Gaussian Classical capacity of Gaussian Quantum channels

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Outline

- Quantum channels for information transmission
  - Classical capacity of quantum channels
- Gaussian channels
  - Additive (classical) noise quantum channel
- Gaussian capacity
- Mathematical challenges
- Conclusion (Gaussian)

Classical capacity of quantum channel

- Quantum channel – completely positive map $\rho^{out} = T\left[\rho^{in}\right]$
- Information encoding into quantum states – modulation

  $\left\{P_i, \rho_i^{in}\right\}$ – input ensemble

- Average input: $\overline{\rho^{in}} = \sum_i P_i \rho_i^{in}$, average output $\overline{\rho} = \sum_i P_i \rho_i^{out}$
- Information content – Von Neumann entropy $S(\rho) = -Tr\left[\rho \cdot Log_2 \rho\right]$
- Information capacity [Holevo-Schumacher-Westmoreland]

$$C_1[T] = \max_{\left\{P_i, \rho_i^{in}\right\}} \left[S(\overline{\rho}) - \sum_i P_i S\left(\rho_i^{out}\right) \right]$$

Additivity problem

- When using entangled inputs is classical capacity superadditive?

$$C_1(T_1 \otimes T_2) > C_1(T_1) + C_1(T_2)$$

– sometimes yes [Hastings]

- Capacity :

$$C(T) = \lim_{n\to\infty} \frac{1}{n} C_1\left(T^{\otimes n}\right)$$

- Memory channels / correlated channel uses :
  - Parallel n-mode channel

$$C(T) = \lim_{n\to\infty} \frac{1}{n} C_1\left(T^{(n)}\right)$$

Bosonic channels – continuous variables

- Quadratures of electromagnetic field mode $\hbar = \omega = 1$

$$\left[\hat{a}, \hat{a}^{\dagger}\right] = 1 \qquad \hat{q} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^{\dagger}\right) \quad \hat{p} = \frac{1}{i\sqrt{2}}\left(\hat{a} - \hat{a}^{\dagger}\right)$$

- *Gaussian states* are fully determined by $\rho_G \rightarrow (\vec{d}, V_{in})$
  - displacement vector $\vec{d} = \left(q, p\right)^T$ – first moments
  - covariance matrix (CM) V – second moments
- *Gaussian channels* transform first and second moments

$$\vec{d}_{out} = X\vec{d}_{in} + \vec{d} \quad , \qquad X \text{ - Real matrix}$$

$$V_{out} = XV_{in}X + Y \, , \qquad Y \text{ - CM}$$

Additive noise channel

- Classical noise $X = I$ $V_{out} = V_{in} + Y$
  - "displacement" with a Gaussian probability distribution
- Input energy constraint $Tr\left[V_{in} + V_m\right] \leq E$
- Non-trivial optimization problem
- Challenge 1: Classical capacity of Gaussian channels is additive
  - True for "entanglement breaking" channels [Holevo 2008]
- Restriction to Gaussian states: Gaussian modulation
  - "displacement" with a Gaussian probability distribution
- Modulated output CM $\overline{V} = V_{in} + V_m + Y$

Gaussian capacity

– Von Neumann entropy is a function of CM

$$C_1\left[T_G\right]=\max_{\{Tr(V_{in}+V_m)\leq E\}}\left[S\left(\bar{V}\right)-\int dq\,dp P\left(q,p\right)S\left(V_{out}\right)\right]$$

– All signal states at the output have the same entropy

$$C_1\left[T_G\right]=\max_{\{Tr(V_{in}+V_m)\leq E\}}\left[S\left(\bar{V}\right)-S\left(V_{out}\right)\right]$$

– Challenge 2: Gaussian capacity is the capacity

- Proven for lossy channel with vacuum noise [V. Giovannetti et al., Phys. Rev. Lett. 92, 027902 (2004)]

One mode problem

- 2x2 CM: Symplectic eigenvalue $V \to diag(v,v)$
- Entropy: $S(V) = g(v - 1/2)$

$$g(x) = (x+1)Log_2(x+1) - (x)Log_2(x)$$

- Optimization in terms of CM

$$C_1[T_G] = \max_{\{Tr(V_{in}+V_m)\le E\}} \left[S(V_{in} + V_m + Y) - S(V_{in} + Y)\right]$$

- If noise CM is diagonalized by a passive symplectic transformation then optimal input and modulation are diagonal In the eigenbasis of the CM of noise

$$\overline{V} \to \overline{v} = \sqrt{(i_q + m_q + e_q)(i_p + m_p + e_p)}$$

Waterfilling solution

– Uniform distribution of the output energy is optimal

– Optimal input is a *minimum output entropy* Gaussian state

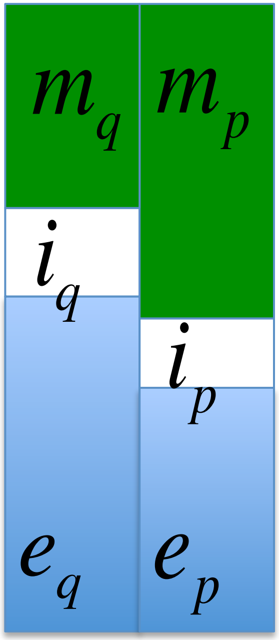
$$C_1[T] = g\left(\left(E + e_q + e_p - 1\right)/2\right) - g\left(\sqrt{e_q e_p}\right)$$

$$\overline{V}_q = \overline{V}_p$$

– Challenge 3: minimum output entropy is achieved on a Gaussian state (vacuum state)

$$\frac{i_q}{i_p} = \frac{e_q}{e_p}$$

$$E > E_{th} = \sqrt{\frac{e_q}{e_q}} + e_q - e_p$$

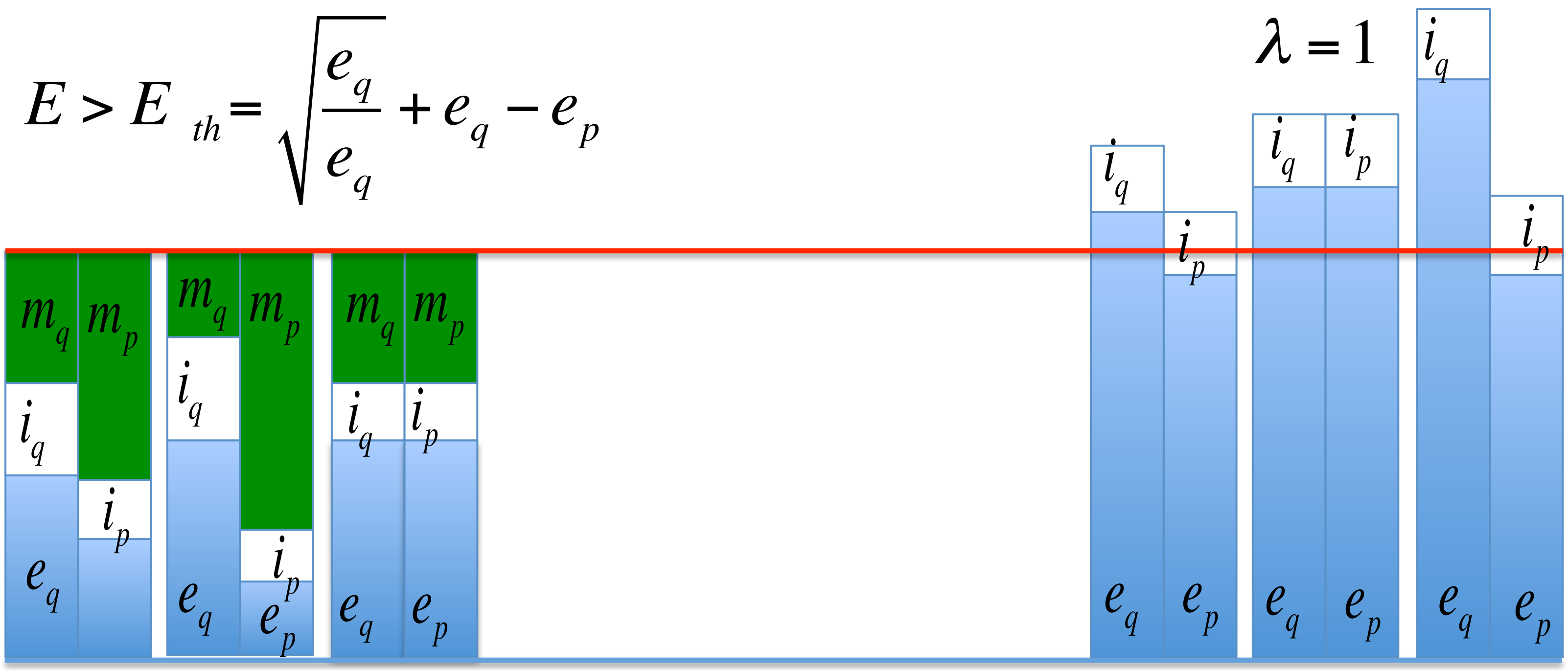


[Holevo, et al. PRA 59, 1820 (1999)]

Optimal states for multimode channel

Non-trivial input energy distribution between the modes
– common Lagrange multiplier – **waterfilling**

$$E > E_{th} = \sqrt{\frac{e_q}{e_q}} + e_q - e_p$$

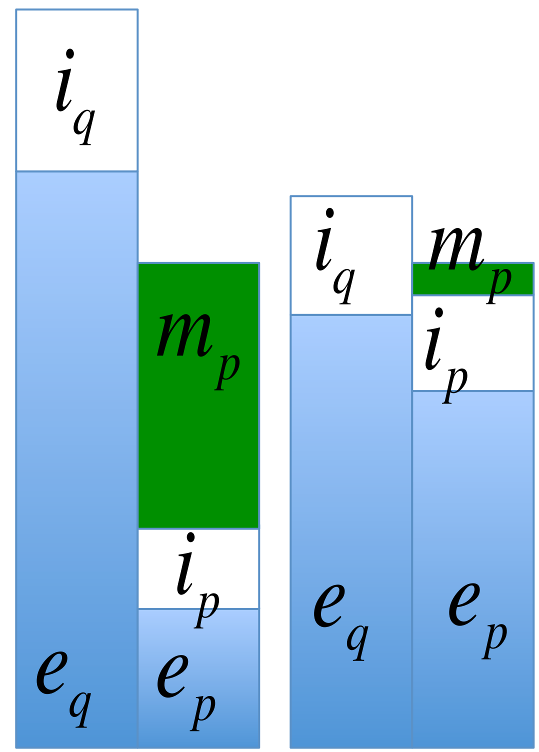


$$\frac{i_q}{i_p} = \frac{e_q}{e_p} \qquad \frac{i_q}{i_p} = 1$$

$$1 < E < E_{th}$$

Solution of 2d type

- Only one quadrature modulated
- Lagrange multipliers – no explicit solution
  - An implicit solution given by a transcendent equation
  - Input energy also spent on the non-modulated quadrature
  - Optimal input state is not the minimum output entropy state



$$1 < \frac{i_q}{i_p} < \frac{e_q}{e_p}$$

[Schäfer, et al. PRA 80, 062313 (2009)]

Solution of 2[d] type

$$\frac{g'\left(\bar{\nu}-1/2\right)}{\bar{\nu}}\left(E+e_p-e_q-2i_q\right)=\frac{g'\left(\nu_{out}-1/2\right)}{\nu_{out}}\left(e_p-e_q\frac{i_p}{i_q}\right)$$

$$g'\left(x\right)=Log_2\left(x+1\right)-Log_2 x$$

$$\bar{\nu}=\sqrt{(i_q+e_q)(E-i_q+e_p)}$$

$$\nu_{out}=\sqrt{(i_q+e_q)(i_p+e_q)}$$

$$i_q+i_p+m_p=E$$

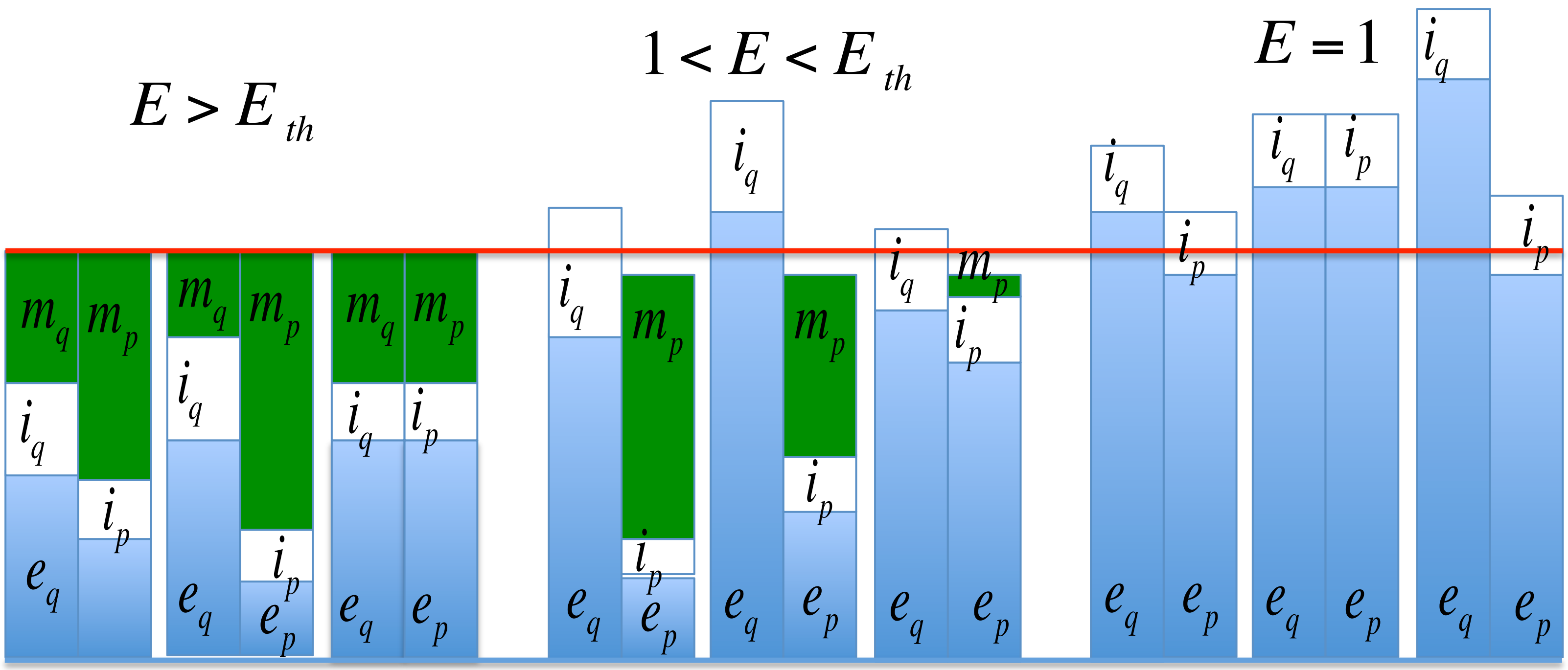
$$i_q i_p=1/4$$

$$\frac{g'\left(\bar{\nu}-1/2\right)}{\bar{\nu}}\left(i_q+e_q\right)=\lambda$$

- Lagrange multiplier

Optimal states for multimode channel

Non-trivial energy distribution among the modes
– common Lagrange multiplier – generalized **waterfilling**



$$\frac{i_q}{i_p} = \frac{e_q}{e_p} \qquad 1 < \frac{i_q}{i_p} < \frac{e_q}{e_p} \qquad \frac{i_q}{i_p} = 1$$

Application - memory channel

- Correlated noise $\longrightarrow$ correlated optimal input
  - multimode problem
- Parallel channel
- Gauss – Markov noise

$$C(T) = \lim_{n\to\infty} \frac{1}{n} C_1\left(T^{(n)}\right)$$

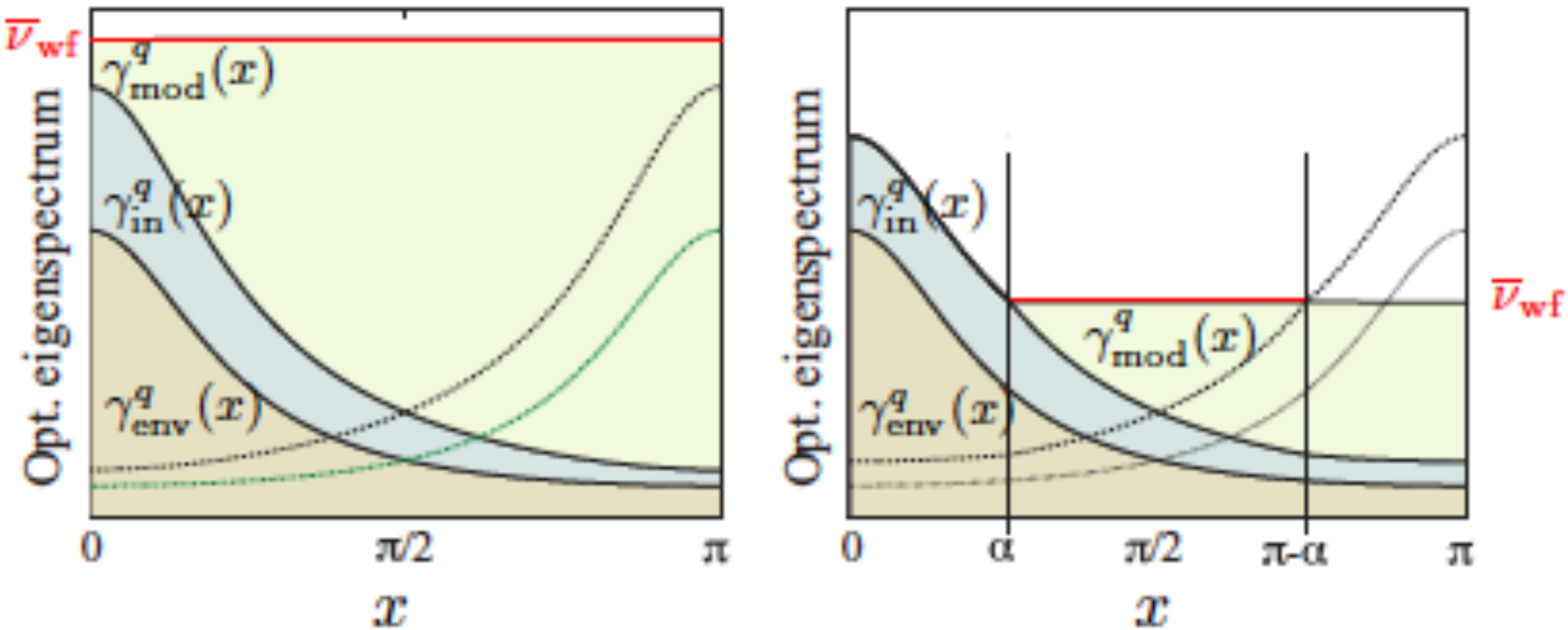
$$\left\{ \mathrm{M}(\phi) \right\}_{i,j} = N\phi^{|i-j|}$$

$$Y = \begin{pmatrix} \mathrm{M}_q(\phi) & 0 \\ 0 & \mathrm{M}_p(-\phi) \end{pmatrix}$$

- Symplectic and orthogonal diagonalization is possible as $\mathrm{M}_q(\phi)$ and $\mathrm{M}_q(-\phi)$ commute in the limit $n \longrightarrow \infty$

Additive Gauss-Markov noise

Diagonalization of infinite noise matrix
– continuous spectrum on a finite domain



Optimal multimode input state in the original basis is entangled – preparation?

Conclusion

- Gaussian capacity of additive noise bosonic channels
- Three types of solution for one mode
- Input state that realizes classical capacity is not always the minimum output entropy state
- Challenge 1: Minimum output entropy for Gaussian channels is achieved on a Gaussian state
- Challenge 2: Classical capacity of memoryless Gaussian quantum channels is realized by Gaussian states
- Challenge 3: Classical capacity of memoryless Gaussian quantum channels is additive

Thank you for your attention !

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