Gaussian Classical capacity of Gaussian Quantum channels

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Outline

• Quantum channels for information transmission
  – Classical capacity of quantum channels
• Gaussian channels
  – Additive (classical) noise quantum channel
• Gaussian capacity
• Mathematical challenges
• Conclusion (Gaussian)
Classical capacity of quantum channel

- Quantum channel – completely positive map \( \rho^{\text{out}} = T[\rho^{\text{in}}] \)
- Information encoding into quantum states – modulation
  \( \{ P_i, \rho_i^{\text{in}} \} \) – input ensemble
- Average input: \( \overline{\rho^{\text{in}}} = \sum_i P_i \rho_i^{\text{in}} \), average output \( \overline{\rho} = \sum_i P_i \rho_i^{\text{out}} \)
- Information content – Von Neumann entropy \( S(\rho) = -\text{Tr}[\rho \cdot \text{Log}_2 \rho] \)
- Information capacity [Holevo-Schumacher-Westmoreland]

\[
C_1[T] = \max_{\{P_i, \rho_i^{\text{in}}\}} \left[ S(\overline{\rho}) - \sum_i P_i S(\rho_i^{\text{out}}) \right]
\]
Additivity problem

• When using entangled inputs is classical capacity superadditive?
  \[ C_1(T_1 \otimes T_2) > C_1(T_1) + C_1(T_2) \]
  – sometimes yes [Hastings]

• Capacity:
  \[ C(T) = \lim_{n \to \infty} \frac{1}{n} C_1(T^{\otimes n}) \]

• Memory channels / correlated channel uses:
  – Parallel n-mode channel
  \[ C(T) = \lim_{n \to \infty} \frac{1}{n} C_1(T^{(n)}) \]
Bosonic channels – continuous variables

- Quadratures of electromagnetic field mode
  \[ \left[ \hat{a}, \hat{a}^\dagger \right] = 1 \quad \hat{q} = \frac{1}{\sqrt{2}} (\hat{a} + \hat{a}^\dagger) \quad \hat{p} = \frac{1}{i \sqrt{2}} (\hat{a} - \hat{a}^\dagger) \]
  
- Gaussian states are fully determined by
  - displacement vector \( \vec{d} = (q, p)^T \) – first moments
  - covariance matrix (CM) \( V \) – second moments

- Gaussian channels transform first and second moments

  \[
  \vec{d}_{\text{out}} = X \vec{d}_{\text{in}} + \vec{d} \quad , \quad X - \text{Real matrix} \\
  V_{\text{out}} = XV_{\text{in}}X + Y \quad , \quad Y - \text{CM}
  \]
Additive noise channel

- Classical noise: $X = I$, $V_{out} = V_{in} + Y$—“displacement” with a Gaussian probability distribution
- Input energy constraint: $\text{Tr} \left[ V_{in} + V_m \right] \leq E$
- Non-trivial optimization problem
- **Challenge 1:** Classical capacity of Gaussian channels is additive
  - True for “entanglement breaking” channels [Holevo 2008]
- Restriction to Gaussian states: Gaussian modulation
  - “displacement” with a Gaussian probability distribution
- Modulated output CM: $\overline{V} = V_{in} + V_m + Y$
Gaussian capacity

– Von Neumann entropy is a function of CM

\[ C_1[T_G] = \max_{\{Tr(V_{in}+V_m) \leq E\}} \left[ S(\overline{V}) - \int dq dp P(q, p) S(V_{out}) \right] \]

– All signal states at the output have the same entropy

\[ C_1[T_G] = \max_{\{Tr(V_{in}+V_m) \leq E\}} \left[ S(\overline{V}) - S(V_{out}) \right] \]

– Challenge 2: Gaussian capacity is the capacity

• Proven for lossy channel with vacuum noise
  [V. Giovannetti et al., Phys. Rev. Lett. 92, 027902 (2004)]
One mode problem

- 2x2 CM: Symplectic eigenvalue \( V \rightarrow \text{diag}(\nu, \nu) \)
- Entropy: \( S(V) = g(\nu - 1/2) \)
  \[
g(x) = (x+1) \log_2(x+1) - (x) \log_2(x)
\]
- Optimization in terms of CM
  \[
  C_1[T_G] = \max_{\{\text{Tr}(V_{in} + V_{m}) \leq E\}} \left[ S(V_{in} + V_{m} + Y) - S(V_{in} + Y) \right]
  \]
- If noise CM is diagonalized by a passive symplectic transformation then optimal input and modulation are diagonal in the eigenbasis of the CM of noise
  \[
  \bar{V} \rightarrow \bar{\nu} = \sqrt{(i_q + m_q + e_q)(i_p + m_p + e_p)}
  \]
Waterfilling solution

- Uniform distribution of the output energy is optimal
- Optimal input is a *minimum output entropy* Gaussian state

\[ C_1[T] = g \left( \frac{(E_e + e_q + e_p - 1)}{2} \right) - g \left( \sqrt{e_q e_p} \right) \]

- Challenge 3: minimum output entropy is achieved on a Gaussian state (vacuum state)

\[ E > E_{th} = \sqrt{\frac{e_q}{e_q}} + e_q - e_p \]

[Holevo, et al. PRA 59, 1820 (1999)]
Optimal states for multimode channel

Non-trivial input energy distribution between the modes – common Lagrange multiplier – **waterfilling**

\[
E > E_{th} = \sqrt{\frac{e_q}{e_q} + e_q - e_p}
\]

\[
\frac{i_q}{i_p} = \frac{e_q}{e_p}
\]

\[
\lambda = 1
\]

\[
\frac{i_q}{i_p} = 1
\]
Solution of 2\textsuperscript{d} type

- Only one quadrature modulated
- Lagrange multipliers – no explicit solution
  - An implicit solution given by a transcendent equation
  - Input energy also spent on the non-modulated quadrature
  - Optimal input state is not the minimum output entropy state

[Schäfer, et al. PRA 80, 062313 (2009)]
Solution of 2\textsuperscript{d} type

\[
\frac{g'(\bar{\nu} - 1/2)}{\bar{\nu}} (E + e_p - e_q - 2i_q) = \frac{g'(\nu_{out} - 1/2)}{\nu_{out}} \left( e_p - e_q \frac{i_p}{i_q} \right)
\]

\[ g'(x) = \log_2 (x + 1) - \log_2 x \]

\[ \bar{\nu} = \sqrt{(i_q + e_q)(E - i_q + e_p)} \]

\[ \nu_{out} = \sqrt{(i_q + e_q)(i_p + e_q)} \]

\[ i_q + i_p + m_p = E \]

\[ i_q i_p = \frac{1}{4} \]

- Lagrange multiplier
Optimal states for multimode channel

Non-trivial energy distribution among the modes
– common Lagrange multiplier – generalized waterfilling

\[ \frac{i_q}{i_p} = \frac{e_q}{e_p} \quad \text{for } E > E_{th} \]

\[ 1 < \frac{i_q}{i_p} < \frac{e_q}{e_p} \quad \text{for } 1 < E < E_{th} \]

\[ \frac{i_q}{i_p} = 1 \quad \text{for } E = 1 \]
Application - memory channel

- Correlated noise $\rightarrow$ correlated optimal input  
  - multimode problem  
- Parallel channel  
- Gauss – Markov noise  
  \[ \{M(\phi)\}_{i,j} = N\phi^{i-j} \]

- Symplectic and orthogonal diagonalization is possible as $M_q(\phi)$ and $M_q(-\phi)$ commute in the limit $n \rightarrow \infty$  
  \[ Y = \begin{pmatrix} M_q(\phi) & 0 \\ 0 & M_p(-\phi) \end{pmatrix} \]  
  \[ C(T) = \lim_{n \rightarrow \infty} \frac{1}{n} C_1(T^{(n)}) \]
Additive Gauss-Markov noise

Diagonalization of infinite noise matrix – continuous spectrum on a finite domain

Optimal multimode input state in the original basis is entangled – preparation?
Conclusion

• Gaussian capacity of additive noise bosonic channels

• Three types of solution for one mode

• Input state that realizes classical capacity is not always the minimum output entropy state

• Challenge 1: Minimum output entropy for Gaussian channels is achieved on a Gaussian state

• Challenge 2: Classical capacity of memoryless Gaussian quantum channels is realized by Gaussian states

• Challenge 3: Classical capacity of memoryless Gaussian quantum channels is additive
Thank you for your attention!