

INFORMATION AND CODING THEORY

Exercise sheet n° 1 :

1-1. A discrete random variable X has a uniform probability distribution $p(x)$ over the set \mathcal{X} of cardinality $|\mathcal{X}| = m$.

- (a) Give an example for \mathcal{X} and calculate the probability distribution $p(x)$ where $x \in \mathcal{X}$.
- (b) Determine the entropy $H(X)$. If $m = 64$, what is $H(X)$?
- (c) How many bits are needed to enumerate the alphabet \mathcal{X} without encoding, where $|\mathcal{X}| = 64$?
- (d) How many symbols taken from a quaternary alphabet are necessary to enumerate the alphabet \mathcal{X} without encoding, where $|\mathcal{X}| = 64$?
- (e) Show that $H(X)$ is greater than the entropy of any other random variable Y which has values on the same set \mathcal{X} ? (*Read the recall on Lagrange multipliers.*)

1-2. A random variable X has the alphabet $\mathcal{X} = \{A, B, C, D\}$.

- (a) Calculate $H(X_P)$, associated to probabilities $P(X) = \{1/2, 1/4, 1/8, 1/8\}$.
- (b) Calculate $H(X_Q)$, associated to probabilities $Q(X) = \{1/4, 1/4, 1/4, 1/4\}$.
- (c) We define a binary code :
 $\{C(X = A)=0, C(X = B)=10, C(X = C)=110, C(X = D)=111\}$.
Calculate the expected length (in bits) of the codewords if X is distributed according to $P(X)$ and $Q(X)$
- (d) Compare the four obtained results.

1-3. A coin (with same probability for “head” or “tail”) is flipped repeatedly until one obtains “head”. Let the random variable X be given by the number of flips which are necessary to obtain for the first time “head”.

- (a) Calculate $H(X)$.
- (b) One repeatedly flips the coin until “head” is obtained for the first time. Find a sequence of questions of type “yes/no” to determine the value of X . Compare the entropy of X with the expected length of the sequence of questions necessary to fully determine X .

Note :

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}, \quad \sum_{n=1}^{\infty} nr^n = \frac{r}{(1-r)^2}, \quad r < 1.$$

1-4. Assume X and Y to be random variables. What is the inequality relating $H(X)$ and $H(Y)$ if,

- (a) $Y = 2^X$, with $\mathcal{X} = \{0, 1, 2, 3, 4\}$ associated to probabilities $\{1/2, 1/4, 1/8, 1/16, 1/16\}$.
- (b) $Y = \cos(X)$, with $\mathcal{X} = \{0, \pi/2, \pi, 3\pi/2, 2\pi\}$ associated to probabilities $\{1/3, 1/3, 1/12, 1/12, 1/6\}$.
- (c) Show that the entropy of any function $f(X)$ of the random variable X is less or equal than the entropy of X . In order to prove this, it is useful to calculate the joint entropy $H(X, f(X))$. In which case the inequality $H(f(X)) \leq H(X)$ is saturated?

1-5. Let X and Y be two random variables, taking values x_1, x_2, \dots, x_r and y_1, y_2, \dots, y_s . Assume a random variable $Z = X + Y$.

- (a) Show that $H(Z|Y) = H(X|Y)$ and $H(Z|X) = H(Y|X)$. Deduce that if X and Y are independently distributed, then $H(X) \leq H(Z)$ and $H(Y) \leq H(Z)$. Thus, summing two random variables can only increase the uncertainty.
- (b) Give an example for the two random variables X and Y (necessarily correlated) such that $H(X) > H(Z)$ and $H(Y) > H(Z)$.
- (c) In which case the equality $H(Z) = H(X) + H(Y)$ is satisfied?

Optional : Give an example of a distribution of two random variables X and Y whose correlation coefficient is zero, although they are not independent. Show that in this case $H(X:Y) \neq 0$, showing that the mutual entropy is a better measure of the dependence of X and Y . Correlation

$$\text{coefficient : } r = \frac{\langle xy \rangle - \langle x \rangle \langle y \rangle}{\sqrt{\langle x^2 \rangle - \langle x \rangle^2} \sqrt{\langle y^2 \rangle - \langle y \rangle^2}}$$

Recall :

1. Change of the base of logarithms

$$\log_a N = \log_b N / \log_b a$$

2. Lagrange theorem :

– We want to maximize

$$u(x_1, x_2, \dots, x_n) \text{ with } k \text{ constraints } g_j(x_1, x_2, \dots, x_n) = a_j, j = 1, \dots, k.$$

– We introduce the Lagrangian :

$$L(x_1, \dots, x_n, \lambda_1, \dots, \lambda_k) = u(x_1, x_2, \dots, x_n) + \sum_{j=1}^k \lambda_j [a_j - g_j(x_1, x_2, \dots, x_n)].$$

– Condition on an extremum :

$$\forall i, \frac{\partial L}{\partial x_i} = 0.$$

– If u is concave, the solution is a maximum.

Website :

<http://quic.ulb.ac.be/teaching/>