Finding a marked node on any graph by continuous-time quantum walk

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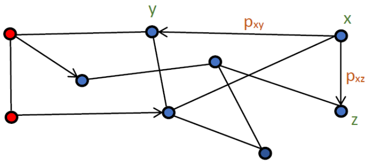
Outline

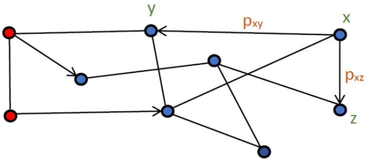
1. Introduction
   - Spatial search by random walk
   - Spatial search by quantum walk
2. Our contributions
   - Overview
   - Childs-Goldstone algorithm
   - Modified Childs-Goldstone algorithm
   - New algorithm using interpolated Markov Chains
3. Conclusion

Classical random walk on a graph



- Classical random walk on a discrete state space X, such that $|X| = n$.
- Described by a $n \times n$ stochastic matrix P such that its $(x, y)^{\text{th}}$ entry is p_{xy}.
- If the row-vector v_0 is the initial state of the walker, after t-steps: $v_t = v_0 P^t$.
- Stationary state: row vector π such that $\pi = \pi P$.
- Assumptions: **P is ergodic and reversible**
  - Eigenvalues of P lie between -1 and 1.
  - π is unique.
  - $\pi_x p_{xy} = \pi_y p_{yx}$ for all (x, y)

Classical Hitting time



Set of marked nodes:
$M \subseteq X$.

Hitting time: Starting from some random node $x \sim \pi$, the expected number of steps to reach some node $\in M$.

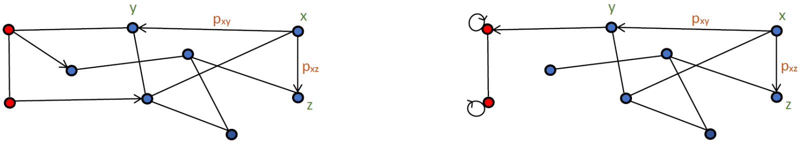
Spatial search (classical)

1. Sample $x \in X$ from π.
2. Check if $x \in M$.
3. If $x \in M$, output x
4. Otherwise update x according to P and go to step 2.

Hitting time of P with respect to M is the expected number of times step 4 is executed.

Classical Hitting time

Spatial search stops when $x \in M \implies$ Walk on an absorbing Markov chain P'



$HT(P, M) =$ Expected number of steps of P' to reach some $x \in M$.

Complexity of spatial search by **quantum walk**?

Discrete-time quantum walk (DTQW)?

Continuous-time quantum walk (CTQW)?

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Discrete-time and continuous-time quantum walks

Complexity of spatial search by DTQW

For any *ergodic*, *reversible* Markov chain P with a set of M marked nodes: $\mathcal{O}\left(\sqrt{HT^+(P,M)}\right)$.
[Krovi, Magniez, Ozols, and Roland 2014]

$HT^+(P,M)$: *Extended hitting time*

- For $|M| = 1$, $HT^+(P,M) = HT(P,M) \implies$ Quadratic speedup for unique marked node.
- For $|M| > 1$, $HT^+(P,M) \geq HT(P,M)$.

Previous talk

Improved to

$$\mathcal{O}\left(\sqrt{HT(P,M)}\log\left(\sqrt{HT(P,M)}\right)\right).$$

[Ambainis, Gilyén, Jeffery and Kokainis 2019]

Discrete-time and continuous-time quantum walks

Complexity of spatial search by DTQW

For any *ergodic*, *reversible* Markov chain P with a set of M marked nodes: $\mathcal{O}\left(\sqrt{HT^+(P,M)}\right)$.
[Krovi, Magniez, Ozols, and R. 2014]

$HT^+(P,M)$: *Extended hitting time*

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- For $|M| > 1$, $HT^+(P,M) \geq HT(P,M)$.

Complexity of spatial search by CTQW

- No such general result is known.
- Childs and Goldstone proposed a CTQW-based algorithm in 2004.
- Has been applied to certain specific graphs such as d-dimensional lattices, hypercubes and others.

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Main results

For any ergodic, reversible Markov chain:

- General conditions for the optimality of the Childs and Goldstone algorithm
- Modified version of the Childs and Goldstone algorithm with better running time
- Spatial search algorithm by CTQW with running time $\Theta(\sqrt{HT^{+}(P,M)})$

Continuous-time quantum walk on a graph

General idea

- Prepare initial state $|\psi_0\rangle$
- Evolve under Hamiltonian H_G, encoding the connectivity of the graph
- Probability of the walker being at node $|x\rangle$, after time t

$$p(t) = |\langle x| e^{-iH_G t} |\psi_0\rangle|^2.$$

Application to spatial search

- What Hamiltonian H_G to consider for a given graph?
- How is the Hamiltonian modified for marked nodes? (Oracle?)

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Spatial search by CTQW (Childs and Goldstone 2004)

Setup

- Hamiltonian: $H_G = A$, adjacency matrix of the graph
  - Hilbert space: spanned by the nodes of the graph $\{|1\rangle, \ldots, |n\rangle\}$
- Node $|w\rangle$ marked by energy penalty: $H_{oracle} = |w\rangle\langle w|$

$\mathcal{CG}$ algorithm

1. Prepare the state $|\psi_0\rangle = \frac{1}{\sqrt{n}} \sum_x |x\rangle$
2. Evolve according to Hamiltonian $H = H_{oracle} + rH_G$ for time $T = \mathcal{O}(\sqrt{n})$
   - where r is a real number that should be optimized
3. Measure in the node basis

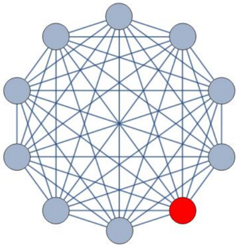
Complete graph:

$$A_{ij} = 1$$

Same as analog Grover

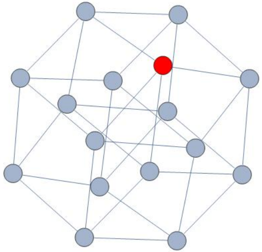
$$H = -|w\rangle\langle w| - |s\rangle\langle s|$$
$$= -|w\rangle\langle w| - \frac{1}{n}A$$

Optimal! $T = \mathcal{O}(\sqrt{n})$



Hypercube:

Optimal! $T = \mathcal{O}(\sqrt{n})$



Square Lattices:

$d \leq 3$

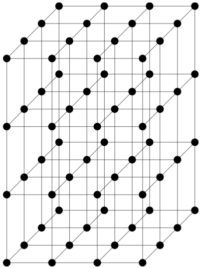
Fails

$d = 4$

$$T = O(\sqrt{n}\log^{3/2} n)$$

$d > 4$

Optimal! $T = \mathcal{O}(\sqrt{n})$



Sufficient condition for the optimality of $\mathcal{CG}$ algorithm

Assume:

- Eigenvalues of H_G:

$$\lambda_n = 1 > \lambda_{n-1} = 1 - \Delta \geq \ldots \geq \lambda_1 \geq 0$$

- State transitive graph

Theorem

Let

$$r = \frac{1}{n} \sum_{i \neq n} \frac{1}{1 - \lambda_i} \qquad \nu = \frac{r}{\sqrt{\frac{1}{n} \sum_{i \neq n} \frac{1}{(1-\lambda_i)^2}}}$$

If $\Delta \gg \frac{\nu}{r\sqrt{n}}$, then for $T = \Theta\left(\frac{\sqrt{n}}{\nu}\right)$, the $\mathcal{CG}$ algorithm prepares a state $|\psi_f\rangle$ such that $|\langle w|\psi_f\rangle| = \Theta(\nu)$.
By repetition, one can find the marked node in total time $T_{search} = \Theta\left(\frac{\sqrt{n}}{\nu^3}\right)$

Discussion

Theorem

If $\Delta \gg \frac{\nu}{r\sqrt{n}}$, then for $T = \Theta\left(\frac{\sqrt{n}}{\nu}\right)$, the $\mathcal{CG}$ algorithm prepares a state $|\psi_f\rangle$ such that $|\langle w|\psi_f\rangle| = \Theta(\nu)$.

By repetition, one can find the marked node in total time $T_{search} = \Theta\left(\frac{\sqrt{n}}{\nu^3}\right)$

Discussion

- If $\Delta \gg \frac{\nu}{r\sqrt{n}}$, the algorithm is optimal for any P such that $\nu = \Theta(1)$
- When $\Delta = \Theta(1)$, we have $\nu = \Theta(1)$

  [Chakraborty, Novo, Ambainis and Omar 2016]
- ν can be $\Theta(1)$ even when $\Delta \neq \Theta(1)$
  - For d-dimensional lattices with $d > 4$, $\Delta = \Theta(n^{-2/d})$ but $\nu = \Theta(1)$.
  - At $d = 4$, $\nu = \Theta(1/\sqrt{\log n}) \implies T_{search} = \mathcal{O}(\sqrt{n}\log^{3/2} n)$.

    [Childs and Goldstone 2004]

Drawbacks of the $\mathcal{CG}$ algorithm

Theorem

If $\Delta \gg \frac{\nu}{r\sqrt{n}}$, then for $T = \Theta\left(\frac{\sqrt{n}}{\nu}\right)$, the $\mathcal{CG}$ algorithm prepares a state $|\psi_f\rangle$ such that $|\langle w|\psi_f\rangle| = \Theta(\nu)$.
By repetition, one can find the marked node in total time $T_{search} = \Theta\left(\frac{\sqrt{n}}{\nu^3}\right)$

Drawbacks

- The condition $\Delta \gg \frac{\nu}{r\sqrt{n}}$ needs to be satisfied.
  - Does not hold for lattices of dimension less than four.
- We show that $\Omega(\sqrt{\Delta}) < \nu < 1$.
  - $HT(P, \{w\}) \leq \frac{n}{\Delta}$. So when $\nu = \sqrt{\Delta} \ll 1$, $T_{search} = \Omega\left(\frac{\sqrt{HT(P,\{w\})}}{\Delta}\right)$
    $\Rightarrow$ No quadratic speed-up!

Example: Movement of rook on a rectangular chessboard

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Questions

How to make quantum walks faster?

Use *coins*!

- DTQW [Ambainis, Kempe and Rivosh 2004]
- CTQW [Childs and Goldstone 2004]

How to introduce coins for general graphs?

Walk on edges of the graph rather than nodes!

- DTQW [Szegedy 2004]
- CTQW [Somma and Ortiz 2010]

Somma-Ortiz Hamiltonian

- Hilbert space: spanned by the edges of the graph $|x, y\rangle$

$$|x^E\rangle = \sum_y \sqrt{p_{xy}}\,|x, y\rangle$$

- Consider unitary S and projector Π^E such that

$$S\,|x, y\rangle = |y, x\rangle \qquad\qquad \Pi^E = \sum_x |x^E\rangle\langle x^E|$$

- The Somma-Ortiz Hamiltonian is defined as

$$H_P = i[S, \Pi^E]$$

- Properties of H_P
  - Eigenstates and eigenvalues related to those of P
  - Spectral gap is $\Theta(\sqrt{\Delta})$
- Applications
  - Gibbs Sampling [Somma and Ortiz 2010]
  - Spatial search by adiabatic evolution [Krovi, Ozols and R. 2010]

Oracle

Problem

Using

$$H = H_{oracle} + rH_P \qquad \text{with} \qquad H_{oracle} = |w^E\rangle\langle w^E|$$

requires to take $r = 0$ $\Rightarrow$ does not work!

Alternative oracle

We set

$$H = H_{oracle} + H_P \qquad \text{with} \qquad H_{oracle} = -|w^E\rangle\langle w^E| H_P - H_P |w^E\rangle\langle w^E|$$

- Previously used for
  - Spatial search on crystal lattices [Childs and Ge 2014]
  - Spatial search on graphene lattices [Foulger et al. 2014]

Modified $\mathcal{CG}$ algorithm ($\mathcal{CG}'$)

Setup

- $H_P = i[S, \Pi^E]$, Somma-Ortiz Hamiltonian for Markov chain P
  - Hilbert space: spanned by the edges of the graph $\{|x,y\rangle\}$
- Node $|w\rangle$ marked by $H_{oracle} = -|w^E\rangle\langle w^E| H_P - H_P |w^E\rangle\langle w^E|$

$\mathcal{CG}'$ algorithm

1. Prepare the state $|\psi_0\rangle = \frac{1}{\sqrt{n}} \sum_x |x^E\rangle$
2. Evolve according to Hamiltonian $H = H_{oracle} + H_P$ for time T
   - where T will be specified on the next slide
3. Measure in the node basis

Modified $\mathcal{CG}$ algorithm ($\mathcal{CG}'$)

Assume:

- Eigenvalues of P:

$$\lambda_n = 1 > \lambda_{n-1} = 1 - \Delta \geq \ldots \geq \lambda_1 \geq 0$$

- State transitive graph

Theorem

Let

$$\mu = \sqrt{\frac{1}{n} \sum_{i \neq n} \frac{1}{(1 - \lambda_i)^2}} \qquad |\widetilde{w}\rangle = \frac{H_P |w^E\rangle}{|| H_P |w^E\rangle ||}$$

If $\Delta \gg \frac{1}{n\mu^2}$, then for $T = \Theta\left(\mu\sqrt{n}\right)$, the $\mathcal{CG}'$ algorithm prepares a state $|\psi_f\rangle$ such that $\left|\langle\widetilde{w}|\psi_f\rangle\right| = \Omega(1/\mu)$.
Moreover, $e^{-it' H_{oracle}} |\widetilde{w}\rangle = |w^E\rangle$, for $t' = \mathcal{O}(\mu)$.
By repetition, one can find the marked node in total time $T_{search} = \Theta\left(\mu^3\sqrt{n}\right)$

Discussion

Theorem

Let

$$\mu = \sqrt{\frac{1}{n}\sum_{i\neq n}\frac{1}{(1-\lambda_i)^2}} \qquad\qquad |\widetilde{w}\rangle = \frac{H_P\,|w^E\rangle}{||H_P\,|w^E\rangle\,||}$$

If $\Delta \gg \frac{1}{n\mu^2}$, then for $T = \Theta\left(\mu\sqrt{n}\right)$, the $\mathcal{CG}'$ algorithm prepares a state $|\psi_f\rangle$ such that $\left|\langle\widetilde{w}|\psi_f\rangle\right| = \Omega(1/\mu)$.
Moreover, $e^{-it' H_{oracle}}\,|\widetilde{w}\rangle = |w^E\rangle$, for $t' = \mathcal{O}(\mu)$.
By repetition, one can find the marked node in total time $T_{search} = \Theta\left(\mu^3\sqrt{n}\right)$

Discussion

- Whenever $\mathcal{CG}$ algorithm is optimal, so is $\mathcal{CG}'$.
- $\mathcal{CG}'$ also works for lattices down to $d = 2$
  - For $d = 2$, we get $T_{search} = \mathcal{O}(\sqrt{n}\log^{3/2} n)$

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Basic idea

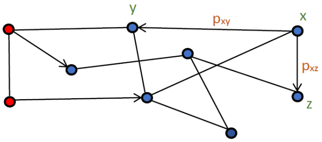
- We use the Somma-Ortiz Hamiltonian as for $\mathcal{CG}'$
- Rather than marking a node by adding a oracle Hamiltonian
- We modify the classical random walk directly

Trick

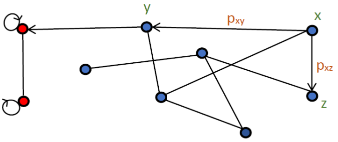
Interpolated Markov Chain

Interpolated Markov Chain

Let $U = X \backslash M$.



$$\pi = (\pi_U \;\; \pi_M)$$



$\pi' \propto (0 \;\; \pi_M)$ (P' is not ergodic)

Interpolated Markov Chains

$$P(s) = (1-s)P + sP', \;\; s \in [0,1]$$

For $0 \leq s < 1$, stationary state:

$$\pi(s) \propto ((1-s)\pi_U \;\; \pi_M)$$

Somma-Ortiz Hamiltonian

Interpolated Markov Chains

$$P(s) = (1-s)P + sP',\ s \in [0,1]$$

For $0 \leq s < 1$, stationary state:

$$\pi(s) \propto ((1-s)\pi_U \ \ \pi_M)$$

Properties of the Somma-Ortiz Hamiltonian $H_{P(s)}$ built on $P(s)$

- Spectral gap: $\Theta(\sqrt{\Delta(s)})$
- Eigenstate with eigenvalue 0: $|v_n(s)\rangle = \sum_x \sqrt{\pi_x(s)}\,|x^E\rangle$
- For $s^* = 1 - \frac{p_M}{1-p_M}$ with $p_M = \sum_{x \in M} \pi_x$,

$$|v_n(s^*)\rangle = \frac{|U\rangle + |M\rangle}{\sqrt{2}}$$

  - where $|U\rangle = \frac{1}{\sqrt{1-p_M}} \sum_{x \notin M} \sqrt{\pi_x}\,|x^E\rangle$ and $|M\rangle = \frac{1}{\sqrt{p_M}} \sum_{x \in M} \sqrt{\pi_x}\,|x^E\rangle$

New algorithm

Observation

- $|v_n(s^*)\rangle$ has a constant overlap with $|M\rangle$
- So all we need to do is prepare $|v_n(s^*)\rangle$ and measure.

New algorithm

1. Prepare the state $|v_n(0)\rangle = \sum_x \sqrt{\pi_x}|x^E\rangle$.
2. Evolve according to Hamiltonian $H_{P(s^*)}$ for a time chosen uniformly at random between $[0, T]$, where
   - $T = \Theta(\sqrt{HT^+(P,M)})$
   - $s^* = 1 - \frac{p_M}{1-p_M}$
3. Measure in the node basis

New algorithm

1. Prepare the state $|v_n(0)\rangle = \sum_x \sqrt{\pi_x}|x^E\rangle$.

2. Evolve according to Hamiltonian $H_{P(s^*)}$ for a time chosen uniformly at random between $[0, T]$, where
   - $T = \Theta(\sqrt{HT^+(P,M)})$
   - $s^* = 1 - \frac{p_M}{1-p_M}$

3. Measure in the node basis

Proof idea: Quantum phase randomization [Boixo, Knill and Somma 2009]

- Decoherence in the eigenbasis of $H_{P(s^*)}$ kills coherence terms between $|v_n(s^*)\rangle$ and its orthogonal eigenstates.

- We obtain a mixed state between $|v_n(s^*)\rangle$ and the rest.

There are other ways to prepare $|v_n(s^*)\rangle$. (Current work).

Conclusion

- Provided general conditions for the optimality of the spatial search algorithm by Childs and Goldstone
- Modified Childs and Goldstone algorithm
  - Applicable to any ergodic, reversible Markov chain
  - Improved running time
- Spatial search algorithm by CTQW that runs in $\Theta(\sqrt{HT^+(P,M)})$ time for any ergodic, reversible Markov chain.
- Together, these three algorithms subsume or improve on most (all?) spatial search algorithms by CTQW in the literature

Open questions:

Extended hitting time vs hitting time for CTQW

Current work:

CTQW for preparing the stationary state of an ergodic, reversible Markov chain.

Thank you for your attention!

For more details see:

S. Chakraborty, L. Novo and J. Roland, Finding a marked node on any graph by continuous time quantum walk, arXiv:1807.05957.