AN INTUITIVE APPROACH FOR THE SIMULATION OF QUANTUM CORRELATIONS

Julien Degorre[*†] and Jérémie Roland[*]

We study the problem of simulating quantum correlations with the help of shared randomness, supplemented with additional resources. More specifically, we focus on the case of local projective measurements on a qubit pair in a maximally entangled state for which different protocols using different resources have been proposed. We introduce a generic method to derive these protocols, which clarifies the link between them and sheds new light on the problem.

INTRODUCTION

The main goal of quantum information theory is to quantify the power of quantum resources. One of the basic resources useful in quantum information processes is entanglement, which was first presented as a paradox in the *gedanken* experiment of Einstein, Podolsky and Rosen [7]. In this experiment, two distant parties, Alice and Bob, share a quantum entangled system and perform an arbitrary quantum measurement on their part of the system. The special case of projective measurements on a qubit pair in the singlet state will be the focus of our work. Bell showed that the correlations exhibited by such a quantum system could not be reproduced by any model based on local variables only, even with the help of an unlimited amount of shared randomness. This establishes the non-locality of quantum mechanics. Nonetheless, Alice and Bob could simulate the quantum correlations if, in addition to shared random variables, they are given other resources. In order to gauge non-locality, it is interesting to identify and quantify these resources. Several protocols have previously been proposed to simulate these correlations, using different additional resources: classical communication between Alice and Bob [9, 2, 3, 6, 12], post-selection [8], and, finally, a resource called a non-local box [4].

The purpose of this paper is to show that three of the main protocols can all be derived from a basic local protocol, which in turn arises naturally from a protocol which requires an infinite amount of communication [11]. Our method gives a coherent

* Laboratoire de Recherche en Informatique, Universit´e Paris-Sud, France

† Laboratoire d'Informatique Th´eorique et Quantique, D´epartement d'Informatique et de Recherche Op´erationnelle, Universit´e de Montr´eal, Canada

view of these different protocols and thus a better understanding of the relationship between them.

QUANTUM CORRELATIONS

We consider the following EPR experiment (see Fig. 1): Alice and Bob share a qubit pair in the singlet state $|\psi\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$; that is a maximally entangled state of two qubits. Alice and Bob then each receive the classical description of a projective measurement they have to perform on their respective qubit. These can be represented by unit vectors $\vec{a}$ and $\vec{b}$ pointing in some direction on the Bloch sphere. The result of Alice's and Bob's measurements, $A \in \{1, -1\}$ and $B \in \{1, -1\}$, are then distributed according to the following probabilities

$$p(A, B) = \frac{1 - AB\vec{a} \cdot \vec{b}}{4}, \tag{1}$$

so that their joint expectation value $\langle AB \rangle$ is given by

$$\langle AB \rangle = -\vec{a} \cdot \vec{b}. \tag{2}$$

If the marginal expectation values $\langle A \rangle$ and $\langle B \rangle$ are 0, which will always be the case in this manuscript, Eq. (1) is equivalent to Eq. (2).

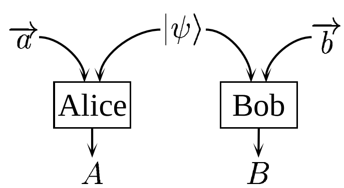

Figure 1: EPR experiment. Alice and Bob share a pair of qubits in the singlet state $|\psi\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$. Both perform a measurement on their qubit, specified by vectors $\vec{a}$ or $\vec{b}$, and obtain results $A = \pm 1$ or $B = \pm 1$.

SIMULATION OF QUANTUM CORRELATIONS

The problem is the following: Alice and Bob do not share an entangled state, but want nevertheless to simulate the above experiment, that is, on inputs $\vec{a}$ and $\vec{b}$, Alice must output a value $A \in \{1, -1\}$ and Bob a value $B \in \{1, -1\}$ distributed according to (1). It is well known that for appropriate measurement choices, the correlations may violate the CHSH Bell inequality [5] and therefore are not reproducible by a local classical model, even with an unlimited source of shared randomness. In the following sections, we show how to solve this problem with additional resources.

Infinite amount of communication

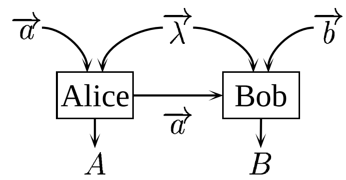

Figure 2: Schatten's protocol. Alice and Bob may simulate the quantum correlations with the help of one shared random vector $\vec{\lambda} \in S_2$ if Alice sends her input $\vec{a}$ to Bob.

It is clearly possible to solve this problem with protocols where Alice sends her input $\vec{a}$ to Bob. Even though such protocols would not be efficient, as Alice needs to send an infinite amount of information to communicate $\vec{a}$ to Bob, it is useful to consider one of them here, proposed by Schatten [11].

This protocol makes use of one shared random variable $\vec{\lambda}$ uniformly distributed on the unit sphere S_2 (see Fig. 2):

- Alice sends $\vec{a}$ to Bob;
- Alice outputs $A = \text{sgn}(\vec{a} \cdot \vec{\lambda})$;
- Bob outputs $B = -\text{sgn}(\vec{b} \cdot (\vec{\lambda} + \vec{a}))\ \text{sgn}(\vec{a} \cdot \vec{\lambda})$;

where $\text{sgn}(x) = 1$ for $x \geq 0$ and $\text{sgn}(x) = -1$ for $x < 0$ ($x \in \mathbb{R}$).

The expectation value $\langle AB \rangle$ then readily follows from the spherical integral

$$\int_{S_2} d\lambda\ \text{sgn}(\vec{v} \cdot (\vec{\lambda} + \vec{w})) = \vec{v} \cdot \vec{w}, \tag{3}$$

where $\vec{\lambda} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ and $d\lambda = \sin\theta\ d\theta\ d\varphi/(4\pi)$.

Four equivalent local protocols

We will now show that the most efficient known protocols can all be derived from four equivalent local protocols. The initial idea is to start from a protocol that achieves the quantum correlations up to a multiplicative constant, without using any communication (local protocol). Finding inspiration in Schatten's protocol, we may define the following protocol where, intuitively, Bob replaces $\overrightarrow{a}$ by a new random variable $\overrightarrow{\lambda_2}$ (Bob "guesses" $\overrightarrow{a}$):

- Alice outputs $A = \text{sgn}(\overrightarrow{a} \cdot \overrightarrow{\lambda_1})$;
- Bob outputs $B = -\text{sgn}(\overrightarrow{b}.(\overrightarrow{\lambda_1} + \overrightarrow{\lambda_2}))$.

Using the following spherical integral

$$\int_{S_2} d\lambda \, (\overrightarrow{v} \cdot \overrightarrow{\lambda}) \, \text{sgn}(\overrightarrow{w} \cdot \overrightarrow{\lambda}) = \frac{\overrightarrow{v} \cdot \overrightarrow{w}}{2}, \tag{4}$$

together with (3), it is straightforward to check that this protocol induces $\langle AB \rangle = -\overrightarrow{a} \cdot \overrightarrow{b}/2$, that is, half of the quantity we need.

Let us now define the following notation:

$$x_\diamond = \text{sgn}(\overrightarrow{a} \cdot \overrightarrow{\lambda_\diamond}) \quad \text{and} \quad y_\diamond = \text{sgn}(\overrightarrow{b} \cdot \overrightarrow{\lambda_\diamond}) \quad \text{for } \diamond = 1, 2, +, -, \tag{5}$$

where $\overrightarrow{\lambda_+} = \overrightarrow{\lambda_1} + \overrightarrow{\lambda_2}$ and $\overrightarrow{\lambda_-} = \overrightarrow{\lambda_1} - \overrightarrow{\lambda_2}$. In this protocol, $A = x_1$ and $B = -y_+$. Applying symmetries $S : (\overrightarrow{\lambda_1}, \overrightarrow{\lambda_2}) \mapsto (\overrightarrow{\lambda_2}, \overrightarrow{\lambda_1})$, $S' : (\overrightarrow{\lambda_+}, \overrightarrow{\lambda_-}) \mapsto (\overrightarrow{\lambda_-}, \overrightarrow{\lambda_+})$ and $S \circ S'$ to this protocol, we obtain four equivalent protocols P_k which all achieve

$$\iint_{S_2} d\lambda_1 d\lambda_2 A_k B_k = -\frac{\overrightarrow{a} \cdot \overrightarrow{b}}{2} \qquad \forall \, k = 1 \ldots, 4, \tag{6}$$

where the four output pairs (A_k, B_k) are given in the following table.

Protocol P_k	P_1	P_2	P_3	P_4
Alice's output A_k	x_1	x_1	x_2	x_2
Bob's output B_k	$-y_+$	$-y_-$	$-y_+$	y_-

Non-local boxes

We now show how to obtain, from the four local protocols above, a protocol [4] that achieves the full quantum correlations, $\langle AB \rangle = -\overrightarrow{a} \cdot \overrightarrow{b}$, if we allow Alice and Bob to share a new resource usually called non-local box.

A PR non-local box [10] is a device shared by Alice and Bob, that has two inputs $x, y \in \{1, -1\}$ for Alice and Bob, respectively, and outputs $a, b \in \{1, -1\}$ for Alice and Bob, respectively, according to the distribution

$$p(a, b|x, y) = \begin{cases} \frac{1}{2} & \text{if } a \cdot b = f(x, y), \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where $f(x,y) = (1+x+y-xy)/2$. This resource has been studied because of its interesting properties. Firstly it is maximally non-local, in the sense that it maximally violates the CHSH Bell inequality. Secondly it is causal, in the sense that Alice's output a is independent of Bob's input y, $p(a|x,y) = p(a|x)$ (and vice-versa). Finally, it is a strictly weaker resource than one bit of communication: due to the causality property, it may not be used to communicate but, on the other hand, it may be shown that one use of a non-local box may be simulated by one bit of communication [4].

First, let us note that summing Eq. (6) over the four protocols,

$$-\vec{a}\cdot\vec{b} = \iint_{S_2} d\lambda_1 d\lambda_2 (-1)\frac{x_1y_+ + x_1y_- + x_2y_+ - x_2y_-}{2}. \tag{8}$$

In this expression, $x_\diamond$'s are known by Alice and $y_\diamond$'s by Bob. If we could factor the integrand, which takes values ± 1, into a product AB where $A = \pm 1$ depends only on $x_\diamond$'s and $B = \pm 1$ only on $y_\diamond$'s, we would have a protocol that achieves the quantum correlations. Of course, by Bell inequalities, this is not possible as is, so what can we do? Trying to factor the expression, we could write for instance

$$\begin{aligned} -\vec{a}\cdot\vec{b} &= \iint_{S_2} d\lambda_1 d\lambda_2 (-x_1y_+)\frac{1+y_+y_- + x_1x_2 - x_1x_2y_+y_-}{2} && (9)\\ &= \iint_{S_2} d\lambda_1 d\lambda_2 (-x_1y_+) f(x_1x_2, y_+y_-). && (10)\end{aligned}$$

If Alice and Bob could use a special resource that on input x outputs a to Alice, and on input y outputs b to Bob, such that $a\cdot b = f(x,y)$, this would finish the job of factoring expression (8). Surprinsingly, this is exactly what a PR non-local box [10] does.

Therefore, if Alice and Bob share a non-local box, they may use it to simulate the quantum correlations with the following protocol, as was proven by Cerf *et al* [4]:

- Alice inputs $x = x_1x_2$ into the box and gets back a;
- Bob inputs $y = y_+y_-$ into the box and gets back b;
- Alice outputs $A = ax_1$;
- Bob outputs $B = -by_+$.

By invariance under symmetries S, S' and $S\circ S'$ in Eq. (8), and choosing to include the (-1) sign into the function f or not, we finally have 8 equivalent protocols that make use of the 8 possible maximally non-local boxes [1], corresponding to functions $f_{\alpha\beta\gamma}(x,y) = \gamma f(\alpha x, \beta y)$ with $\alpha, \beta, \gamma = \pm 1$.

One bit of communication

Toner and Bacon [12] were the first to give a protocol that simulates quantum correlations with one bit of communication. Since a non-local box may always be simulated by one bit of communication, another protocol can be derived from the protocol in the previous section. Here, we show specifically how to derive Toner and Bacon's protocol in our setting.

Starting from expression (8), we may write

$$-\overrightarrow{a}\cdot\overrightarrow{b}=\iint_{S_2}d\lambda_1 d\lambda_2\left[(-x_1y_+)\frac{1+y_+y_-}{2}+(-x_2y_+)\frac{1-y_+y_-}{2}\right]. \tag{11}$$

We see that the integrand takes value $-x_1y_+$ when $y_+y_-=1$, and $-x_2y_+$ when $y_+y_-=-1$. Therefore, we can obtain a new protocol, where Bob always outputs y_+, while Alice outputs $-x_1$ or $-x_2$ depending on the value of $y=y_+y_-$. As initially only Bob knows y, he must send its value to Alice, which requires one bit of communication. Hence, we find Toner and Bacon's protocol [12]:

- Bob computes $y=y_+y_-$ and sends its value to Alice,
- Bob outputs $-y_+$,
- Alice outputs x_1 if $y=1$ or x_2 if $y=-1$.

Let us note that reorganizing the terms of Eq. (11), we may find 8 different but equivalent protocols, 4 where Bob sends a bit to Alice and 4 the other way around.

Detector inefficiency

In a real experiment, Alice's and Bob's detectors could be partially inefficient. Their efficiency, parameterized by $\eta\leq 1$, is the probability of producing an output. By exploiting this inefficiency (sometimes called detection loophole), we will derive a protocol which reproduces the quantum correlations with post-selection, where Alice or Bob are allowed to occasionally not produce an output [8].

Starting back from Eq. (6), we have

$$\begin{aligned}-\overrightarrow{a}\cdot\overrightarrow{b} &= \iint_{S_2}d\lambda_1 d\lambda_2(-x_1y_+)(1+y_+y_-) && (12)\\ &= \iint_{S_2}d\lambda_1 d\lambda_2(-x_1y_+)p(\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}), && (13)\end{aligned}$$

where $p(\overrightarrow{\lambda_1}, \overrightarrow{\lambda_2}) = 1 + y_+ y_-$, that is,

$$p(\overrightarrow{\lambda_1}, \overrightarrow{\lambda_2}) = \begin{cases} 2 & \text{if } \mathrm{sgn}(\overrightarrow{b} \cdot \overrightarrow{\lambda_+}) = \mathrm{sgn}(\overrightarrow{b} \cdot \overrightarrow{\lambda_-}), \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

It is easy to check that $p(\overrightarrow{\lambda_1}, \overrightarrow{\lambda_2}) \geq 0$ $(\forall \overrightarrow{\lambda_1}, \overrightarrow{\lambda_2})$ and $\iint_{S_2} d\lambda_1 d\lambda_2 p(\overrightarrow{\lambda_1}, \overrightarrow{\lambda_2}) = 1$, and thus that $p(\overrightarrow{\lambda_1}, \overrightarrow{\lambda_2})$ may be considered as a density function. Therefore, Eq. (13) means that Alice and Bob could reproduce the quantum correlations by outputting $A = x_1$ and $B = -y_+$ provided that they share random variables $\overrightarrow{\lambda_1}$ and $\overrightarrow{\lambda_2}$ distributed according to $p(\overrightarrow{\lambda_1}, \overrightarrow{\lambda_2})$. One way of generating this distribution follows directly from (14): start from uniformly distributed $\overrightarrow{\lambda_1}$ and $\overrightarrow{\lambda_2}$, check whether $\mathrm{sgn}(\overrightarrow{b} \cdot \overrightarrow{\lambda_+}) = \mathrm{sgn}(\overrightarrow{b} \cdot \overrightarrow{\lambda_-})$, and discard the pair if it is not the case. However, only Bob can perform the test, so a workaround is to allow him sometimes not to produce any output:

- Alice outputs $A = x_1$,
- Bob checks whether $y_+ = y_-$. If so, he outputs $B = -y_+$. Otherwise, he does not produce any output[1].

Let us note that when Bob outputs, $y_+ = y_-$, so we may rewrite $B = -\mathrm{sgn}(\overrightarrow{b} \cdot \overrightarrow{\lambda_+}) = -\mathrm{sgn}(\overrightarrow{b} \cdot \overrightarrow{\lambda_+} + \overrightarrow{b} \cdot \overrightarrow{\lambda_-}) = -\mathrm{sgn}(\overrightarrow{b} \cdot \overrightarrow{\lambda_1})$. Therefore, B depends on a single random variable $\overrightarrow{\lambda_1}$, just as Alice's output $A = \mathrm{sgn}(\overrightarrow{a} \cdot \overrightarrow{\lambda_1})$ does. So our protocol is equivalent to that of Gisin and Gisin [8] since the marginal distribution of $\overrightarrow{\lambda_1}$ is given by $p(\overrightarrow{\lambda_1}) = 2|\overrightarrow{b} \cdot \overrightarrow{\lambda_1}|$. One advantage of our approach, in addition to clarifying the relation of this protocol with the previous ones, is to give a nice method to generate $p(\overrightarrow{\lambda_1})$ by starting from two uniform random vectors $\overrightarrow{\lambda_1}$ and $\overrightarrow{\lambda_2}$ and performing a test.

CONCLUSION

In the particular problem of simulating a projective measurement on the bipartite singlet state, we have shown that the most efficient protocols known using different resources [8, 12, 4] can all be derived from a local protocol. This gives a new and coherent approach for the problem, which could be helpful to solve generalized problems, such as the simulation of multiparty quantum correlations, or to improve the known results for simulating general measurements.

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[1] In this protocol, Bob's detector efficiency is only $\eta_B = 50\%$, while Alice has a perfect detector, $\eta_A = 1$. With a slight modification, we can get $\eta_A = \eta_B = 67\%$ (see [8] for details).

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