Superadditivity of classical capacity revisited

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We introduce new type of superadditivity for classical capacity of quantum channels, which involves the properties of channels' environment. By imposing different restrictions on the total energy contained in channels environment we can consider different types of superadditivity. Using lossy bosonic and additive noise quantum channels as the examples, we demonstrate that they can be either additive or superadditive depending on the values of channels parameters. The parameters corresponding to transition between the additive and superadditive cases are related with recently found critical and supercritical parameters of Gaussian channels.

- Quantum channel is a map $\Phi:\hat\rho_{\rm in}\to\hat\rho_{\rm out}$

$$\hat\rho_{\rm out}={\rm Tr}_{\rm env}\left[\hat U\left(\hat\rho_{\rm in}\otimes\hat\rho_{\rm env}\right)\hat U^{+}\right]$$

- Input is dual to environment: $\hat\rho_{\rm in}\leftrightarrow\hat\rho_{\rm env}$
- Additivity of Holevo χ quantity for discrete channels [Hastings, Shor]: $\chi(\Phi_1\otimes\Phi_2)=\chi(\Phi_1)+\chi(\Phi_2)$

$$\chi(\Phi)=\max_{\{\hat\rho^{(\alpha)},P_\alpha\}}\left\{S\left[\int\Phi_n\left(\hat\rho^{(\alpha)}\right)P_\alpha d\alpha\right]-\int S\left[\Phi\left(\hat\rho^{(\alpha)}\right)\right]P_\alpha d\alpha\right\}$$

- If input and environment are factorized, what are the optimal states $\hat\rho^{(1)}_{\rm env}$ and $\hat\rho^{(2)}_{\rm env}$?

$$\hat\rho_{\rm out}={\rm Tr}_{\rm env}\left\{\hat U\left[\left(\hat\rho^{(1)}_{\rm in}\otimes\hat\rho^{(2)}_{\rm in}\right)\otimes\left(\hat\rho^{(1)}_{\rm env}\otimes\hat\rho^{(2)}_{\rm env}\right)\right]\hat U^{+}\right\}$$

- How to generalize this additivity to continuous variables channels? Classical capacity of two channels ($n=2$) is finite only if input energy is finite:

$$\frac{1}{2n}\int{\rm Tr}\left[\hat\rho^{(\alpha)}_{\rm in}\hat a^\dagger\hat a\right]P_\alpha\,d\alpha=N+\frac12<\infty$$

- How to compare channels acting together and separately? E.g. capacity of two pure lossy channels:

$$C=g(\eta_1N_1)+g(\eta_2N_2),\quad\text{where}\quad N_1+N_2=N$$

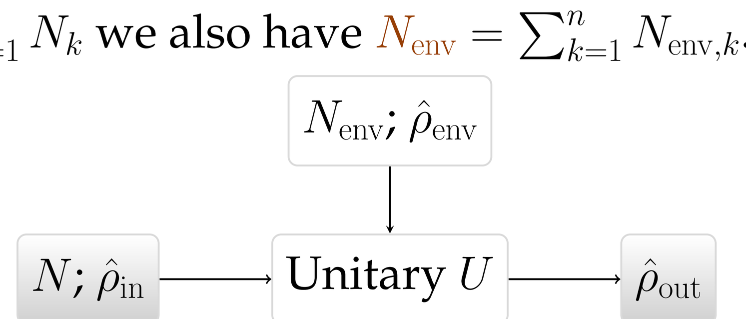
If N_1, N_2 are optimally chosen, they are not equal to $N/2\Rightarrow C>g(\eta_1N/2)+g(\eta_2N/2)\Rightarrow$ even pure lossy channel (vaccuum environment) is superadditive. Classical Gaussian channels are superadditive too (waterfilling solution is non-equal distribution of N between the modes). Any two channels with different environments are superadditive! $\Rightarrow$ This is *bad definition* of the superadditivity.

- What is about additivity of product channel? Is $\chi(\Phi\otimes\Phi)=2\chi(\Phi)$ correct? We proved it for lossy bosonic [1] and additive noise [2] channels assuming Gaussian conjecture to be valid. Capacity of both channels are concave functions $C(N)\Rightarrow$ they cannot be superadditive. Nevertheless, if we will find some continuous variables channel with non-concave dependence $C(N)$, we immediately get another superadditive channel in addition to Hastings' example(!).
- What do we have now? Product of channels is
  - Additive (?), if channels are identical.
  - Supperadditive, if channels are different.

How to find reasonable definition of additivity? We propose a solution: let us allow redistribution of energy between channel environments too!

$$\frac{1}{2n}{\rm Tr}\left[\hat\rho_{\rm env}\hat a^\dagger\hat a\right]=N_{\rm env}+\frac12,\ \text{where } N_{{\rm env},1}+N_{{\rm env},2}=N_{\rm env}$$

- Now the channel is completely specified by unitary $\hat U:\{\hat\rho_{\rm in},\hat\rho_{\rm env}\}\to\hat\rho_{\rm out}$ – binary operation on the set of quantum states, where in addition to restriction $N=\sum_{k=1}^n N_k$ we also have $N_{\rm env}=\sum_{k=1}^n N_{{\rm env},k}$.



Thus, our new formulation of the superadditivity problem for multimode channel:

Given the values N and $N_{\rm env}$

- *What is the channel capacity?*
- *What is the optimal channel input state?*
- *What is the optimal state of the channel environment?*

- Optimal distribution $\{N_{{\rm env},k}\}$ for fixed N gives *"optimal channel memory"*.

- Restriction of $N_{\rm env}$ is not really necessary: $C<\infty$ $\forall N_{\rm env}$. Moreover, even in this case the optimal $\{N_{{\rm env},k}\}$ is not trivial, and vacuum environment ($N_{\rm env}=0$) is never optimal(!).
- What is about phase-insensitive Gaussian channels? It is proved, that they:

A. Never optimal for information transmission [1].

B. Have the lowest capacity for some range of parameters. E.g. it is the case for lossy channel, if

$$N\geqslant\left[\sqrt{3/2+5/(2\sqrt3)}-1\right]/2\approx0.3578$$

and beam-splitter transmissivity $\eta\leqslant1-1/\sqrt3$ [1]. It is also the case for the additive noise channel, if $N_{\rm env}\geqslant1/\sqrt{12}$ [3].

C. Have the lowest capacity than any phase-sensitive channel, if the squeezing of its environment [1]

$$0<s\leqslant\ln(2N+1)$$

D. "The Holy Grail of Quantum Optical Communication" (minimum output entropy conjecture) which should finally prove well-known formula for its capacity (here example of lossy channel)

$$C=g\left[\eta N+(1-\eta)\mathcal N_{\rm env}\right]-g\left[(1-\eta)\mathcal N_{\rm env}\right]$$

is still not proved, but capacity of phase-sensitive channel (squeezed environment) is already found, if input energy is above some threshold value and environment is pure [4].

E. Gaussian states corresponding to minimum output entropy are not optimal if channel is phase-sensitive [1], [3].

F. If channel environment containes non-zero energy ($N_{\rm env}>0$), its optimal memory will never be realized by phase-insensitive channel.

- Any Gaussian quantum channel with squeezed environment can be experimentally realized by a box operating with "input" and "environment" light beams. Implementation of the majority of Gaussian channels requires only the beam-splitter and a two-mode squeezer.
- Welcome to the wonderful world of Gaussian channels with squeezed environment, whose (Gaussian) capacity was recently well studied [1], [2].
- Environment squeezing s, amount of thermal photons $\mathcal N_{\rm env}$ – entries of environment covariance matrix:

$$V_{\rm env}:=V(\mathcal N_{\rm env},s)=\left[\mathcal N_{\rm env}+\frac12\right]\begin{pmatrix}e^s&0\\0&e^{-s}\end{pmatrix},$$

$$\frac12{\rm Tr}\,V_{\rm env}=N_{\rm env}+\frac12,\qquad N_{\rm env}=\left(\mathcal N_{\rm env}+\frac12\right)\cosh s-\frac12.$$

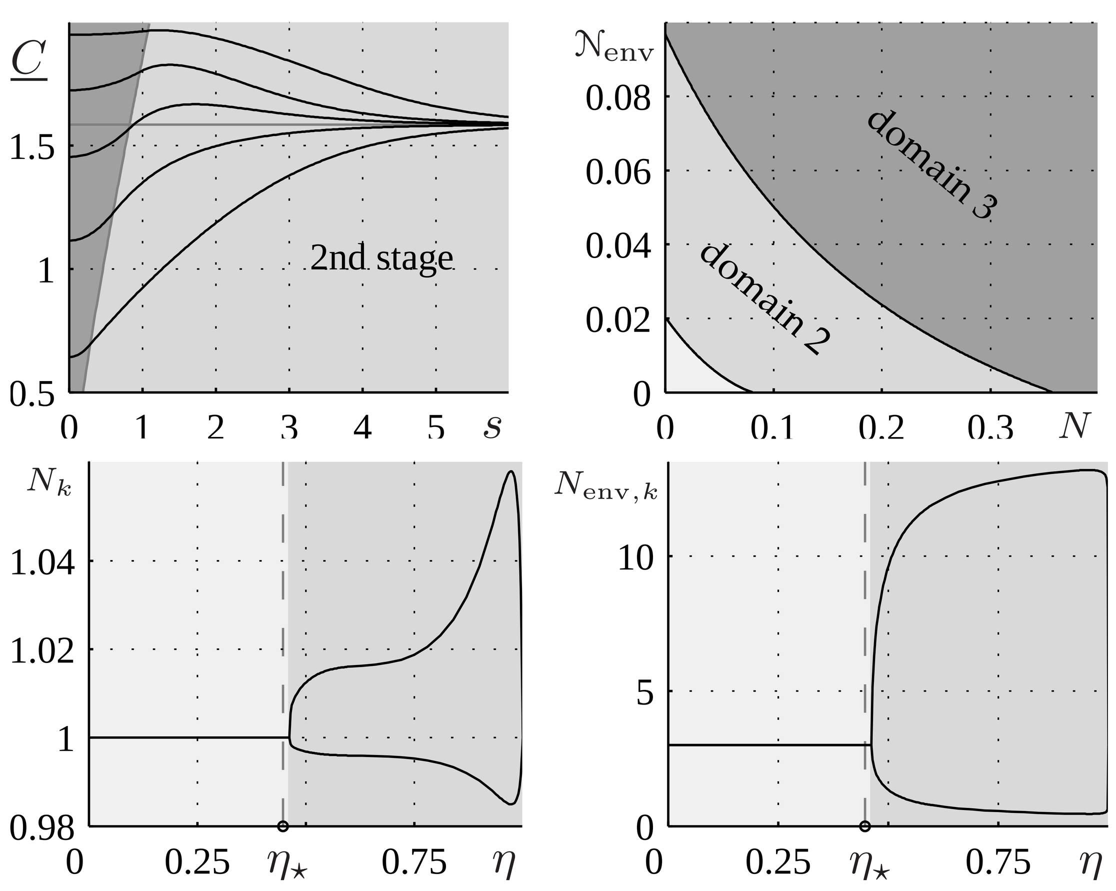
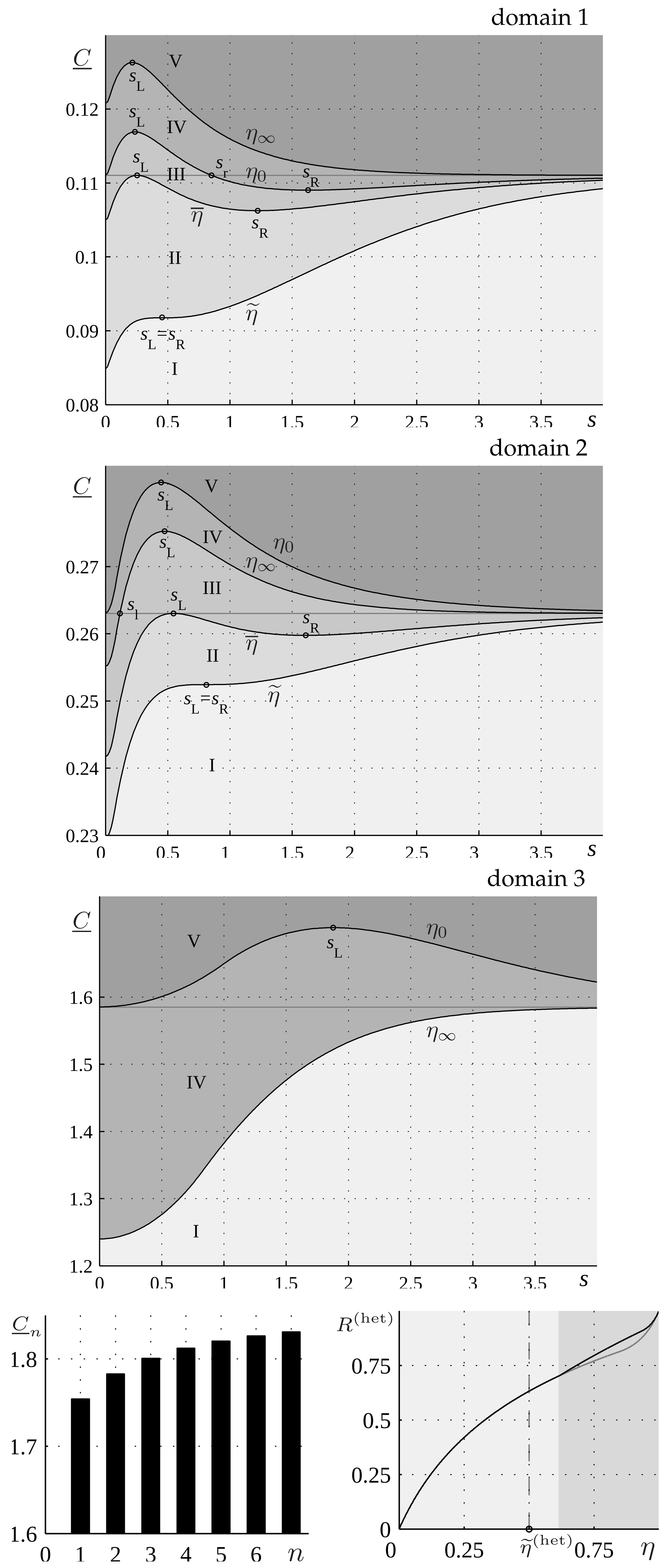


Figure 2: Lossy bosonic channel. Top-left: capacity vs environment squeezing for different values of transmissivity. Bottom: optimal $\{N_k\}$ and $\{N_{{\rm env},k}\}$ – transition from additive to superadditive region.



Transition between domains of parameters happens if

$$(1-\tilde\eta_\infty)\left(\mathcal N_{\rm env}+\frac12\right)=\frac{1}{\sqrt{12}},\quad\tilde\eta_\infty=\sqrt{\frac{2}{15}\left(1+(2N+1)^{-2}\right)}.$$

You can find many other analytical results, proofs and theorems (also for memory channel) in [1].

References

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