Optimal environment for quantum bosonic Gaussian channels

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Outline

• Classical capacity of quantum channels
• Gaussian capacity
• Correlated noise – optimal memory
• Conclusion
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$C$ – maximal rate of information transmission
(asymptotically errorless for infinite number of channel uses)
• Classical capacity of quantum channels

\[ C = \frac{1}{2} \log_2 \left(1 + \frac{P}{N}\right) \]

Shannon:

- \( C \) – maximal rate of information transmission
  (asymptotically errorless for infinite number of channel uses)
- \( N \) – noise power (Gaussian noise – a good approximation of natural noise)
- \( P \) – input signal power

Tokyo, January 10, 2012 – NEC Corporation (NEC) 1.15-Tb/s ultra-long haul optical superchannel technology over 10,000 kilometers
INFOANOTECH.COM
• **Classical capacity of quantum channels**

\[ C = \frac{1}{2} \log_2 \left[ 1 + \frac{P}{N} \right] \]

\[ = \frac{1}{2} \left( \log_2 [P + N] - \log_2 [N] \right) \]

\[ P \leq E \]

- \( C \) – maximal rate of information transmission (asymptotically errorless for infinite number of channel uses)
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**Shannon:**

\[ C = \frac{1}{2} \log_2 \left[ 1 + \frac{P}{N} \right] \]

\[ = \frac{1}{2} \left( \log_2 [P + N] - \log_2 [N] \right) \]

**Input power constraint!**
Information encoding into quantum states

- Quantum (letter) states $\rho_i$ introduce additional (quantum noise):
  - Probability of $\rho_i$: $P_i$ → input ensemble $\{P_i, \rho_i\}$
  - Input ensemble: $\rho_{in} = \sum P_i \rho_i$

- Quantum channel – completely positive linear map:
  - Output ensemble: $\rho_{out} = T[\rho_{in}]$

- Classical capacity of quantum channel: $T$

\[
C_{\chi}[T] = \max_{\{P_i, \rho_i\}} \left[ S(T[\rho_{in}]) - \sum_i P_i S(T[\rho_i]) \right] \quad \text{[Holevo-Schumacher-Westmoreland]}
\]

- Regularization (additivity problem)

\[
C[T] = \lim_{n \to \infty} \frac{1}{n} C_{\chi}[T^\otimes n] \quad \text{where} \quad T^\otimes n[\rho^{(n)}] \quad \text{and} \quad \rho^{(n)} \in H^\otimes n
\]

Von Neumann entropy: $S(\rho) = -\text{Tr}[\rho \cdot \text{Log}_2 \rho]$
Encoding with continuous variables

- Quadratures of electromagnetic field mode ($\hbar = \omega = 1$)
  \[ \hat{q} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger) \]
  \[ \hat{p} = \frac{1}{i\sqrt{2}}(\hat{a} - \hat{a}^\dagger) \]
  \[ [\hat{a}, \hat{a}^\dagger] = 1 \]

- Phase space description – Wigner quasidistributions:
  - displacement vector of the first moments of the quadratures:
    \[ q_i = \langle \hat{q} \rangle_{\rho_i} = \text{Tr}[\hat{q}\rho_i] \]
    \[ p_i = \langle \hat{p} \rangle_{\rho_i} = \text{Tr}[\hat{p}\rho_i] \]
  - encoding of real values into quantum states
    \[ (q_i, p_i) \rightarrow \rho_i(q_i, p_i) \]
  - continuous encoding
    \[ \rho_{in} = \int dq dp P(q, p)\rho(q, p) \]
Encoding with continuous variables

• Quadratures of electromagnetic field mode (\(\hbar = \omega = 1\))
  \[
  \hat{q} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^+) \\
  \hat{p} = \frac{1}{i\sqrt{2}}(\hat{a} - \hat{a}^+)
  \]

• Phase space description – Wigner quasidistributions:
  \[
  W_{\rho_i}(q, p)
  \]
  – displacement vector of the first moments of the quadratures:
  \[
  q_i = \langle \hat{q} \rangle_{\rho_i} = \text{Tr}[\hat{q}\rho_i] \\
  p_i = \langle \hat{p} \rangle_{\rho_i} = \text{Tr}[\hat{p}\rho_i]
  \]
  – encoding of real values into quantum states
  \[
  (q_i, p_i) \rightarrow \rho_i(q_i, p_i)
  \]
  – continuous encoding
  \[
  \bar{\rho}_{in} = \int dq dp P(q, p) \rho(q, p)
  \]

\[
C_x[T] = \max_{\{P, \rho\}} \left[ S(T[\bar{\rho}_{in}]) - \int dq dp P(q, p) S(T[\rho(q, p)]) \right]
\]

complicated....
Good news for Gaussian channels

- *Gaussian conjecture is proven!*
  for a large class of Gaussian channels


- Vacuum (coherent) input state achieves the minimum output entropy
  
  \[ C_\chi [T] = \max_{\{P, \rho\}} \left[ S(T[\bar{\rho}_{in}]) - \int dq dp P (q, p) S(T[\rho(q, p)]) \right] \]

- Vacuum(coherent) input state achieves the classical capacity

- Classical capacity of Gaussian channels is additive

Restriction to *Gaussian inputs*
Gaussian channels

- Gaussian noise – a good approximation of natural noise
- Gaussian channels transform Gaussian states to Gaussian states
- Gaussian states are states with a Gaussian Wigner function

Completely defined by
- the first moments \( \langle \hat{a} \rangle_\rho, \langle \hat{p} \rangle_\rho \) – displacement vector
- the second moments \( \gamma \) – covariance matrix (CM)

- Von Neumann entropy is a function of symplectic eigenvalues of CM:
  \[
  S(\gamma) = \sum_{i=0}^{n} g(\nu_i - 1/2) \quad \gamma \rightarrow \text{diag}(\nu_1, \nu_1, \nu_2, \nu_2, \ldots, \nu_n, \nu_n)
  \]
  \[
  g(x) = (x + 1) \log_2 (x + 1) - (x) \log_2 (x)
  \]

- **Gaussian capacity** – Classical capacity with maximization over Gaussian inputs
Parameterizing the Gaussian channels

- *Fiducial* channel (single mode)

\[ \gamma_{out} = T_F [\gamma_{in}] = |\tau| \gamma_{in} + |1 - \tau| \gamma_{env} \]

\[ \sqrt{\det \gamma_{env}} = |1 - \tau| \left( N_{env} + \frac{1}{2} \right) \]

Any single-mode Gaussian channel \( T_G \) has its *fiducial* counterpart \( T_F \) such that

\[ C_\chi [T_G] = C_\chi [T_F] \]


\[ \tau \in ]-\infty, +\infty[ \]
Gaussian capacity

- **Fiducial channel (single mode)**

\[
\gamma_{out} = T_F \left[ \gamma_{in} \right] = |\tau| \gamma_{in} + |1 - \tau| \gamma_{env} \quad \tau \in \mathbb{R}
\]

\[
\gamma_{env} = \left( N_{env} + \frac{1}{2} \right) \begin{bmatrix} e^{2s} & 0 \\ 0 & e^{-2s} \end{bmatrix}
\]

- **Classical capacity**

\[
C_{\chi} [T_F] = g \left( \left| \tau \right| \frac{E}{2} + y \cosh(2s) - \frac{1}{2} \right) - g \left( y + \frac{|\tau| - 1}{2} \right)
\]

\[
y = |1 - \tau| \left( N_{env} - \frac{1}{2} \right)
\]

- **Minimum output entropy input state** – the vacuum squeezed as the noise

\[
\gamma_{in} = \frac{1}{2} \begin{bmatrix} e^{2s} & 0 \\ 0 & e^{-2s} \end{bmatrix}
\]

\[
\text{Tr}[\gamma_{in}] = \cosh(2s)
\]

\[
\text{Tr}[\gamma_{vac} + \gamma_{m}] \leq E
\]

- **If not** Lagrange multipliers optimization:

*Optimal input is not minimum output entropy state!*
Correlated noise – squeezed noise

- Noise correlations over $n$ successive uses of the channel:
  \[
  \gamma_{env} = \begin{pmatrix}
  M^{(q)}(\phi) & 0 \\
  0 & M^{(p)}(-\phi)
  \end{pmatrix}
  \quad M_{ij}(\phi) = N\phi^{|i-j|}
  \]

- Broadband channel of $n$ modes:
  \[
  C(T) = \lim_{n \to \infty} \frac{1}{n} C_T\left(T^{(n)}(\rho^{(n)})\right)
  \quad \rho^{(n)} \in H^\otimes n
  \]

- CM is diagonalized by a passive symplectic transformation in the infinite limit of number of uses
  \[
  \tilde{M}_{ij}(\pm\phi) = N \frac{1 - \phi^2}{1 + \phi^2 \mp 2\phi \cos(x)} \delta_{ij}
  \]

(Difference in $q$ and $p$ quadratures → squeezing)

[Schäfer, et al. PRA 84, 032318 (2010)]
Noise correlations (squeezing) increase the classical capacity

- Capacity of the additive noise channel in the limit of “full” correlations

\[
\gamma_{env} = \begin{pmatrix}
M^{(q)}(\phi) & 0 \\
0 & M^{(p)}(-\phi)
\end{pmatrix}
\]

attains the capacity of an ideal channel

\[
M_{ij}(\phi) = N\phi^{|i-j|}
\]

\[
\phi \to 1
\]

- The limit of the distribution of corresponding diagonal noise is a delta-like function with two “infinitely” squeezed modes

\[
\lim_{\phi \to 1} C_{\chi} = g\left(\frac{E-1}{2}\right)
\]

\[
\tilde{M}(\phi)
\]

- **NB:** For all \( \phi \), the energy of the noise is the same!

- **Problem:** what is the optimal noise distribution given the noise energy of a Gaussian channel?

Energy constraint on the noise

- Capacity optimization problem with two constraints
  
  \[
  \text{Tr}[\gamma_{\text{vac}} + \gamma_m] \leq E \quad \text{Tr}[\gamma_{\text{env}}] \leq E_{\text{env}}
  \]

  - Optimal environment mode squeezing in a single mode
  - Optimal distribution of environment energy between different modes

- Additive noise channel (classical noise) – optimal environment

  \[
  \bar{\gamma}_{\text{out}} = \gamma_{\text{in}} + \gamma_m + \gamma_{\text{env}}
  \]

  - Single-mode – infinite “squeezing” (channel B1 in Holevo classification)

  \[
  \gamma_{\text{env}} = E_{\text{env}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
  \]

  - Two modes of B1 channel:
    (Gaussian conjecture is not proven !)

  \[
  E_{\text{env,1}} = E_{\text{env}} \quad E_{\text{env,2}} = 0
  \]
Noise energy constraint for the *fiducial* channel

- Quantum noise

\[ \bar{\gamma}_{out} = |\tau|(\gamma_{in} + \gamma_m) + |1 - \tau|\gamma_{env} \]

- Infinite squeezing for a finite \( E_{env} \) is impossible, a non-trivial optimization is required

- Worst case noise – an opposite problem:
  - Thermal noise for a single mode?

\[ \gamma_{env} = \left( N_{env} + \frac{1}{2} \right) \begin{bmatrix} e^{2s} & 0 \\ 0 & e^{2s} \end{bmatrix} \]

- A uniform distribution between the modes?
Conclusion and outlook

• Gaussian capacity of *fiducial* bosonic channel is the classical capacity if there is enough input energy

• Optimal environment (memory) problem under the noise energy constraint

• Additive noise channel (classical noise): full correlations or “infinite” squeezing and concentration of all noise in one mode (quadrature) are optimal

• Noise optimization for a general *fiducial* channel requires further study

• Is the thermal noise the “worst case environment”? 
Thank you for your attention!
Gaussian capacity

- Restriction to Gaussian

\[
\rho(q,p) = \hat{D}(q,p) \rho_G \hat{D}^\dagger(q,p) \quad \overline{\rho}_{\text{in}} = \int dq dp P_G(q,p) \rho(q,p)
\]

\[
C_G[T_G] \geq \max_{\{P_G, \rho_G\}} \left[ S(T[\overline{\rho}_{\text{in}}]) - \int dq dp P_G(q,p) S(T[\rho(q,p)]) \right]
\]

\[
C_G[T_G] \geq \max_{\{P_G, \rho_G\}} \left[ S(T[\overline{\rho}_{\text{in}}]) - \int dq dp P_G(q,p) S(T[\rho_G]) \right]
\]

\[
C_G[T_G] = \max_{\{P_G, \rho_G\}} \left[ S(T[\overline{\rho}_{\text{in}}]) - S(T[\rho_G]) \right]
\]

\[
C_G[T_G] = \max_{\{\gamma_m, \gamma_{\text{in}}\}} \left[ S(\overline{\gamma}_{\text{out}}) - S(\gamma_{\text{out}}) \right]
\]
Gaussian capacity

• Restriction to Gaussian input states and encodings

\[ \rho(q, p) = \hat{D}(q, p) \rho_G \hat{D}^+(q, p) \quad \bar{\rho}_{in} = \int dq dp P_G(q, p) \rho(q, p) \]

\[ C_G[T_G] = \max_{\{P_G, \rho_G\}} \left[ S(T[\bar{\rho}_{in}]) - \int dq dp P_G(q, p) S(T[\rho(q, p)]) \right] \]

\[ C_G[T_G] = \max_{\{P_G, \rho_G\}} \left[ S(T[\bar{\rho}_{in}]) - \int dq dp P_G(q, p) S(T[\rho_G]) \right] \]

\[ C_G[T_G] = \max_{\{P_G, \rho_G\}} \left[ S(T[\bar{\rho}_{in}]) - S(T[\rho_G]) \right] \]

\[ C_G[T_G] = \max_{\{\gamma_m, \gamma_{in}\}} \left[ S(\bar{\gamma}_{out}) - S(\gamma_{out}) \right] \]
Gaussian capacity

- Restriction to Gaussian

\[ \rho(q, p) = \hat{D}(q, p) \rho_G \hat{D}^+(q, p) \]

\[ \bar{\rho}_{in} = \int dq dp P_G(q, p) \rho(q, p) \]

\[ C_G[T_G] = \max_{\{P_G, \rho_G\}} \left[ S(T[\bar{\rho}_{in}]) - \int dq dp P_G(q, p) S(T[\rho(q, p)]) \right] \]

\[ C_G[T_G] = \max_{\{P_G, \rho_G\}} \left[ S(T[\bar{\rho}_{in}]) - \int dq dp P_G(q, p) S(T[\rho_G]) \right] \]

\[ C_G[T_G] = \max_{\{P_G, \rho_G\}} \left[ S(T[\bar{\rho}_{in}]) - S(T[\rho_G]) \right] \]

\[ C_G[T_G] = \max_{\{\gamma_m, \gamma_{in}\}} \left[ S(\bar{\gamma}_{out}) - S(\gamma_{out}) \right] \]

\[ \text{Input energy constraint!} \]

\[ \text{Tr}[\gamma_{in} + \gamma_m] \leq E \]
Gaussian capacity

- “Naïve” optimization

\[ C_G[T_G] = \max_{\{\gamma_m, \gamma_{in}\}} \left[ S(\gamma_{out}) - S(\gamma_{out}) \right] \]

  - Conjecture – minimum output entropy

\[ \min_{\{\gamma_m, \gamma_{in}\}} \left[ S(\gamma_{out}) \right] = S(T[\rho_{\text{vac}}]) \]

  - Choose modulation

\[ \max_{\{\gamma_m, \gamma_{\text{vac}}\}} \left[ S(\gamma_{out}) \right] = S\left(\left(\overline{N}_{th} + \frac{1}{2}\right)I\right) \]

Is it always possible to choose modulation satisfying the input energy constraint?

\[ \text{Tr}[\gamma_{\text{vac}} + \gamma_m] \leq E \]
Waterfilling solution (single-mode)

• Optimizing \( C_1[T_G] = \max \{ \gamma_{in}, \gamma_m \} \left[ S(\gamma_{in} + \gamma_m + \gamma_e) - S(\gamma_{in} + \gamma_e) \right] \)
  
  – Uniform distribution of the output energy is optimal \( \overline{\gamma}_q = \overline{\gamma}_p \)
  
  – Optimal input is a minimum output entropy Gaussian state

\[
C[T] = g \left( \frac{(\lambda + e_q + e_p - 1)}{2} \right) - g \left( \sqrt{e_q e_p} \right)
\]

[Holevo, et al. PRA 59, 1820 (1999)]
Waterfilling solution

- Optimizing $C_1[T_G] = \max_{\{\gamma_{in}, \gamma_m\}} \left[ S(\gamma_{in} + \gamma_m + \gamma_e) - S(\gamma_{in} + \gamma_e) \right]$
  - Uniform distribution of the output energy is optimal
  - Optimal input is a minimum output entropy Gaussian state

- $C[T] = g\left(\frac{\lambda + e_q + e_p - 1}{2}\right)

- $\lambda > \lambda_{th} = \sqrt{e_q/e_p} + e_q - e_p$
Solution of 2^d type

• Only one quadrature modulated
• Lagrange multipliers – no explicit solution
  – An implicit solution via a transcendent equation
  – Input energy also spent on the non-modulated quadrature
  – Optimal input state is not the minimum output entropy state*

\[ 1 < \frac{i_q}{i_p} < \frac{e_q}{e_p} \]

[Schäfer, et al. PRA 80, 062313 (2009)]
\[ g'(\bar{\nu} - 1/2)(\bar{\gamma}_p - \bar{\gamma}_q) / \bar{\nu} = g'(\nu_{out} - 1/2)(\gamma_p - \gamma_q / (4i_q^2)) / \nu_{out} \]

\[ g'(x - 1/2) = \log_2 \left[ (x + 1/2) / (x - 1/2) \right] \]

\[ \bar{\nu} = \sqrt{\bar{\gamma}_q \bar{\gamma}_p} \quad \nu_{out} = \sqrt{\gamma_q \gamma_p} \]

\[ \bar{\gamma}_{q,p} = i_{q,p} + m_{q,p} + e_{q,p} \quad \gamma_{q,p} = i_{q,p} + e_{q,p} \]

\[ i_q + i_p + m_q + m_p = \lambda \quad i_q i_p = 1/4 \]
Optimal states for multimode channel

Non-trivial energy distribution among the modes – common Lagrange multiplier – **waterfilling**

\[ \lambda > \lambda_{th} \]

\[ \frac{i_q}{i_p} = \frac{e_q}{e_p} \]

\[ \frac{i_q}{i_p} = 1 \]
Optimal states for multimode channel

Non-trivial energy distribution among the modes – common Lagrange multiplier – generalized waterfilling

\[ \lambda > \lambda_{th} \]

\[ 1 < \frac{i_q}{i_p} < \frac{e_q}{e_p} \]

\[ \frac{i_q}{i_p} = 1 \]