Estimating capacities and rates of Gaussian quantum channels

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Motivations

- Most of the performed studies e.g. on classical capacity concern simple settings (memoryless and vacuum environment)
- No general methods available for evaluating, e.g. classical capacity
- Rates usually derived in a different way with respect to capacity
- Consider lossy bosonic channel as a paradigm of Gaussian channels
- Introduce a generic model for multiple channel uses and devise a method to evaluate the Holevo function (turns out to be useful for classical capacity as well as for dyne rates)
- Maximization problem can be split it into “inner” one and “outer” one

based on Pilyavets, Lupo & Mancini, arXiv0907.1532 (provisionally accepted by IT Trans)

Outline

- Gaussian channels
  - Lossy bosonic channel
- Classical capacity and rates
- Single channel use (bosonic mode)
  - The “inner” optimization problem
  - Solution
  - Its properties (critical parameters)
- Multiple channel uses (bosonic modes)
  - The “outer” optimization problem
  - Solution
  - Its properties (and applications)
- Conclusions and outlook

Gaussian channels

They map Gaussian states into Gaussian states; for single use:

$$\{a, V\} \mapsto \{X^T a + d, X^T V X + Y\}$$

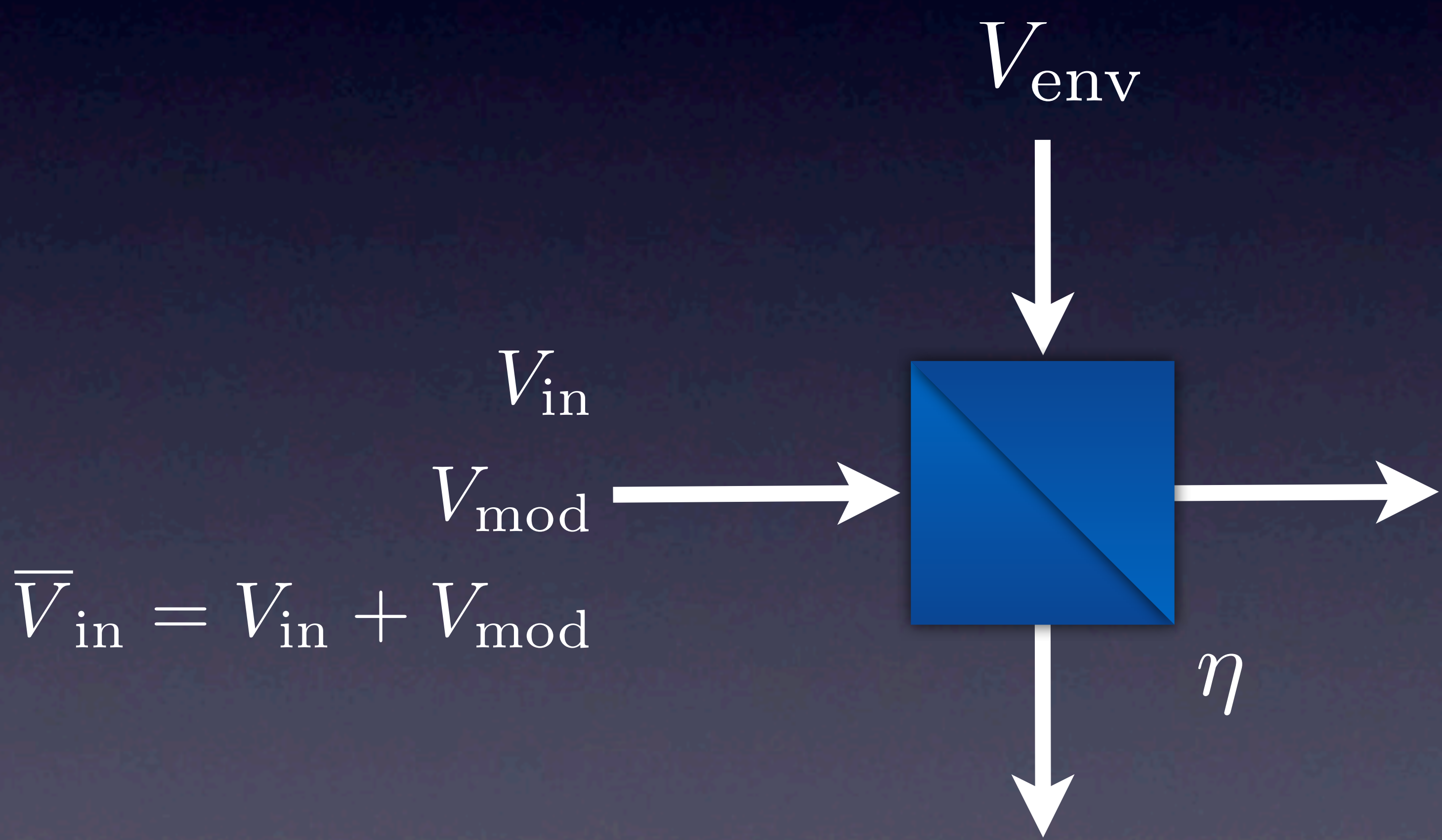
Channel defined by the triad: (d, X, Y)

For *n* uses channel defined by a triad:

$$(d_n, X_n, Y_n) = \begin{cases} = (\oplus^n d, \oplus^n X, \oplus^n Y) & \text{memoryless} \\ \neq (\oplus^n d, \oplus^n X, \oplus^n Y) & \text{memory} \end{cases}$$

The lossy channel

$$X = \sqrt{\eta} I, \quad Y = (1-\eta) V_{\mathrm{env}}$$



$$\begin{aligned} V_{\mathrm{out}} &= \eta V_{\mathrm{in}} + (1-\eta) V_{\mathrm{env}} \\ \overline{V}_{\mathrm{out}} &= \eta (V_{\mathrm{in}} + V_{\mathrm{mod}}) + (1-\eta) V_{\mathrm{env}} \end{aligned}$$

The eigenvalues of the various matrices will be denoted by $(e_u, i_u, \bar{i}_u, m_u, o_u, \bar{o}_u)$

Classical capacity and rates

$$C_n := \frac{1}{n} \max_{V_{\rm in}, V_{\rm mod}} \chi_n^G$$

$$\chi_n^G := \sum_{k=1}^{n} \left[g\left(\bar{\mathfrak{d}}_k - \frac{1}{2} \right) - g\left(\mathfrak{d}_k - \frac{1}{2} \right) \right]$$

$$g(x) := (x+1)\log(x+1) - x \log x$$

$$\frac{{\rm Tr}\overline{V}_{\rm in}}{2n} \leq \overline{N}_{\rm in} + \frac{1}{2}$$

To the logarithmic approximation of g

$$C^{\log} = \frac{1}{n} \max_{V_{\rm in}, V_{\rm mod}} \sum_{k=1}^{n} \log \frac{\bar{\mathfrak{d}}_k}{\mathfrak{d}_k}$$

$$R_n^{\rm hom} = C_n^{\log}$$

$$R_n^{\rm het} = C_n^{\log}[V_{\rm env} \to V_{\rm env}^{\rm het}]$$

Single channel use

Theorem
The max of Holevo function over Gaussian states is achieved for V_{in}, V_{mod}, V_{env} simultaneously diagonalizable and the optimal V_{in} corresponds to a pure state

Corollary
If V_{in}, V_{mod}, V_{env} are simultaneously diagonalizable, the maximum of dyne rates is achieved by input pure states

Covariance matrices parametrized as

$$V = \left(\mathcal{N} + \frac{1}{2}\right)\left(\begin{array}{cc} e^{s} & 0 \\ 0 & e^{-s} \end{array}\right) \qquad \frac{\mathrm{Tr}V}{2} \leq N + \frac{1}{2}$$

The “inner” optimization problem

Maximize χ_1^G

With
$$i_u > 0 \qquad (i_{u_\star} = 1/(4i_u))$$
$$m_u,\ m_{u_\star} \geq 0$$
$$i_u + \frac{1}{4i_u} + m_u + m_{u_\star} = 2\overline{N}_{\text{in}} + 1$$

Definition

Solution belongs to the 1st stage if m_u, $m_{u^*}=0$ are optimal

Solution belongs to the 2nd stage if only $m_u=0$ (or m_{u^*}) is optimal

Solution belongs to the 3rd stage if m_u, $m_{u^*}>0$ are optimal

Remark

Stages are crossed (from 1st to 3rd) by increasing the input energy

1st stage capacity equal to zero

$$\overline{N}_{\rm in}(1 \to 2) = 0$$

2nd stage solution for i_u of the transcendent equation

$$\overline{\mathfrak{o}}g'\left(\overline{\mathfrak{o}} - \frac{1}{2}\right)\left(\frac{1}{o_u} - \frac{1}{\overline{o}_{u_\star}}\right) - \mathfrak{o}g'\left(\mathfrak{o} - \frac{1}{2}\right)\left(\frac{1}{o_u} - \frac{1}{4i_u^2 o_{u_\star}}\right) = 0$$

$$\overline{N}_{\rm in}(2 \to 3) = \frac{1}{2}\left(\sqrt{\frac{e_u}{e_{u_\star}}} - 1\right) - \frac{1-\eta}{\eta}\left(N_{\rm env} - e_u + \frac{1}{2}\right)$$

3rd stage

$$C_1 = g\left(\eta\overline{N}_{\rm in} + (1-\eta)N_{\rm env}\right) - g\left((1-\eta)\mathcal{N}_{\rm env}\right)$$

Properties of the solution

Theorem:

C_1 is a concave and increasing function of $\overline{N}_{\rm in}$

The one-shot capacity for fixed e_u, e_{u^*}, η can be considered as a black-box returning C_1 upon inputting $\overline{N}_{\rm in}$, while preserving the concavity

$$\overline{N}_{\rm in} \longrightarrow \boxed{C_1 = C_1\left(\overline{N}_{\rm in}\right)} \longrightarrow C_1$$

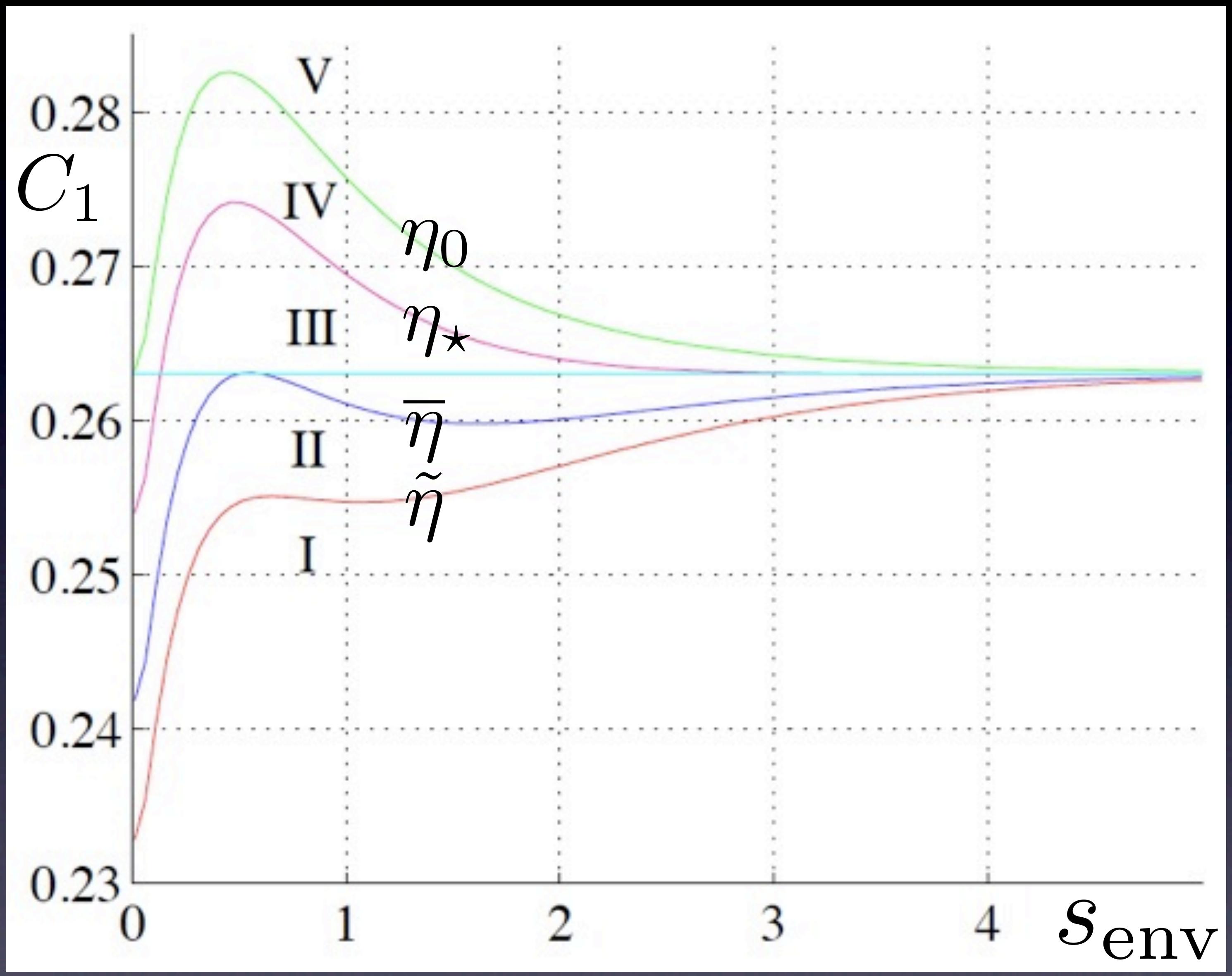
Corollary:

C_1 is additive

Theorem:

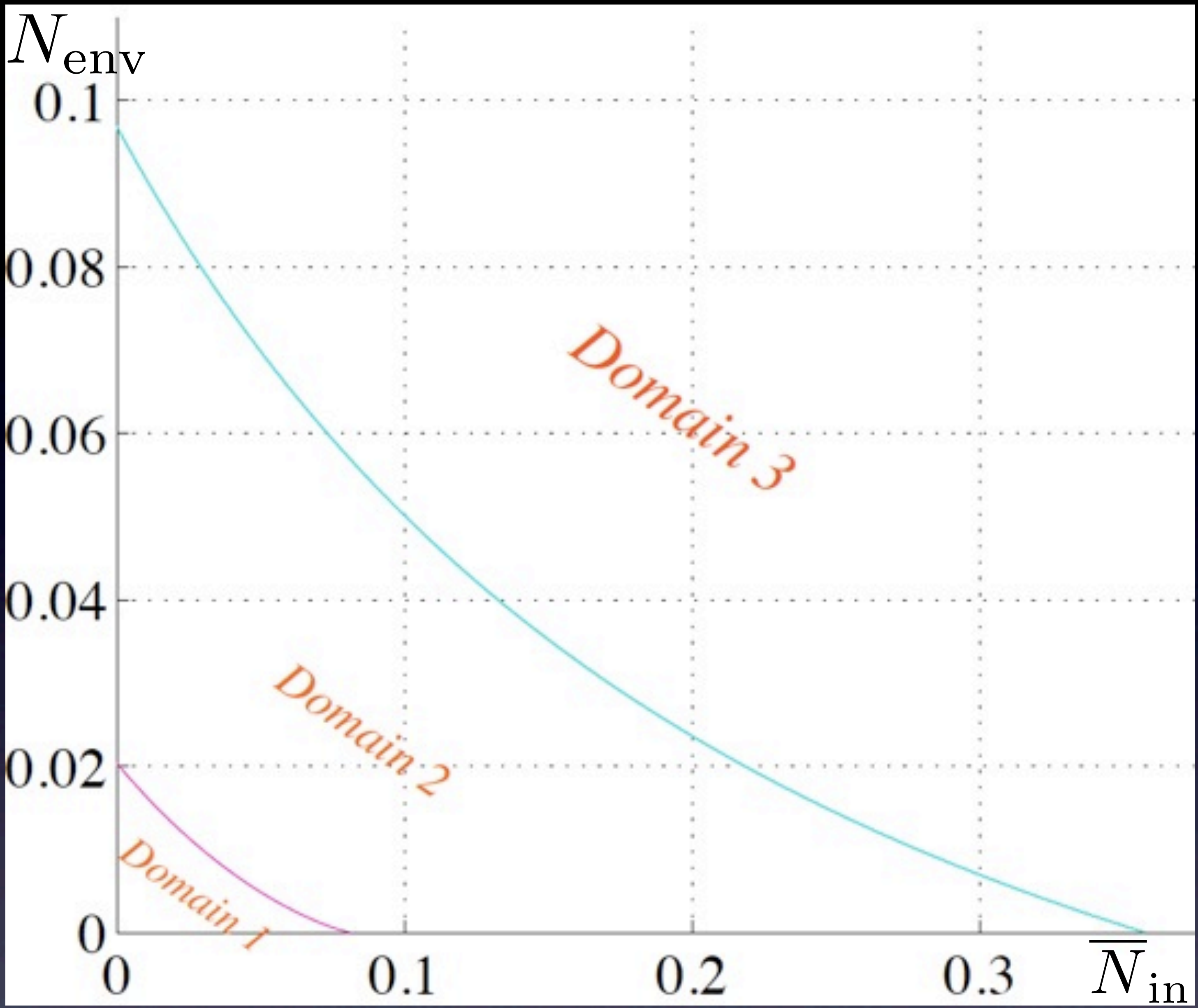
C_1 is a monotonic function of all its parameters $\left(\eta, \overline{N}_{\rm in}, s_{\rm env}, \mathcal{N}_{\rm env}\right)$ except s_{env}

Regimes



Critical parameters at boundaries of regimes, e.g. $\eta_\star = 1 - \frac{1}{\sqrt{3}}$

Domains



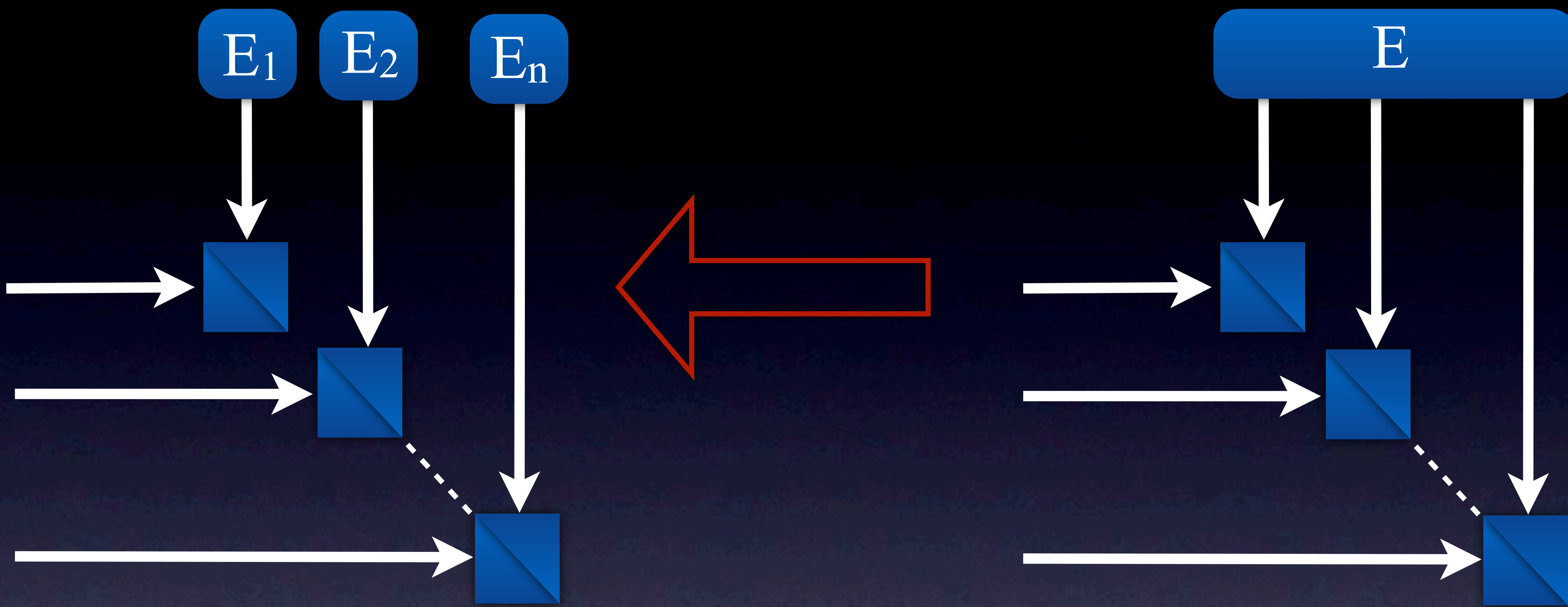
In the domain 1: $\tilde{\eta} < \overline{\eta} < \eta_0 < \eta_*$

In the domain 2: $\tilde{\eta} < \overline{\eta} < \eta_* < \eta_0$

In the domain 3: $\nexists\ \tilde{\eta}, \overline{\eta}$

Critical parameters at boundaries of domains, e.g. $\overline{N}_{\mathrm{in}}^{\star} = \sqrt{\frac{3\sqrt{3}+5}{8\sqrt{3}}} - \frac{1}{2}$

Multiple channel uses



Different single channel uses come from *memory unravelling*

Lupo & Mancini, PRA 81, 052314 (2010)

The action of E could be reduced to that of E_1, E_1,..., E_n by finding suitable Gaussian encoding/decoding unitaries

$$(0, E_n, 0),\ (0, D_n, 0)\,|\,D_n X_n E_n = \oplus_{k=1}^{n} X^{(k)};\ D_n Y_n D_n^T = \oplus_{k=1}^{n} Y^{(k)};\ E_n^T E_n = I_n$$

Always possible for E pure, or thermal squeezed!

The "outer" optimization problem

To maximize χ_n^G it now suffices to consider:

$$\overline{N}_{\text{in},1} \longrightarrow \boxed{C_1^{(1)} = C_1^{(1)}\left(\overline{N}_{\text{in},1}\right)} \longrightarrow C_1^{(1)}$$

$$\overline{N}_{\text{in},2} \longrightarrow \boxed{C_1^{(2)} = C_1^{(2)}\left(\overline{N}_{\text{in},2}\right)} \longrightarrow C_1^{(2)}$$

$$\vdots$$

$$\overline{N}_{\text{in},n} \longrightarrow \boxed{C_1^{(n)} = C_1^{(n)}\left(\overline{N}_{\text{in},n}\right)} \longrightarrow C_1^{(n)}$$

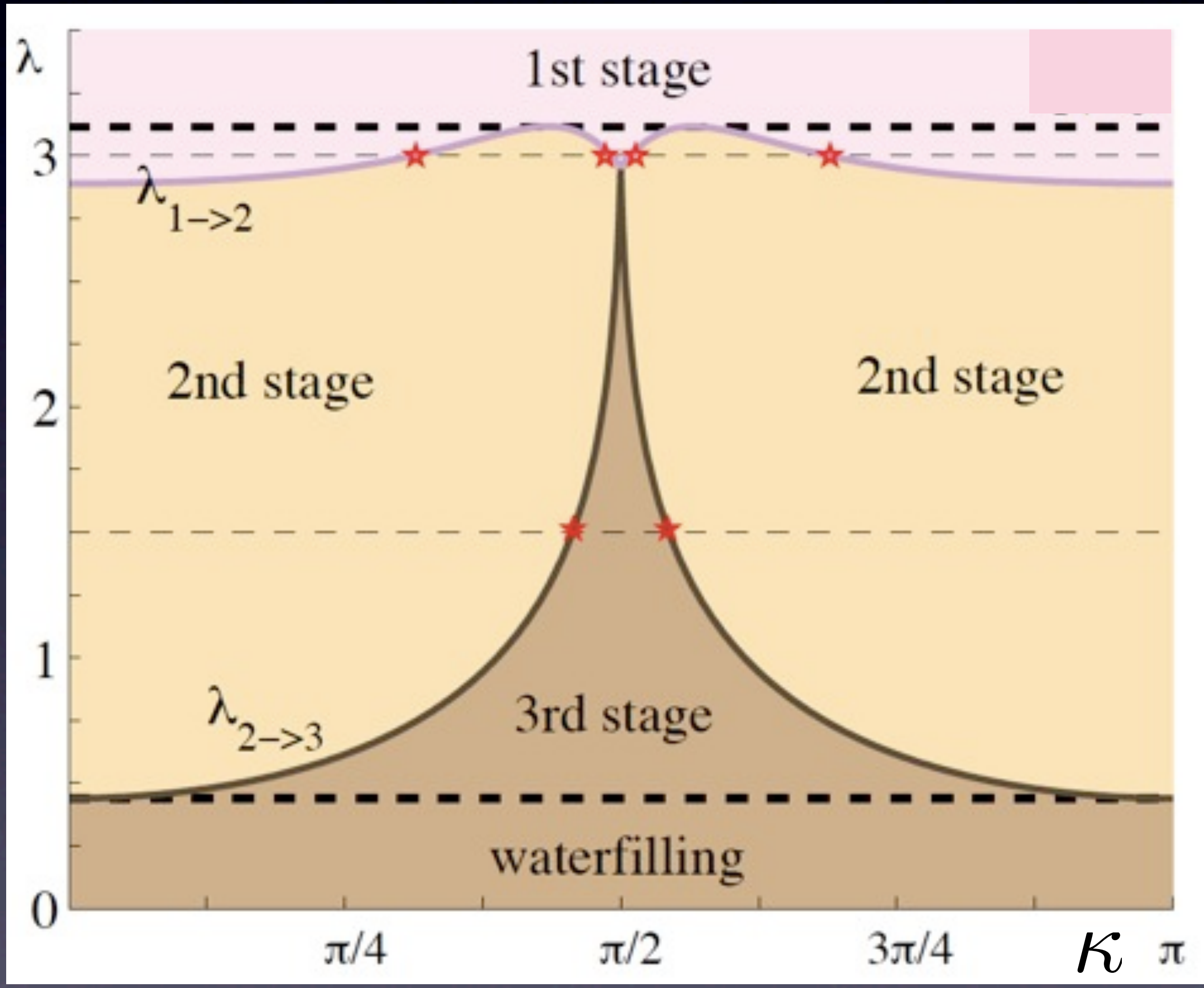
Find the distribution of $\overline{N}_{\text{in},k}$ $\left(\sum_{k=1}^{n} \overline{N}_{\text{in},k} = n\overline{N}_{\text{in}}\right)$ giving the maximum of $\sum_{k=1}^{n} C_1^{(k)}$

This "outer" optimization problem can be interpreted as the search for the optimal distribution of modes across stages

Algorithm

Due to the properties of C_1 it's possible to def. $\lambda_{\max} := \max_k \frac{\partial C_1^{(k)}}{\partial \overline{N}_{\text{in},k}} \left(\overline{N}_{\text{in},k} = 0\right) < +\infty$

$$\lambda_{1\to 2}(k) = \frac{\partial C_1^{(k)}}{\partial \overline{N}_{\text{in},k}} \left(\overline{N}_{\text{in},k}(1 \to 2)\right) ; \lambda_{2\to 3}(k) = \frac{\partial C_1^{(k)}}{\partial \overline{N}_{\text{in},k}} \left(\overline{N}_{\text{in},k}(2 \to 3)\right)$$



Look for $\overline{N}_{\text{in},k} \Big| \sum_{k=1}^{n} \overline{N}_{\text{in},k} = n\overline{N}_{\text{in}}$

Convex separable programming guarantees uniqueness and optimality of the solution together with convergence of the algorithm

In the stage 1: $\overline{N}_{\mathrm{in},k} = 0$

In the stage 2: $$\overline{\mathcal{N}}_{\mathrm{out},k} = \frac{1}{e^{\omega_k/T} - 1}$$

$$\overline{\mathcal{N}}_{\mathrm{out},k} = \bar{\mathfrak{o}}_k - 1/2, \quad \omega_k = \bar{\mathfrak{o}}_k / \bar{o}_{u,k}, \quad T = \eta/\lambda$$

$\bar{\mathfrak{o}}_k$, $\bar{o}_{u,k}$ can be expressed by means of $\overline{N}_{\mathrm{in},k}$

upon solving the “inner” problem

In the stage 3: $\overline{N}_{\mathrm{in},k} = \frac{1}{\eta}\left[\frac{1}{e^{\lambda/\eta}-1} - (1-\eta)N_{\mathrm{env},k}\right]$

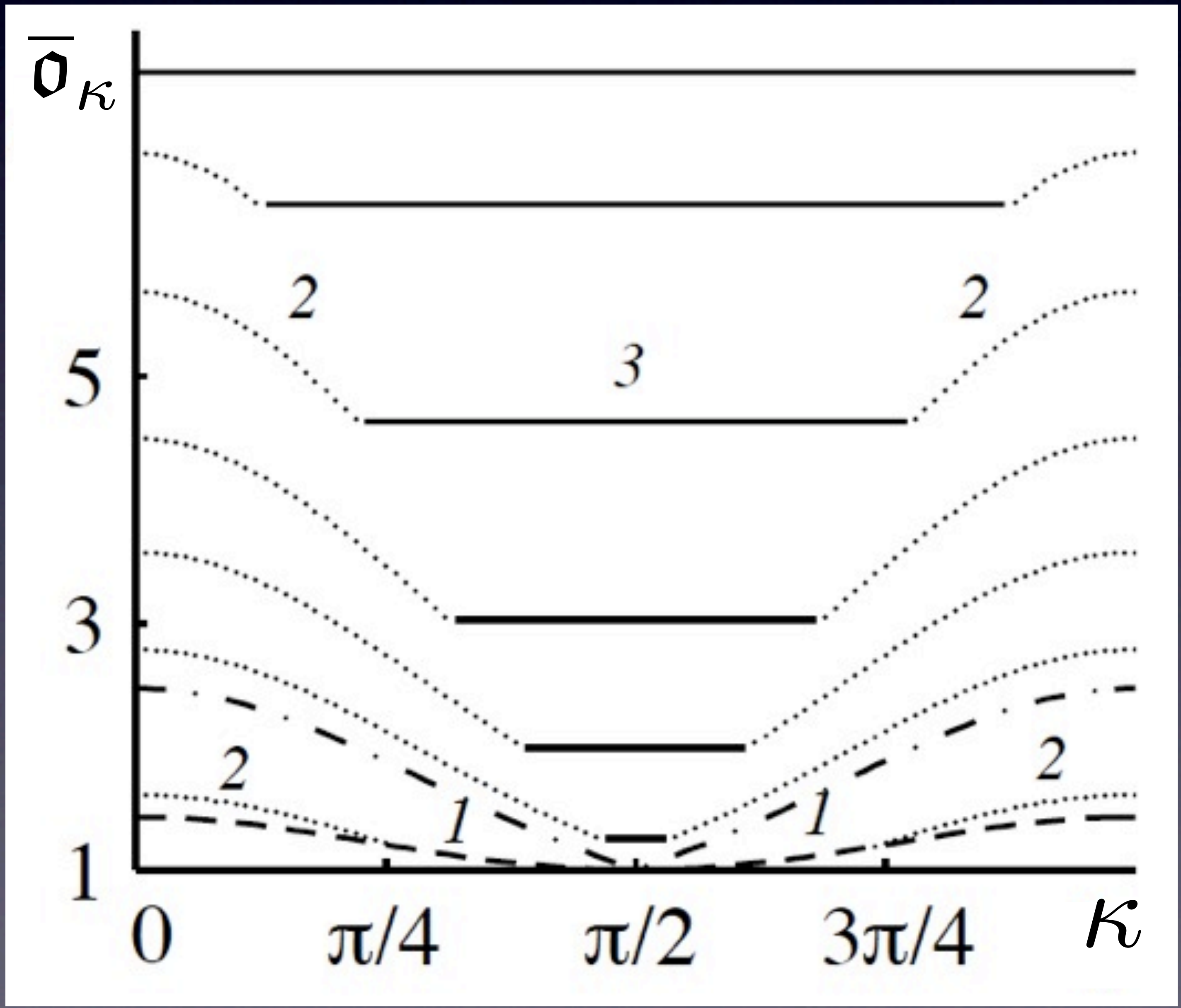
If all modes belong to the 3rd stage

$$C_n = g\left(\eta\overline{N}_{\mathrm{in}} + (1-\eta)N_{\mathrm{env}}\right) - \frac{1}{n}\sum_{k=1}^{n} g\left((1-\eta)\mathcal{N}_{\mathrm{env},k}\right)$$

Quantum water filling

$$V_{\text{env}} = \left(\mathcal{N}_{\text{env}} + \frac{1}{2}\right) \begin{pmatrix} e^{\Omega s_{\text{env}}} & 0 \\ 0 & e^{-\Omega s_{\text{env}}} \end{pmatrix}$$

$$\Omega_{i,j} = \delta_{i,j+1} + \delta_{i,j-1}$$



Super-additivity

Memoryless	Memory
$\overline{N}_{\text{in}} \longrightarrow \boxed{C_1 = C_1\left(\overline{N}_{\text{in}}\right)} \longrightarrow C_1$	$\overline{N}_{\text{in},1} \longrightarrow \boxed{C_1^{(1)} = C_1^{(1)}\left(\overline{N}_{\text{in},1}\right)} \longrightarrow C_1^{(1)}$
$\overline{N}_{\text{in}} \longrightarrow \boxed{C_1 = C_1\left(\overline{N}_{\text{in}}\right)} \longrightarrow C_1$	$\overline{N}_{\text{in},2} \longrightarrow \boxed{C_1^{(2)} = C_1^{(2)}\left(\overline{N}_{\text{in},2}\right)} \longrightarrow C_1^{(2)}$
$\vdots$	$\vdots$
$\overline{N}_{\text{in}} \longrightarrow \boxed{C_1 = C_1\left(\overline{N}_{\text{in}}\right)} \longrightarrow C_1$	$\overline{N}_{\text{in},n} \longrightarrow \boxed{C_1^{(n)} = C_1^{(n)}\left(\overline{N}_{\text{in},n}\right)} \longrightarrow C_1^{(n)}$

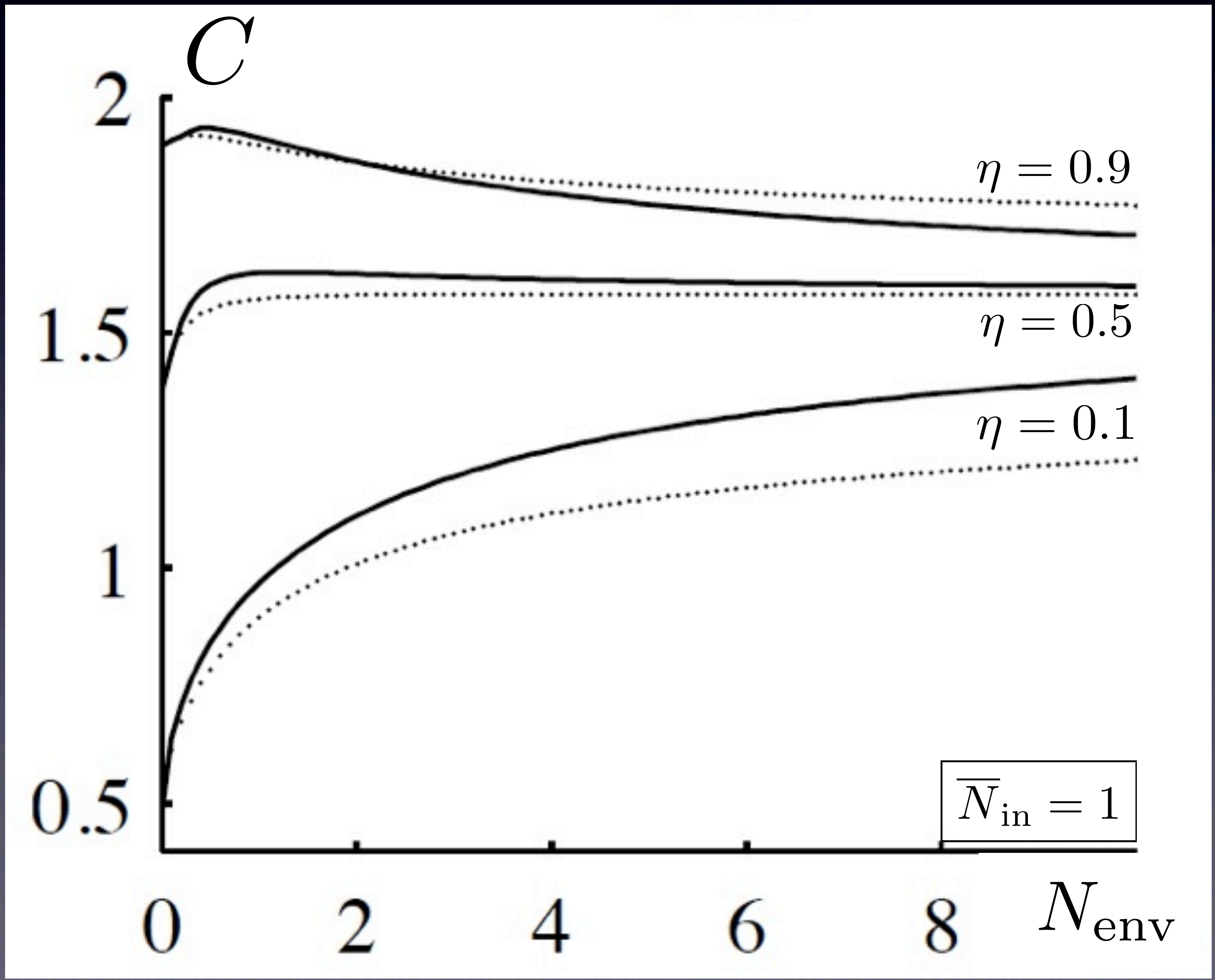
For a fixed N_{env}, sufficient condition to have

$$\sum_{k=1}^{n} C_1^{(k)} < nC_1 \,\Bigg|\, \sum_{k=1}^{n} \overline{N}_{\text{in},k} = n\overline{N}_{\text{in}}$$

is $\eta < \eta_\star \,, \quad \sum_{k=1}^{n} \overline{N}_{\text{in},k} > n\overline{N}_{\text{in}}^{\star}$

$$V_{\text{env}} = \left(\mathcal{N}_{\text{env}} + \frac{1}{2}\right) \begin{pmatrix} e^{\Omega s_{\text{env}}} & 0 \\ 0 & e^{-\Omega s_{\text{env}}} \end{pmatrix}$$

$$\Omega_{i,j} = \delta_{i,j+1} + \delta_{i,j-1}$$



Conclusions and outlook

- Optimization methods for capacity and rates
- Full characterization of the single-mode lossy channel
- Concavity (and then additivity) of the one-shot capacity
- Full characterization of the multiple use lossy channel
- Superadditivity for memory channel related to critical parameters
- Application to other Gaussian channels [additive noise, J. Schafer et al. arXiv1011.4118]
- Application to other capacities
- Open questions: optimality of Gaussian input states; coding theorems for generic memory channels