Superadditivity of classical capacity revisited
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We introduce new type of superadditivity for classical capacity of quantum channels, which involves the properties of channels’ environment. By imposing different restrictions on the total energy contained in channels environment we can consider different types of superadditivity. Using lossy bosonic and additive noise quantum channels as the examples, we demonstrate that they can be either additive or superadditive depending on the values of channels parameters. The parameters corresponding to transition between the additive and superadditive cases are related with recently found critical and supercritical parameters of Gaussian channels.

- Quantum channel is a map \( \Phi : \rho_{in} \rightarrow \rho_{out} \)
- Input is dual to environment: \( \rho_{in} \leftrightarrow \rho_{env} \)
- Additivity of Holevo \( \chi \) quantity for discrete channels [Hastings, Shor]:
\[
\chi(\rho) = \max_{\phi(\rho_1) = \rho_1, \phi(\rho_2) = \rho_2} \left\{ S \left[ \Phi(\rho_{in}) \right] \right\} - \left\{ S \left[ \rho_{in} \right] \right\}
\]
- How much they are? Product channels is - Additive (\( \alpha \)), if channels are identical.
- Superadditive, if channels are different.

Restriction of \( N_{env} \) is not really necessary: \( C < \infty \) \( \forall N_{env} \). Moreover, even in this case the optimal \( \{ N_{env} \} \) is not trivial, and vacuum environment \( \{ N_{env} = 0 \} \) is never optimal (!).

- What is about phase-insensitive Gaussian channels? It is proved, that they:
  A. Never optimal for information transmission [1].
  B. Have the lowest capacity for some range of parameters. E.g. it is the case for lossy channel, if
\[
N \geq \frac{1}{3} \left[ \sqrt{4/3} + 2(\pi T - 1) \right] / 0.3578
\]
and beam-splitter transmissivity \( \eta \leq 1 - 1/\sqrt{\eta} \) [1].

- How to generalize this additivity to continuous variables channels? Classical capacity of two channels \( n = 2 \) is finite only if input energy is finite:
\[
\frac{1}{2n} \int \text{Tr} \left[ \rho_{in} \rho_{env} \right] \rho_{in} \rho_{env} = N \frac{1}{2} < \infty
\]
- How to compare channels acting together and separately? E.g. capacity of two pure lossy channels:
\[
C = g(\gamma N_1) + g(\gamma N_2), \quad \text{where} \quad N_1 + N_2 = N
\]

If \( N_1, N_2 \) are optimally chosen, they are not equal to \( N/2 \) \( \Rightarrow C = g(\gamma N/2) + g(\gamma N/2) \equiv \) even pure lossy channel (vacuum environment) is superadditive. Classical Gaussian channels are superadditive too (waterfilling solution is non-equial distribution of \( N \) between the modes). Any two channels with different environments are superadditive! \( \Rightarrow \) This is bad definition of the superadditivity.

- What is about additivity of product channel? Is \( \chi(\Phi \otimes \Psi) = 2\chi(\Phi) \) correct? We proved it for lossy bosonic [1] and additive noise [2] channels assuming Gaussian conjecture to be valid. Capacity of both channels \( \{ C(N) \} \) \( \Rightarrow \) they cannot be superadditive. Nevertheless, if we will find some continuous variables channel with non-concave dependence \( C(N) \), we immediately get another superadditive channel in addition to Hastings’ example (!).

- What is the optimal state? Product channels is - Additive (\( \alpha \)), if channels are identical.
- Superadditive, if channels are different.

To find reasonable definition of additivity? We propose a solution: let us allow redistribution of energy between channels environments too!
\[
\frac{1}{2n} \text{Tr} \left[ \rho_{in} \rho_{env} \right] = N_{env}^0 \frac{1}{2}, \quad \text{where} \quad N_{env,1} + N_{env,2} = N_{env,0}
\]
- Now the channel is completely specified by unitary \( U_{op} \).
- Superadditive (\( \alpha \)), binary operation on the set of quantum states, where in addition to restriction \( N = \sum \{ N_i \} \), we also have \( N_{env} = \sum N_{env,i} \).

\[
\begin{align*}
N_{\rho_{in}} &\quad \text{Unitary} \quad U_{op} \quad \rho_{out} \\
n_{\rho_{in}, \rho_{out}} &\quad \rho_{out} \\
n_{\rho_{in}} &\quad \text{Unitary} \quad U_{op} \\
n_{\rho_{in}, \rho_{out}} &\quad \rho_{out}
\end{align*}
\]

Thus, our new formulation of the superadditivity problem for multimode channel:

Given the values \( N_{env} \) and \( N_{max} \):
- What is the channel capacity?
- What is the optimal channel input state?
- What is the optimal state of the channel environment?
- Optimal distribution \( \{ N_{env} \} \) for fixed \( N \) gives “optimal channel memory”.

References

Figure 2: Lossy bosonic channel. Top-left: capacity vs environment squeezing for different values of transmissivity. Bottom: optimal \( \{ N_{env} \} \) and \( \{ N_{env} \} \) - transition from additive to superadditive region.