The channel environment that makes classical capacity superadditive

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We introduce new types of superadditivity for classical capacity of quantum channels which involves the properties of channels' environment. By imposing different restrictions on the total energy contained in channels' environment we can consider different types of superadditivity. Using lossy bosonic and classical additive noise quantum channels as examples, we demonstrate that their capacities can be either additive or superadditive depending on the values of channels parameters. The parameters corresponding to transition between the additive and superadditive cases are related with recently found critical and supercritical parameters for Gaussian quantum channels.

Gaussian classical channels

• n parallel channels:

$$Y_j = X_j + Z_j, \qquad Z_j \sim \mathcal{N}(0, N_j), \qquad j = 1, \ldots, n$$

Capacity:

$$C = \frac{1}{2}\sum_{j=1}^{n} \log_2\left(1 + \frac{P_j}{N_j}\right), \quad \sum_{j=1}^{n} P_j = P, \quad \mathbb{E}X_j^2 = P_j$$

• What is the best distribution of N_j if total amount of noise is fixed? What is the *optimal workpoint* of the parallel channels in this case?

$$\text{If } \sum_{j=1}^{n} N_j = N, \text{ then } \max_{N_j} C \ — \ ? \qquad (1)$$

• Given no any other restrictions,

$$\max_{N_j} C = \infty, \quad \text{if} \quad \exists\, j \mid N_j = 0$$

It is non-physical result. Actually, classical information theory cannot properly address this question.

• How to pose this problem correctly? Let an extra restriction be $N_j \geqslant \min N_j = N_{\min} > 0$, then the optimal distribution:

$$\begin{cases} N_1 = N - (n-1)N_{\min} & \leftarrow \text{one "garbage" channel} \\ N_2, \ldots, N_n = N_{\min} & \leftarrow n-1 \text{ "noiseless" channels} \end{cases}$$

Gaussian quantum channels

• Gaussian quantum channel with classical additive noise is a quantum extension of the classical Gaussian channel. Can it address the problem (1) correctly? Yes! [1] The non-vanishing vacuum noise always exist. In terms of number of photons we get:

$$\begin{cases} N_1 = N & \leftarrow \text{one "garbage mode"} \\ N_2, \ldots, N_n = 0 & \leftarrow n-1 \text{ "noiseless modes"} \end{cases} \qquad (2)$$

• What is about *other* Gaussian quantum channels? E.g. lossy bosonic? Is the solution (2) universal? No, but... is very similar! Depending on the parameters of the channel and N it has either *superadditive* type [2]

$$\begin{cases} N_1 = N - (n-1)N_{\min} & \leftarrow \text{one "garbage mode"} \\ N_2, \ldots, N_n = N_{\min} & \leftarrow n-1 \text{ "noiseless modes"} \end{cases}$$

or *additive* type

$$N_1, \ldots, N_n = N/n$$

So, by varying some parameter we can go from one type of solution to another. E.g., if we consider lossy bosonic channel and vary beam-splitter transmissivity η, we can see the transitions between the additive and superadditive cases:

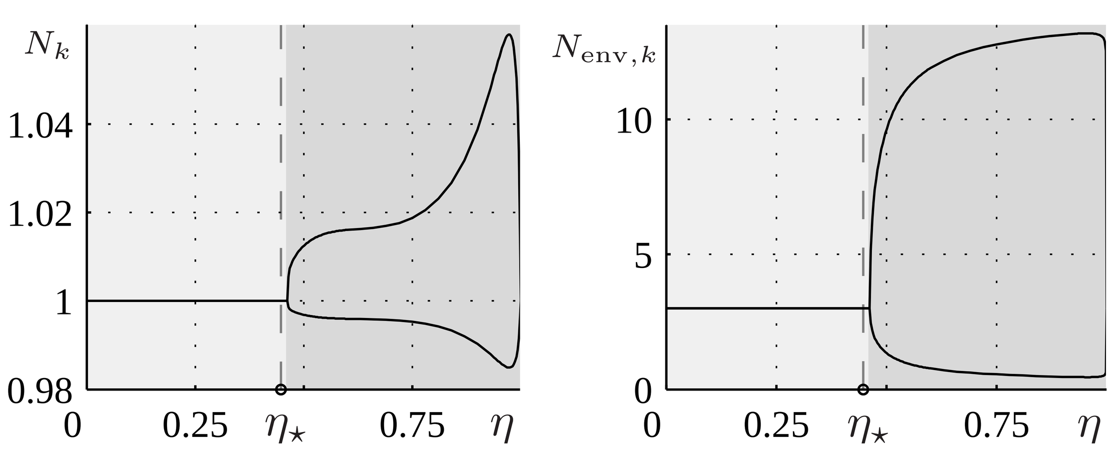


Fig.1 The optimal $\{N_k\}$ (corresponds to P_k of classical channel) and the optimal $\{N_{\mathrm{env},k}\}$ (corresponds to N_k of classical channel) for lossy bosonic quantum channel.

• Can we pose the problem of "environment" (super)additivity for arbitrary quantum channel? E.g. for arbitrary Gaussian [3]? Yes!

Superadditivity over inputs. Can we see it?

• Classical capacity is *additive* (conventional definition) if

$$C_1(\Phi \otimes \Psi) = C_1(\Phi) + C_1(\Psi)$$

Here C_1 is *one-shot* capacity. It is supremum of Holevo χ quantity over the input quantum ensembles.

• The capacity is

$$C = \lim_{n\to\infty} \frac{C_1(\Phi^{\otimes n})}{n},$$

where $C = C_1(\Phi)$ if C_1 is additive.

• Cause of superadditivity: the channels are allowed to share a *joint* input state which may be *correlated between the inputs* or even entangled.

• Continuous variables channels always require energy restriction at the input.

• Given the restrictions N_1 and N_2 for channels Φ and Ψ, respectively, the capacity is *strongly additive* [4, 5] if

$$C_1(\Phi_{N_1} \otimes \Psi_{N_2}) = C_1(\Phi_{N_1}) + C_1(\Psi_{N_2}).$$

• Suppose that the restriction is imposed only on $N = \frac{1}{2}(N_1 + N_2)$ but N_1 and N_2 may vary, then the capacity is *weakly* additive if [4]

$$C_1\big((\Phi \otimes \Psi)_{2N}\big) = \max_{N_1+N_2=2N}\big[C_1(\Phi_{N_1}) + C_1(\Psi_{N_2})\big].$$

• Both strong and weak additivities imply that the optimal input (symbol) states can be realized by the states factorized over the channels.

• One-shot capacity C_1 of any bosonic channel Φ_N is a concave function of the amount of input photons N [4, 5].

• If bosonic channel Φ_N is strongly additive, the optimal distribution of the photons between the channels' inputs is uniform $\Rightarrow$ the capacity is additive for all n [6]:

$$C_1(\Phi_{nN}^{\otimes n}) = n\, C_1(\Phi_N)$$

Despite strong additivity is not proved in general, it holds, e.g., for the entanglement breaking bosonic channels [7].

• There is a strong believe that classical capacity of *all* bosonic Gaussian channels is *always* additive over the input energy N $\Rightarrow$ most probably, there is *no superadditivity over inputs*.

Superadditivity over environments

• Stinespring dilation of quantum channel:

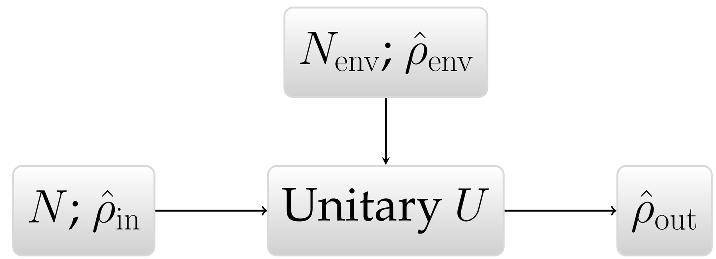
$$\Phi(\hat\rho_{\mathrm{in}}) = \mathrm{Tr}_{\mathrm{env}}[\hat U(\hat\rho_{\mathrm{in}} \otimes \hat\rho_{\mathrm{env}})\hat U^\dagger].$$

• Instead of associating Φ with the pair $(\hat U, \hat\rho_{\mathrm{env}})$ let us associate Φ *only with unitary* $\hat U$ saying that Alice and Bob may choose $\hat\rho_{\mathrm{env}}$ which will be the same for all information transmissions and will be the best for that.

• What happens with the capacity of n parallel channels if their "environments" are allowed to be in a joint arbitrary n-partite state similarly to the channels' inputs? (This problem is known to appear in the study of memory channels realized by correlated noise).

• For a certain types of noise, the correlations between the environment modes may increase the capacity compared to the non-correlated environment with the same number of "environment" photons per mode [2] (here the energy restriction for the multimode environment appeares nataurally).

• Let us define the capacity of product channels as a maximum over both (I) the distribution of the amount of photons N between channels' inputs and (II) the distribution of the amount of photons in environment N_{env} between the channels' environments. Hence, the capacity becomes a function of two variables: $C = C(N, N_{\mathrm{env}})$.



• Then, we propose a new formulation of the superadditivity problem:

Given the average numbers of photons for the channels input nN and for their environment nN_{env},

- *What is the channel capacity?*
- *What is the optimal channel input state?*
- *What is the optimal state of the channel environment?*

In particular, we call the capacity *additive over the environment* if

$$\max_{\Phi^{(2)}}\left[C_1\left(\left(\Phi^{(2)}\right)_{2N}^{2N_{\mathrm{env}}}\right)\right] = 2C_1\left(\Phi_N^{N_{\mathrm{env}}}\right) \qquad (3)$$

where the maximum at the left side is taken over all possible channels $\Phi^{(2)}$ whose dimensions are the same as for the product $\Phi\otimes\Phi$ and whose amount of photons in the environment is equal to $2N_{\mathrm{env}}$ (lower and upper indecies denote the amount of input and environment photons granted for the channel, respectively).

• We assume that the entangled inputs are not necessary for bosonic Gaussian channels to achieve the maximum in the left side of (3), therefore it can be rewritten as

$$\max_{\Phi^{(2)}}\left[C_1\left(\left(\Phi^{(2)}\right)_{2N}^{2N_{\mathrm{env}}}\right)\right] = \max_{N_{\mathrm{env},1}+N_{\mathrm{env},2}=2N_{\mathrm{env}}}\left[\max_{N_1+N_2=2N}\left(C_1\left(\Psi_{N_1}^{N_{\mathrm{env},1}}\right) + C_1\left(\Xi_{N_2}^{N_{\mathrm{env},2}}\right)\right)\right]$$

(the channels $\Phi^{(2)}$ and $\Psi \otimes \Xi$ act in the same space).

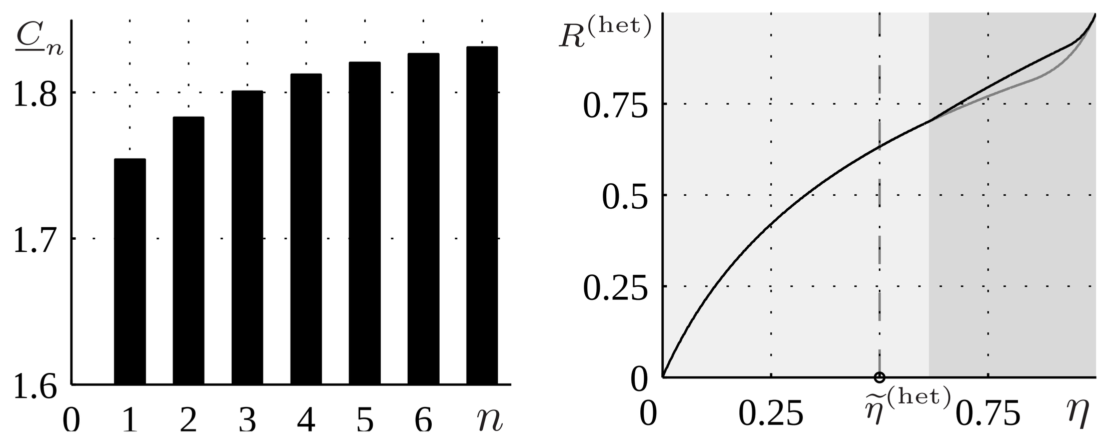


Fig.2 Classical capacity (left) and heterodyne rate (right) of lossy bosonic Gaussian quanum channel optimized with respect to its environment. It is plotted vs the amount of channel modes n (left) and beam-splitter transmissivity η (right). The grey curve is the heterodyne rate without optimization over the environment.

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