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Relentless passion for innovation

Adiabatic quantum optimization
and
Anderson localization

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October 22, 2010

Why quantum computing?

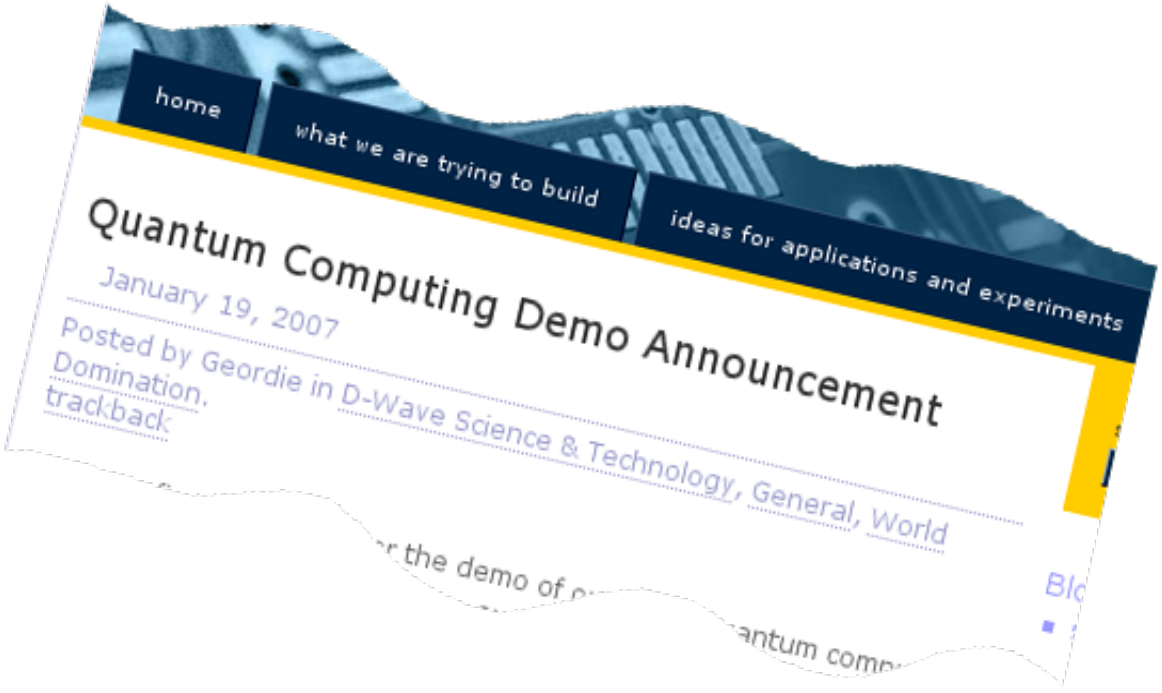
Quantum computing provides speed-up for specific problems

- Factoring
- Discrete logarithms
- etc...

What about NP-complete problems?

- 3-SAT:

$$(x_1 \vee \bar{x_2} \vee x_5) \wedge (\bar{x_1} \vee x_3 \vee \bar{x_5}) \wedge (\bar{x_2} \vee x_4 \vee x_5)$$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what we are trying to build
ideas for applications and experiments
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January 19, 2007
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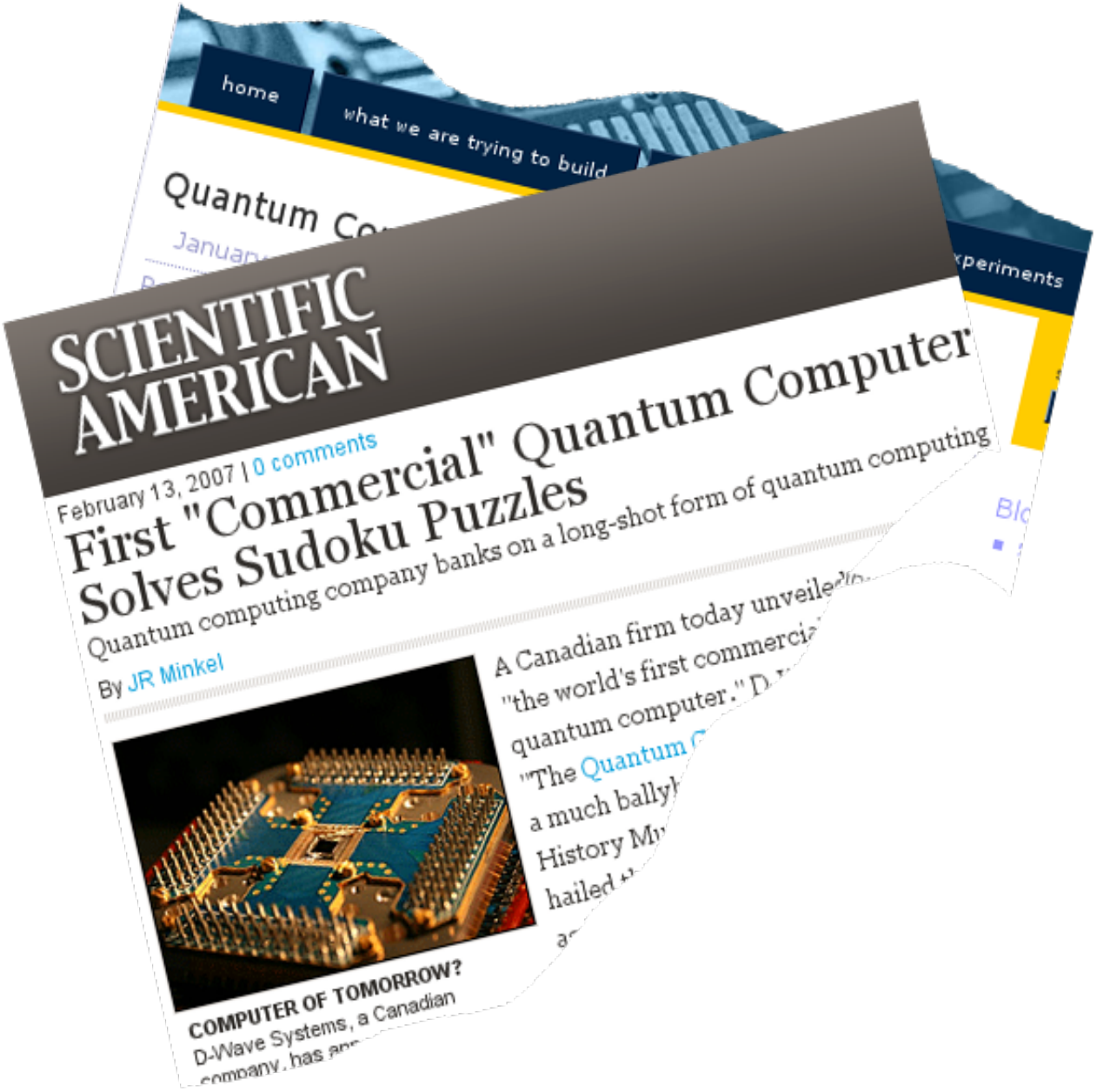
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February 13, 2007 | 0 comments
First "Commercial" Quantum Computer Solves Sudoku Puzzles
Quantum computing company banks on a long-shot form of quantum computing
By JR Minkel
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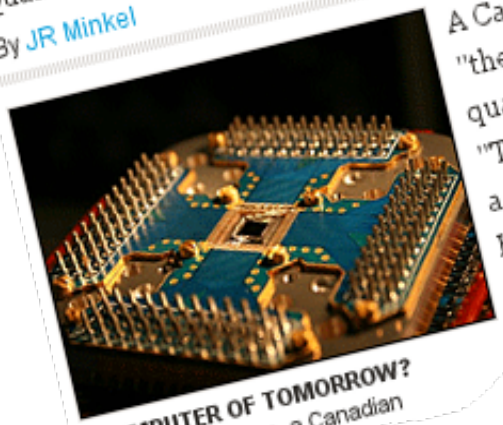

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The Economist

...puters. Ro... potentially faster 1,000-qubit version should be available by the en... of 2008.

As it turns out quantum technology is particularly adept at tackling what are known in mathematics as "NP-complete" problems. NP standing for nondeterministic polynomial time, these are problems where the massi... of com... ta and variables prevent ...ason... ...t of quantum computer.

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On paper at least, quantum computers promise to reduce dramatically the time needed to solve a range of mathematical tasks known as NP-complete problems. One famous example is the ...problem--findin... shortest route between

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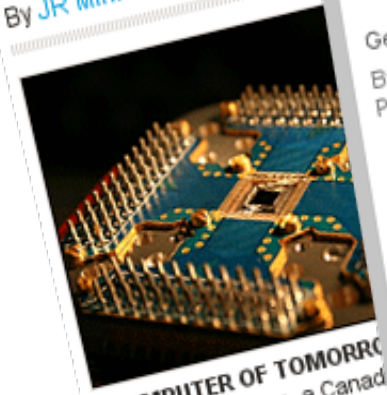

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Geordie Rose, the founder of D-Wave Systems, says he has created a commercial quantum com

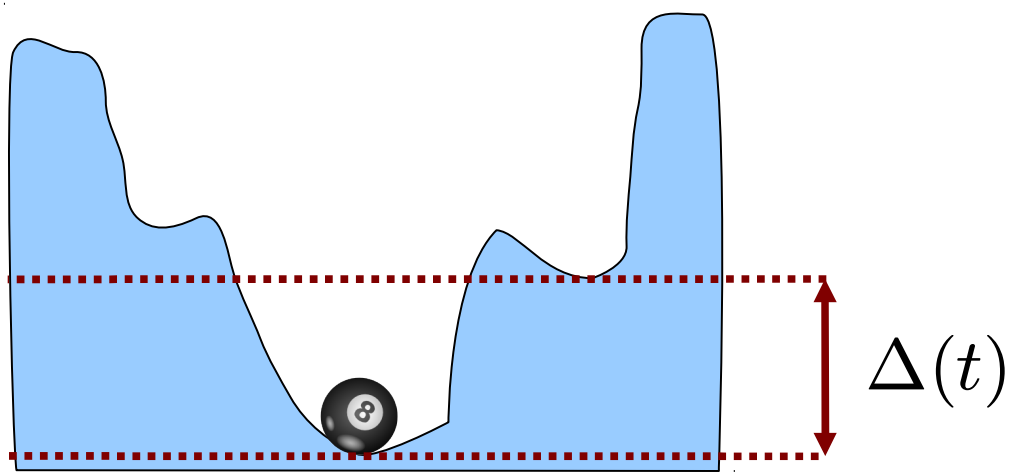
By JASON PONTIN
Published: April 8, 2007

DID **D-Wave Systems** achieve the incredible — a startling advance in computing that would radically expand human capacities for industrial activity and scientific discovery, long before experts believed it possible?

It says it did

concurred.

Adiabatic evolution



Slow evolution → Stay in ground state (=lowest energy)

Prob. of jumping depends on:

- Total time T (slower is better)
- Gap $\Delta(t)$ (larger gap is better)

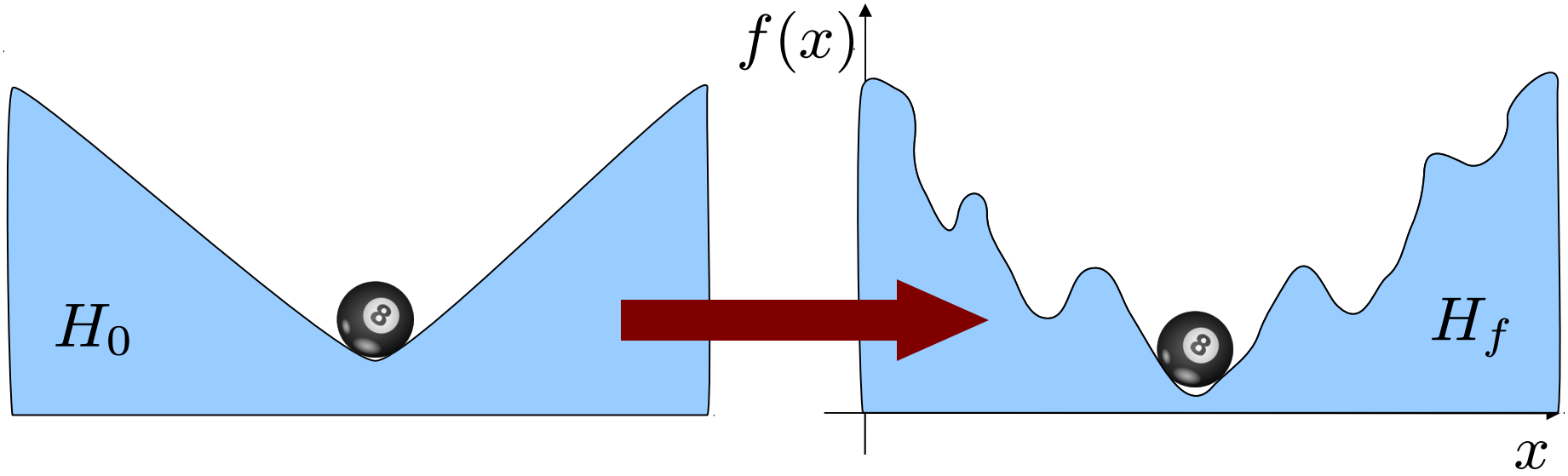
$$T \gg \frac{1}{\Delta_{\min}^2}$$

Quantum approach to optimization

- Problem: Find minimum of a function $f(x)$

1) Choose physical system with known minimum energy state

2) Modify energy function to match $f(x)$



This is "Adiabatic Quantum Computation"

[Farhi et al. '00]

How powerful is it?

- It is quantum! Unstructured search in time $O(\sqrt{N})$ (cf Grover)

[vanDam-Mosca-Vazirani'01,Roland-Cerf'02]

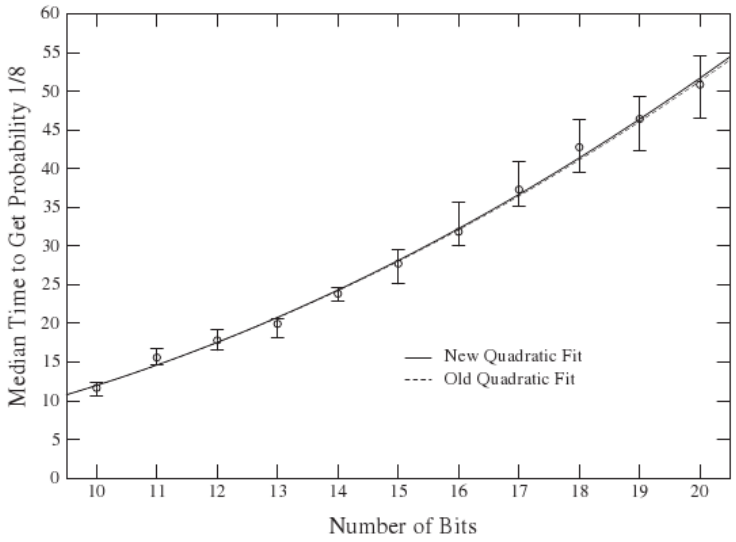
- It is universal for quantum computation

[Aharonov *et al.*'05]

Good, but what about NP-complete problems?

- Numerical simulations: promising scaling

[Farhi *et al.*'00,Hogg'03,Banyuls *et al.*'04,Young *et al.'08*]



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Good, but what about NP-complete problems?

- Numerical simulations: promising scaling 

[Farhi *et al.*'00,Hogg'03,Banyuls *et al.*'04,Young *et al.'08*]

- But exponentially small gap
  - for bad choice of initial Hamiltonian [Znidaric-Horvat'06,Farhi *et al.*'08]
  - for specifically designed hard instances [vanDam-Vazirani'03,Reichardt'04]

But maybe typical gaps are only polynomial?

Exact-Cover 3 (EC3)

- NP complete problem (similar to 3-SAT)
- N bits $\vec{x} = (x_1, \ldots, x_N)$
- M clauses of 3 bits:

$$(x_{i_C}, x_{j_C}, x_{k_C}) \text{ satisfied } \Leftrightarrow x_{i_C} + x_{j_C} + x_{k_C} = 1$$
$$\Leftrightarrow 100, 010 \text{ or } 001$$

- Problem: Find assignment $\vec{x}$ satisfying all clauses

➡ Minimize function $f(\vec{x}) = \sum_C (x_{i_C} + x_{j_C} + x_{k_C} - 1)^2$

$$= M - \sum_i B_i x_i + 2 \sum_{i,j} J_{ij} x_i x_j$$

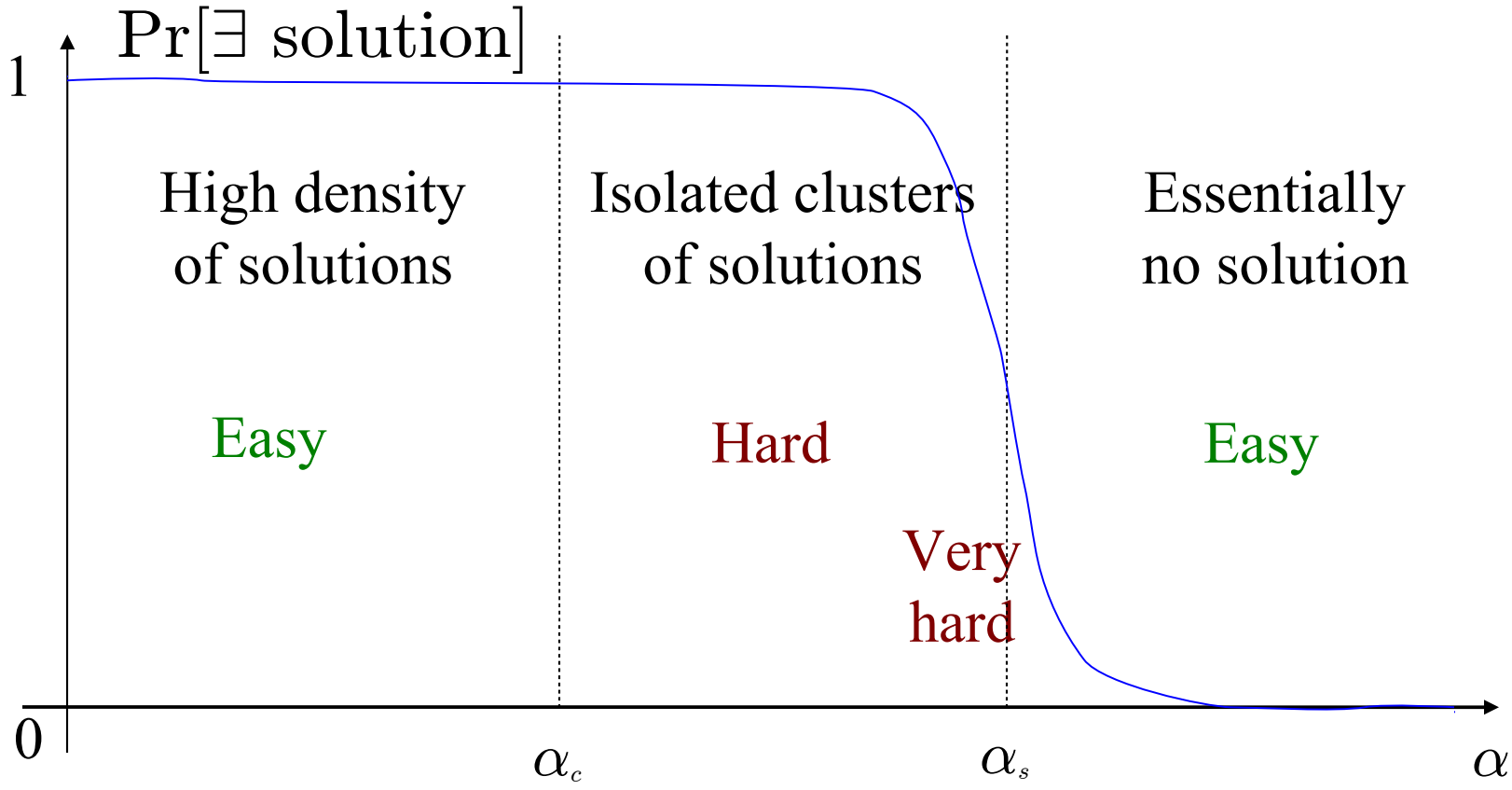
#clauses

#clauses with bit i

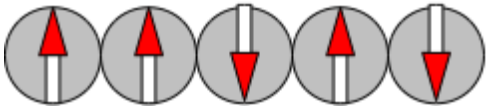
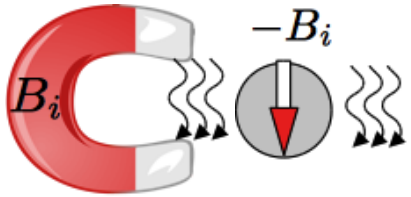
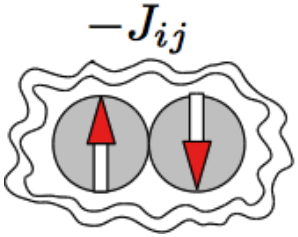
#clauses with bits i,j

Random instances

- Pick M clauses uniformly at random
- Hardness depends on clauses-to-bit ratio: $\alpha = \frac{M}{N}$

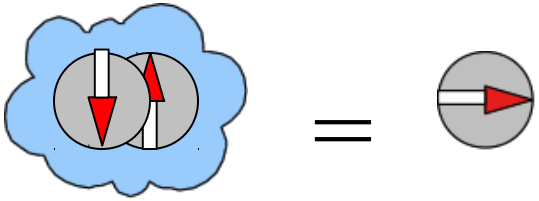


From bits to spins

Bits x_i	Spins
We use "physicist's bits" → $+1$	
-1	
Bit strings: $\vec{x} = (+1, +1, -1, +1, -1)$	
$f(\vec{x}) = M - \frac{1}{2}\sum_i B_i x_i + \frac{1}{2}\sum_{i,j} J_{ij} x_i x_j$	Hamiltonian H_f ("energy function")
Energy term $B_i x_i$ →magnetic field	
Energy term $J_{ij} x_i x_j$ →magnetic coupling	

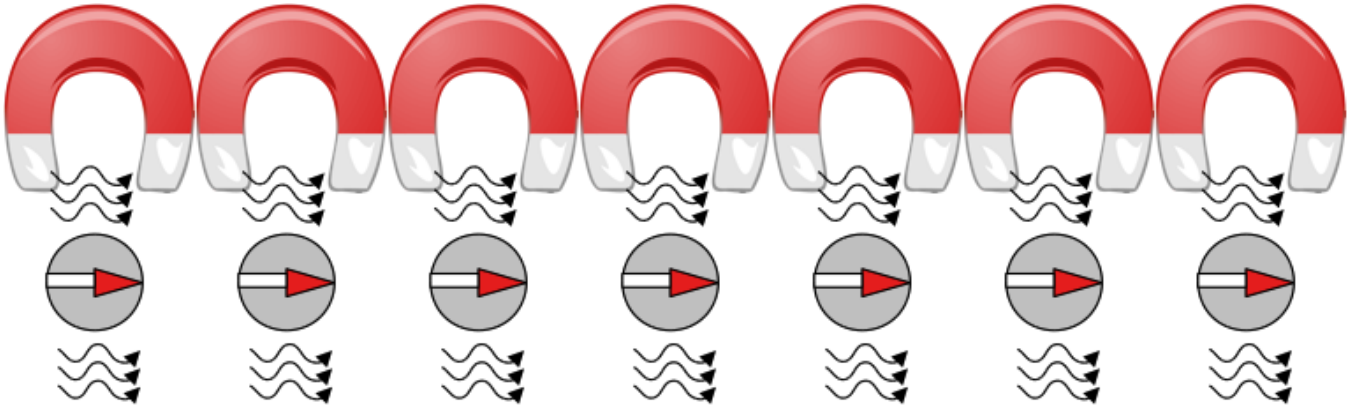
From annealing to adiabatic QC

Quantum Mechanics: spins can be "both up and down"



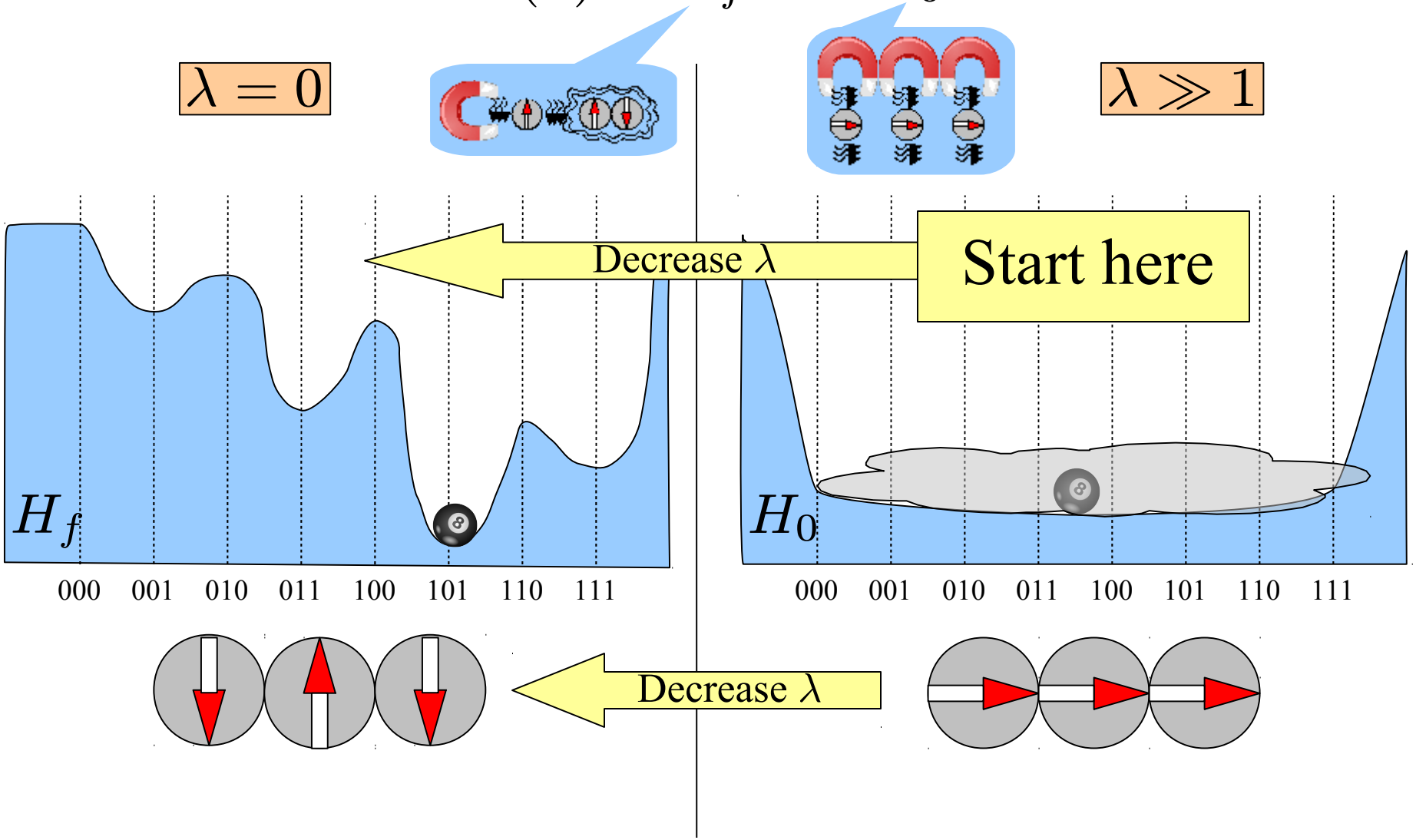
Initialization: Instead of high temperature

→ strong transverse magnetic field H_0



Adiabatic quantum algorithm

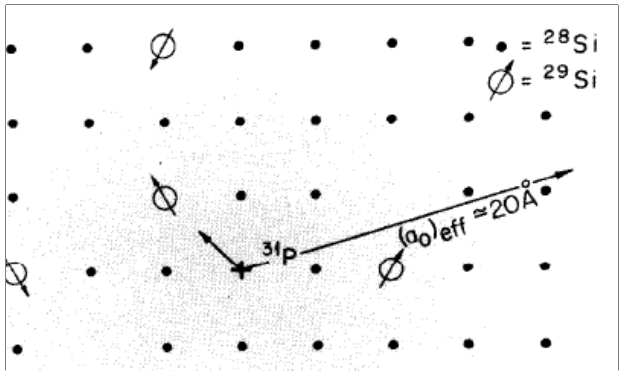
Consider Hamiltonian $H(\lambda) = H_f + \lambda\ H_0$



Anderson localization

"Extended states become localized due to disorder"

Model:
- Grid with coupling λ
- Random energies



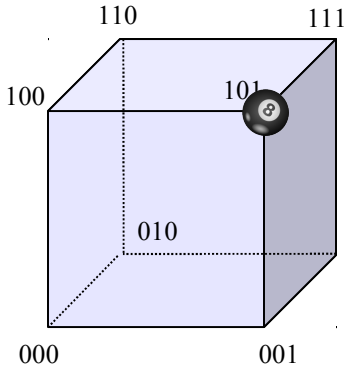
P. Anderson
Nobel Prize
Physics 1977

$\lambda > \lambda_c \rightarrow$ Extended state $\rightarrow$ Metal

$\lambda < \lambda_c \rightarrow$ Localized state $\rightarrow$ Insulator

In our case:

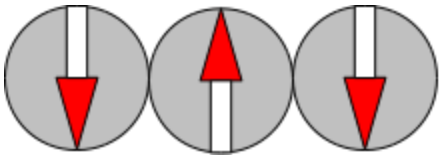
- Hypercube with coupling λ
- Energies from random Exact-Cover 3



Localized and extended states

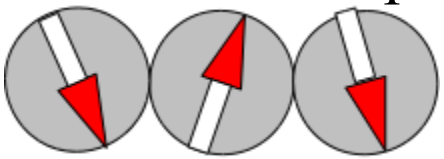
$\lambda = 0$

- State is localized



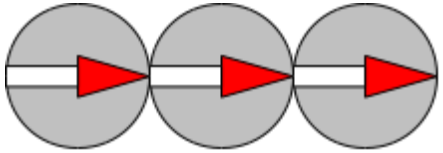
$\lambda > 0$

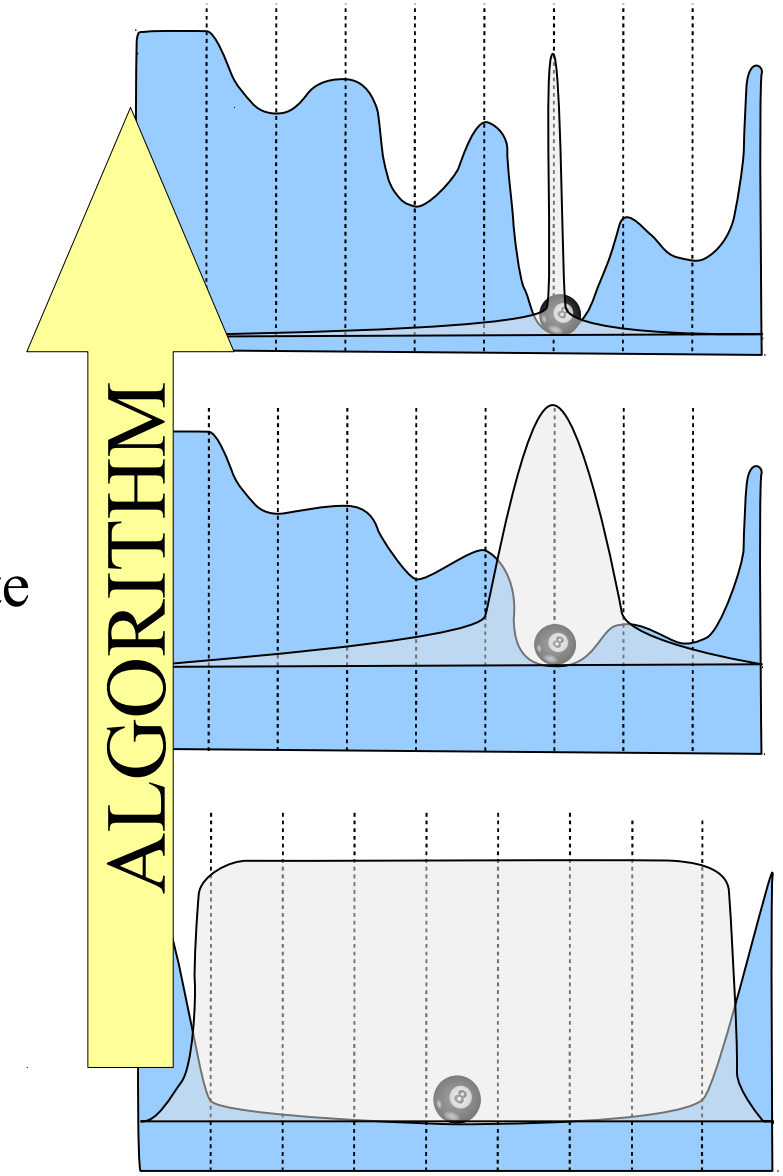
- Transverse field "spreads" the state



$\lambda \gg 1$

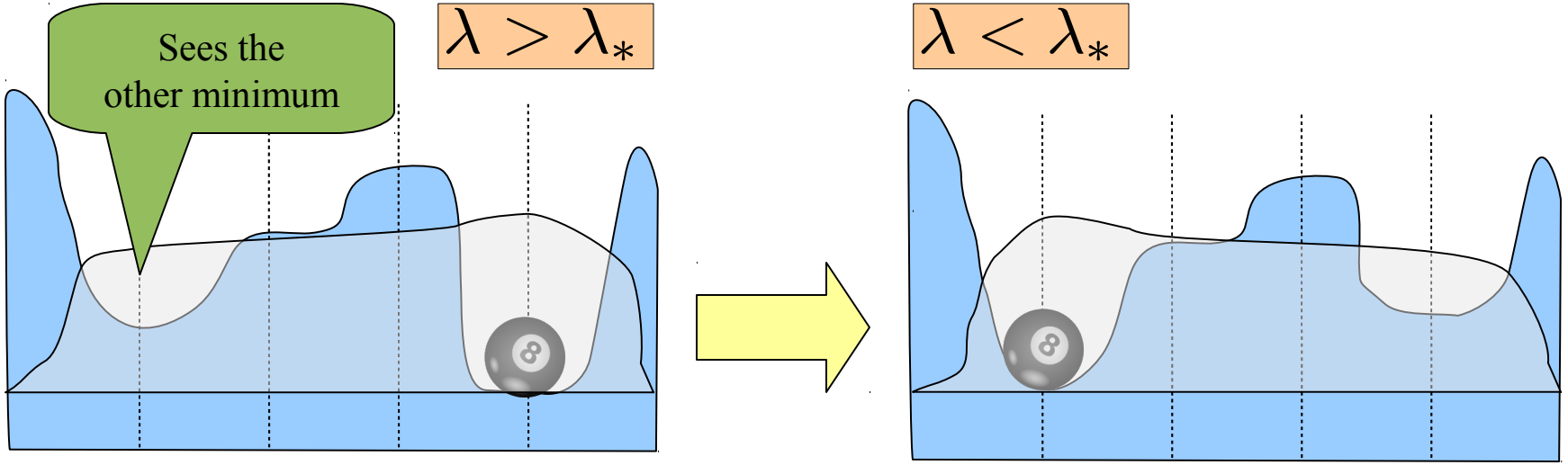
- State is extended

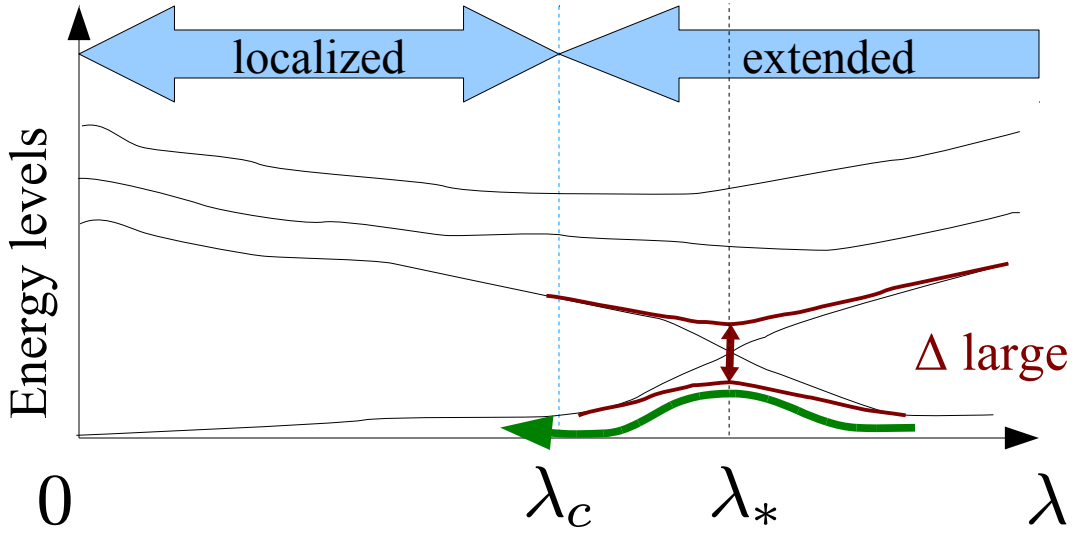




Tunneling: extended state

What if a local minimum later becomes the global minimum?



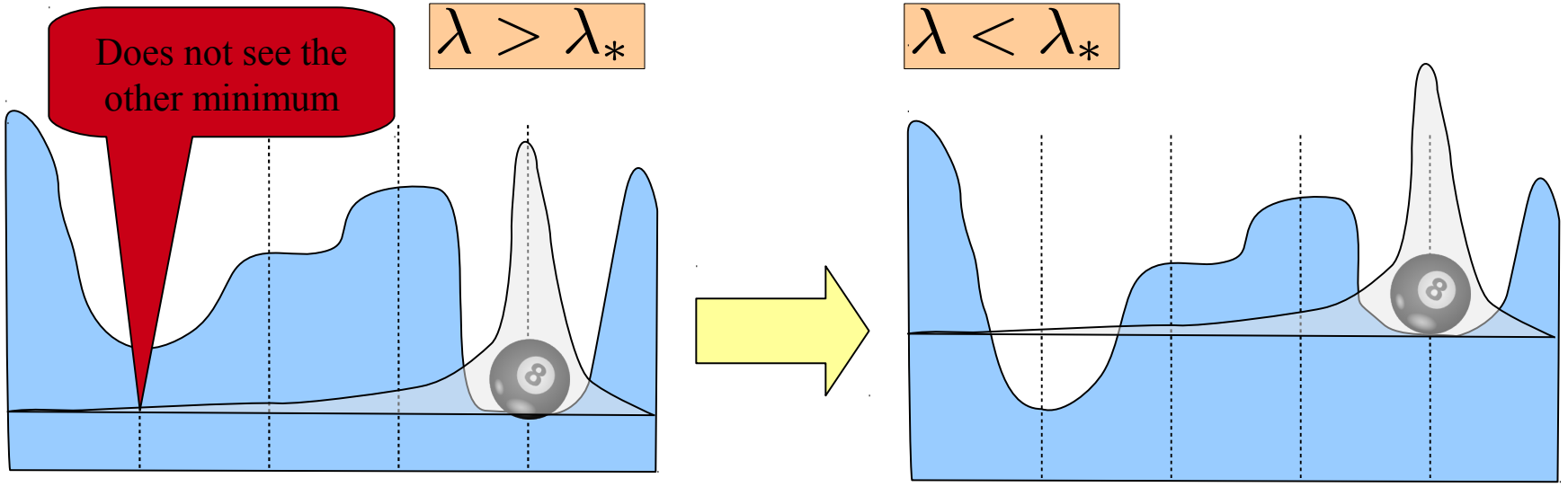


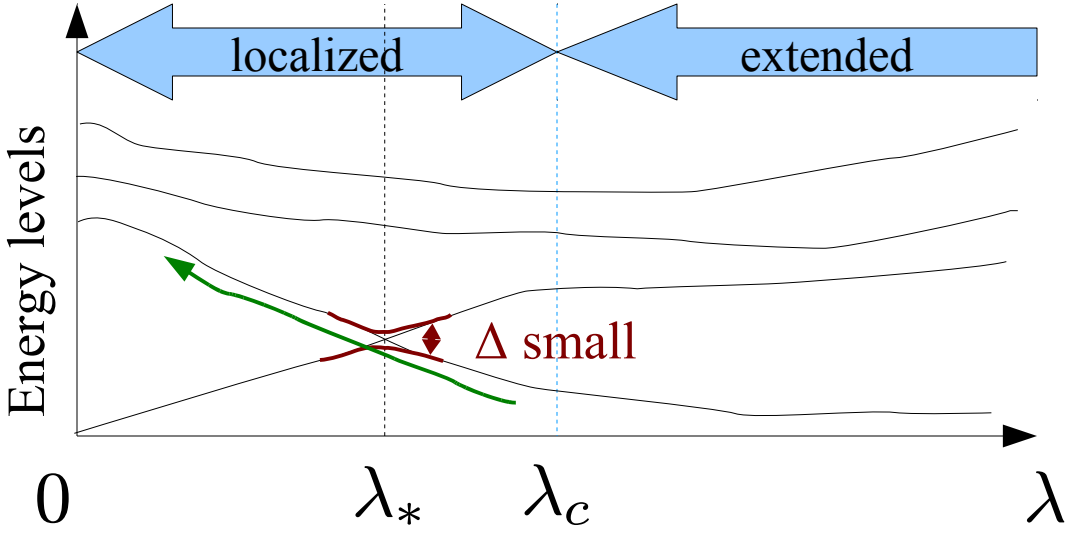
Crossing avoided due to "spreading"

→ Tunneling

Tunneling: localized state

What if a local minimum later becomes the global minimum?

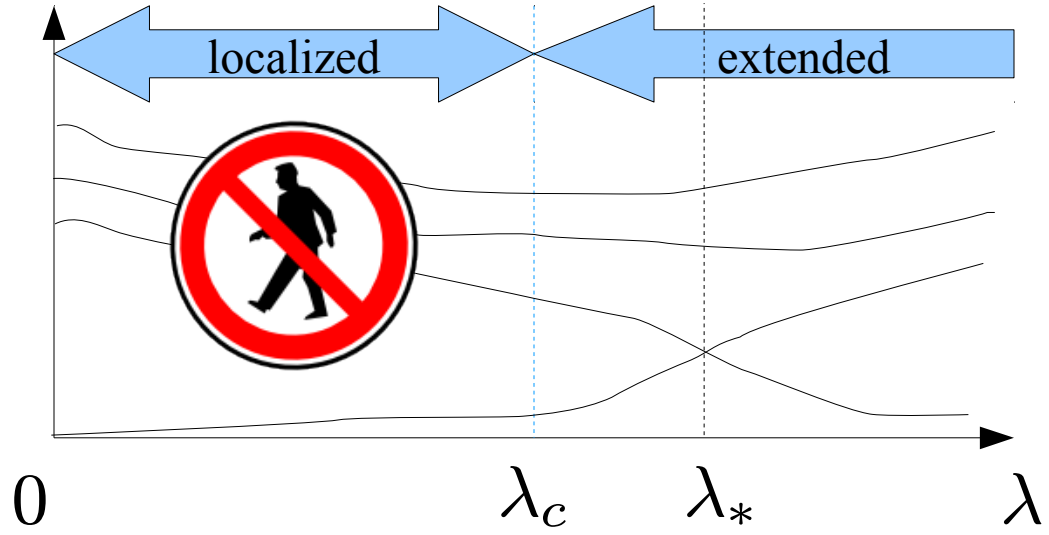




Little "spreading"

→ Inefficient tunneling

Our result



As the size of the problem N increases

1) Anderson localization would imply $\lambda_c = \Omega(1/\log N)$

2) Anti-crossings for smaller and smaller $\lambda_* = (CN)^{-1/8}$

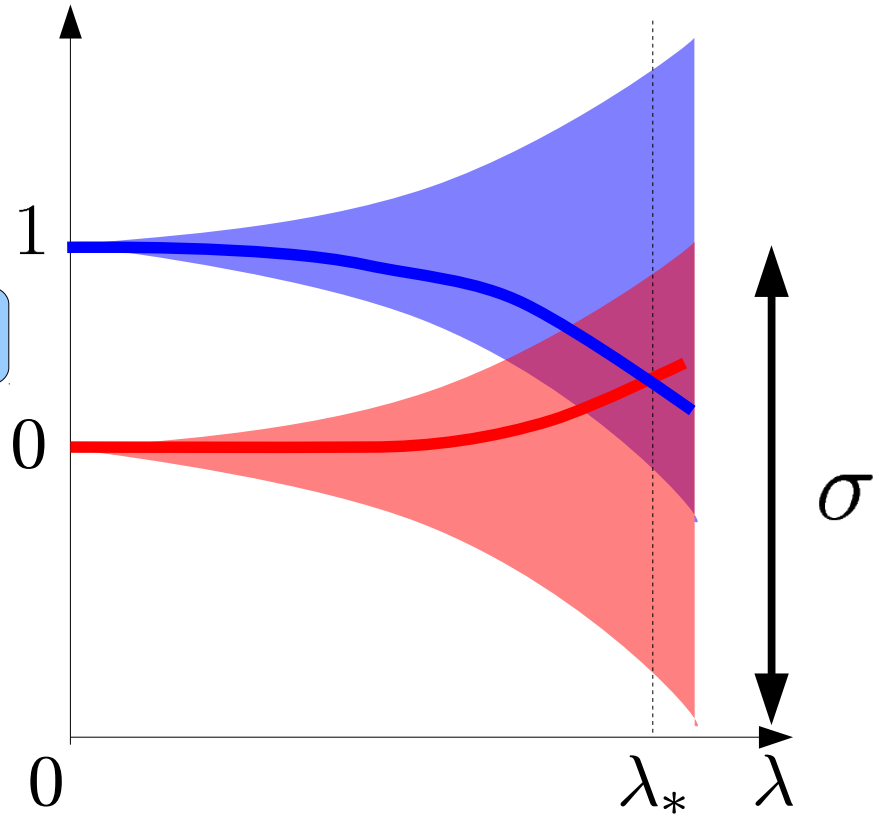
For $N > \dfrac{1}{C\lambda_c^8}$ we have $\lambda_* < \lambda_c$

The algorithm fails (stuck in a local minimum)

Level anti-crossings

Study slopes by perturbation theory

Random instances ⇒ random slopes!



As N increases,

- Standard deviation σ increases
- Position of anti-crossing λ_* goes to 0

Level anti-crossings

Consider EC3 instance with 2 solutions $\vec{x}_1, \vec{x}_2$

$$E_1(0) = E_2(0) = 0$$

Suppose

$$E_1(\lambda_*) - E_2(\lambda_*) > 4$$

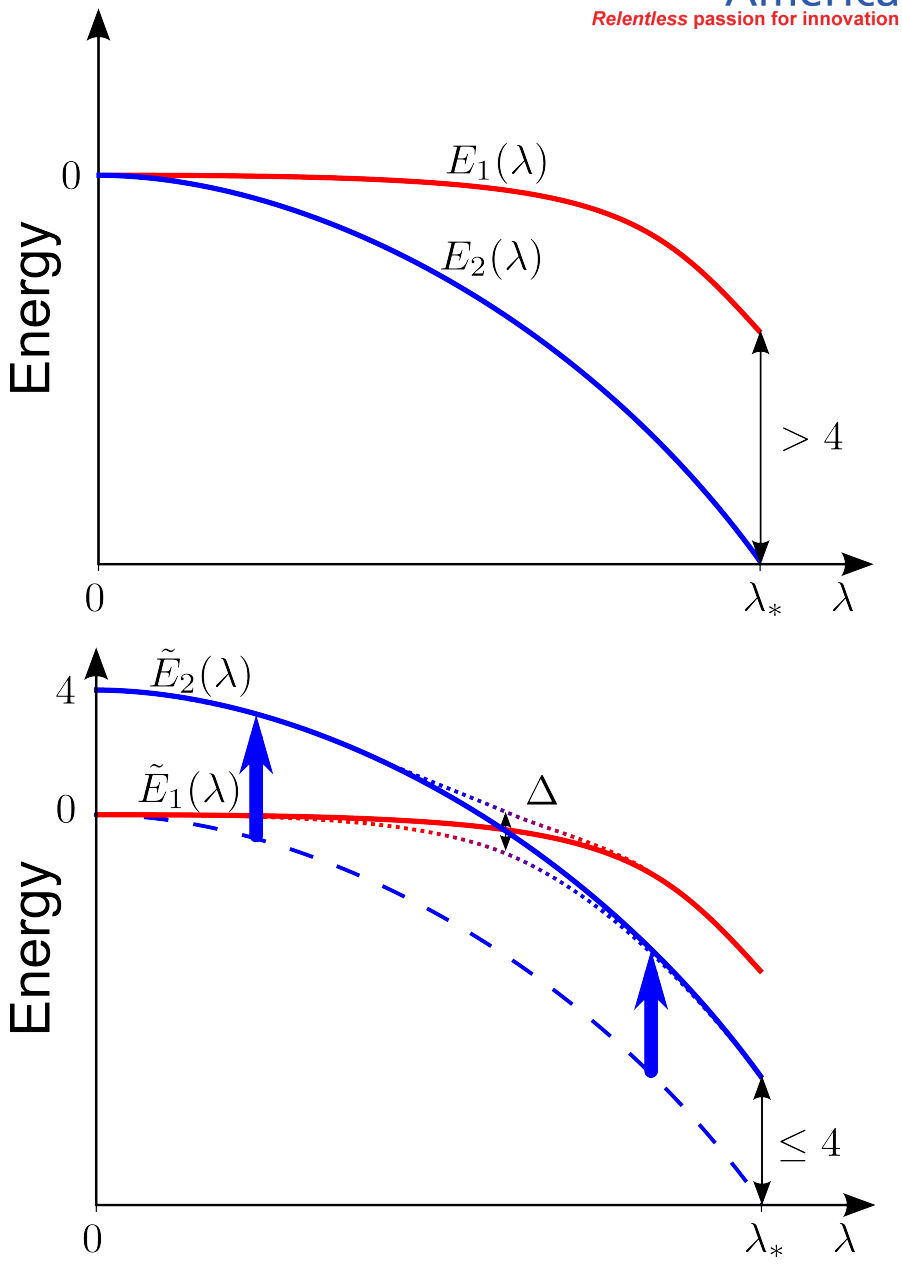
Add a clause

- satisfied by $\vec{x}_1$ $\quad \tilde{E}_1(0) = 0$
- violated by $\vec{x}_2$ $\quad \tilde{E}_2(0) = 4$

anti-crossing

$$d(\vec{x}_1, \vec{x}_2) = n$$

$\rightarrow$ Gap $\quad \Delta \sim \lambda_*^n$



Perturbation theory

We compute $E_{1,2}(\lambda)$ by perturbation theory

$$E_{\vec{x}}(\lambda) = E_{\vec{x}}(0) + \sum_{m=1}^{\infty} \lambda^{2m} F_{\vec{x}}^{(m)}$$

We prove:

$$\boxed{F_{\vec{x}}^{(m)} = O(N)\ \forall m}$$

For 2 solutions, the difference has zero mean, so

$$\boxed{(F_1^{(m)} - F_2^{(m)})^2 = O(N)\ \forall m}$$

Numerical simulations

- We generated EC3 random instances with >2 solutions
- then computed $E_1(\lambda) - E_2(\lambda)$ by order 4 perturbation theory

Leading order because:
- Odd orders are zero
- Order 2 is solution-independent for EC3

Numerical simulations

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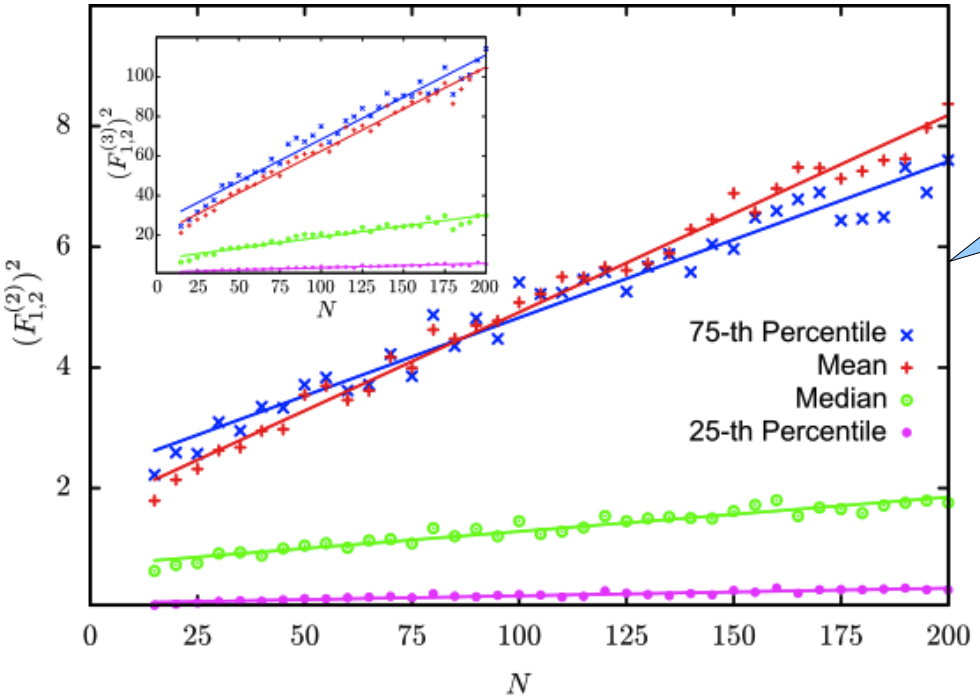
- then computed $E_1(\lambda) - E_2(\lambda)$ by order 4 perturbation theory



Each data point computed from 2500 instances

$$(E_1(\lambda) - E_2(\lambda))^2 \approx CN\lambda^8$$

We have $E_1(\lambda) - E_2(\lambda) > 4$ for $\lambda > \sqrt{2}(CN)^{-1/8}$

How small is the gap?

We show that up to leading order in perturbation theory:

$$\boxed{\Delta < (2\lambda_*)^n}$$

Since: 1) level crossings appear at $\lambda_* = O(N^{-1/8})$

2) typical distance between solutions is $n = \Theta(N)$

We have: $\boxed{\Delta = O(\exp(-N \log N))}$

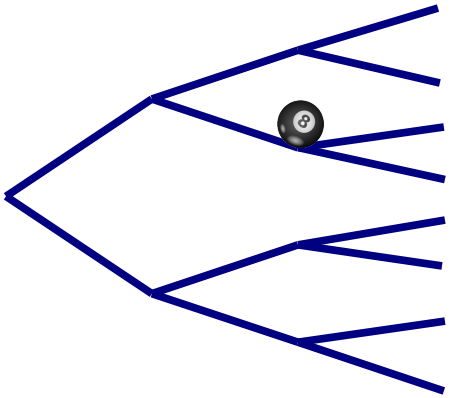
Can we trust perturbation theory?

Anderson localization theory

$\Rightarrow$ Perturbation theory valid as long as states are localized

Cayley tree with branching number K :

$$\lambda_c = \Theta\left(\frac{E}{K \log K}\right)$$



Here: Energy E and degree K are $\Theta(N)$, which would imply

$$\lambda_c = \Theta((\log N)^{-1}) \gg \Theta(N^{-1/8}) = \lambda_*$$

However, $F_{\vec{x}}^{(m)} = O(N) \quad \forall m$ suggests $\lambda_c = \Theta(1)$

Degeneracy of the ground state

- Our estimation: $\sigma \sim \sqrt{N}\lambda^4$

- Ground state is degenerate

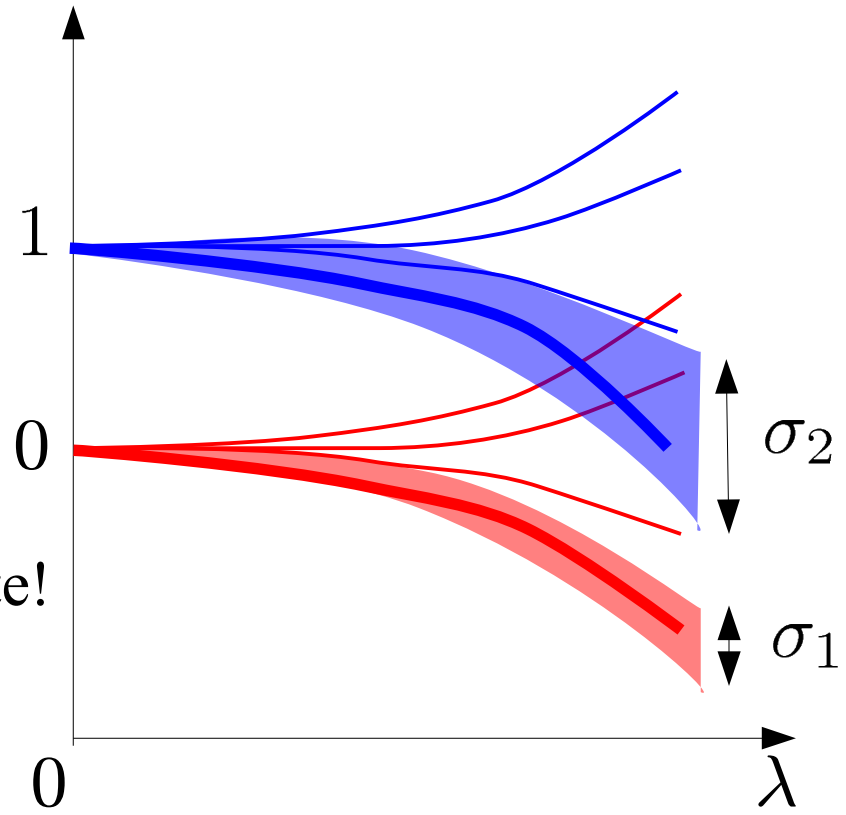
$$S_1 \sim e^{\eta N} \quad \Rightarrow \quad \sigma_1 \sim \lambda^4$$

[Knysh-Smelyanskiy'10]

BUT

- First excited state is more degenerate!

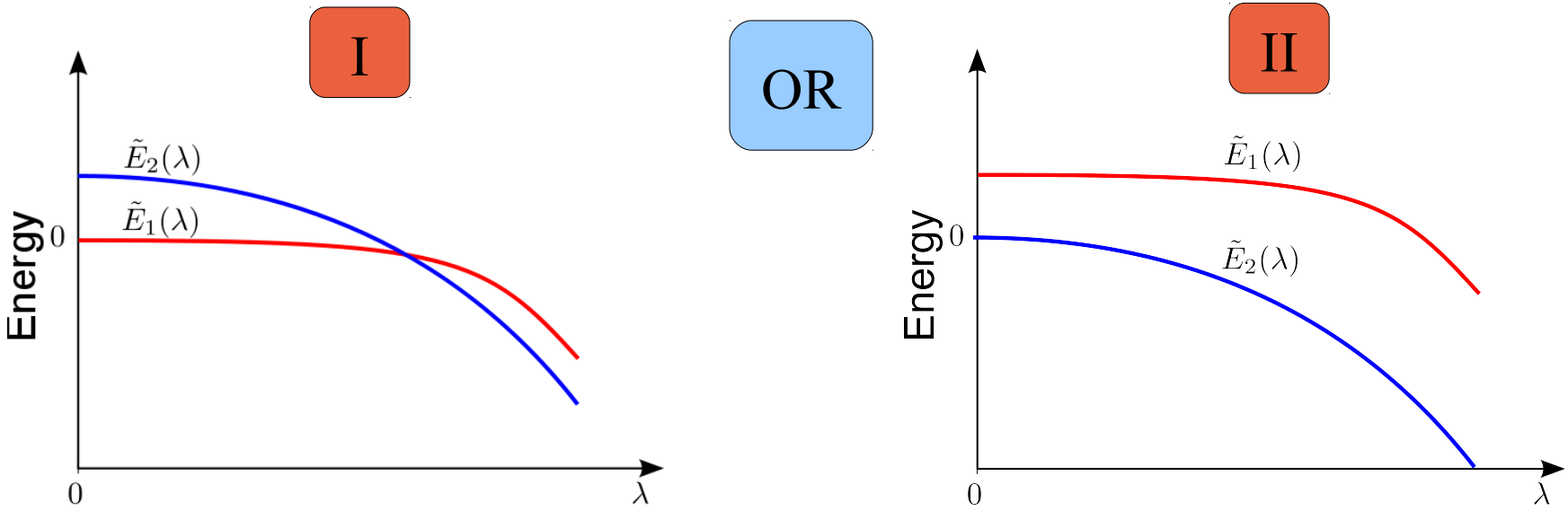
$$S_2 \sim N\ e^{\eta N} \quad \Rightarrow \quad \sigma_2 \sim \frac{\log N}{\sqrt{\eta}}\lambda^4$$



- Also: $\eta \to 0$ as $\alpha \to \alpha_s$

$\Rightarrow$ Effect of degeneracy only appears for large N

Effect of path change

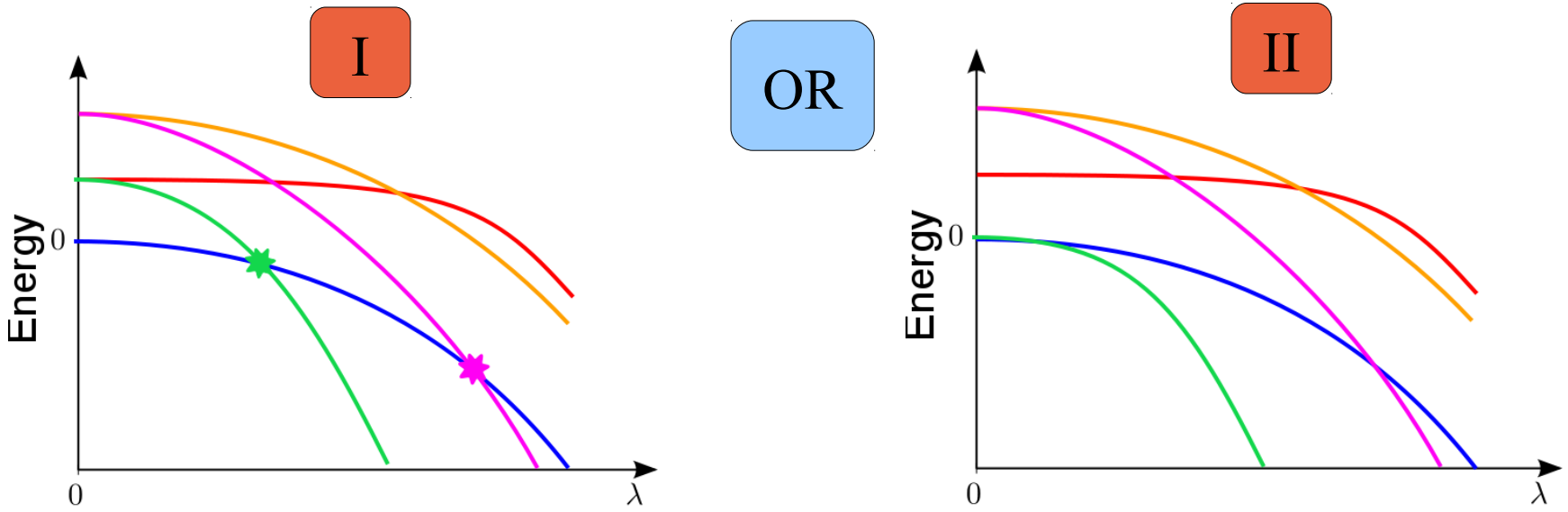


- Idea: Pick random $H(s)$ ("path change") to obtain case II

[Farhi *et al.*'09]

- Avoid 1 crossing: $Pr[\text{“II”}] = \text{constant}$

Effect of path change



- Idea: Pick random $H(s)$ ("path change") to obtain case II

[Farhi *et al.*'09]

- Avoid 1 crossing: $Pr[\text{"II"}] = \text{constant}$
- Avoid poly # of crossings: $Pr[\text{"II"}] = 1/\text{poly}$
- Estimated # of crossings: $\exp(N/\log^8 N)$

Conclusion

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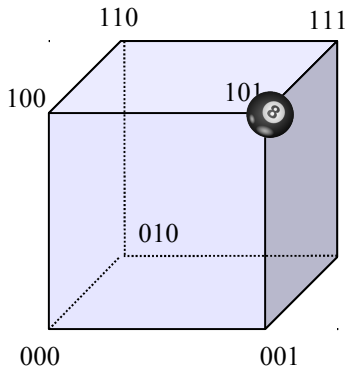
- Anderson localization causes exponentially small gaps in adiabatic quantum optimization
- Does not depend on the particular problem (same for 3SAT)
- Does not depend on the particular path $H(s)$ either

  (as long as $H(s)$ is local)
- Important assumption: Localization on the hypercube



⇒ should be studied more closely

Thank you!