A Lossy Bosonic Quantum Channel with Non-Markovian Memory

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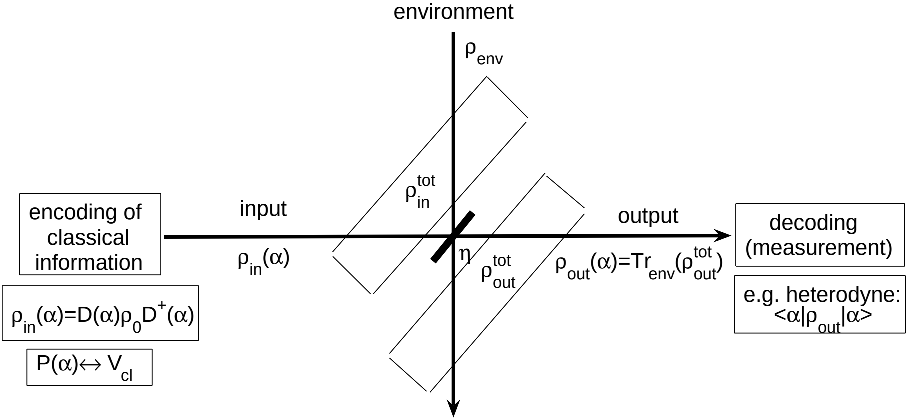
Introduction

- Quantum channels
  - Gaussian channels
    - Lossy bosonic Gaussian channels
- Characteristics of quantum channel
  - Quantum capacity
  - Classical capacity
  - Rates
    - Homodyne rate
    - Heterodyne rate

General definitions

- Any state can be labeled as ρ or V as all states are Gaussian.
  - V_{in} - covariance matrix for input state
  - V_{env} - covariance matrix for environment state
  - V_{cl} - covariance matrix for classical distribution of coherent amplitude α.
  - V_{out} - covariance matrix for output state of the channel
  - $\overline{V}_{\text{out}}$ - covariance matrix for output state of the channel avaraged over classical distribution (encoding of information) V_{cl}
- Capacities and rates
  - $C_n = \max_{\text{states}} \frac{1}{n}\chi_n$ - classical capacity for n uses of channel
  - $C = \max_{n\to\infty} C_n$ - classical capacity on infinite amount of channel uses
- Conjectures
  - Capacity for lossy bosonic channel can be achieved on Gaussian states
  - Maximizing of Holevo χ leads to capacity for memory channel too

Scheme for 1 use of lossy bosonic channel



Energy restriction: $Tr(V_{in}+V_{cl})/(2n)=N+1/2$

Beam splitter action: $\rho_{tot}^{out}=U(\rho_{in}\otimes\rho_{env})U^{+}=U\rho_{tot}^{in}U^{+}$

1 use capacity: notations and known results

- It is sufficient to consider only diagonal matrices (!).
- Abitrary covariance matrices in diagonal form for 1 use:

$$V_{\rm env} = (N_{\rm env} + 1/2)\begin{pmatrix} e^{s} & 0 \\ 0 & e^{-s} \end{pmatrix} \qquad V_{\rm in} = (N_{\rm in} + 1/2)\begin{pmatrix} e^{r} & 0 \\ 0 & e^{-r} \end{pmatrix}$$

- Already known capacities:
  - If environment is in ground (vacuum) state:

$$C = g[\eta N]$$

  - If environment is in termal state:

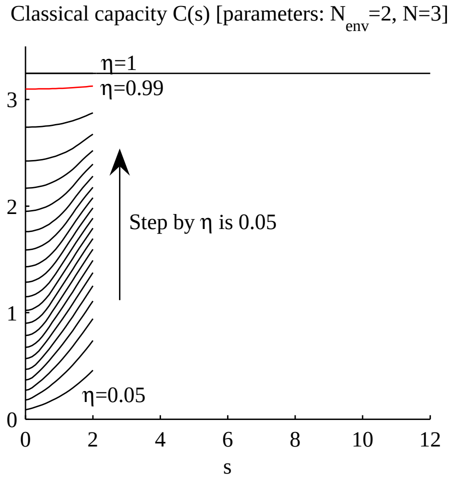
$$C = g[\eta N + (1-\eta)N_{\rm env}] - g[(1-\eta)N_{\rm env}]$$

- Environment in termal and squeezed state:

$$C = g\left[\eta N + (1-\eta)\big((N_{\rm env} + 1/2)\cosh(s) - 1/2\big)\right] - g[(1-\eta)N_{\rm env}]$$

$$g(x) = (x+1)\,\log_2(x+1) - x\,\log_2 x$$

1 use capacity: classical capacity



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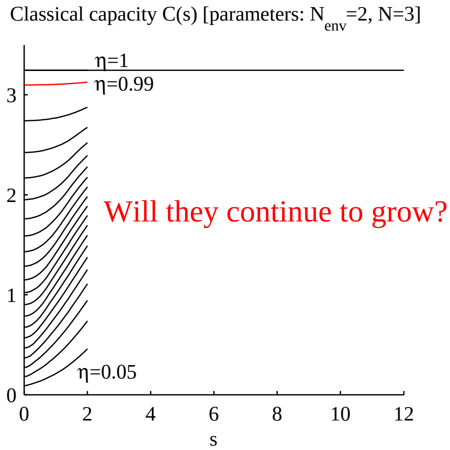

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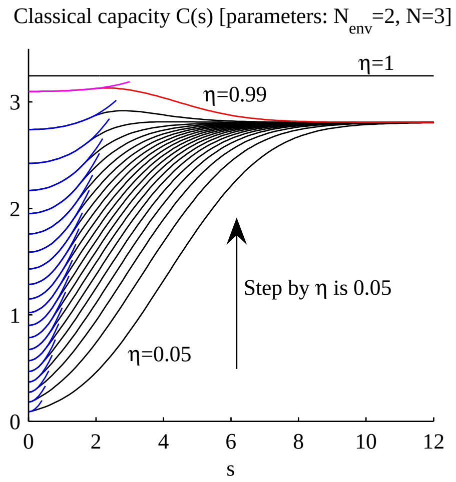

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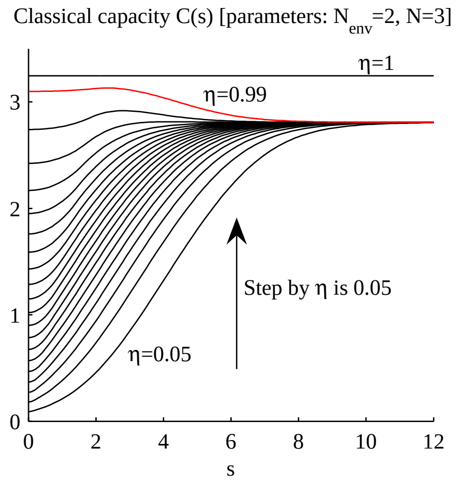

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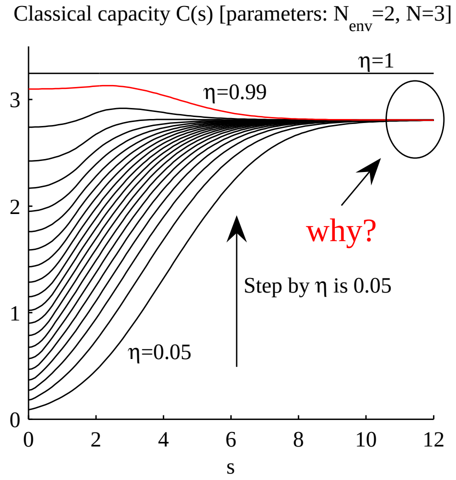

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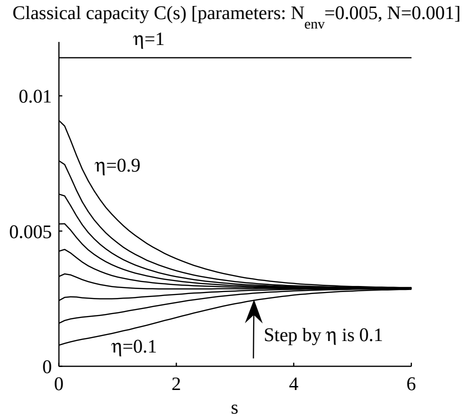

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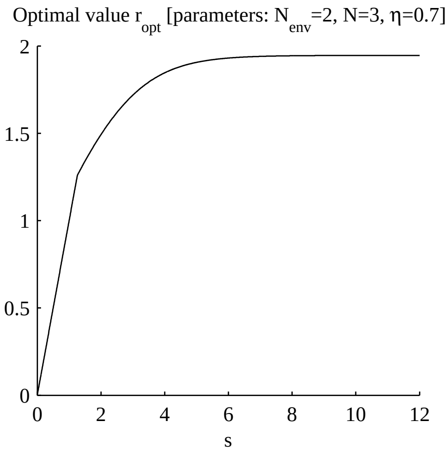
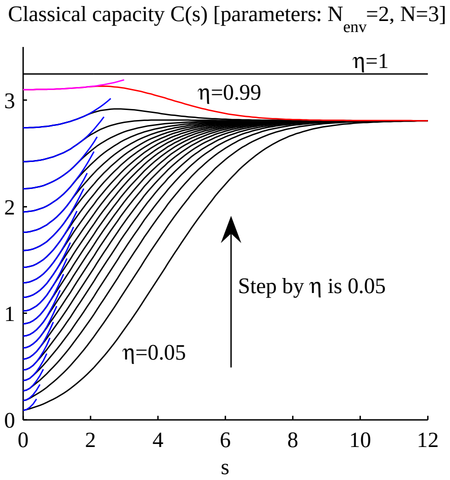
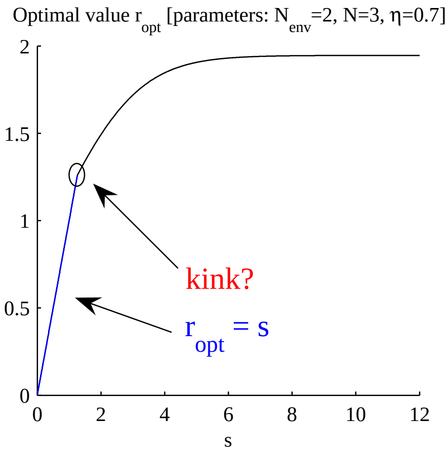
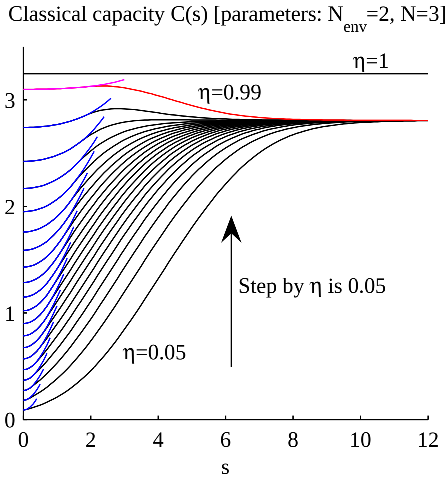




1 use capacity: classical capacity





1 use capacity: complete analytical solution

- Suppose that eigenvalues of $V_{\rm env}$ matrix are e_1, e_2. Then, eigenvalues of matrix $V_{\rm cl}$ (which are c_1, c_2) and $V_{\rm in}$ (which are i_1, i_2) can be found from the following relations if <u>both</u> c_1 and c_2 are positive:

$$c_1 = N + \frac{1}{2} - \frac{1}{2}\sqrt{\frac{e_1}{e_2}} + \frac{1}{2}\left(1 - \frac{1}{\eta}\right)(e_1 - e_2)$$

$$c_2 = N + \frac{1}{2} - \frac{1}{2}\sqrt{\frac{e_2}{e_1}} + \frac{1}{2}\left(1 - \frac{1}{\eta}\right)(e_2 - e_1)$$

$$i_1 = \frac{1}{2}\sqrt{\frac{e_1}{e_2}}, \quad i_2 = \frac{1}{2}\sqrt{\frac{e_2}{e_1}}$$

In this case capacity can be expressed in explicit form and is equal to

$$C = g\big[\eta N + (1-\eta)\big((N_{\rm env} + 1/2)\cosh(s) - 1/2\big)\big] - g[(1-\eta)N_{\rm env}]$$

1 use capacity: complete analytical solution

- If c_k (according to previous relations) is negative, then $c_k = 0$, $c_m = 2N + 1 - i_k - 1/(4i_k)$, $i_m = 1/(4i_k)$, and i_k is a solution of the following transcedental equation ($\{k, m\} = \{1, 2\}$ or $\{k, m\} = \{2, 1\}$):

$$\frac{a_m - a_k}{\sqrt{a_m a_k}} \log_2 \frac{\sqrt{a_m a_k} + 1/2}{\sqrt{a_m a_k} - 1/2} = \frac{o_m i_k - o_k i_m}{i_k \sqrt{o_m o_k}} \log_2 \frac{\sqrt{o_m o_k} + 1/2}{\sqrt{o_m o_k} - 1/2}$$

where

$$o_1 = \eta i_1 + (1 - \eta) e_1 \qquad a_1 = \eta(i_1 + c_1) + (1 - \eta) e_1$$

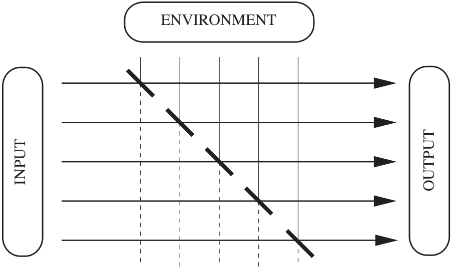
$$o_2 = \eta i_2 + (1 - \eta) e_2 \qquad a_2 = \eta(i_2 + c_2) + (1 - \eta) e_2$$

No explicit relation for capacity.

- Is there something new for n uses of the channel?
- Is there new "physics" there?
- Can we say that entanglemet is useful for information transmission for many uses of the channel?

Let us see...

Scheme for n uses of lossy bosonic channel



- In the case of our type of memory it is sufficient to consider only commuting matrices (!).
- Suppose that eigenvalues of V_{env} matrix are e_{qk}, e_{pk}, $k = 1, ..., n$. Then, eigenvalues of matrix V_{cl} (which are c_{qk}, c_{pk}) and V_{in} (which are i_{qk}, i_{pk}) can be found from the following relations if for all k both c_{qk} and c_{pk} are positive:

$$c_{qk} = N + \frac{1}{2} - \frac{1}{2}\sqrt{\frac{e_{qk}}{e_{pk}}} + \frac{1-\eta}{\eta}\left(\frac{\mathrm{Tr}\, V_{\mathrm{env}}}{2n} - e_{qk}\right)$$

$$c_{pk} = N + \frac{1}{2} - \frac{1}{2}\sqrt{\frac{e_{pk}}{e_{qk}}} + \frac{1-\eta}{\eta}\left(\frac{\mathrm{Tr}\, V_{\mathrm{env}}}{2n} - e_{pk}\right)$$

$$i_{qk} = \frac{1}{2}\sqrt{\frac{e_{qk}}{e_{pk}}}, \quad i_{pk} = \frac{1}{2}\sqrt{\frac{e_{pk}}{e_{qk}}}$$

- In this case the capacity can be expressed in explicit form. It is equal to

$$C_n = g\left[\eta N + (1-\eta)\left(\frac{\mathrm{Tr}\, V_{\mathrm{env}}}{2n} - \frac{1}{2}\right)\right] - \frac{1}{n}\sum_{k=1}^{n}[(1-\eta)(\sqrt{e_{qk}e_{pk}} - 1/2)]$$

$$C = g\left[\eta N + (1-\eta)\left(\lim_{n\to\infty}\frac{\mathrm{Tr}\, V_{\mathrm{env}}}{2n} - \frac{1}{2}\right)\right] - \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}[(1-\eta)(\sqrt{e_{qk}e_{pk}} - 1/2)]$$

- If c_{qk} or c_{pk} is negative we don't have explicit relation for the capacity
- Capacity is always achieved on states V_{in} minimizing uncertainty relation (!)

n uses capacity: Ω-model of channel memory

- Environment matrix:

$$V_{\text{env}} = \frac{1}{2} \begin{bmatrix} \exp(s\Omega) & 0 \\ 0 & \exp(-s\Omega) \end{bmatrix}$$

where

$$\Omega = \begin{pmatrix} 0 & 1 & \multicolumn{3}{c}{\dots\dots\dots\dots} & 0 \\ 1 & 0 & 1 & \multicolumn{2}{c}{\dots\dots} & 0 \\ \vdots & 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & \ddots & 1 \\ 0 & 0 & \multicolumn{2}{c}{\dots\dots} & 1 & 0 \end{pmatrix}$$

n uses capacity: Ω-model of channel memory

- Correlations decay exponentially over channel uses, it is quite "realistic" (correlations are non-Markovian)
- Energy constraint:
$$\frac{\mathrm{Tr}(V_{\mathrm{in}} + V_{\mathrm{cl}})}{2n} = N + \frac{1}{2}$$
- Ω-model allows us to test entanglement
- Capacity for Ω-model:

$$C = g\left[\eta N + \frac{1}{2}(1-\eta)(I_0(2s) - 1)\right]$$

n uses capacity: Ω-model, maximization over set

- Input state:

$$V_{\rm in} = \frac{1}{2}\left[\begin{array}{cc} \exp(r\Omega) & 0 \\ 0 & \exp(-r\Omega) \end{array}\right]$$

- Covariance matrix for classical distribution used to encode an information:

$$V_{\rm cl} := \frac{2nN(1-\theta_n)}{{\rm Tr}(Y)}\, Y$$

where

$$Y = \frac{1}{2}\left[\begin{array}{cc} \exp(y\Omega) & 0 \\ 0 & \exp(-y\Omega) \end{array}\right]$$

$$\theta_n := \frac{{\rm Tr}(V_{\rm in}) - n}{2nN}$$

- Region of possible values of r is restricted by input energy constraint (by roots of equation $\theta_n = 1$)

Rates

- Heterodyne rate:

$$\begin{aligned} I[\mathbf{Z}:\mathbf{A}] &= H[\mathbf{Z}] - H[\mathbf{Z}|\mathbf{A}] \\ &= \frac{1}{2}\log_2 \det\left[\left(\overline{V}_{\rm out}+\frac{1}{2}\right)\left(V_{\rm out}+\frac{1}{2}\right)^{-1}\right] \end{aligned}$$

- Homodyne rate (measurement of quadratures q in all modes):

$$\begin{aligned} I[\Re\mathbf{Z}:\Re\mathbf{A}] &= H[\Re\mathbf{Z}] - H[\Re\mathbf{Z}|\Re\mathbf{A}] \\ &= \frac{1}{2}\log_2 \det\left[\left(\overline{V}_{\rm out}^{(11)}\right)\left(V_{\rm out}^{(11)}\right)^{-1}\right] \end{aligned}$$

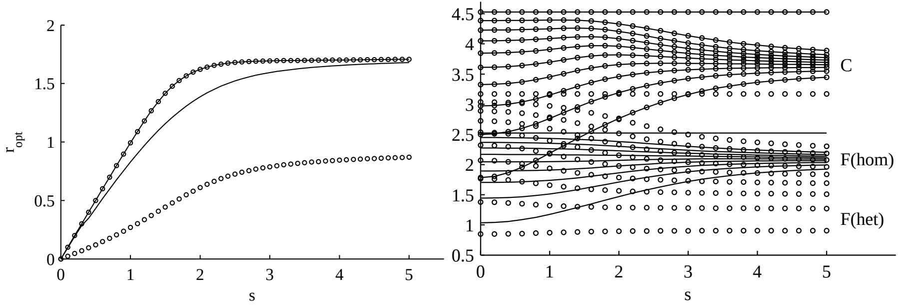
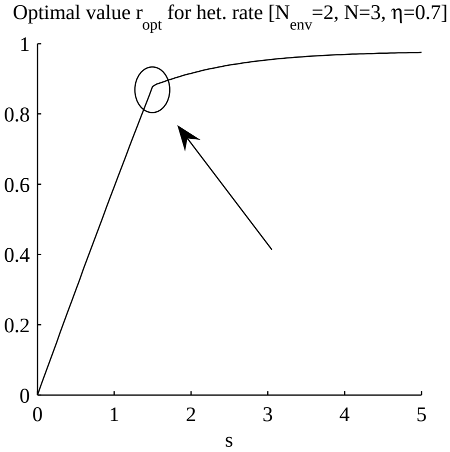
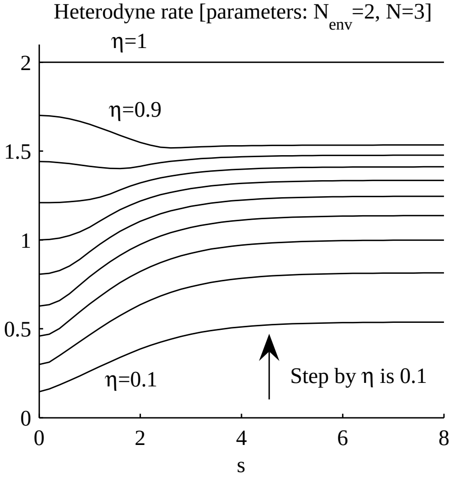


Figure: On the left, the optimal value of r for the quantities C (solid line with circles), F (heterodyne rate, line with circles only) and F (homodyne rate, pure solid line) is shown versus s. The values of the other parameters are $N = 8, \eta = 0.7$. On the right, the quantities C (solid lines with circles), F (heterodyne rate, lines with circles only) and F (homodyne rate, pure solid lines) are plotted versus s for values of η going from 0.1 (bottom curve) to 1 (top curve) with step 0.1. The value of the other parameter is $N = 8$.

Heterodyne rates for 1 use of the channel





Conclusions

- Squeezing can enhance the capacity in some cases
- Optimal input squeezing is related to the environment squeezing
- Entanglemet is useful for information transmission as it always comes with squeezing
- Almost all interesting “physics” (behavior) can be found in already 1 use of the channel

Conclusions

THANK YOU!