Estimating capacities and rates of Gaussian quantum channels

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Motivations

- Most of the performed studies e.g. on classical capacity concern simple settings (memoryless and vacuum environment)
- No general methods available for evaluating, e.g. classical capacity
- Rates usually derived in a different way with respect to capacity
- Consider lossy bosonic channel as a paradigm of Gaussian channels
- Introduce a generic model for multiple channel uses and devise a method to evaluate the Holevo function (turns out to be useful for classical capacity as well as for dyne rates)
- Maximization problem can be split it into “inner” one and “outer” one

based on Pilyavets, Lupo & Mancini, arXiv0907.1532 (provisionally accepted by IT Trans)
Outline

- Gaussian channels
  - Lossy bosonic channel
- Classical capacity and rates
- Single channel use (bosonic mode)
  - The “inner” optimization problem
  - Solution
  - Its properties (critical parameters)
- Multiple channel uses (bosonic modes)
  - The “outer” optimization problem
  - Solution
  - Its properties (and applications)
- Conclusions and outlook
Gaussian channels

They map Gaussian states into Gaussian states; for single use:

\[ \{a, V\} \mapsto \{X^T a + d, X^T V X + Y\} \]

Channel defined by the triad: \((d, X, Y)\)

For \(n\) uses channel defined by a triad:

\[
(d_n, X_n, Y_n) = \begin{cases} 
= (\oplus^n d, \oplus^n X, \oplus^n Y) & \text{memoryless} \\
\neq (\oplus^n d, \oplus^n X, \oplus^n Y) & \text{memory}
\end{cases}
\]
The lossy channel

\[ X = \sqrt{\eta} I, \quad Y = (1 - \eta)V_{\text{env}} \]

The eigenvalues of the various matrices will be denoted by \((e_u, i_u, \bar{i}_u, m_u, o_u, \bar{o}_u)\)
Classical capacity and rates

\[ C_n := \frac{1}{n} \max_{V_{\text{in}}, V_{\text{mod}}} \chi^G_n \]

\[ \chi^G_n := \sum_{k=1}^{n} \left[ g(\bar{o}_k - \frac{1}{2}) - g(o_k - \frac{1}{2}) \right] \]

\[ g(x) := (x + 1) \log(x + 1) - x \log x \]

\[ \frac{\text{Tr} V_{\text{in}}}{2n} \leq \overline{N}_{\text{in}} + \frac{1}{2} \]

To the logarithmic approximation of \( g \)

\[ C^{\log} = \frac{1}{n} \max_{V_{\text{in}}, V_{\text{mod}}} \sum_{k=1}^{n} \log \frac{\bar{o}_k}{o_k} \]

\[ R_n^{\text{hom}} = C^{\log}_n \]

\[ R_n^{\text{het}} = C^{\log}_n [V_{\text{env}} \rightarrow V_{\text{env}}^{\text{het}}] \]
Single channel use

**Theorem**
The max of Holevo function over Gaussian states is achieved for $V_{in}$, $V_{mod}$, $V_{env}$ simultaneously diagonalizable and the optimal $V_{in}$ corresponds to a pure state

**Corollary**
If $V_{in}$, $V_{mod}$, $V_{env}$ are simultaneously diagonalizable, the maximum of dyne rates is achieved by input pure states

Covariance matrices parametrized as

$$V = \left( \mathcal{N} + \frac{1}{2} \right) \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}$$

$$\frac{\text{Tr}V}{2} \leq N + \frac{1}{2}$$
The “inner” optimization problem

Maximize $\chi_1^G$

With

\[ i_u > 0 \quad (i_{u*} = 1/(4i_u)) \]
\[ m_u, m_{u*} \geq 0 \]
\[ i_u + \frac{1}{4i_u} + m_u + m_{u*} = 2\bar{N}_{in} + 1 \]

**Definition**
Solution belongs to the **1st stage** if $m_u, m_{u*}=0$ are optimal
Solution belongs to the **2nd stage** if only $m_u =0$ (or $m_{u*}$) is optimal
Solution belongs to the **3rd stage** if $m_u, m_{u*}>0$ are optimal

**Remark**
Stages are crossed (from 1st to 3rd) by increasing the input energy
1st stage capacity equal to zero

\[
\overline{N}_{in}(1 \to 2) = 0
\]

2nd stage solution for \( i_u \) of the transcendent equation

\[
\bar{o}g' \left( \bar{o} - \frac{1}{2} \right) \left( \frac{1}{o_u} - \frac{1}{\bar{o}_{u*}} \right) - \sigma g' \left( \sigma - \frac{1}{2} \right) \left( \frac{1}{o_u} - \frac{1}{4i_u^2 o_{u*}} \right) = 0
\]

\[
\overline{N}_{in}(2 \to 3) = \frac{1}{2} \left( \sqrt{\frac{e_u}{e_{u*}}} - 1 \right) - \frac{1-\eta}{\eta} \left( N_{env} - e_u + \frac{1}{2} \right)
\]

3rd stage

\[
C_1 = g \left( \eta \overline{N}_{in} + (1 - \eta) N_{env} \right) - g \left( (1 - \eta)N_{env} \right)
\]
Properties of the solution

Theorem:
\( C_1 \) is a concave and increasing function of \( \bar{N}_{in} \)

The one-shot capacity for fixed \( e_u, e_u^*, \eta \) can be considered as a black-box returning \( C_1 \) upon inputting \( \bar{N}_{in} \), while preserving the concavity

\[
\bar{N}_{in} \rightarrow C_1 = C_1 (\bar{N}_{in}) \rightarrow C_1
\]

Corollary:
\( C_1 \) is additive

Theorem:
\( C_1 \) is a monotonic function of all its parameters \((\eta, \bar{N}_{in}, s_{env}, N_{env})\) except \( s_{env} \)
Critical parameters at boundaries of regimes, e.g. $\eta_\star = 1 - \frac{1}{\sqrt{3}}$
Domains

In the domain 1: $\tilde{\eta} < \bar{\eta} < \eta_0 < \eta^*$

In the domain 2: $\tilde{\eta} < \bar{\eta} < \eta^* < \eta_0$

In the domain 3: $\exists \tilde{\eta}, \bar{\eta}$

Critical parameters at boundaries of domains, e.g. $N_{in}^* = \sqrt{\frac{3\sqrt{3}+5}{8\sqrt{3}}} - \frac{1}{2}$
Multiple channel uses

Different single channel uses come from memory unravelling
Lupo & Mancini, PRA 81, 052314 (2010)

The action of $E$ could be reduced to that of $E_1, E_1, \ldots, E_n$ by finding suitable Gaussian encoding/decoding unitaries

\[
(0, E_n, 0), (0, D_n, 0) \mid D_n X_n E_n = \bigoplus_{k=1}^{n} X^{(k)}; \quad D_n Y_n D_n^T = \bigoplus_{k=1}^{n} Y^{(k)}; \quad E_n^T E_n = I_n
\]

Always possible for $E$ pure, or thermal squeezed!
The “outer” optimization problem

To maximize $\chi_n^G$ it now suffices to consider:

\[
\begin{align*}
N_{\text{in},1} & \rightarrow \quad C_1^{(1)} = C_1^{(1)} \left( N_{\text{in},1} \right) \rightarrow C_1^{(1)} \\
N_{\text{in},2} & \rightarrow \quad C_1^{(2)} = C_1^{(2)} \left( N_{\text{in},2} \right) \rightarrow C_1^{(2)} \\
& \quad \vdots \\
N_{\text{in},n} & \rightarrow \quad C_1^{(n)} = C_1^{(n)} \left( N_{\text{in},n} \right) \rightarrow C_1^{(n)}
\end{align*}
\]

Find the distribution of $N_{\text{in},k} \quad \left( \sum_{k=1}^{n} N_{\text{in},k} = n \bar{N}_{\text{in}} \right)$

giving the maximum of $\sum_{k=1}^{n} C_1^{(k)}$

This “outer” optimization problem can be interpreted as the search for the optimal distribution of modes across stages.
Due to the properties of $C_1$ it’s possible to define $\lambda_{\text{max}} := \max_k \frac{\partial C_1^{(k)}}{\partial N_{\text{in},k}} (N_{\text{in},k} = 0) < +\infty$

$$\lambda_{1 \to 2}(k) = \frac{\partial C_1^{(k)}}{\partial N_{\text{in},k}} (N_{\text{in},k}(1 \to 2)); \lambda_{2 \to 3}(k) = \frac{\partial C_1^{(k)}}{\partial N_{\text{in},k}} (N_{\text{in},k}(2 \to 3))$$

Convex separable programming guarantees uniqueness and optimality of the solution together with convergence of the algorithm.
In the stage 1: \( \overline{N}_{in,k} = 0 \)

In the stage 2: \( \overline{N}_{out,k} = \frac{1}{e^{\omega_k/T} - 1} \)

\[ \overline{N}_{out,k} = \bar{\sigma}_k - 1/2, \quad \omega_k = \bar{\sigma}_k/\bar{o}_{u,k}, \quad T = \eta/\lambda \]

\( \overline{N}_{out,k} \) can be expressed by means of \( \overline{N}_{in,k} \)

upon solving the “inner” problem

In the stage 3: \( \overline{N}_{in,k} = \frac{1}{\eta} \left[ \frac{1}{e^\lambda/\eta - 1} - (1 - \eta)N_{env,k} \right] \)

If all modes belong to the 3rd stage

\[ C_n = g \left( \eta \overline{N}_{in} + (1 - \eta)N_{env} \right) - \frac{1}{n} \sum_{k=1}^{n} g ((1 - \eta)N_{env,k}) \]
Quantum water filling

\[ V_{\text{env}} = \left( N_{\text{env}} + \frac{1}{2} \right) \begin{pmatrix} e^{\Omega s_{\text{env}}} & 0 \\ 0 & e^{-\Omega s_{\text{env}}} \end{pmatrix} \]

\[ \Omega_{i,j} = \delta_{i,j+1} + \delta_{i,j-1} \]
Super-additivity

For a fixed $N_{env}$, sufficient condition to have

$$\sum_{k=1}^{n} C_1^{(k)} < nC_1 \left| \sum_{k=1}^{n} N_{in,k} = nN_{in} \right.$$ 

is $\eta < \eta^*$, $\sum_{k=1}^{n} N_{in,k} > nN_{in}^*$
\[ V_{\text{env}} = \left( N_{\text{env}} + \frac{1}{2} \right) \begin{pmatrix} e^{\Omega s_{\text{env}}} & 0 \\ 0 & e^{-\Omega s_{\text{env}}} \end{pmatrix} \]

\[ \Omega_{i,j} = \delta_{i,j+1} + \delta_{i,j-1} \]
Conclusions and outlook

- Optimization methods for capacity and rates
- Full characterization of the single-mode lossy channel
- Concavity (and then additivity) of the one-shot capacity
- Full characterization of the multiple use lossy channel
- Superadditivity for memory channel related to critical parameters
- Application to other Gaussian channels [additive noise, J. Schafer et al. arXiv1011.4118]
- Application to other capacities
- Open questions: optimality of Gaussian input states; coding theorems for generic memory channels