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# Optimal environment for quantum bosonic Gaussian channels


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# Outline

- Classical capacity of quantum channels
- Gaussian capacity
- Correlated noise – optimal memory
- Conclusion

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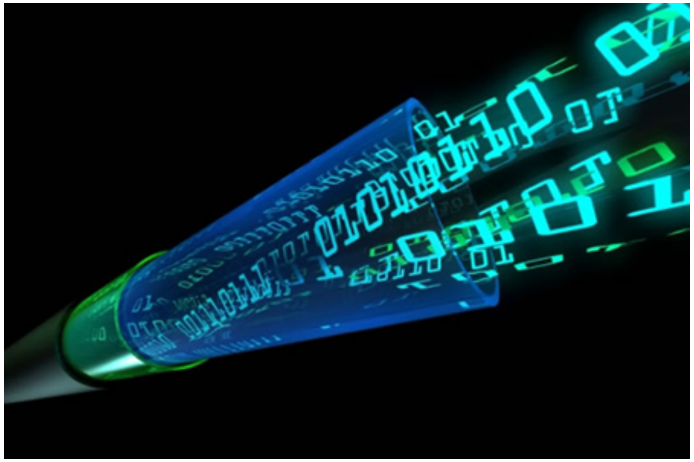
Squeezing

- Classical capacity of quantum channels



Tokyo, January 10, 2012 – NEC Corporation (NEC) 1.15-Tb/s ultra-long haul optical superchannel technology over 10,000 kilometers

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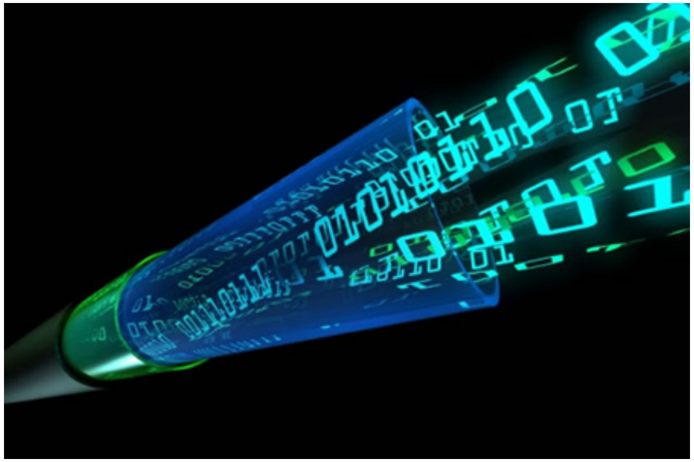
- Classical capacity of quantum channels

  C – maximal rate of information transmission
  (asymptotically errorless for infinite number of channel uses)



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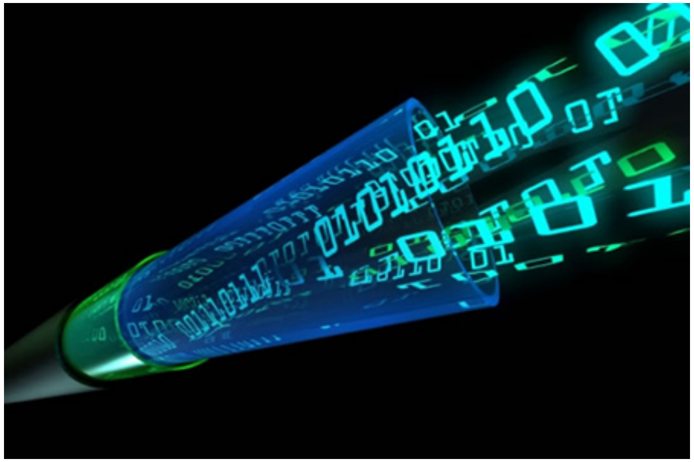
  *C* – maximal rate of information transmission
  (asymptotically errorless for infinite number of channel uses)

  *N* – noise power ( Gaussian noise – a good approximation of natural nose)

  *P* – input signal power

Shannon: $$C = \frac{1}{2}\log_2\left[1+\frac{P}{N}\right]$$

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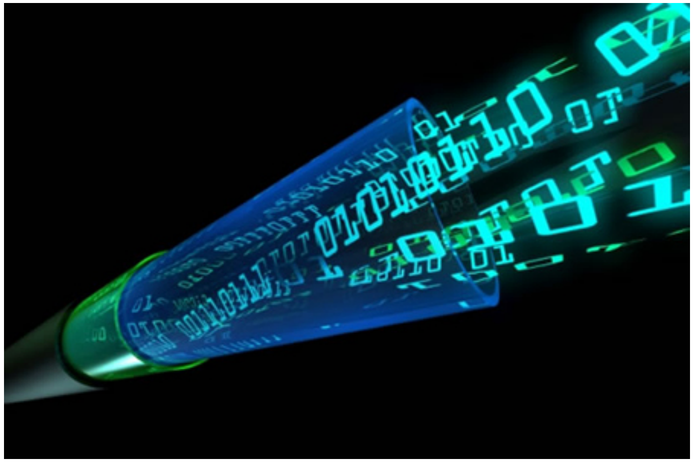
Shannon: $$C = \frac{1}{2}\log_2\left[1+\frac{P}{N}\right] = \frac{1}{2}\left(\log_2\left[P+N\right]-\log_2\left[N\right]\right)$$



Input power constraint ! $P \leq E$

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# Information encoding into quantum states

- Quantum (letter) states $\rho_i$ introduce additional (quantum noise):

$P_i$ – probability of $\rho_i$ ⇨ $\{P_i, \rho_i\}$ – input ensemble $\overline{\rho}_{in} = \sum P_i \rho_i$

- Quantum channel – completely positive linear map : $\rho_{out} = T[\rho_{in}]$
- Classical capacity of quantum channel : $T$

$$C_\chi[T] = \max_{\{P_i, \rho_i\}} \left[ S\left(T[\overline{\rho}_{in}]\right) - \sum_i P_i S\left(T[\rho_i]\right) \right]$$

[Holevo-Schumacher-Westmoreland]

Von Neumann entropy : $S(\rho) = -\mathrm{Tr}[\rho \cdot Log_2 \rho]$

- Regularization (additivity problem)

$C[T] = \lim_{n\to\infty} \frac{1}{n} C_\chi[T^{\otimes n}]$ where $T^{\otimes n}[\rho^{(n)}]$ and $\rho^{(n)} \in H^{\otimes n}$

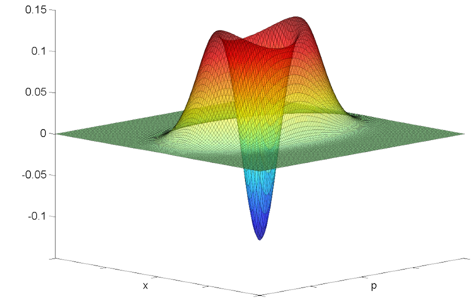
# Encoding with continuous variables

- Quadratures of electromagnetic field mode ( $\hbar = \omega = 1$ ) $\left[\hat{a}, \hat{a}^{\dagger}\right] = 1$

$$\hat{q} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^{\dagger}\right) \qquad \hat{p} = \frac{1}{i\sqrt{2}}\left(\hat{a} - \hat{a}^{\dagger}\right)$$

- Phase space description – Wigner quasidistributions : $W_{\rho_i}\left(q, p\right)$
  - displacement vector of the first moments of the quadratures:

$$q_i = \left\langle \hat{q} \right\rangle_{\rho_i} = \mathrm{Tr}[\hat{q}\rho_i]$$
$$p_i = \left\langle \hat{p} \right\rangle_{\rho_i} = \mathrm{Tr}[\hat{p}\rho_i]$$




  - encoding of real values into quantum states $(q_i, p_i) \rightarrow \rho_i(q_i, p_i)$
  - continuous encoding $\overline{\rho}_{in} = \int dq\, dp P(q,p)\rho(q,p)$

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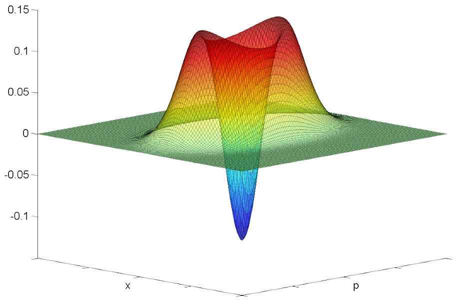


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$$C_\chi\left[T\right] = \max_{\{P,\ \rho\}}\left[S\left(T[\overline{\rho}_{in}]\right) - \int dq\, dp P\left(q,p\right) S\left(T[\rho(q,p)]\right)\right]$$

compilcated....

# Good news for Gaussian channels

- *Gaussian conjecture is proven !*

  for a large class of Gaussian channels

  [V. Giovannetti, R. García-Patrón, N. J. Cerf and A. S. Holevo, *Nature Photonics*, 8, 796 (2014).]

  - *Vacuum (coherent) input state* achieves the minimum output entropy

$$C_\chi[T] = \max_{\{P,\ \rho\}}\left[S\left(T[\bar{\rho}_{in}]\right) - \int dq\, dp P\,(q,p) S\left(T[\rho(q,p)]\right)\right]$$

  - Vacuum(coherent) input state achieves the classical capacity
  - Classical capacity of Gaussian channels is additive

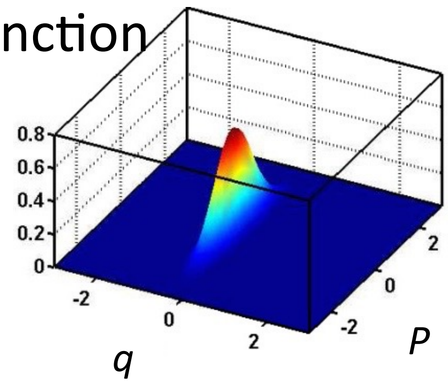
 Restriction to *Gaussian inputs*

# Gaussian channels

- Gaussian noise – a good approximation of natural noise
- Gaussian channels transform Gaussian states to Gaussian states

- Gaussian states are states with a Gaussian Wigner function

  Completely defined by
  - the first moments $\left(\langle\hat{q}\rangle_\rho,\langle\hat{p}\rangle_\rho\right)^T$ – displacement vector
  - the second moments $\gamma$ – covariance matrix (CM)




- Von Neumann entropy is a function of symplectic eigenvalues of CM:

$$S(\gamma)=\sum_{i=0}^{n} g(\nu_i-1/2) \qquad \gamma\to diag(\nu_1,\nu_1,\nu_2,\nu_2,\ldots,\nu_n,\nu_n)$$

$$g(x)=(x+1)\log_2(x+1)-(x)\log_2(x)$$

- *Gaussian capacity* – Classical capacity with maximization over Gaussian inputs

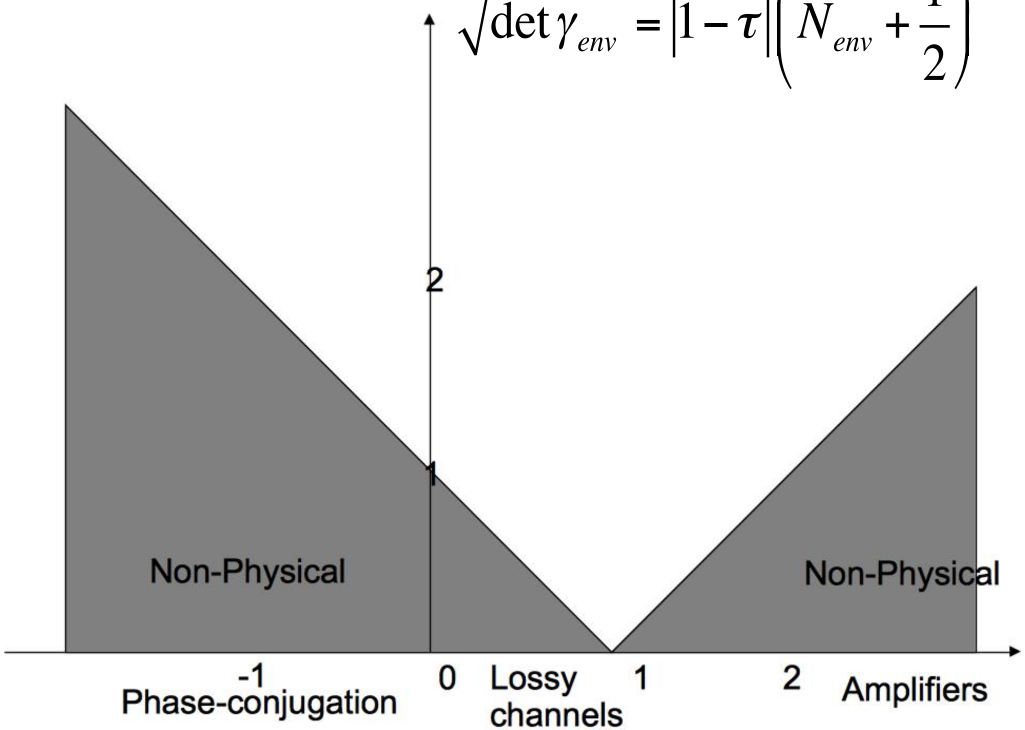
# Parameterizing the Gaussian channels

- *Fiducial* channel (single mode)

$$\gamma_{out} = T_F\left[\gamma_{in}\right] = |\tau|\gamma_{in} + |1-\tau|\gamma_{env}$$

$$\gamma_{env} = \left(N_{env} + \frac{1}{2}\right)\begin{bmatrix} e^{2s} & 0 \\ 0 & e^{2s} \end{bmatrix}$$

$$\sqrt{\det \gamma_{env}} = |1-\tau|\left(N_{env} + \frac{1}{2}\right)$$




$$\tau \in \left]-\infty, +\infty\right[$$

Any single-mode Gaussian channel $T_G$ has its *fiducial* counterpart $T_F$ such that $C_\chi\left[T_G\right] = C_\chi\left[T_F\right]$

[J. Schäfer. et al. PRL 111 (2013) 030503]

# Gaussian capacity

- *Fiducial* channel (single mode)

$$\gamma_{out} = T_F[\gamma_{in}] = |\tau|\gamma_{in} + |1-\tau|\gamma_{env}$$

$$\tau \in ]-\infty, +\infty[$$

$$\gamma_{env} = \left(N_{env} + \frac{1}{2}\right)\begin{bmatrix} e^{2s} & 0 \\ 0 & e^{-2s} \end{bmatrix}$$

- Classical capacity

$$C_\chi[T_F] = g\left(|\tau|\frac{E}{2} + y\cosh(2s) - \frac{1}{2}\right) - g\left(y + \frac{|\tau|-1}{2}\right)$$

$$y = |1-\tau|\left(N_{env} - \frac{1}{2}\right)$$

- Minimum output entropy input state – the vacuum squeezed as the noise

$$\gamma_{in} = \frac{1}{2}\begin{bmatrix} e^{2s} & 0 \\ 0 & e^{-2s} \end{bmatrix} \rightarrow \mathrm{Tr}[\gamma_{in}] = \cosh(2s)$$

does work if $\mathrm{Tr}[\gamma_{vac} + \gamma_m] \leq E$ !

- If not → Lagrange multipliers optimization :

*Optimal input is not minimum output entropy state !*

# Correlated noise – squeezed noise

- Noise correlations over *n* successive uses of the channel:

$$\gamma_{env} = \begin{pmatrix} M^{(q)}(\phi) & 0 \\ 0 & M^{(p)}(-\phi) \end{pmatrix}$$

$$M_{ij}(\phi) = N\phi^{|i-j|}$$

- Broadband channel of *n* modes :

$$C(T) = \lim_{n\to\infty} \frac{1}{n} C_\chi\left(T^{(n)}(\rho^{(n)})\right)$$

$$\rho^{(n)} \in H^{\otimes n}$$

- CM is diagonalized by a passive symplectic transformation in the infinite limit of number of uses

$$\tilde{M}_{ij}(\pm\phi) = N \frac{1-\phi^2}{1+\phi^2 \mp 2\phi\cos(x)} \delta_{ij}$$

$$x \in [0, 2\pi]$$

(Difference in *q* and *p* quadratures → squeezing)

[Schäfer, et al. PRA 84, 032318 (2010)]

# Noise correlations (squeezing) increase the classical capacity

- Capacity of the additive noise channel in the limit of "full" correlations

$$\gamma_{env} = \begin{pmatrix} M^{(q)}(\phi) & 0 \\ 0 & M^{(p)}(-\phi) \end{pmatrix}$$

$$M_{ij}(\phi) = N\phi^{|i-j|} \qquad \phi \to 1$$

attains the capacity of an ideal channel

$$\lim_{\phi\to 1} C_{\chi} = g\left(\frac{E-1}{2}\right)$$

- The limit of the distribution of corresponding diagonal noise $\tilde{M}(\phi)$ is a delta-like function with two "infinitely" squeezed modes
- *NB: For all* $\phi$ *the energy of the noise is the same !*

- **Problem :** what is the optimal noise distribution given the noise energy of a Gaussian channel?

[O. Pilyavets, et al. IEEE Trans. Inf. Theory 58, 6126 (2012)]

# Energy constraint on the noise

- Capacity optimization problem with two constraints

$$\mathrm{Tr}\left[\gamma_{vac} + \gamma_m\right] \leq E \qquad \mathrm{Tr}\left[\gamma_{env}\right] \leq E_{env}$$

  - Optimal environment mode squeezing in a single mode
  - Optimal distribution of environment energy between different modes

- Additive noise channel (classical noise) – optimal environment

$$\overline{\gamma}_{out} = \gamma_{in} + \gamma_m + \gamma_{env}$$

  - Single-mode – infinite "squeezing" (channel B1 in Holevo classification)

$$\gamma_{env} = E_{env}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

  - Two modes of B1 channel : (Gaussian conjecture is not proven !)

$$E_{env,1} = E_{env} \qquad E_{env,2} = 0$$

# Noise energy constraint for the *fiducial* channel

- Quantum noise

$$\bar{\gamma}_{out} = |\tau|(\gamma_{in} + \gamma_m) + |1-\tau|\gamma_{env}$$

$$\gamma_{env} = \left(N_{env} + \frac{1}{2}\right)\begin{bmatrix} e^{2s} & 0 \\ 0 & e^{2s} \end{bmatrix}$$

  – Infinite squeezing for a finite $E_{\mathrm{env}}$ is impossible, a non-trivial optimization is required

- <u>Worst case noise</u> – an opposite problem :

  – Thermal noise for a single mode ?

$$\gamma_{env} = \left(N_{env} + \frac{1}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

  – A uniform distribution between the modes ?

# Conclusion and outlook

- Gaussian capacity of *fiducial* bosonic channel is the classical capacity **if** there is enough input energy
- Optimal environment (memory) problem under the noise energy constraint
- Additive noise channel (classical noise) : full correlations or “infinite” squeezing and concentration of all noise in one mode (quadrature) are optimal
- Noise optimization for a general *fiducial* channel requires further study
- Is the thermal noise the “worst case environment” ?

Thank you for your attention !

# Gaussian capacity

- Restriction to Gaussian input states and encodings

$$\rho(q,p) = \hat{D}(q,p)\,\rho_G\,\hat{D}^\dagger(q,p) \qquad \overline{\rho}_{in} = \int dq\,dp\,P_G(q,p)\rho(q,p)$$

$$C_G[T_G] = \max_{\{P_G,\rho_G\}}\left[S\left(T[\overline{\rho}_{in}]\right) - \int dq\,dp\,P_G(q,p)\,S\left(T[\rho(q,p)]\right)\right]$$

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$$C_G[T_G] = \max_{\{P_G,\rho_G\}}\left[S\left(T[\overline{\rho}_{in}]\right) - S\left(T[\rho_G]\right)\right]$$

$$C_G[T_G] = \max_{\{\gamma_m,\gamma_{in}\}}\left[S(\overline{\gamma}_{out}) - S(\gamma_{out})\right]$$

# Gaussian capacity

- Restriction to Gaussian input states and encodings

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$$C_G[T_G] = \max_{\{\gamma_m,\gamma_{in}\}}\left[S\left(\overline{\gamma}_{out}\right) - S\left(\gamma_{out}\right)\right]$$

# Gaussian capacity

- Restriction to Gaussian input states and encodings

$$\rho(q,p) = \hat{D}(q,p)\,\rho_G\,\hat{D}^\dagger(q,p) \qquad \bar{\rho}_{in} = \int dq\,dp\,P_G(q,p)\,\rho(q,p)$$

$$C_G\left[T_G\right] = \max_{\{P_G,\rho_G\}}\left[S\left(T[\bar{\rho}_{in}]\right) - \int dq\,dp P_G\left(q,p\right) S\left(T[\rho(q,p)]\right)\right]$$

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$$C_G\left[T_G\right] = \max_{\{\gamma_m,\gamma_{in}\}}\left[S\left(\bar{\gamma}_{out}\right) - S\left(\gamma_{out}\right)\right]$$

Input energy constraint !

$$\mathrm{Tr}\left[\gamma_{in} + \gamma_m\right] \leq E$$

# Gaussian capacity

- "Naïve" optimization

$$C_G\left[T_G\right] = \max_{\{\gamma_m,\gamma_{in}\}}\left[S\left(\overline{\gamma}_{out}\right) - S\left(\gamma_{out}\right)\right]$$

  - Conjecture – minimum output entropy

$$\min_{\{\gamma_m,\gamma_{in}\}}\left[S\left(\gamma_{out}\right)\right] = S\left(T[\rho_{vac}]\right)$$

  - Choose modulation

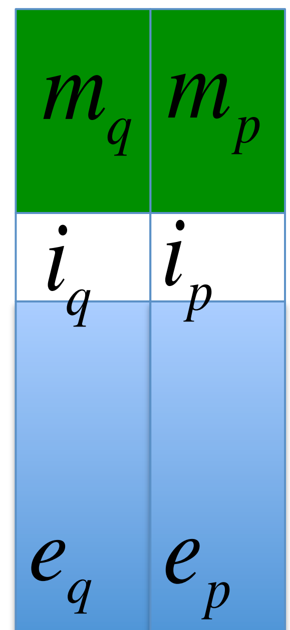
$$\max_{\{\gamma_m,\gamma_{vac}\}}\left[S\left(\overline{\gamma}_{out}\right)\right] = S\left(\left(\overline{N}_{th} + \frac{1}{2}\right)I\right)$$

Is it always possible to choose modulation satisfying the input energy constraint ?

$$\mathrm{Tr}\left[\gamma_{vac} + \gamma_m\right] \leq E$$

# Waterfilling solution (single-mode)

- Optimizing $C_1[T_G] = \max_{\{\gamma_{in},\gamma_m\}} \left[ S(\gamma_{in} + \gamma_m + \gamma_e) - S(\gamma_{in} + \gamma_e) \right]$
  - Uniform distribution of the output energy is optimal $\bar{\gamma}_q = \bar{\gamma}_p$
  - Optimal input is a minimum output entropy Gaussian state $\dfrac{i_q}{i_p} = \dfrac{e_q}{e_p}$




$$C[T] = g\left(\left(\lambda + e_q + e_p - 1\right)/2\right) - g\left(\sqrt{e_q e_p}\right)$$

[Holevo, et al. PRA 59, 1820 (1999)]

# Waterfilling solution

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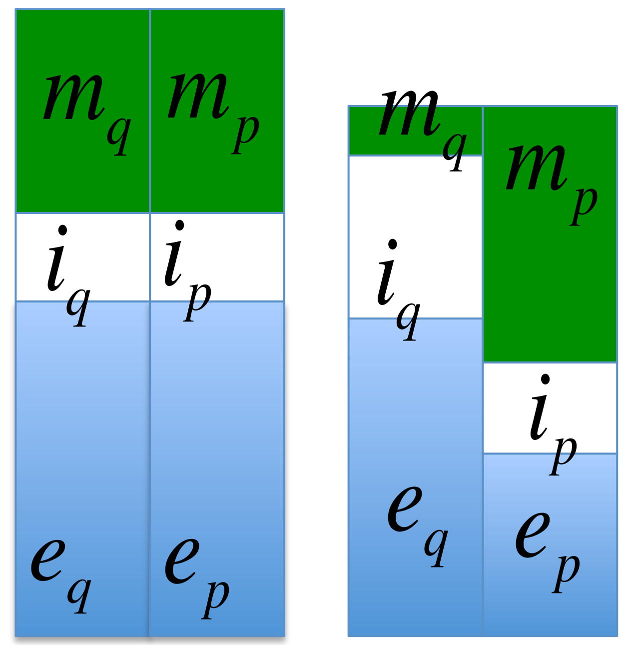
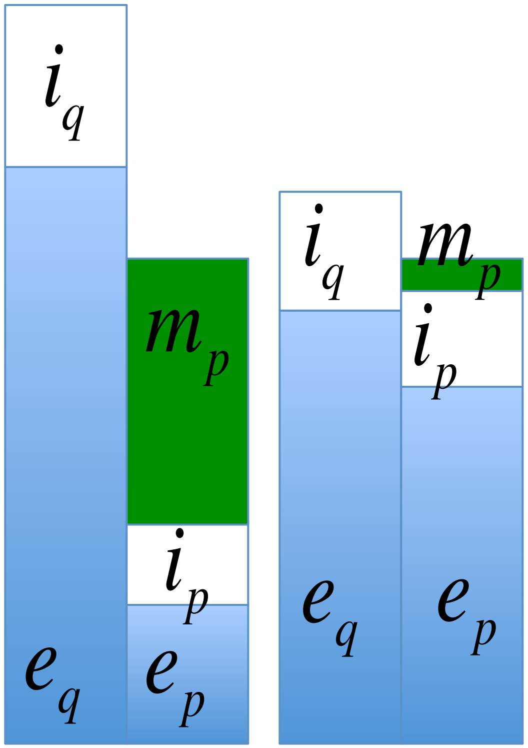



$$\lambda > \lambda_{th} = \sqrt{e_q / e_p} + e_q - e_p$$

$$1 < \lambda < \lambda_{th}$$

# Solution of 2^d type

- Only one quadrature modulated
- Lagrange multipliers – no explicit solution
  - An implicit solution via a transcendent equation
  - Input energy also spent on the non-modulated quadrature
  - Optimal input state is not the minimum output entropy state*




$$1 < \frac{i_q}{i_p} < \frac{e_q}{e_p}$$

[Schäfer, et al. PRA 80, 062313 (2009)]

$$g'(\bar{v}-1/2)(\bar{\gamma}_p-\bar{\gamma}_q)/\bar{v}=g'(v_{out}-1/2)\left(\gamma_p-\gamma_q/\left(4i_q^2\right)\right)/v_{out}$$

$$g'(x-1/2)=Log_2\left[(x+1/2)/(x-1/2)\right]$$

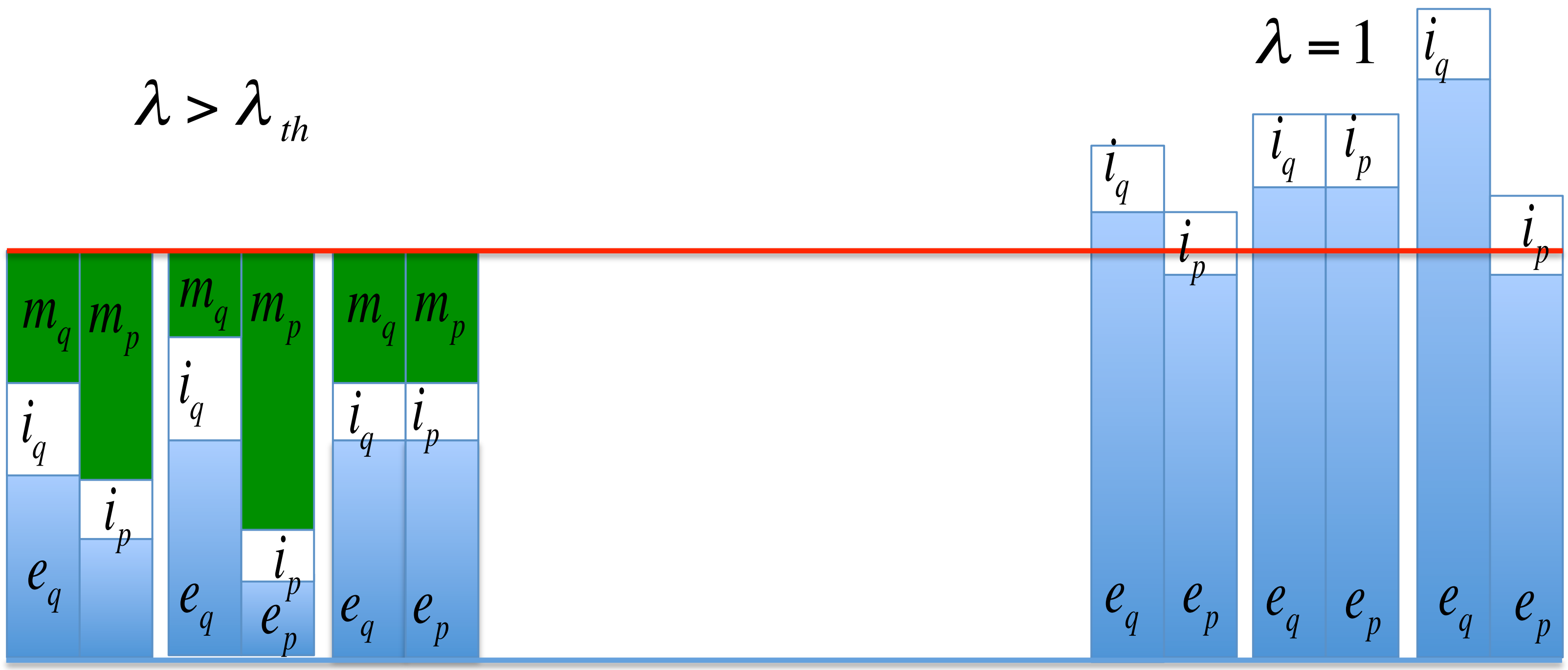
$$\bar{v}=\sqrt{\bar{\gamma}_q\bar{\gamma}_p} \qquad v_{out}=\sqrt{\gamma_q\gamma_p}$$

$$\bar{\gamma}_{q,p}=i_{q,p}+m_{q,p}+e_{q,p} \qquad \gamma_{q,p}=i_{q,p}+e_{q,p}$$

$$i_q+i_p+m_q+m_p=\lambda \qquad i_qi_p=1/4$$

# Optimal states for multimode channel

Non-trivial energy distribution among the modes
– common Lagrange multiplier – **waterfilling**

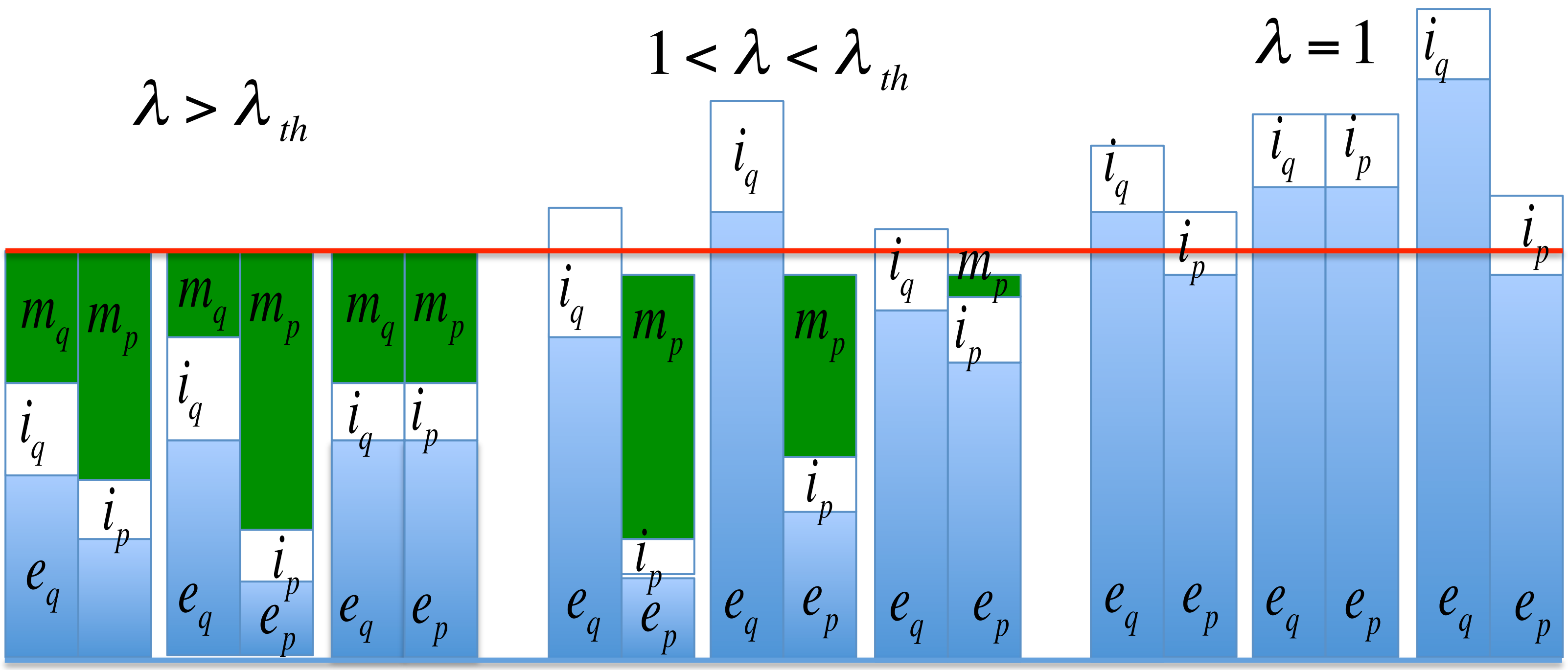



$$\frac{i_q}{i_p} = \frac{e_q}{e_p}$$

$$\frac{i_q}{i_p} = 1$$

# Optimal states for multimode channel

Non-trivial energy distribution among the modes
– common Lagrange multiplier – generalized **waterfilling**

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