Superadditivity of classical capacity revisited

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We introduce new type of superadditivity for classical capacity of quantum channels, which involves the properties of channels' environment. By imposing different restrictions on the total energy contained in channels environment we can consider different types of superadditivity. Using lossy bosonic and additive noise quantum channels as the examples, we demonstrate that they can be either additive or superadditive depending on the values of channels parameters. The parameters corresponding to transition between the additive and superadditive cases are related with recently found crytical and supercritical parameters of Gaussian channels.

Characterization of quantum channels capacities is one of the most important tasks in the theory of quantum communications. It attracted much interest because of possible superadditivity owing to the use of entangled states. A memoryless quantum channel is the simplest basic channel model. Its n invocations (uses) act on the total n-partite input state $\hat{\rho}_{\rm in}$ as a sequence $\Phi^{\otimes n}$ of identical maps Φ. Alternatively, these invocations can be thought of as a single multidimensional channel or n parallel identical channels acting at once. It was believed that the capacity of such channels is additive. However, in 2008 it was shown by Hastings that this hypothesis does not hold in general, *i.e.* non-correlated channels acting together may have a higher capacity than they have acting separately. This superadditivity arises because the channels are allowed to share an arbitrary joint n-partite input state.

We focus on continuous variables (CV) bosonic Gaussian channels. They are highly relevant to experimental implementations. For these channels the capacity is defined under the energy restriction on the input state which can be translated into the average amount of photons nN given to n uses of the channel:

$$nN = \sum_{k=1}^{n} N_k.$$

Then the capacity C for a kth single use of the memoryless channel becomes a function of the number of photons N_k given to this use:

$$\begin{array}{l} N_1 \longrightarrow \boxed{C_1 = C_1(N_1)} \longrightarrow C_1 \\ \qquad\cdots\cdots\cdots\cdots\cdots\cdots \\ N_n \longrightarrow \boxed{C_n = C_n(N_n)} \longrightarrow C_n \end{array}$$

In this setting another type of superadditivity might be possible. Indeed, if $C(N)$ is not concave, then the superadditivity may arise just due to non-uniform distribution of the average amount of photons N_k between the uses.

For lossy [1] and additive noise [2] bosonic Gaussian channels we have found that the capacity restricted to Gaussian encoding and modulation is concave. Therefore for these channels this superadditivity does not exist. However, if one finds a CV channel with non-concave dependence $C(N)$ its capacity will be superadditive. Note that our result on the concavity of $C(N)$ for the lossy and additive noise channels allowed us to find the capacity for both of these channels in the presence of memory by using convex separable programming [1].

The action of the channel may be equivalently represented by a unitary transformation applied to the product state of channel's input and environment:

$$\hat{\rho}_{\rm out} = {\rm Tr}_{\rm env}\left[U\left(\hat{\rho}_{\rm in}\otimes\hat{\rho}_{\rm env}\right)U^{+}\right].$$

Using this fact in the context of superadditivity we make a natural step further and pose a question: what happens with the capacity of n parallel channels if their "environments" are allowed to be in a joint arbitrary n-partite state similarly to the channels inputs. Then, the average amount of photons in the environments $nN_{\rm env}$ plays the role similar to that of input [1]. Now the capacity becomes a function of two variables: $C = C(N, N_{\rm env})$, *i.e.*

$$\begin{array}{l} N_1, N_{\rm env,1} \longrightarrow \boxed{C_1 = C_1(N_1, N_{\rm env,1})} \longrightarrow C_1 \\ \qquad\cdots\cdots\cdots\cdots\cdots\cdots \\ N_n, N_{\rm env,n} \longrightarrow \boxed{C_n = C_n(N_n, N_{\rm env,n})} \longrightarrow C_n \end{array}$$

and we propose a new formulation of the superadditivity problem:

Given the values nN and $nN_{\rm env}$

- *What is the channel capacity?*
- *What is the optimal channel input state?*
- *What is the optimal state of the channel environment?*

In this setting the problem originally formulated for n identical parallel channels is transformed to a problem for n parallel channels of the same type which may differ only by their noise (average amount of photons in their environments). Thus, the optimal environment and input will be defined by average photon number distributions between "black boxes" represeting parallel channels characterized by $C(N, N_{\rm env})$.

Interestingly, for the lossy channel we have shown that the optimal photon number distribution between the environments of the channels may be non-uniform which can be interpreted as a "violation of mode symmetry" [1]. In addition, this solution presents superadditivity in the form we are proposing. The reason is that despite C is concave as a function of N for any fixed $N_{\rm env}$, it is generally neither a concave nor a monotonic function of $N_{\rm env}$ for fixed N.

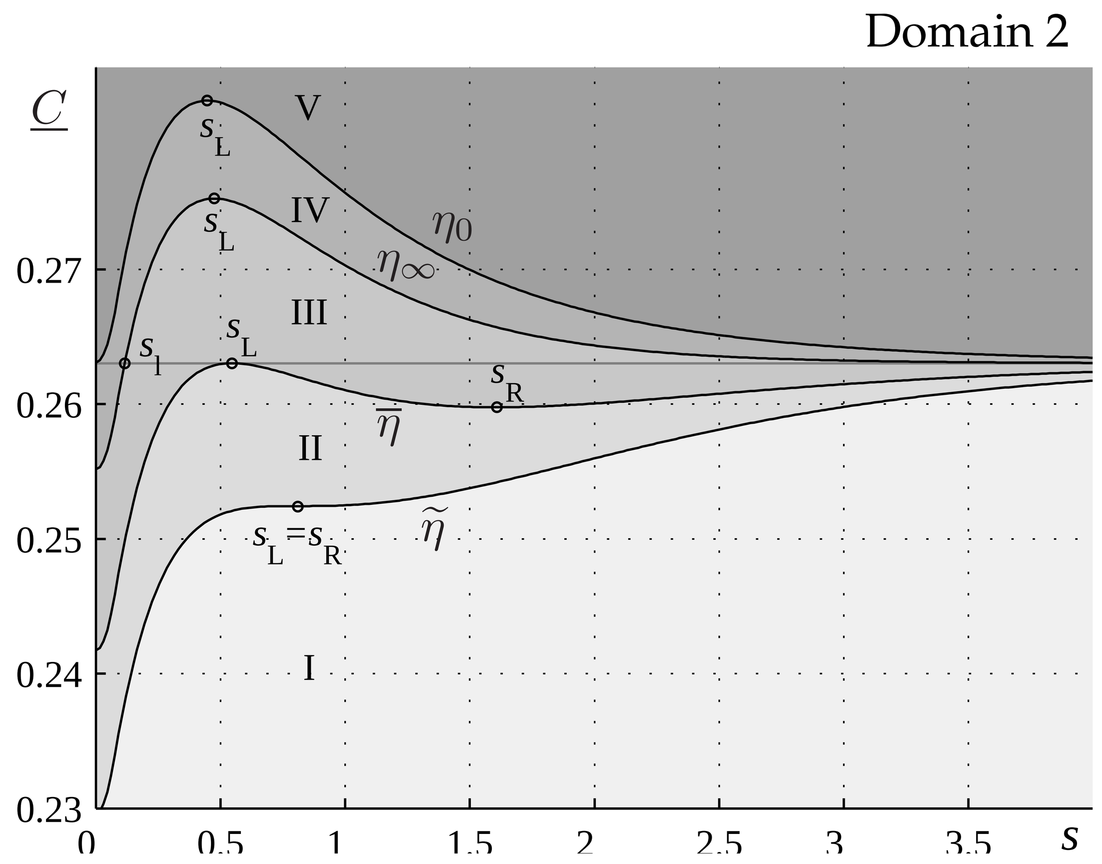

Figure 1: $C(s)$ **for different values of transmissivity** η**.**

An environment squeezing s and average amount of thermal photons $\mathcal{N}_{\rm env}$ are entries of environment covariance matrix

$$V_{\rm env} := V(\mathcal{N}_{\rm env}, s) = \left[\mathcal{N}_{\rm env}+\frac{1}{2}\right]\begin{pmatrix} e^{s} & 0 \\ 0 & e^{-s}\end{pmatrix}.$$

where

$$\frac{1}{2}\,{\rm Tr}\,V_{\rm env} = N_{\rm env}+\frac{1}{2},$$
$$N_{\rm env} = \left(\mathcal{N}_{\rm env}+\frac{1}{2}\right)\cosh s - \frac{1}{2}.$$

The solution of the optimal environment problem may be applied to the problem of "optimal channel memory", because some important memory channels can be *unravelled*, so that the capacity of a memory channel becomes equal to the capacity of parallel independent channels with the non-uniform distribution of photons between the environments. The optimality of non-uniform distribution is translated to superiority of the capacity of a memory channel over the memoryless one under the same energy constraints for both input and environment. Since information transmission rates may also be non-monotonous functions of $N_{\rm env}$, the problem of optimal channel memory can be posed for those quantities as well [1].

We have shown that the non-uniform distribution is not always optimal and the transition between the "uniform" and "non-uniform" optimal solutions is governed by so-called "critical" and "supercritical" parameters introduced for both lossy [1] and additive noise [2] channels. Transition between domains "2" and "3" happens if

$$(1-\tilde{\eta}_\infty)\left(\mathcal{N}_{\rm env}+\frac{1}{2}\right) = \frac{1}{\sqrt{12}},$$

where the *effective supercritical transmissivity* $\tilde{\eta}_\infty$ is

$$\tilde{\eta}_\infty = \sqrt{\frac{2}{15}\left(1+(2N+1)^{-2}\right)}.$$

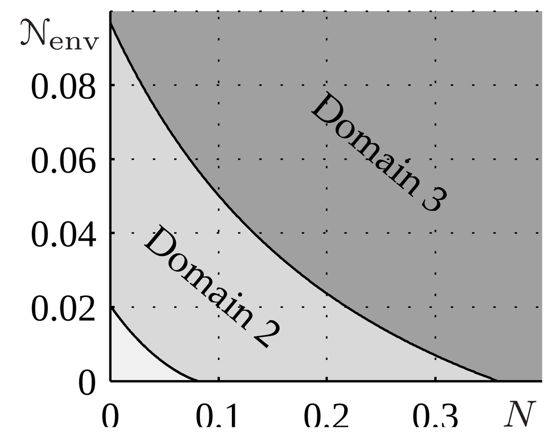

Figure 2: Lossy channel, the supercritical parameters.

Analytical results:

- $C(s)$ is always monotonic over $0 < s < +\infty$ if both

$$N \geqslant \left[\sqrt{3/2+5/(2\sqrt{3})}-1\right]/2 \approx 0.3578,$$
$$\eta \leqslant 1-1/\sqrt{3} \qquad (1)$$

and has no more than one maximum in this interval otherwise.

- $C(s)$ has no more than one maximum if

$$\mathcal{N}_{\rm env} \geqslant \left[\left(\sqrt{3}-2/\sqrt{5}\right)^{-1}-1\right]/2 \approx 0.0969.$$

- The equality $C(0) = C(\infty)$ for $N \to \infty$ is possible only if

$$\eta \geqslant \frac{2}{e},$$

the equality holds for pure environment ($\mathcal{N}_{\rm env} = 0$).

- The condition similar to Eq. (1) also exist for additive noise channel [2]:

$$\mathcal{N}_{\rm env} \geqslant \frac{1}{\sqrt{12}}$$

We expect that these constants can also be used to characterize the boundaries between the additive and superadditive cases.

The heterodyne rate $R^{\rm (het)}$ is also non-monotonic function of $N_{\rm env}$ (or environment squeezing s, equivalently).

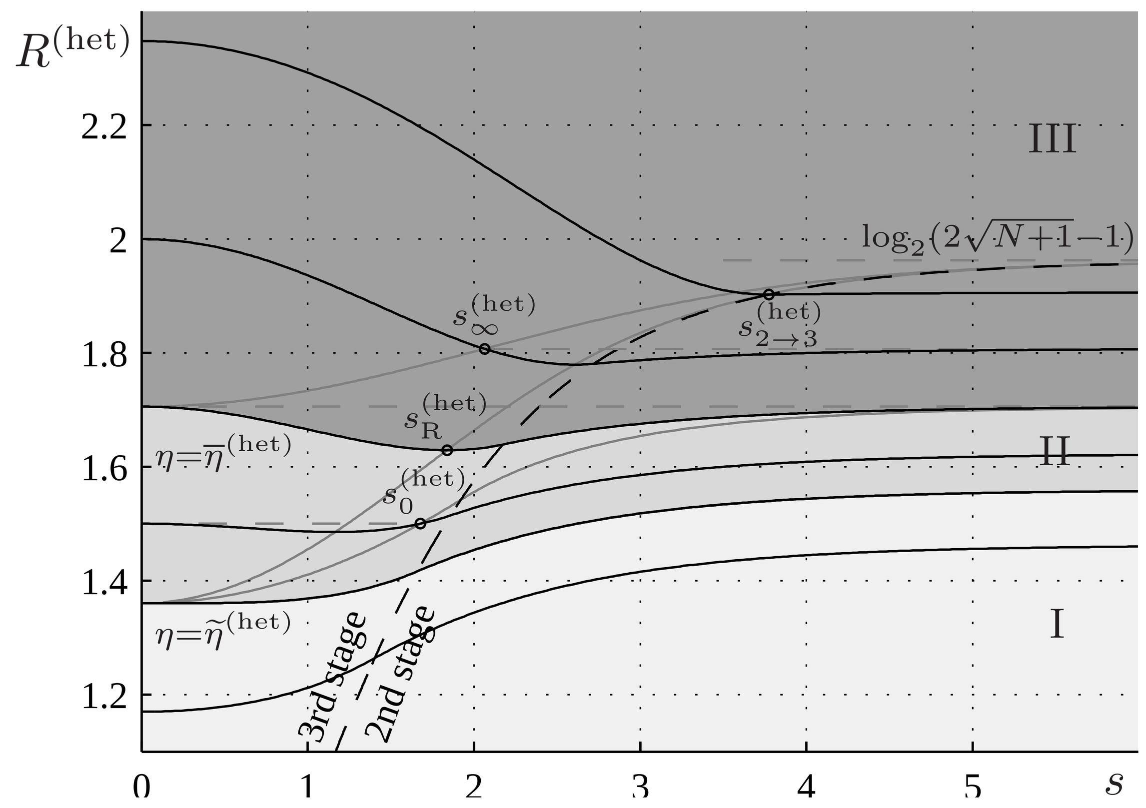

Figure 3: The heterodyne rate $R^{\rm het}(s)$ **for different values of transmissivity** η**.**

Thus, the superadditivity of new type can be posed for the heterodyne rate as well. In the case of pure environment, the transitions between the regions I, II and III correspond the the following values of transmissivity:

$$\tilde{\eta}^{\rm (het)} = \frac{1}{1+N}, \qquad \bar{\eta}^{\rm (het)} = \frac{2}{2+N}$$

The "universal limits" for the heterodyne rate and capacity are

$$\lim_{s\to\infty} R^{\rm (het)} = \log_2\left[\sqrt{1+2(2N+1)\eta+\eta^2}-\eta\right],$$
$$\lim_{s\to\infty} C = \log_2(2N+1).$$

References

[1] O. V. Pilyavets, C. Lupo and S. Mancini, arXiv:0907.1532v3 (2011). To appear in *IEEE Tran. Inf. Th.* **58**:12 (2012).

[2] J. Schäfer, E. Karpov and N. J. Cerf, *Proc. of SPIE* **7727** (Bellingham, WA), 77270J (2010).

[3] J. Schäfer, E. Karpov and N. J. Cerf, *Phys. Rev.* A **84**, 032318 (2011).