Transmission of Classical Information through Gaussian Quantum Channels with Memory

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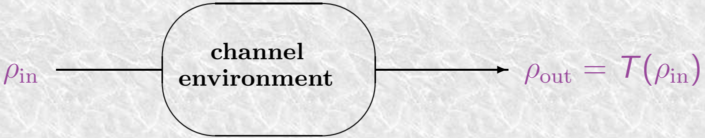
I. Basic definitions from quantum channels theory

- *Quantum channel* T is a (trace-preserving) *quantum map*:

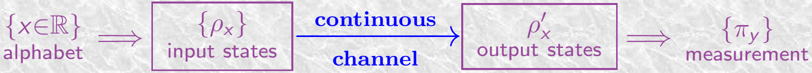
  i.e. quantum state $\longrightarrow$ quantum state,

  which is *completely positive*:

  i.e. $T \otimes \mathrm{Id}$ is also quantum map.



- If we *encode information into quantum state* in input, we can decode that information by measuring quantum state at the output of channel. This is *classical information transmission* by quantum channels:



  The aim at decoding (measurement) is to distinguish states with different x.

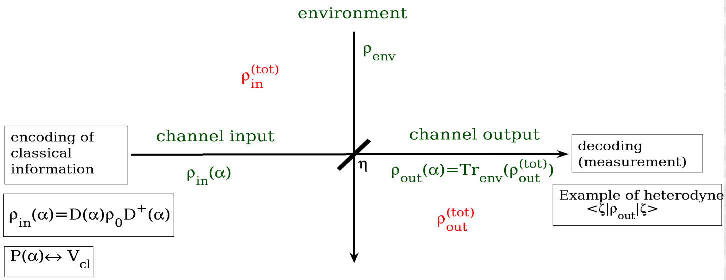
- The channel which acts independently on each use ($T = T^{\otimes n}$) is called *memoryless*. Otherwise, we call it *memory channel*.

I. Basic definitions from quantum channels theory

- *Achievable rate* of information transmission is the speed at which information can be reliably transferred through channel for fixed encoding and decoding.
- *Maximal achievable rate* of information transmission (considering all possible types of encoding and decoding) over channel is called *capacity*.
- We take electromagnetic fields E and H ("field quadratures") as our continuous variables, therefore our channel is called *bosonic channel* (as we work with bosonic field modes).
- Achievable rates and capacity for bosonic channel are finite only if there is an *energy restriction* at channel input.
- *Gaussian channels* are those continuous channels which map Gaussian states into Gaussian states (quantum state is Gaussian if, e.g., its representation is given by Gaussian distribution for some variables).

I. Definition of LBMC

- *Stinespring's dilation theorem* allows quantum channel to be modelled as
$$E(\rho) = \mathrm{Tr}_E[U(\rho \otimes \rho_E)U^\dagger]$$
- Example — (Gaussian) *lossy bosonic quantum channel*, where losses are introduced by interaction with extra "environment modes" on beam-splitter with transmissivity η:

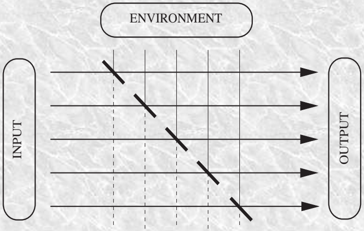


Energy restriction $\mathrm{Tr}(V_{in}+V_{cl})/(2n)=N+1/2$

Beam-splitter action $\rho_{tot}^{out}=U(\rho_{in}\otimes\rho_{env})U^{+}=U\rho_{tot}^{in}U^{+}$

I. Definition of LBMC

- Schematic representation of *multimode* (*multiuse*) lossy bosonic channel (*LBC*), where one can introduce memory — correlations between channel uses:



- Any Gaussian quantum state ρ can be completely described by covariance matrix V for quadratures $\mathbf{x} := (q_1, \ldots, q_n, p_1, \ldots, p_n)$ entering in its *Wigner function*:

$$\rho \longleftrightarrow W(\mathbf{x}) = \frac{1}{\sqrt{\det V}} \exp\left[-\frac{1}{2}\left(\mathbf{x} - \boldsymbol{\alpha}, V^{-1}(\mathbf{x} - \boldsymbol{\alpha})\right)\right]$$

- LBC can be reduced to the following relation between covariance matrices for input, environment and output states:

$$\boxed{V_{\mathrm{out}} = \eta V_{\mathrm{in}} + (1-\eta) V_{\mathrm{env}}}$$

I. Quantum channel capacity: definition

- Classical symbols to encode are distrubuted as Gaussian with covariance matrix $V_{\rm cl}/2$:

$$P(\boldsymbol{\alpha}) = \frac{1}{\pi^n \sqrt{\det V_{\rm cl}}} \exp\left[-\left(\boldsymbol{\alpha}, V_{\rm cl}^{-1}\boldsymbol{\alpha}\right)\right]$$

thus, *averaged output* channel state is

$$\boxed{\overline{V}_{\rm out} = \eta\left(V_{\rm in} + V_{\rm cl}\right) + (1-\eta)V_{\rm env}}$$

- *n-uses channel capacity* C_n can be estimated by its *Holevo bound* χ_n maximized over all possible encodings and decodings, as *Holevo coding theorem* [J.P. Gordon (1964), A.S. Holevo (1973)] states (S is von Neumann entropy):

$$C_n \leqslant \max_{\rho_{\rm in}, \rho_{\rm cl}} \frac{\chi_n}{n}, \qquad \chi_n = S\left(\int \rho_{\rm out}^{(\boldsymbol{\alpha})} P(\boldsymbol{\alpha}) d\boldsymbol{\alpha}\right) - \int S\left(\rho_{\rm out}^{(\boldsymbol{\alpha})}\right) P(\boldsymbol{\alpha}) d\boldsymbol{\alpha}$$

Below *we conjecture achievability of maximum on the set of Gaussian states* and call (for simplicity) this maximum as capacity:

$$C_n \equiv \max_{V_{\rm in}, V_{\rm cl}} \frac{\chi_n}{n}$$

II. LBMC: uncertainty relation and memory model

- Quantum state must satisfy HUR: $2n \times 2n$ covariance matrix V is admissable iff $V + i\Omega \geqslant 0$ where Ω is commutation matrix for canonical variables (quadratures):

$$\Omega = \begin{pmatrix} 0_n & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0_n \end{pmatrix}$$

Ω is called *symplectic form*. Quantum mechanics makes phase space *geometry* to be *symplectic*. $\Rightarrow$ HUR can be rewritten in terms of *symplectic eigenvalues* ν_k:

$$\boxed{\nu_k \geqslant 1/2}$$

> **Definition:** $\nu_k = \nu_k(V)$, $k = 1, \ldots, n$ are *symplectic eigs* of
> $$V = \begin{pmatrix} V_{qq} & V_{qp} \\ V_{qp}^\top & V_{pp} \end{pmatrix} \text{ if } \pm i\nu_k \text{ are eigs of } \widetilde{V} = \Omega^{-1} V = \begin{pmatrix} -V_{qp}^\top & -V_{pp} \\ V_{qq} & V_{qp} \end{pmatrix}$$

- Von Neumann entropy of Gaussian state is function of its symplectic eigs:

$$S(\rho) = \sum_{k=1}^{n} g\left(\nu_k - \frac{1}{2}\right), \quad \text{where} \quad g(v) = (v+1)\log_2(v+1) - v\log_2 v$$

- We restrict a class of *environment models* to study by matrices:

$$V_{\mathrm{env}} = \begin{pmatrix} V^{(qq)} & 0 \\ 0 & V^{(pp)} \end{pmatrix}, \quad \text{where } V^{(qq)} \text{ and } V^{(pp)} \text{ commute}$$

$\Longrightarrow$ The *problem becomes spectral* (all matrices can be taken in diagonal form).

II. LBMC: calculating of capacity

- Thus, *mathematical problem we need to solve*:

find $\boxed{C = \lim_{n\to\infty} C_n}$ where $C_n = \max_{\substack{i_{uk},\, i_{u_\star k} \\ c_{uk},\, c_{u_\star k}}} \frac{1}{n} \sum_{k=1}^{n} \left[g\left(\overline{\nu}_k - \frac{1}{2}\right) - g\left(\nu_k - \frac{1}{2}\right) \right]$

with symplectic eigs $\nu_k = \nu_k\left(V_{\mathrm{out}}\right)$ and $\overline{\nu}_k = \overline{\nu}_k\left(\overline{V}_{\mathrm{out}}\right)$:

$$\nu_k = \sqrt{o_{qk}\, o_{pk}} \qquad o_{uk} = \eta i_{uk} + (1-\eta) e_{uk}$$

$$\overline{\nu}_k = \sqrt{a_{qk}\, a_{pk}} \qquad a_{uk} = \eta (i_{uk} + c_{uk}) + (1-\eta) e_{uk}$$

- Eigs of matrices [$u \in \{q, p\}$; if $u = q \Rightarrow u_\star = p$; if $u = p \Rightarrow u_\star = q$; $k = 1, \ldots, n$]:
  - $i_{uk} \leftrightarrow V_{\mathrm{in}}$ — input *seed state* (we encode information in it)
  - $e_{uk} \leftrightarrow V_{\mathrm{env}}$ — environment state
  - $c_{uk}/2 \leftrightarrow V_{\mathrm{cl}}/2$ — distribution of encoded variable α ("modulation of signal")
  - $o_{uk} \leftrightarrow V_{\mathrm{out}}$ — output state
  - $a_{uk} \leftrightarrow \overline{V}_{\mathrm{out}}$ — output state averaged over encoding (over modulation)
- The maximum above is taken for fixed $e_{uk}, e_{u_\star k}, \eta, N$ and is *constrained* by:
  - Energy restriction: $\frac{1}{2n} \mathrm{Tr}(V_{\mathrm{in}} + V_{\mathrm{cl}}) = \frac{1}{2n} \sum_{k=1}^{n} \sum_{u\in\{q,p\}} [i_{uk} + c_{uk}] = N + \frac{1}{2}$
  - HUR: $\nu_k(V_{\mathrm{in}}) \geqslant \frac{1}{2} \implies i_{uk} > 0,\ i_{uk} i_{u_\star k} \geqslant \frac{1}{4}$
  - positivity: $c_{uk} \geqslant 0$

What can be proved *without* solving above maximization problem:
Suppose, we consider LBMC with all covariance matrices to be diagonal. Then:

- **Theorem:** *Maximum of Holevo bound is always achieved on pure state* $V_{\rm in}$*:*
  $$\boxed{i_{uk}\, i_{u_\star k} = 1/4}$$
  [J. Schäfer, D. Daems, E. Karpov, N. J. Cerf, PRA 80, 062313 (2009)]

$\Longrightarrow$ $i_{u_\star k}$ is *already found*, *i.e.* actual restrictions: $c_{uk}, c_{u_\star k} \geqslant 0$ and $i_{uk} > 0$.
Thus, we can use *Lagrange multipliers method* to find maximum of χ_n.

Proof: $\sqsupset$ $i_{uk}\, i_{u_\star k} > 1/4 \Rightarrow i_{u_\star k} = i'_{u_\star k} + \delta_k$, where $i'_{u_\star k} = 1/(4 i_{uk})$, $\delta_k > 0$.
Let us change varibales to make $V_{\rm in}$ pure (it preserves energy constraint N):

$$i'_{u_\star k} = i_{u_\star k} - \delta_k \qquad i'_{uk} = i_{uk}$$
$$c'_{u_\star k} = c_{u_\star k} + \delta_k \qquad c'_{uk} = c_{uk}$$

$\Rightarrow \overline{\nu}'_k = \overline{\nu}$, $\nu'_k < \nu_k$. Because of g_0 is monotonically growing function of its argument
$C_k(i'_{uk}, i'_{u_\star k}, c'_{uk}, c'_{u_\star k}) > C_k(i_{uk}, i_{u_\star k}, c_{uk}, c_{u_\star k})$ ■

II. LBMC: channel capacity for one use

- We will see that Lagrange multipliers method always results in $i_{uk} > 0 \Rightarrow$ we have to satisfy only to $c_{uk}, c_{u_\star k} \geqslant 0$. *Below there is a way to find such solution.*
- There are 3 possibilities for 1-use (1-mode) channel depending on energy restriction N and threshold value

$$N_{\rm thr}^{2\to 3}(e_u > e_{u_\star}) = \frac{1}{2}\left[\sqrt{\frac{e_u}{e_{u_\star}}} - 1 - \frac{1-\eta}{\eta}(e_{u_\star} - e_u)\right], \qquad 0 < N_{\rm thr}^{2\to 3} < \infty$$

  - Both $c_u, c_{u_\star} > 0$ — *3rd stage* — what holds if $N > N_{\rm thr}^{2\to 3} \Rightarrow$ capacity can be found in enclosed form:

$$C = g[\eta N + (1-\eta)M_{\rm env}] - g\left[(1-\eta)N_{\rm env}\right]$$

  - Case of $c_u = 0, c_{u_\star} > 0$ — *2nd stage* — what holds if $N \leqslant N_{\rm thr}^{2\to 3} \Rightarrow$ capacity depends on solution of one *transcendent equation* for i_u.
  - $c_u = 0, c_{u_\star} = 0$ — *1st stage* — what holds only if $N = 0$ (channel is not used for information transmission: $C = 0$).

$\Longrightarrow$ 1 use capacity for fixed values of $e_u, e_{u_\star}$ and η can be mentioned as a (concave) function:

$$N \longrightarrow \boxed{C = C(N)} \longrightarrow C$$

II. LBMC: channel capacity for one use

- In the following it will be useful to introduce function $g_k(v) = v^k g^{(k)}(v - 1/2)$. Thus, $g_0(v) = g(v - 1/2)$, $g_1(v) = vg'(v - 1/2)$ and so on. It also has simple rules to take derivatives, e.g.: $g_1' = (g_1 + g_2)/v$, $g_2' = (2g_2 + g_3)/v$, $g_1'' = (2g_2 + g_3)/v^2$.
- E.g. solution for 2nd stage is $c_u = 0$, $c_{u_\star} = 2N + 1 - i_u - 1/(4i_u)$, $i_{u_\star} = 1/(4i_u)$ and i_u is a root of the *mode transcendent equation* $F(i_u) = 0$, where

$$F(i_u) = \frac{\partial C}{\partial i_u} = \frac{\eta}{2}\left[g_1(\overline{\nu})\left(o_u^{-1} - a_{u_\star}^{-1}\right) - g_1(\nu)\left(o_u^{-1} - (4i_u^2 o_{u_\star})^{-1}\right)\right]$$

One can show that it is always form the interval

$$N + \frac{1}{2} - \sqrt{N^2 + N} < i_u < N + \frac{1}{2} + \sqrt{N^2 + N}$$

- g-function can be expanded on *quantum-admissable region* $v \geqslant 1/2$ [A.S. Holevo, 1999]:

$$g\left(v - \frac{1}{2}\right) = \log_2 v + \frac{1}{\ln 2}\left[1 - \frac{1}{2}\sum_{j=1}^{\infty}\frac{(2v)^{-2j}}{j(2j+1)}\right]$$

Thus, in zeroth-order approximation $g_1(v) = 1$ and the solution of mode transcendent equation is

$$i_u \approx i_u^{(0)} = \frac{1}{2}\left[\sqrt{1 + (2N+1)\phi + \phi^2/4} - \phi/2\right], \quad \text{where} \quad \phi = e_{u_\star}^{-1}\eta/(1-\eta)$$

.

II. LBMC: channel capacity for one use

- Let us prove concavity of function $C(N)$. Formally,

$$\frac{dC}{dN} = \frac{\partial C}{\partial N} + \frac{\partial C}{\partial i_u}\frac{\partial i_u}{\partial N}$$

However, we are only interested in variables, maximizing $\chi \Rightarrow \frac{\partial C}{\partial i_u} = 0$ and

$$\boxed{\frac{dC}{dN} = \frac{\partial C}{\partial N}}$$

- One can show that for all values of N and for both 2nd and 3rd stages

$$\boxed{\frac{dC}{dN} = \frac{\eta}{a_{u_\star}} g_1(\overline{\nu})}$$

and in particular [for $\mathrm{thr}^{1\to 2}$ we take $i_u = i_{u_\star} = 1/2$ and convention: u is that quadratrure for which $c_u = 0$ after perturbation of N — equivalent to $e_u > e_{u_\star}$]:

$$\mathrm{thr}^{1\to 2} \equiv \frac{dC}{dN}(N=0) = \frac{\eta}{o_{u_\star}} g_1(\nu) = \eta\sqrt{\frac{o_u}{o_{u_\star}}} g'(\nu - 1/2)$$

$$\mathrm{thr}^{2\to 3} \equiv \frac{dC}{dN}(N_{\mathrm{thr}}^{2\to 3}) = \eta\, g_1(\overline{\nu}) = \eta g'[\eta\,(i_u - 1/2) + (1-\eta)(e_u - 1/2)]$$

II. LBMC: channel capacity for one use

- Analogously,
$$\frac{\partial^2 C}{\partial N^2} = \eta^2 g_2(\overline{\nu})/\overline{\nu}^2 < 0$$
in the third stage (new update: the second stage is correctly considered in arXiv:0907.1532)
- Because of always $g_2 < 0$, $g_1 > 0$ concavity is preserved on the whole region of $N \in [0, \infty)$:
$$\frac{\partial^2 C}{\partial N^2}(N_{\mathrm{thr}}^{2\to 3} - 0) < \frac{\partial^2 C}{\partial N^2}(N_{\mathrm{thr}}^{2\to 3} + 0)$$
while first derivative is continuous in this point $\blacksquare$.
- Note, that
$$\max_N \frac{\partial C}{\partial N} = \frac{\partial C}{\partial N}(N = 0) < \infty$$
(euqal to ∞ only for $e_u = e_{u_\star} = 1/2$).

II. LBMC: purity theorems (purity of $V_{\rm env}$)

- **Theorem:** *Maximum of capacity over set of environment states* $\{V_{\rm env}\}$ *with* *fixed* *average amount of photons:*
  $$\frac{1}{2n}\,\mathrm{Tr}\,V_{\rm env} - \frac{1}{2} = M_{\rm env} \to \text{fixed}$$
  can be achieved on pure state $V_{\rm env}$*:*
  $$\boxed{e_{uk}e_{u_\star k} = 1/4}$$

$\Longrightarrow$ Optimal environment state $V_{\rm env}$ can be always chosen(?) to be pure.

Proof: At first, note that it takes place the following

- **Lemma** (credit to V. Zborovskii): $\sqsupset$ $a, b, c, d > 0$, $d > b$, $a - b > c - d$ *and* $f(x)$ *is monotonically growing concave function in interval* $x \in (0, \infty)$, *then*
  $$f(a) - f(b) > f(c) - f(d).$$

Our environment model

$$V_{\rm env} = \bigoplus_{k=1}^{n} V_{\rm env}^{(k)}, \quad \text{where} \quad V_{\rm env}^{(k)} = \left(N_{\rm env}^{(k)} + \frac{1}{2}\right)\begin{bmatrix} e^{s_k} & 0 \\ 0 & e^{-s_k} \end{bmatrix}$$

$$M_{\rm env} = \frac{1}{n}\sum_{k=1}^{n} M_{\rm env}^{(k)}, \quad \text{where} \quad M_{\rm env}^{(k)} = \left(N_{\rm env}^{(k)} + \frac{1}{2}\right)\cosh(s_k) - \frac{1}{2}$$

$\sqsupset$ kth mode $\in$ 1st. $\Rightarrow C_k \equiv 0$. $\sqsupset$ kth mode $\in$ 3st. $\Rightarrow \max_{N_{\rm env}^{(k)}} C_k = C_k(N_{\rm env}^{(k)} = 0)$

$\Rightarrow$ it is optimal to make kth mode pure.

II. LBMC: purity theorems (purity of $V_{\rm env}$)

- $\sqsupset$ kth mode $\in$ 2st. and $c_{qk} = 0$, where we operate with quadratures maximizing C_k. Note, that $c_{qk} = 0 \Rightarrow o_{qk} > a_{pk}$. Proof: suppose contradiction: $o_{qk} < a_{pk} \Rightarrow$ one can "redistribute energy" from c_{pk} in a way that $a'_{qk} > o_{qk}$, $a'_{pk} < a_{pk}$, *i.e.* $|a'_{qk} - a'_{pk}| < |o_{qk} - a_{pk}|$ while $a'_{qk} + a'_{pk} = o_{qk} + a_{pk}$, what means that $\overline{\nu}'_k > \overline{\nu}_k$, $\nu'_k = \nu_k \Rightarrow C'_k > C_k$ what contradicts to assumption that all qaudratures are already optimal. Also, evidently, $o_{qk} > a_{pk} \Rightarrow o_{qk} > o_{pk}$.
- One can show (taking into account procedure of calculating $N_k^{2\to 3}$) that from $c_{qk} = 0 \Rightarrow e_{qk} > e_{pk}$ [it makes a rule "we always encode in less noisy quadrature"].
- Let us change varibales for kth mode preserving $M_{\rm env}^{(k)}$ and making new $N'^{(k)}_{\rm env} = 0$ (we keep i_{uk} and c_{uk} the same): $e_{qk} \to e'_{qk}$, $e_{pk} \to e'_{pk}$. This will bring us to $o'_{qk} > o_{qk}$ and $o'_{pk} < o_{pk}$, *i.e.* $o'_{qk} - o'_{pk} > o_{qk} - o_{pk}$ while $o'_{qk} + o'_{pk} = o_{qk} + o_{pk}$. This means that $\nu'_k < \nu_k$.
- One can write down for the variable change made above:

$$o'_{qk}(o'_{pk} + \eta c_{pk}) - o'_{qk} o'_{pk} > o_{qk}(o_{pk} + \eta c_{pk}) - o_{qk} o_{pk}$$

  Taking into account inequalities above and applying Lemma for $f(x) = \sqrt{x}$ we get that $\overline{\nu}'_k - \nu'_k > \overline{\nu}_k - \nu_k$. Applying the Lemma again for function $f(x) = g_0(x - 1/2)$ we get that $C'_k > C_k$. PROFIT! ■

II. LBMC: channel capacity for n uses

- n-uses capacity is actually a *sum of convex functions, each of them depending on one varibale* (here energy restriction $N = \sum_{k=1}^{n} N_k$):

$$C(N) = \sum_{k=1}^{n} X_k(N_k) = \frac{1}{n} \sum_{k=1}^{n} C_k(e_{uk}, e_{u_\star k})$$

  This optimization problem is called "*convex separable programming*" and was solved in [S.M. Stefanov, *Comp. Opt. and App.* **18**, 27–48 (2001)].

- Thus, finding of multiuse capacity can be represented as search of optimal distribution $P(N_k)$ over "boxes" — monomodal (1-use) channels — to get "optimal summary output" $\sum_{k=1}^{n} X_k$:

$$N_1 \longrightarrow \boxed{X_1 = X_1(N_1)} \longrightarrow X_1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$N_n \longrightarrow \boxed{X_n = X_n(N_n)} \longrightarrow X_n$$

- To solve this type of "external maximization" mathematically one need to find such modes that

$$\frac{\partial C_k}{\partial N_k} = \text{const}(k), \quad \text{but} \quad \frac{\partial C_k}{\partial N_k} \leqslant \frac{\partial C_k}{\partial N_k}(N = 0) < \infty$$

III. LBMC: definition of Ω-model

- Demonstration of suggested method on the example of Ω *memory model* (capacity is presented only for the case of $n \to \infty$):

$$V_{\text{env}} = \left(N_{\text{env}} + \frac{1}{2}\right) \begin{bmatrix} e^{s\Omega} & 0 \\ 0 & e^{-s\Omega} \end{bmatrix}$$

Here N_{env} is average number of *thermal* photons per mode in channel environment, $s \in \mathbb{R}$ and Ω is matrix of $n \times n$-dimension:

$$\Omega = \begin{pmatrix} 0 & 1 & \ldots\ldots\ldots\ldots & & & 0 \\ 1 & 0 & 1 & \ldots\ldots\ldots & & 0 \\ \vdots & 1 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & \ddots & 1 \\ 0 & 0 & \ldots\ldots\ldots & & 1 & 0 \end{pmatrix}$$

- Properties of Ω-model (*quite realistic*):
  - Memory is *non-Markovian*
  - Correlations between channel uses decrease with time
  - Decay of correlations is *exponentional*

III. LBMC: capacity for Ω-model — 3rd stage case

- Capacity C for Ω-model, when all modes are in *3rd stage*:

$$C = g[\eta N + (1-\eta)M_{\mathrm{env}}] - g\left[(1-\eta)N_{\mathrm{env}}\right]$$

where

$$M_{\mathrm{env}} = \left(N_{\mathrm{env}} + \frac{1}{2}\right) I_0(2s) - \frac{1}{2}$$

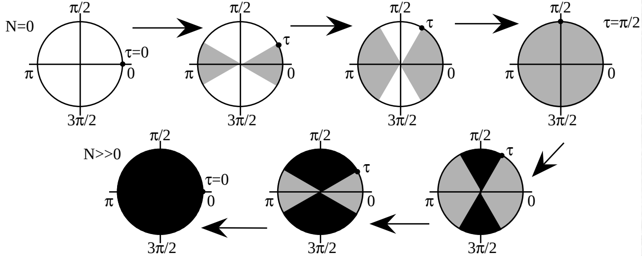


Fig.: Schematic representation of *evolution of stages distribution* with growth of N for Ω-model [for zeroth-order approximation only].

III. LBMC: capacity for Ω-model — general case

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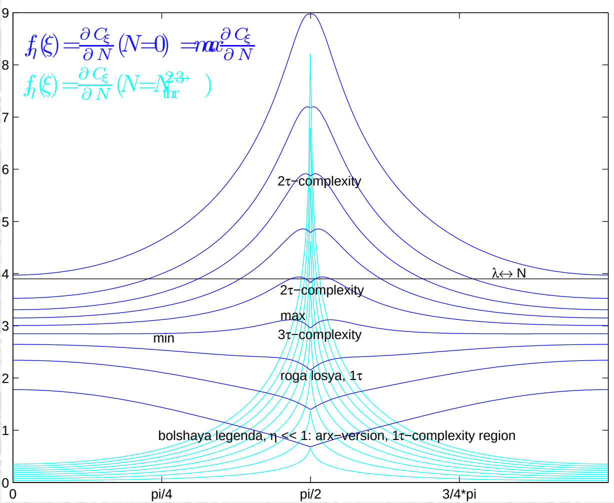


Figure: $\mathrm{thr}_\Omega^{1\to2}(\xi) = \frac{\partial C_\xi}{\partial N_\xi}(N=0)$, $\mathrm{thr}_\Omega^{2\to3}(\xi) = \frac{\partial C_\xi}{\partial N_\xi}(N=N_{\mathrm{thr}}^{2\to3})$

III. LBMC: capacity for Ω-model — general case

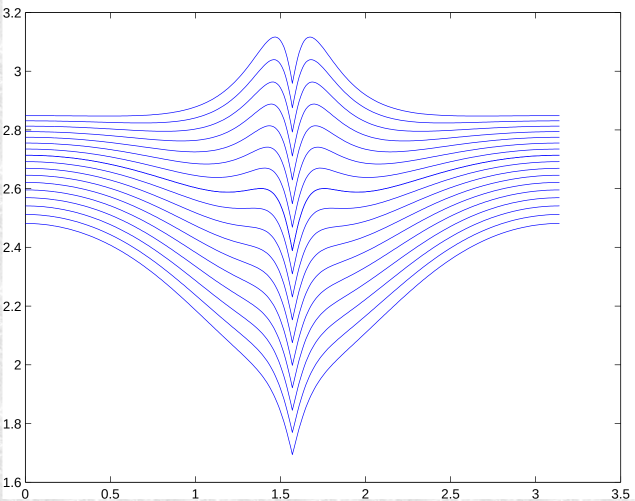


Figure: In another (larger) scale: $\mathrm{thr}_{\Omega}^{1\to 2}(\xi) = \frac{\partial C_\xi}{\partial N_\xi}(N = 0)$

III. LBMC: capacity for Ω-model — general case

- One need to analyze as function of $\xi \in (0, \pi/2)$ and depending on parameters $s, \eta, N_{\rm env}$:

$$\mathrm{thr}_{\Omega}^{1\to 2}(\xi, N=0) = \eta \sqrt{\frac{\frac{\eta}{2} + (1-\eta)(N_{\rm env} + \frac{1}{2})e^{2s\cos\xi}}{\frac{\eta}{2} + (1-\eta)(N_{\rm env} + \frac{1}{2})e^{-2s\cos\xi}}}\, g'\left(\nu_{\xi}(N=0) - \frac{1}{2}\right)$$

- Current success: let us introduce notations ($x = e^{2s\cos\xi}$ — this eliminate variable s and changes ξ to x, below also $\mu_x \equiv \nu_x^2$):

$$O_{qx} = \frac{\eta}{2} + (1-\eta)\left(N_{\rm env} + \frac{1}{2}\right)x, \qquad O'_{px} = -(1-\eta)\left(N_{\rm env} + \frac{1}{2}\right)\frac{1}{x^2}$$

$$O_{px} = \frac{\eta}{2} + (1-\eta)\left(N_{\rm env} + \frac{1}{2}\right)\frac{1}{x}, \qquad O''_{px} = 2(1-\eta)\left(N_{\rm env} + \frac{1}{2}\right)\frac{1}{x^3}$$

$$\mu'_x = \frac{\eta(1-\eta)}{2}\left(N_{\rm env} + \frac{1}{2}\right)\left(1 - \frac{1}{x^2}\right), \qquad \mu''_x = \eta(1-\eta)\left(N_{\rm env} + \frac{1}{2}\right)\frac{1}{x^3}$$

$$\mu_x = \frac{\eta^2}{4} + (1-\eta)^2\left(N_{\rm env} + \frac{1}{2}\right)^2 + \frac{\eta(1-\eta)}{2}\left(N_{\rm env} + \frac{1}{2}\right)\left(x + \frac{1}{x}\right)$$

Threasholds (all g_k-functions are with argument $\sqrt{\mu_x}$):

$$\mathrm{thr}_{1\to2}(x) = \frac{\eta}{O_{px}} g_1$$

$$\mathrm{thr}'_{1\to2}(x) = \frac{\eta}{O_{px}} \left[\frac{g_1+g_2}{2\mu_x}\mu'_x - \frac{O'_{px}}{O_{px}} g_1 \right]$$

$$\mathrm{thr}''_{1\to2}(x) = \frac{\eta}{O_{px}} \left[\frac{g_2-g_1+g_3}{4\mu_x^2}\mu_x'^2 + \frac{g_1+g_2}{2\mu_x}\left(\mu''_x - \frac{O'_{px}}{O_{px}}(\mu'_x+1)\right) - g_1 \left(\frac{O''_{px}}{O_{px}} - \frac{2O'^2_{px}}{O^2_{px}}\right)\right]$$

E.g. saddle-point corresponds to x which is a root of

$$\begin{cases} \mathrm{thr}'_{1\to2}(x) = 0 \\ \mathrm{thr}''_{1\to2}(x) = 0 \end{cases}$$

III. LBMC: capacity for Ω-model — general case

In general case for $n \to \infty$ (*0th-order approximation*) [O.V. Pilyavets, C. Lupo, S. Mancini, arxiv:0907.1532]:

$$C = \left(1 - \frac{2}{\pi}\tau_3\right)\left[g\left(x - \frac{1}{2}\right) - g((1-\eta)N_{\rm env})\right] + \frac{2}{\pi}\int_0^\tau \left[g\left(\sqrt{x\mathcal{O}_{q\xi}} - \frac{1}{2}\right) - g\left(\sqrt{\mathcal{O}_{q\xi}\mathcal{O}_{p\xi}} - \frac{1}{2}\right)\right] d\xi,$$

where $\xi \in [0, \pi/2]$,

$$\mathcal{O}_{q\xi} = \eta\mathcal{I}_{q\xi} + (1-\eta)\mathcal{E}_{q\xi}, \qquad \mathcal{E}_{q\xi} = \left(N_{\rm env} + \frac{1}{2}\right) e^{2s\cos\xi},$$

$$\mathcal{O}_{p\xi} = \frac{\eta}{4}\mathcal{I}_{q\xi}^{-1} + (1-\eta)\mathcal{E}_{p\xi}, \qquad \mathcal{E}_{p\xi} = \left(N_{\rm env} + \frac{1}{2}\right) e^{-2s\cos\xi},$$

Function $\mathcal{I}_{q\xi}$ (spectral density of matrix $V_{\rm in}$) can be found through solution of the next functional equation:

$$\left(\frac{1}{\mathcal{O}_{q\xi}} - \frac{1}{x}\right)\overline{\nu}_\xi \frac{\partial g(\overline{\nu}_\xi - 1/2)}{\partial \overline{\nu}_\xi} = \left(\frac{1}{\mathcal{O}_{q\xi}} - \frac{\mathcal{I}_{p\xi}}{\mathcal{I}_{q\xi}\mathcal{O}_{p\xi}}\right)\nu_\xi \frac{\partial g(\nu_\xi - 1/2)}{\partial \nu_\xi},$$

where $\overline{\nu}_\xi = \sqrt{x\mathcal{O}_{q\xi}}$, $\nu_\xi = \sqrt{\mathcal{O}_{q\xi}\mathcal{O}_{p\xi}}$, $\mathcal{C}_{p\xi} = (x - (1-\eta)\,\mathcal{E}_{p\xi})/\eta - \mathcal{I}_{p\xi}$ and $\mathcal{I}_{p\xi} = 1/(4\mathcal{I}_{q\xi})$.

III. LBMC: capacity for Ω-model — general case

Unknown variable τ can be found from the next functional equation:

$$\eta\left[N+\frac{\tau_1}{\pi}-\frac{1}{\pi}\int_0^{\tau}\mathcal{I}_{q\xi}\,d\xi\right]+\frac{1-\eta}{\pi}\int_0^{\tau_2}\mathcal{E}_{p\xi}\,d\xi=\frac{\tau_2}{\pi}x,$$

where $x=\eta e^{2s\cos\tau}/2+(1-\eta)\mathcal{E}_{q\tau}$ in the case (2,3,2), and $x=\eta/2+(1-\eta)\mathcal{E}_{p\tau}$ for the case of (2,1,2). Values $\tau_j, j=1,2,3$ are shown in table:

	τ_1	τ_2	τ_3
(2,1,2)	τ	τ	$\pi/2$
(2,3,2)	$\pi/2$	$\pi-\tau$	τ

Last question: how to choose correct type of stages distribution: (2,3,2) or (2,1,2). $\Longrightarrow$ We need to find $N=N_2$ from the equation above, substituting $\tau=\pi/2$. If N_2 found more or less than actual our N we have, correspondingly, (2,3,2)-case or (2,1,2)-case.

III. LBMC: capacity for Ω-model — general case

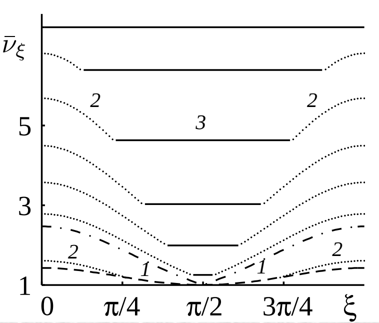


Fig.: *Spectral densities $\overline{\nu}_\xi$* for the case of $\Omega_{ij} = \delta_{i,j+1} + \delta_{i,j-1}$ are shown for parameters $N = 0$; 0.05; 0.67; 1; 2; 3.5; 6; 9; 11 going *from down to top* curve.

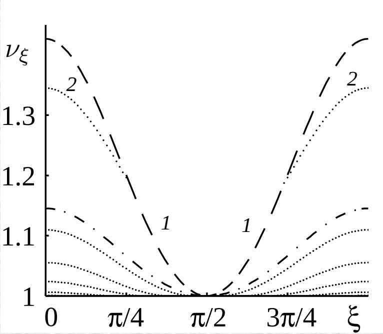


Fig.: *Spectral densities ν_ξ* for the case of $\Omega_{ij} = \delta_{i,j+1} + \delta_{i,j-1}$ are shown for parameters $N = 0$; 0.05; 0.67; 1; 2; 3.5; 6; 9; 11 going *from top to down* curve.

The values of other parameters: $N_{\rm env} = s = 1$, $\eta = 0,5$. These graphs can be interpreted as visualization of *"quantum waterfilling"* for $\overline{\nu}_\xi$ ν_ξ.

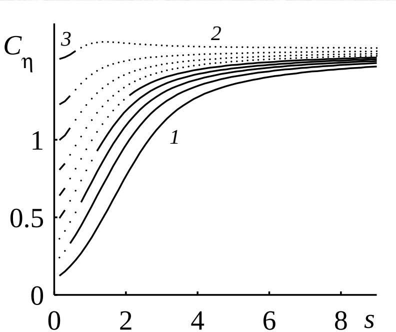


Fig.: *Capacity C* for the case of $\Omega_{ij} = \delta_{i,j+1} + \delta_{i,j-1}$ as a function of squeezing s for values of η starting from 0.1 (down curve) and up to 0.9 (top curve) with step 0.1. Values of other parameters: $N = N_{\rm env} = 1$.

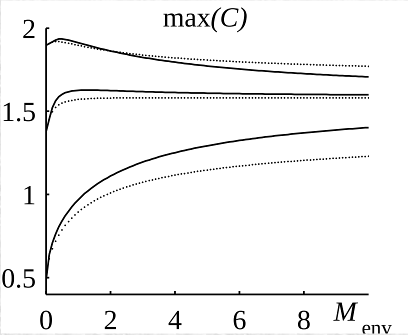


Fig.: *Maximum of capapcity C* over parameters of model $V_{\rm env}$ is shown as a function of $M_{\rm env}$ for values of $\eta = 0,1; 0,5; 0,9$ (we count from down to top).

IV. LBMC: achievable rates

- *Achievable rate* is maximum of averaged *mutual information* I shared between distributions of encoded (in our case — by means of *state displacement* in phase space) symbols α and decoded symbols ζ:

$$F_n = \frac{1}{n} \max_{V_{\rm in}, V_{\rm cl}} I(\zeta, \alpha), \quad I(\zeta : \alpha) = H(\zeta) - H(\zeta|\alpha), \quad H[\phi] = -\int P(\phi) \log_2 P(\phi) d\phi$$

- Examples of considered rates ($|\zeta_j\rangle$ is coherent state for j-th mode):
  - *Homodyne* (quadrature measurement):

$$\otimes_{j=1}^{n} \left(|\,\mathrm{Re}(\zeta_j)\rangle\langle \mathrm{Re}(\zeta_j)|\right) \Rightarrow I(\mathrm{Re}(\boldsymbol{\zeta}) : \mathrm{Re}(\boldsymbol{\alpha}))$$

  - *Heterodyne* (joint quadrature measurement in sense of *POVM*):

$$\otimes_{j=1}^{n} \left(|\zeta_j\rangle\langle\zeta_j|/\pi \right) \Rightarrow I(\boldsymbol{\zeta} : \boldsymbol{\alpha})$$

- Calculations routines for heterodyne $[\mathbf{x} = \boldsymbol{\zeta}/\sqrt{2} = (q_1, \ldots, q_n, p_1, \ldots, p_n)/\sqrt{2}]$ — *mnemonic rule*:

$$P(\boldsymbol{\zeta}|\boldsymbol{\alpha}) = \frac{1}{\pi^n} \langle \boldsymbol{\zeta}|\rho_{\rm out}^{(\boldsymbol{\alpha})}|\boldsymbol{\zeta}\rangle \rightarrow \boxed{V_{\boldsymbol{\zeta}|\boldsymbol{\alpha}} = V_{\rm out} + \frac{1}{2}} \Longleftrightarrow \boxed{V_{\boldsymbol{\zeta}|\boldsymbol{\alpha}} = V_{\rm out}} \leftarrow W_{\rm out}^{(\boldsymbol{\alpha})}(\mathbf{x}) = \widetilde{P}(\mathbf{x}|\boldsymbol{\alpha})$$

$$P(\boldsymbol{\zeta}) = \frac{1}{\pi^n} \langle \boldsymbol{\zeta}|\overline{\rho}_{\rm out}|\boldsymbol{\zeta}\rangle \rightarrow \boxed{V_{\boldsymbol{\zeta}} = \overline{V}_{\rm out} + \frac{1}{2}} \Longleftrightarrow \boxed{V_{\boldsymbol{\zeta}} = \overline{V}_{\rm out}} \leftarrow \overline{W}_{\rm out}(\mathbf{x}) = \widetilde{P}(\mathbf{x})$$

IV. LBMC: homodyne and heterodyne rates

- Calculating mutual information we get *heterodyne rate* published in [1] O.V. Pilyavets, V.G. Zborovskii, S. Mancini, *Phys. Rev.* A **77** 05234 (2008):

$$F_n^{(\mathrm{het})} = \frac{1}{2n} \max_{V_{\mathrm{in}}, V_{\mathrm{cl}}} \log_2 \det \left[\left(\overline{V}_{\mathrm{out}} + \frac{1}{2}\,\mathrm{Id}_{2n} \right) \left(V_{\mathrm{out}} + \frac{1}{2}\,\mathrm{Id}_{2n} \right)^{-1} \right]$$

what leads to explicit solution when all modes are in *3rd stage*:

$$F_n^{(\mathrm{het})} = \log_2 \left[\eta N + (1-\eta) M_{\mathrm{env}} + 1 \right] - \frac{1}{2n} \sum_{k=1}^{n} \sum_{u \in \{q,p\}} \log_2 \left[\frac{\eta}{2} \sqrt{\frac{1/2 + (1-\eta) e_{uk}}{1/2 + (1-\eta) e_{u_\star k}}} + (1-\eta) e_{uk} + \frac{1}{2} \right]$$

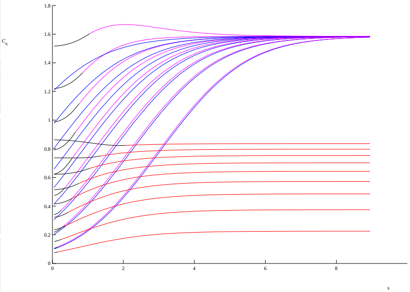
- *Homodyne rate* was also found in [1] (example of $|q\rangle\langle q|$-measurement):

$$F_n^{(\mathrm{hom})} = \frac{1}{2n} \max_{V_{\mathrm{in}}, V_{\mathrm{cl}}} \log_2 \det \left[\left(\overline{V}_{\mathrm{out}}^{(qq)} \right) \left(V_{\mathrm{out}}^{(qq)} \right)^{-1} \right]$$

IV. LBMC: homodyne and heterodyne rates

- *Surprizingly*, Holevo bound *in 0th-order approximation* is equal to homodyne rate if all modes are in *2nd stage* (only quadratures used for encoding are measured):

$$\chi_n^{(0)} \equiv F_n^{(\mathrm{hom})}$$



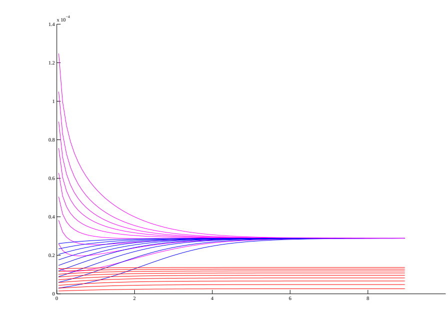


Fig.: Capacity, heterodyne and homodyne rates for 1-use channel.

Concluding remarks and summary of results

On classical capacity of LBMC:

- Method to calculate capacity *analytically* is found when problem is spectral. Algorithm uses some of recent achievments (2001) in optimization theory (problem of convex separable minimization).
- Capacity does not depend on any parameters except of energy constraint N when average amount of environment photons M_{env} tends to infinity.
- There is a *critical* beam-splitter *transmissivity* $\eta^{\star}$ such that optimal squeezing is finite above it and infinite below it.
- There is a *violation of quadrature and mode symmetry* in LBMC what makes squeezing and memory useful in some cases.

On generalization of results to achievable rates and other channel capacities:

- *Analytical* relations which express *heterodyne and homodyne rates* are found. Generalization of results found for capacity is strightforward for the case of rates.
- Heterodyne and homodyne rates do not reach (in general) capacity.
- Proposed approach can be easily extended for other capacities and Gaussian channels when problem is spectral (*noisy channel* is in process now in ULB group).

List of publications

1. O. V. Pilyavets, V. G. Zborovskii, S. Mancini, "*A Lossy Bosonic Quantum Channel with Non-Markovian Memory*", *Phys. Rev. A*, **77**:5 052324 (2008);
   in: H. Imai (ed.), *Proceedings of conference: 8th Asian Conference on Quantum Information Science* (Seoul, Korea, 25-31 August 2008), Korea Institute for Advanced Study, Seoul (2008), pp. 93–94;
   in: A. Lvovsky (ed.), *Proceedings of conference: The Ninth International Conference on Quantum Communication, Measurement and Computing (QCMC)* (Calgary, Canada, 19-24 August 2008), *AIP Conf. Proc.*, **1110**:1 123–126 (2009).
2. C. Lupo, O. V. Pilyavets, S. Mancini, "*Capacities of Lossy Bosonic Channel with Correlated Noise*", *New J. of Phys.*, **11**:6 063023 18pp (2009).
3. O. V. Pilyavets, C. Lupo, S. Mancini, "*Methods for Estimating Capacities of Gaussian Quantum Channels*", arXiv:0907.1532 (2009), [submitted to *IEEE Trans. Inf. Th.*].

In preparation:

4. E. Karpov, J. Schäfer, O. V. Pilyavets, N. Cerf, "*Classical Capacity of Noisy Quantum Channel with Memory*".
5. O. V. Pilyavets, S. Mancini, "*Achievable Rates for Classical Capacity in Lossy Bosonic Channel with Memory*".

thank you!