sqrt{\overline{\gamma_{\text{out},i}^q \gamma_{\text{out},i}^p}}, \quad \gamma_{\text{out},i}^{q,p} = \gamma_{\text{in},i}^{q,p} + \gamma_{\text{env},i}^{q,p}.\end{aligned}\tag{12}$$

The energy (or mean photon number) constraint (2) can now be written as the sum of all *input* and *modulation* eigenvalues

$$\lambda \equiv \sum_{i=1}^n \left(\gamma_{\text{in},i}^q + \gamma_{\text{in},i}^p + \gamma_{\text{mod},i}^q + \gamma_{\text{mod},i}^p \right) = n(2\bar{n} + 1)\tag{13}$$

The total energy $\bar{\lambda}$ of the n -mode channel is the sum of the input energy λ and the energy of the noise λ_{env} (environment)

$$\bar{\lambda} = \lambda + \lambda_{\text{env}}, \quad \lambda_{\text{env}} = \sum_i (\gamma_{\text{env}}^q + \gamma_{\text{env}}^p).\tag{14}$$

The ‘‘one-shot’’ capacity of n parallel classical Gaussian channels is found by the so called *water-filling* solution, which imposes the distribution of the input energy between the channels in such a way that the total energy of all channels, which is the sum of the input energy and of the added noise, is equal for all channels (the channels, which have the noise variance higher than this *water-filling* level, do not participate in the transmission).¹⁸ This is a consequence of the fundamental fact that the Shannon entropy is maximal for uniform

distribution. In the classical case this directly translates into the equality of the total channel energies. It was shown that in the quantum case the situation is similar.¹⁷ However, due to quantum nature of signals carrying the information the quadratures cannot be considered as independent channels. As a result it appears that under certain conditions one quadrature may be not involved in the information transmission but still consumes some input energy necessary to create the quantum state required by the optimal solution. In the present paper we extend the results obtained for Gaussian quantum channels with input energy satisfying the global *water-filling* condition to input energies below the threshold and fully characterize the optimal solution for these channels for all range of input energies for a finite number of modes.

3. OPTIMIZATION PROBLEM

Here we discuss n parallel independent one-mode channels. We will determine the maximum of the Holevo quantity by the method of Lagrange multipliers under the condition of finite (fixed) input energy (13) and pure input states implying

$$\gamma_{\text{in},i}^q \gamma_{\text{in},i}^p = \frac{1}{4}. \quad (15)$$

Taking into account the constraints (13) and (15) we construct the Lagrange function in the form

$$\begin{aligned} L = & \sum_{i=1}^n \left(g \left(\bar{\nu} - \frac{1}{2} \right) - g \left(\nu_{\text{out}} - \frac{1}{2} \right) \right) - \sum_{i=1}^n \sigma_i \left(\gamma_{\text{in},i}^q \gamma_{\text{in},i}^p - \frac{1}{4} \right) \\ & - \mu \left(\sum_{i=1}^n \left(\gamma_{\text{in},i}^q + \gamma_{\text{in},i}^p + \gamma_{\text{mod},i}^q + \gamma_{\text{mod},i}^p \right) - \lambda \right) \end{aligned}$$

with $4n$ independent eigenvalues $\gamma_{\text{in},i}^{q,p}$ and $\gamma_{\text{mod},i}^{q,p}$, n multipliers σ_i , and one multiplier μ .

For the concave Lagrangian function the extremum is unique and is determined by the point in the space of the $4n$ variables (the *input* and the *modulation* eigenvalues) where the gradient of the Lagrangian function vanishes:

$$\left(\nabla_{\gamma_{\text{in},i}^q}, \nabla_{\gamma_{\text{in},i}^p}, \nabla_{\gamma_{\text{mod},i}^q}, \nabla_{\gamma_{\text{mod},i}^p} \right)^T L = 0 \quad (16)$$

where $\nabla_{\gamma_{\text{in},i}^{q,p}}$ and $\nabla_{\gamma_{\text{mod},i}^{q,p}}$ are derivatives with respect to $\gamma_{\text{in},i}^{q,p}, \gamma_{\text{mod},i}^{q,p}$ with $i = 1, \dots, n$. By developing Eq. (16) we obtain a system of $4n$ equations

$$\frac{g'(\bar{\nu}_i - \frac{1}{2})}{2\bar{\nu}_i} \frac{1}{\gamma_i^{p,q}} - \frac{g'(\nu_{\text{out},i} - \frac{1}{2})}{2\nu_{\text{out},i}} \gamma_{\text{out},i}^{p,q} - \mu - \sigma_i \gamma_{\text{in},i}^{p,q} = 0 \quad (17)$$

$$\frac{g'(\bar{\nu}_i - \frac{1}{2})}{2\bar{\nu}_i} \frac{1}{\gamma_i^{p,q}} - \mu = 0 \quad (18)$$

where we used a convention that indexes p, q imply two equations such that index p is applied in the the first equation and index q is applied in the second one. The derivative of $g(x)$ with respect to its argument is denoted as $g'(x)$. In order to solve our system of equations we insert Eqs. (18) for q quadratures into Eqs. (17) for both q and p quadratures with corresponding i . Then $2n$ equations (17) become

$$\frac{g'(\nu_{\text{out},i} - \frac{1}{2})}{2\nu_{\text{out},i}} \gamma_{\text{out},i}^q + \sigma_i \gamma_{\text{in},i}^q = 0 \quad (19)$$

$$\frac{g'(\bar{\nu}_i - \frac{1}{2})}{2\bar{\nu}_i} \left(\overline{\gamma_i^p} - \overline{\gamma_i^q} \right) - \frac{g'(\nu_{\text{out},i} - \frac{1}{2})}{2\nu_{\text{out},i}} \gamma_{\text{out},i}^p - \sigma_i \gamma_{\text{in},i}^p = 0. \quad (20)$$

We further express σ_i from Eq. (19) and insert it into Eq. (20) and arrive at n equations

$$\frac{g'(\bar{\nu}_i - \frac{1}{2})}{2\bar{\nu}_i} \left(\overline{\gamma_i^p} - \overline{\gamma_i^q} \right) - \frac{g'(\nu_{\text{out},i} - \frac{1}{2})}{2\nu_{\text{out},i}} \left(\gamma_{\text{out},i}^p - \frac{\gamma_{\text{in},i}^p}{\gamma_{\text{in},i}^q} \gamma_{\text{out},i}^q \right) = 0. \quad (21)$$

We will not further consider Eq. (19) because it does not contribute to the evaluation of other unknown variables. We will just remember that once all other unknown variables are determined, Eq. (19) may serve for determining σ_i if necessary. If we add the n purity conditions (15) of the “letter” states and one on the input energy constraint (13) to the system of $3n$ Eqs. (18,21), we get in total $4n + 1$ equations which determine $4n + 1$ unknown variables including $4n$ input and modulation eigenvalues, one Lagrange multiplier μ , which links different modes.

As we will see below for some parameters of the problem the solution may lead to negative modulations eigenvalues of some modes. This means that the extremum of the Lagrangian lays outside of the valid domain of positive eigenvalues. In such cases the maximum which we are looking for lays at the border of the valid domain. This border is a collection of hyperplanes with at least one vanishing modulation eigenvalue. Therefore, in the solution different modes in general may contain different number of vanishing and non-vanishing modulation eigenvalues. For each mode there exist only three possible cases:

1. Both (q and p) modulation eigenvalues vanish:

$$\gamma_{\text{mod},i}^q = \gamma_{\text{mod},i}^p = 0.$$

2. Only one modulation eigenvalue vanishes:

$$\gamma_{\text{mod},i}^q = 0, \quad \gamma_{\text{mod},i}^p > 0.$$

We will always use this convention (and not *vice versa*) without loss of generality because for diagonal covariance matrices relabeling q and p of any mode does neither change the input energy nor output entropies.

3. Both modulation eigenvalues are non-vanishing:

$$\gamma_{\text{mod},i}^q > 0, \quad \gamma_{\text{mod},i}^p > 0.$$

Then, the solution of our system of equations corresponds to a particular distribution of n modes into three different sets corresponding to one of three cases realized for the modes. We denote the sets corresponding to three cases by: \mathcal{N}_1 , \mathcal{N}_2 , and \mathcal{N}_3 . We will use the same notations for three sets of integers defined such that if a mode with index i belongs to one of the sets defined above then its index i belongs to the set of integers labelled by the same symbol. Then we will call by *set* both, the sets of modes and the sets of indexes. We denote the number of modes in the sets as n_1 , n_2 , and n_3 respectively. Let us consider in the following the sets one by one.

3.1 Set \mathcal{N}_3 : *water-filling* solution

We start our consideration with the set \mathcal{N}_3 . First of all we note that due to Eq. (18) the Lagrange multiplier μ makes a strong link between the quadratures of all modes: From this equation it is straightforward to conclude that

$$\overline{\gamma_i^{q,p}} = \overline{\gamma_j^{q,p}}, \quad \forall i, j \in \mathcal{N}_3. \quad (22)$$

This means that at the output of the channel each mode is in the same thermal state. The energy level that is common for all quadratures reads

$$\bar{\nu}_{\text{wf}} \equiv \overline{\gamma_i^{q,p}} = \frac{\bar{\lambda}_3}{2n_3}, \quad \forall i \in \mathcal{N}_3, \quad (23)$$

and may be called the *water-filling* level as if the *input* and *modulation* energies “fill” the “vessel” formed by the noise eigenvalues up to this level.^{18,21} The total energy $\overline{\gamma_i^{q,p}}$ of the modes from the set \mathcal{N}_3 is given by $\bar{\nu}_{\text{wf}}$ multiplied by the number of quadratures $2n_3$ belonging to \mathcal{N}_3 . Due to Eq. (18), the *water-filling* level x is simply related to the Lagrange multiplier μ by

$$g' \left(x - \frac{1}{2} \right) = 2\mu \quad \Rightarrow \quad \bar{\nu}_{\text{wf}} = \frac{1}{2} \coth(\mu \log(2)). \quad (24)$$

We find the *input* eigenvalues by inserting Eq. (22) into Eq. (21), which leads to

$$\frac{\gamma_{\text{env},i}^q}{\gamma_{\text{env},i}^p} = \frac{\gamma_{\text{in},i}^q}{\gamma_{\text{in},i}^p}, \quad (25)$$

and with the help of the purity condition (15) finally obtain

$$\gamma_{\text{in},i}^{q,p} = \frac{1}{2} \sqrt{\frac{\gamma_{\text{env},i}^q}{\gamma_{\text{env},i}^p}} \quad (26)$$

$$\gamma_{\text{mod},i}^{q,p} = \bar{\nu}_{\text{wf}} - \gamma_{\text{env},i}^{q,p} - \frac{1}{2} \sqrt{\frac{\gamma_{\text{env},i}^{q,p}}{\gamma_{\text{env},i}^p}}. \quad (27)$$

Observing that this solution always leads to positive $\gamma_{\text{in},i}^{q,p}$ we have to make sure that the obtained $\gamma_{\text{mod},i}^{q,p}$ are also non-negative. A simple reasoning based on the definition of the *water-filling* level (23) shows that non-negativity of the modulation eigenvalues $\gamma_{\text{mod},i}^{q,p}$ (27) implies a lower bound on the input energy allocated to the mode, i.e.

$$\lambda_i \geq \lambda_{\text{thr},i} = \gamma_{\text{env},i}^q - \gamma_{\text{env},i}^p + \sqrt{\frac{\gamma_{\text{env},i}^q}{\gamma_{\text{env},i}^p}}, \quad (28)$$

were we assume without loss of generality that $\gamma_{\text{env},i}^q > \gamma_{\text{env},i}^p$. Therefore the input energy allocated to each mode belonging to \mathcal{N}_3 must exceed its corresponding threshold value $\lambda_{\text{thr},i}$.

If the input energy satisfies condition (28) for all quadratures of all modes then the whole n -mode channel belongs to set \mathcal{N}_3 and we have

$$\bar{\lambda}_i = \bar{\lambda}_3, \quad \forall i \in \mathcal{N}_3. \quad (29)$$

In this case, Eqs. (23), (26), and (27) determine the *global water-filling* solution.

If the condition (28) is not satisfied at least for one quadrature of some mode then this *global water-filling* solution has no physical meaning because in this case the corresponding optimal modulation eigenvalue is negative. Then the global maximum is placed on the boundary of the domain. It means that we have to set the corresponding modulation eigenvalues to zero. As we have already discussed we attribute a mode to the set \mathcal{N}_2 or \mathcal{N}_1 depending on whether one or both modulation eigenvalues of the mode vanish.

3.2 Set \mathcal{N}_2 : single quadrature-modulated modes

In the case 2 only one quadrature has a vanishing modulation eigenvalue

$$\gamma_{\text{mod},i}^q = 0. \quad (30)$$

However, this already changes our system of equations because the assignment (30) eliminates the corresponding partial derivative from Eq. 16. As a result Eq. (18) does not exist for the p quadrature of mode i and is effectively replaced by Eq. (30). Then the corresponding quadrature becomes “detached” from all others that are connected through the common multiplier μ .

The system of equations for this set includes n equations (18) with $\overline{\gamma_i^q}$, and n equations (21), n purity conditions and one energy constraint which determine $2n$ input eigenvalues $\gamma_{\text{in},i}^{q,p}$, n modulation eigenvalues $\gamma_{\text{mod},i}^p$ and one Lagrange multiplier μ .

3.3 Set \mathcal{N}_1 : modes excluded from information transmission

Finally, if its modulation eigenvalues for both quadratures vanish (30) the mode does not contribute at all to the Holevo quantity and consequently to the information transmission. The vanishing modulation eigenvalues $\gamma_{\text{mod},i}^{q,p} = 0$ imply

$$\overline{\gamma_i^{q,p}} = \gamma_{\text{out},i}^{q,p}. \quad (31)$$

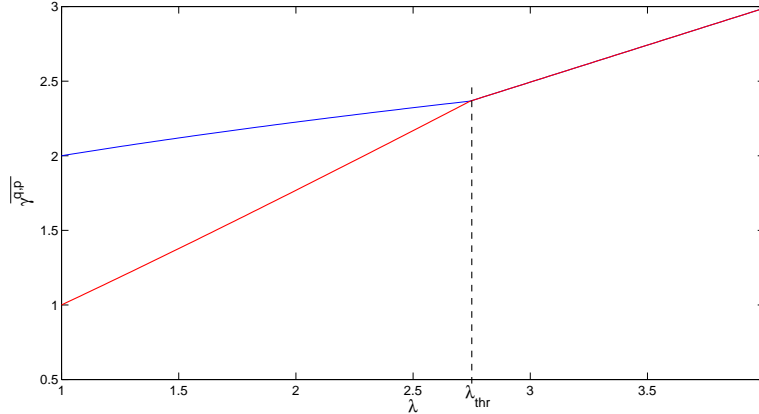


Figure 1. Total energies of quadratures $\overline{\gamma^q}$ (above) and $\overline{\gamma^p}$ (below) for the one-mode channel as functions of the input energy λ . The two eigenvalues coincide from $\lambda_{\text{thr}} = 2\overline{n}_{\text{thr}} + 1 = 2.73$, with noise eigenvalues $\gamma_{\text{env}}^q = 1.5$, $\gamma_{\text{env}}^p = 0.5$.

Therefore in order to satisfy Eq. (21) the input eigenvalues of both quadratures must be also equal: $\gamma_{\text{in},i}^q = \gamma_{\text{in},i}^p$. Due to (26) this implies that the optimal input eigenvalues correspond to the vacuum or non-squeezed (and non-displaced) coherent state

$$\gamma_{\text{in},i}^{q,p} = \frac{1}{2}. \quad (32)$$

4. ANALYSIS OF THE OPTIMAL SOLUTION FOR ONE MODE

If no input energy is devoted to the mode, the solution is trivial: the mode belongs to the set \mathcal{N}_1 and does not contribute to the capacity. Given a positive input energy it is easy to verify the existence of the *local water-filling* solution for the mode by verifying the threshold condition (28). If the input energy is above the threshold, the solution is also simple, i.e. given by Eqs. (26) and (27), and the mode belongs to \mathcal{N}_3 .

If the input energy is below the threshold equation one has to solve Eq. (21). Although there is no explicit solution of this transcendental equation, it provides an implicit function, which uniquely determines the optimal value of $\gamma_{\text{in},i}^q$ for given input energy λ_i and as a consequence of all other eigenvalues. This corresponds to a mode in \mathcal{N}_2 . By solving numerically Eq. (21) below the threshold λ_{thr} (28) and using Eq. (23) above the threshold we plot in Fig. 1 the total energy of q - and p -quadratures of one mode as a function of the input energy λ for one mode with noise eigenvalues $\gamma_{\text{env}}^q > \gamma_{\text{env}}^p$.

We see that for $\lambda = 1$ the mode belongs to set \mathcal{N}_1 and the total energies of the quadratures are sums of the noise eigenvalues with the vacuum energy of the quadratures. With growing λ the mode passes to the set \mathcal{N}_2 and the solution of Eq. (21) leads to an almost linear increase of $\gamma_{\text{env}}^q = 2$, $\gamma_{\text{env}}^p = 1$, suggesting that some simple and at the same time rather good approximations of the exact solution may exist. When the input energy exceeds the threshold λ_{thr} the mode belongs to the *water-filling* set \mathcal{N}_3 where $\overline{\gamma^q}$ and $\overline{\gamma^p}$ become equal.

In agreement with this picture, the analysis of Eq. (21) shows that the sets, \mathcal{N}_1 ($\lambda = 1$) and \mathcal{N}_3 ($\lambda \geq \lambda_{\text{thr}}$) correspond to limiting cases of this equation. Indeed, for the set \mathcal{N}_3 , the input energy is enough for the *water-filling* solution to hold. Therefore the left hand side of Eq. (21) vanishes due to the equality of the total energies of q - and p -quadratures, $\overline{\gamma_i^p} = \overline{\gamma_i^q}$. Simple algebra shows that the input eigenvalues given by Eq. (26) let also the second term of Eq. (21) vanish. Note that the transition point $\lambda = \lambda_{\text{thr}}$ is also described by this solution. In the limit $\lambda \rightarrow 1$ we easily recover the set \mathcal{N}_1 where the input energy is not used for modulation in both q - and p -quadratures and the total energy of each quadrature is equal to its corresponding output energies, Eq. (31).

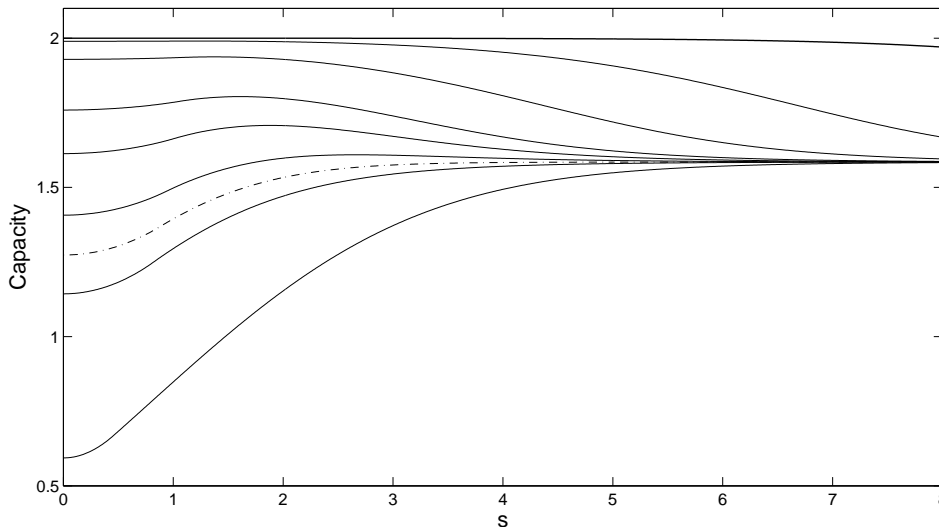


Figure 2. Capacity of the one mode channel vs. s . The dashed curve shows the capacity with $N_{\text{env}} = \frac{1}{2\sqrt{3}}$. The other values for N_{env} (solid curves) read from top to bottom $\{10^{-5}, 10^{-3}, 0.01, 0.05, 0.1, 0.2, 0.4, 0.5\}$. The input energy is $\lambda = 3$.

5. OPTIMAL ENVIRONMENT FOR ONE MODE

Here we consider the problem of optimization of the Holevo quantity not only with respect to the input energy distribution but also with respect to the noise variances.¹⁶ We parametrize the noise eigenvalues as

$$\gamma_{\text{env}}^{q,p} = N_{\text{env}} e^{\pm s} \quad (33)$$

where $N_{\text{env}} = \sqrt{\gamma_{\text{env}}^q \gamma_{\text{env}}^p}$ is the symplectic eigenvalue (or *temperature*) of the noise and s is a parameter characterizing the squeezing of the noise. In this parametrization we will look for the optimal value of s for fixed N_{env} .

First of all, numerically solving Eq. (21) we plot in Fig. 2 the capacity C as a function of s for various N_{env} and fixed input energy λ . Each point of this graph corresponds to the optimal choice of the input and modulation eigenvalues. We emphasize the following properties:

1. All curves have the same asymptotics for large s .
2. There exist two types of curves: for small N_{env} (upper curves) one finds a maximum of capacity at a finite s which goes to larger s with increasing N_{env} ; for N_{env} greater than certain value the curves become monotonous such that C increases up to the maximum value attained asymptotically at infinite s .
3. At small s all curves have a positive curvature and start to grow with increasing s and then in each curve there exists a discontinuous change of curvature from positive to negative.

The last observation is explained by the fact that at $s = 0$ both noise quadratures are equivalent and the input energy is equally distributed among them. It is easy to see from Eq. (28) that in this case the threshold value for the input energy corresponds to the vacuum and for any $\lambda > 1$ the channel belongs the set \mathcal{N}_2 . With increasing s the threshold value λ_{thr} starts to grow, however, by continuity in a certain interval of s the condition (28) holds. When λ_{thr} exceeds our chosen λ , which is fixed, the channel passes to the set \mathcal{N}_3 . At the transition point the curvature changes its sign. The apparent discontinuity of the first derivative of C is in agreement with the discontinuity of the first derivative of $\gamma_i^{q,p}$ in Fig. 1 at the transition point.

In order to analyze further this behavior and to find the position of the maximum of C we have to modify our optimization problem (16). We include in the gradient ∇ the partial derivative with respect to s (as we consider only one mode we drop the index i). As a result we obtain one additional equation.

$$\frac{g'(\bar{\nu}_i - \frac{1}{2})}{2\bar{\nu}_i} (e^s \overline{\gamma}_i^p - e^{-s} \overline{\gamma}_i^q) - \frac{g'(\nu_{\text{out},i} - \frac{1}{2})}{2\nu_{\text{out},i}} (e^s \gamma_{\text{out},i}^p - e^{-s} \gamma_{\text{out},i}^q) = 0. \quad (34)$$

In fact if Eq. (34) is satisfied then the partial derivative of the capacity with respect to s is zero: $\partial C / \partial s = 0$. This corresponds to both an extremum (maximum) of C as a function of s and the horizontal asymptotics of C at large s . Together, Eq. (21) and Eq. (34) form a system of equation determining both the optimal input energy distribution and optimal squeezing s^* which achieve the maximal capacity. Although this system does not have explicit solution we were able to find an exact expression for the optimal input energy distribution at the extremal point s^*

$$\gamma_{\text{in}}^q = \frac{1}{2}\lambda, \quad \gamma_{\text{in}}^p + \gamma_{\text{mod}}^p = \frac{1}{2}\lambda. \quad (35)$$

Being inserted into Eq. (21) this condition results in a compact equation

$$\frac{g'(\bar{\nu}_i - \frac{1}{2})}{2\bar{\nu}_i} e^z \sinh(s) = \frac{g'(\nu_{\text{out},i} - \frac{1}{2})}{2\nu_{\text{out},i}} \sinh(s - r). \quad (36)$$

whith the new notation $\gamma_{\text{in}}^{q,p} = \frac{1}{2}e^{\pm r}$

In order to find out when the finite maximum s^* does exist we analyze Eq. (36) using the following expansion

$$g' \left(x - \frac{1}{2} \right) = \frac{1}{x \log 2} \sum_{k=0}^{\infty} \frac{(2x)^{-2k}}{2k+1}. \quad (37)$$

The first two terms of the expansion (37) used in Eq. (34) result in the equation, which gives a unique value of N_{env} in the limit of large s where all positive powers of e^{-s} can be neglected:

$$N_{\text{env}}^{\text{lim}} = \frac{1}{2\sqrt{3}}. \quad (38)$$

At this value the maximum of capacity C as function of s tends to the infinite squeezing of the noise. One can see at Fig. 2 that this tendency takes place only for $N_{\text{env}} \leq \frac{1}{2\sqrt{3}}$ because for higher values of the thermal contribution to the noise there is no maximum in s and the capacity monotonously approaches the asymptotic value from below. Therefore, we conclude that the value given by Eq. (38) is a critical value which separates two regimes: one with monotonous increase of the capacity with s and another with a maximum of the capacity at a finite s . This can be further observed in Fig. 3, where the optimal squeezing of the environment s^* is plotted vs. N_{env} . One confirms that the squeezing is highly diverging when $N_{\text{env}} \rightarrow N_{\text{env}}^{\text{lim}}$.

Although Eq. (38) was obtained from first two terms of the expansion (37) we have shown that all other terms do not contribute to the terms of zero order in e^{-s} , which results in Eq. (38) in the limit of infinite s .

6. CONCLUSION

We studied the one-shot classical capacity of quantum optical Gaussian channels with additive phase-dependent noise. For the case of a single mode, we described in detail the optimal energy distribution among the quadratures as a function of the noise spectrum and of the amount of input energy allocated to the mode. We have shown that in agreement with the previous results¹⁵⁻¹⁷ there is an input energy threshold above which the optimal energy distribution corresponds to a thermal channel with equal total energy in both quadratures. This corresponds to the *water-filling* solution known for classical Gaussian channels. In the quantum case, this implies a pure input state with the squeezing parameter equal to the squeezing parameter of the noise. We have analyzed the solution below the threshold, where only one quadrature (the one with lower value of the noise energy) is modulated. We observed that with increasing input energy, the optimal quantum state evolves from a coherent state to a

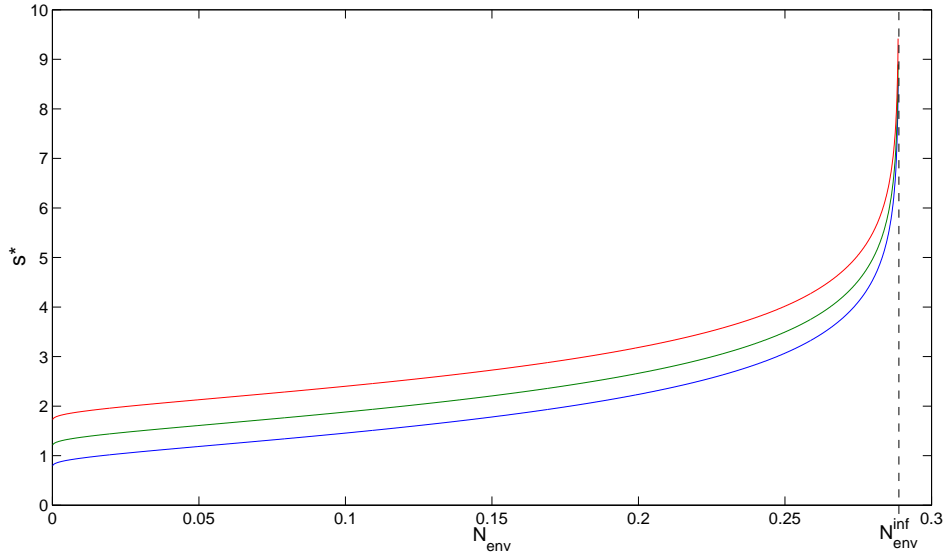


Figure 3. Optimal squeezing s^* vs. “temperature” of the environment N_{env} . From top to bottom the input energies were fixed to $\lambda = \{5, 3, 2\}$.

squeezed state with a squeezing parameter growing from one at vacuum input to the squeezing parameter of the noise at the threshold value of input energy. Curiously, the growth of input energy eigenvalues of both quadratures is almost linear although the solution is given by a transcendental equation, which does not possess any explicit solution.

We studied also in detail how the one-shot classical capacity of one mode depends on the squeezing parameter of the noise for different noise “temperatures” given by the symplectic eigenvalue of the noise. Our result shows that there exist a threshold value which splits the phase-dependent noise models in two types. In the modes with the symplectic eigenvalue of the noise exceeding the threshold value, the capacity is increasing monotonously with the noise squeezing and achieves an asymptotic value for infinite squeezing. This asymptotic value is universal as it does not depend on the symplectic eigenvalue of the noise. In the modes with the symplectic eigenvalue of the noise lower than the threshold value, the capacity attains a maximum at a finite squeezing of the noise and the same universal asymptotic value of the capacity at infinite noise squeezing holds. We have exactly determined the threshold value for the symplectic eigenvalue (or “temperature”) of the noise and the relation between the given input energy and the optimal squeezing of the quantum input state.

We expect our results to be useful for the study of multimode phase-dependent channels and ultimately of the capacity of such channels as well as for the study of the optimal environment (noise model) which provides the maximal capacity for a given distribution of the noise “temperature”. This may be useful for the development of optical connections of future “all-optical” information processing systems.

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