Classical Capacity of a Lossy Bosonic Channel with Memory

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The classical capacity (CC) of lossy bosonic channels with non-Markovian memory is studied. Heuristics is found as a good approximation for a CC. Solution for heuristics is found analytically, while solution for CC itself is found numerically in general, and analytically where it is possible. Phase transition effects in the context of bosonic channels with memory are also discussed.

Introduction

A lot of real-world quantum channels can imply a strongly correlated noise (memory) between uses of the channel. Recently a lot of efforts have been dedicated to the development of quantum models that encompass memory effects. The main motivation that has led to investigate such effects in quantum channels has been the possibility to enhance their CC by means of entangled inputs. Such a possibility has been recently put forward in channels with continuous alphabet. These make use of bosonic field modes whose phase space quadratures enable for continuous variable encoding/decoding. Since the notion of capacity is intimately related with the asymptotic behavior of a channel, there is a persistent wish to move on from small to large (towards infinite) number of channel uses. The lossy bosonic channel (LBC), which consists of a collection of bosonic modes that lose energy en route from the transmitter to the receiver, belongs to the class of Gaussian channels which provide a fertile testing ground for the general theory of quantum channels' capacities [1] and are easy to implement experimentally [2].

General theory, notations and conjectures

General scheme of LBC is depicted in Fig. 1. Each input mode (left-right line), representing one use of the channel, interacts with the corresponding environment mode (top-bottom line) through a beam-splitter with transmissivity η. To introduce memory effects, environment modes are initially considered in a correlated state.

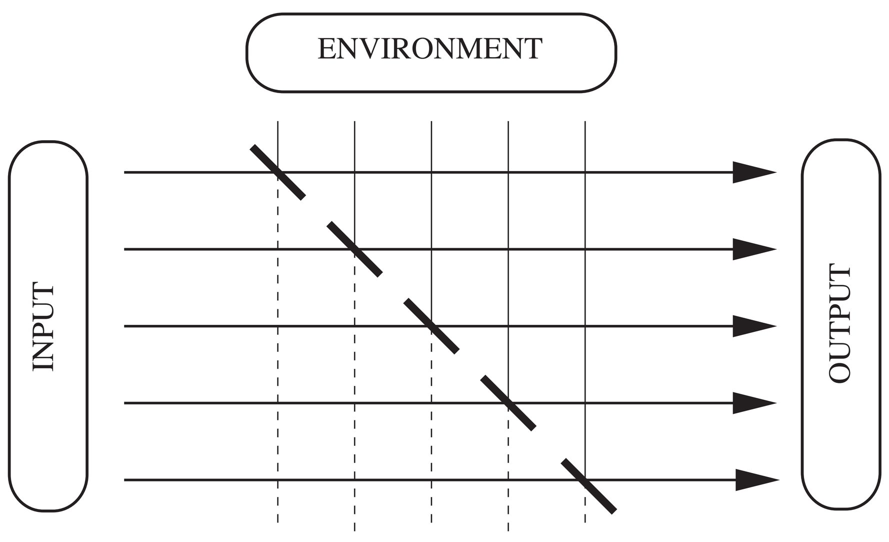

Figure 1: The model for a lossy bosonic channel.

Any state can be labeled as ρ or V as all states involved are multimode squeezed Gaussian ones. The next $2n \times 2n$ covariance matrices [quadratic forms on vectors $(q_1, \dots, q_n, p_1, \dots, p_n)^\top$] and their eigenvalues are stated for:

- $V_{\rm in}$, i_{rk} - input state
- $V_{\rm env}$, e_{rk} - environment state
- $V_{\rm cl}/2$, $c_{rk}/2$ - classical distribution of coherent amplitude $\boldsymbol{\alpha}$.
- $V_{\rm out}$, o_{rk} - output state
- $\overline{V}_{\rm out}$, a_{rk} - output state averaged over classical distribution (encoding of information) $V_{\rm cl}$

Indices $k = 1, \dots, n$, and $r = q, p$. Also, let us introduce formal rule: if $r = q$, $\bar{r} = p$ and vice versa.

Classical variable $\boldsymbol{\alpha}$ is encoded via random displacements of a Gaussian seed state $\rho_{\rm in}$, that is

$$\rho_{\rm in}^{\boldsymbol{\alpha}} = \left[\otimes_{j=1}^{n} D_j(\alpha_j)\right] \rho_{\rm in} \left[\otimes_{j=1}^{n} D_j(\alpha_j)\right]^\dagger .$$

Energy restriction at the input of channel is N photons per mode in avarage:

$$\frac{{\rm Tr}(V_{\rm in} + V_{\rm cl})}{2n} = N + \frac{1}{2}.$$

Channel (beam splitter) action can be reduced to the next relations:

$$\begin{aligned} V_{\rm out} &= \eta\, V_{\rm in} + (1-\eta)\, V_{\rm env}, \\ \overline{V}_{\rm out} &= \eta\, (V_{\rm in} + V_{\rm cl}) + (1-\eta)\, V_{\rm env}. \end{aligned}$$

Holevo χ-quantity for n uses can be calculated as

$$\chi_n = \sum_{k=1}^{n} \left[g\left(\nu_k^{(\overline{V}_{\rm out})} - \frac{1}{2}\right) - g\left(\nu_k^{(V_{\rm out})} - \frac{1}{2}\right) \right],$$

where $\nu_k^{(\overline{V}_{\rm out})}$ and $\nu_k^{(V_{\rm out})}$ are the symplectic eigenvalues of corresponding matrices and

$$g\,(x) := (x+1)\log_2(x+1) - x\log_2 x.$$

CC for n uses C_n and for infinite uses C can be calculated as

$$C_n = \max_{\{V_{\rm in}, V_{\rm cl}\}} \frac{\chi_n}{n}, \qquad C = \max_{n\to\infty} C_n.$$

Conjectures used to calculate the CC:

- CC for LBC can be achieved on Gaussian states.
- Maximizing of Holevo-χ over $V_{\rm in}$, $V_{\rm cl}$ leads to a CC for memory channel also.
- All blocks V_{ij} of matrix

$$V_{\rm env} = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^\top & V_{22} \end{pmatrix}$$

commute each with other. One can show that this conjecture is sufficient to always reach CC on commuting each with other matrices $V_{\rm in}$, $V_{\rm cl}$ and $V_{\rm env}$, therefore all matrices used below considered in diagonalized form.

One use of LBC

Abitrary covariance matrices can be represented in the form:

$$V_{\rm env} = (N_{\rm env}+1/2)\begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}, \quad V_{\rm in} = (N_{\rm in}+1/2)\begin{pmatrix} e^r & 0 \\ 0 & e^{-r} \end{pmatrix}.$$

CC can be found in general using Karush-Kuhn-Tucker (KKT) conditions (generalization of Lagrange multipliers method to include inequalities, which appear due to uncertainty relation). KKT shows that CC is always achieved on pure input state $V_{\rm in}$. In this paragraph we omit index k for eigenvalues of matrices as we have only 1-use case.

Holevo-χ case

Maximizing Holevo-χ over $V_{\rm in}$, $V_{\rm cl}$ we found the following. Suppose, that eigenvalues are the next:

$$\begin{aligned} c_q &= N + \frac{1}{2} - \frac{1}{2}\sqrt{\frac{e_q}{e_p}} + \frac{1}{2}\left(1 - \frac{1}{\eta}\right)(e_q - e_p), \\ c_p &= N + \frac{1}{2} - \frac{1}{2}\sqrt{\frac{e_p}{e_q}} + \frac{1}{2}\left(1 - \frac{1}{\eta}\right)(e_p - e_q), \\ i_q &= \frac{1}{2}\sqrt{\frac{e_q}{e_p}}, \qquad i_p = \frac{1}{2}\sqrt{\frac{e_p}{e_q}}. \end{aligned}$$

If both $c_q, c_p \geqslant 0$ we have a *case of 3rd phase* and explicit relation for capacity:

$$C = g[\eta N + (1-\eta)((N_{\rm env}+1/2)\cosh(s) - 1/2)] - g[(1-\eta)N_{\rm env}]$$

If some c_r according to previous relations is negative we have a *case of 2nd phase*, then $c_r = 0$, $c_{\bar r} = 2N + 1 - i_r - 1/(4i_r)$, $i_{\bar r} = 1/(4i_r)$ and i_r can be found as the solution of the transcedent equation

$$\frac{a_{\bar r} - a_r}{\sqrt{a_{\bar r} a_r}} \log_2 \frac{\sqrt{a_{\bar r} a_r} + 1/2}{\sqrt{a_{\bar r} a_r} - 1/2} = \frac{o_{\bar r} i_r - o_r i_{\bar r}}{i_r \sqrt{o_{\bar r} o_r}} \log_2 \frac{\sqrt{o_{\bar r} o_r} + 1/2}{\sqrt{o_{\bar r} o_r} - 1/2}.$$

No any explicit relation for CC can be found in this case.

The third possibility is both $c_q, c_p \leqslant 0$, which is a *case of 1st phase*, however, it exists for 1 use only in degenerated form: $c_q = 0$, $c_p = 0$ what corresponds to $N = 0, C = 0$. Thus, starting from $N = 0$ and increasing it we pass 3 phases and 2 phase transitions.

Heuristics-ϵ case

One can get that solution of KKT is changed very small (less than 1% variation, and usually less than 0.05%) if instead of Holevo-χ we maximize quantity

$$\epsilon_n = \sum_{k=1}^{n} \left[\log_2\left(\nu_k^{(\overline{V}_{\rm out})}\right) - \log_2\left(\nu_k^{(V_{\rm out})}\right) \right].$$

This discovering allows to solve 2nd phase case as explicit analytical relation for 1 and 2 uses of channel, and find analytically CC for n uses expressing it through a root of one transcedent equation depending of one variable. Also, high potential for analytical investigation is expected applying ϵ_n to a problem of other capacities.

Case of 3rd phase gives the same as Holevo-χ. For the case of 2nd phase when $c_r \leqslant 0$ (in original equation) we have

$$E = \frac{\eta/4}{\eta(2N+1) + (1-\eta)e_{\bar r}}, \quad B = \frac{e_{\bar r}(1-\eta)/4}{\eta(2N+1) + (1-\eta)e_{\bar r}},$$
$$i_{\bar r} = E + \sqrt{E^2 + B}, \quad i_r = 1/(4 i_{\bar r}), \quad c_{\bar r} = 2N + 1 - i_r - i_{\bar r}.$$

Multiple uses of LBC

The case of multiple uses can be solved completely analagously to 1 use case. Explicit analytical solution is possible only when $c_{qk}, c_{pk} \geqslant 0$ for $\forall k$:

$$C_n = g\left[\eta N + (1-\eta)\left(\frac{{\rm Tr}\, V_{\rm env}}{2n} - \frac{1}{2}\right)\right] - \frac{1}{n}\sum_{k=1}^{n}[(1-\eta)(\sqrt{e_{qk} e_{pk}} - 1/2)]$$

In the general case applying heuristics-ϵ approach we get the following transcedent equation on variable x (this is simply equation of modes balance):

$$\eta\, {\rm Tr}_{\{1,2|c_{rj}\neq 0\}}\, \overline{V}_{\rm in} + (1-\eta)\, {\rm Tr}_{\{1,2|c_{rj}\neq 0\}}\, V_{\rm env} = (2n_1 + n_2)x,$$

where ${\rm Tr}_{\{1,2|c_{rj}\neq 0\}}$ stands for summation over all 1st and 2nd phase quadratures except of that j for which $c_{rj} = 0$. Parameters n_k is amount of modes which belong to kth phase. $\overline{V}_{\rm in} = V_{\rm in} + V_{\rm cl}$. In explicit way

$${\rm Tr}_{\{1,2|c_{rj}\neq 0\}}\, \overline{V}_{\rm in} = 2n(N + 1/2) - n_3 - \textstyle\sum_{\{j\in 2_{\rm ph}|c_{rj}=0\}} i_{rj}(x),$$
$$i_{rj}(x) = \frac{E_{\bar r j}}{\eta}\left(\sqrt{1 + 2x/E_{\bar r j}} - 1\right), \qquad E_{rj} = \frac{\eta^2}{8(1-\eta)e_{rj}}.$$

Also, $n = n_1 + n_2 + n_3$.

KKT equations gives the chain:

$$\begin{aligned} x :&= a_{pn} = a_{qn} = a_{pm} = a_{qm} = \dots \\ &= a_{qk} = o_{qk}\frac{i_{pk}}{i_{qk}} = a_{pj} = o_{pj}\frac{i_{qj}}{i_{pj}} = \dots \end{aligned}$$

Here modes $n, m \in 1_{\rm ph}$, while $k, j \in 2_{\rm ph}$ where $c_{pk} = 0$ and $c_{qj} = 0$. This chain allows express to CC in explicit way as a function of x. In spite of quite difficult analytical expressions which can be obtained by this approach they essentially simplify numerical calculation of capacity.

Ω-model

As particular case of matrix $V_{\rm env}$ one can consider

$$V_{\rm env} = \frac{1}{2}\begin{bmatrix} \exp(s\Omega) & 0 \\ 0 & \exp(-s\Omega) \end{bmatrix},$$

where

$$\Omega = \begin{pmatrix} 0 & 1 & \dots & \dots & \dots & 0 \\ 1 & 0 & 1 & \dots & \dots & 0 \\ \vdots & 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & \ddots & 1 \\ 0 & 0 & \dots & \dots & 1 & 0 \end{pmatrix}.$$

CC for infinite uses for such LBC ($V_{\rm env}$ is entangled) and for the case of $N \gg s$ is

$$C = g\left[\eta N + \frac{1}{2}(1-\eta)(I_0(2s) - 1)\right],$$

what clearly shows growth of CC with growing degree of memory s.

References

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