Cloning of Continuous Quantum Variables

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The cloning of quantum variables with continuous spectra is analyzed. A Gaussian quantum cloning machine is exhibited that copies equally well the states of two conjugate variables such as position and momentum. It also duplicates all coherent states with a fidelity of 2/3. More generally, the copies are shown to obey a no-cloning Heisenberg-like uncertainty relation.

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Most of the concepts of quantum computation have been initially developed for discrete quantum variables, in particular, binary quantum variables (quantum bits or qubits). Recently, however, a lot of attention has been devoted to the study of *continuous* quantum variables in informational or computational processes, as they might be intrinsically easier to manipulate than their discrete counterparts. Variables with a continuous spectrum such as the position of a particle or the amplitude of an electromagnetic field have been shown to be useful to perform quantum teleportation [1], quantum error correction [2], or, even more generally, quantum computation [3]. Also, quantum cryptographic schemes relying on continuous variables have been proposed [4], while the concept of entanglement purification has been extended to continuous variables [5]. In this context, a promising feature of quantum computation over continuous variables is that it can be carried out in quantum optics experiments by manipulating squeezed states with *linear* optics elements such as beam splitters [6].

In this Letter, the problem of copying the state of a system with continuous spectrum is investigated, and it is shown that a particular unitary transformation, called *cloning,* can be found that copies the position and momentum states equally well. Let us first state the problem in physical terms. Consider, as an example of a continuous variable, the position x of a particle in a one-dimensional space, and its canonically conjugate variable p. If the wave function is a Dirac delta function—the particle is fully localized in *position* space, then x can be measured exactly, and several perfect copies of the system can be prepared. However, such a cloning process fails to exactly copy nonlocalized states, e.g., momentum states. Conversely, if the wave function is a plane wave with momentum p—the particle is localized in *momentum* space, then p can be measured exactly and one can again prepare perfect copies of this plane wave. However, such a "plane-wave cloner" is then unable to copy position states exactly. In short, it is impossible to copy perfectly the eigenstates of two conjugate variables such as position and momentum, or the quadrature amplitudes of an electromagnetic field. This is the content of the famous no-cloning theorem [7].

In what follows, it is shown that a unitary *cloning* transformation can, nevertheless, be found that provides two copies of a system with a continuous spectrum, but at the price of a nonunity cloning fidelity. More specifically, we define a class of cloning machines that yield two imperfect copies of a continuous variable, say x, the underlying cloning transformation being *displacement covariant.* By this, we mean that any two input states that are related by a displacement [or translation in phase space (x, p)] result in copies that are related in the same way. Hence, the resulting cloning fidelity is invariant under displacements in phase space. Moreover, the qualities of the two copies obey a no-cloning uncertainty relation akin to the Heisenberg relation, implying that the product of the x-error variance on the first copy times the p-error variance on the second one remains bounded by $(\hbar/2)^2$—it cannot be zero. Within this class, a symmetric *rotation-covariant* cloner can be found that provides two identical copies of a continuous system with the *same* error distribution for position *and* momentum states. This cloner is named "Gaussian" as it effects Gaussian-distributed position and momentum errors on the input variable. It can be viewed as the continuous counterpart of the universal qubit cloner [8], as its cloning fidelity is invariant under rotations in phase space. In fact, it also duplicates in a same manner the eigenstates of linear combinations of $\hat{x}$ and $\hat{p}$, such as Gaussian wave packets or coherent states. The latter states are shown to be cloned with a fidelity that is equal to 2/3.

In the following, we shall work in position basis, whose states $|x\rangle$ are normalized according to $\langle x|x'\rangle = \delta(x - x')$. We assume $\hbar = 1$, so that the momentum eigenstates are given by $|p\rangle = (2\pi)^{-1/2} \int dx e^{ipx}|x\rangle$. We define the maximally entangled states of two continuous variables,

$$|\psi(x, p)\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx' e^{ipx'} |x'\rangle_1 |x' + x\rangle_2 , \quad (1)$$

where 1 and 2 denote the two variables, while x and p are two real parameters. Equation (1) is akin to the original Einstein-Podolsky-Rosen (EPR) state [9], but parametrized by the center-of-mass position and momentum. It is easy to check that $|\psi(x, p)\rangle$ is maximally entangled as

$\mathrm{Tr}_1(|\psi\rangle\langle\psi|) = \mathrm{Tr}_2(|\psi\rangle\langle\psi|) = \mathbb{1}/(2\pi)$ for all values of x and p, with $\mathrm{Tr}_{1,2}$ denoting partial traces with respect to variables 1 and 2, respectively. The states $|\psi\rangle$ are orthonormal, i.e., $\langle\psi(x', p')|\psi(x, p)\rangle = \delta(x - x')\delta(p - p')$, and satisfy a closure relation

$$\iint_{-\infty}^{\infty} dx\, dp\, |\psi(x,p)\rangle\langle\psi(x,p)| = \mathbb{1}_1 \otimes \mathbb{1}_2 \tag{2}$$

so they form an orthonormal basis of the joint Hilbert space of variables 1 and 2. Interestingly, applying some unitary operator on *one* of these two entangled variables makes it possible to transform the EPR states into each other. Specifically, let us define the set of *displacement* operators $\hat{D}$ parametrized by x and p,

$$\hat{D}(x,p) = e^{-ix\hat{p}} e^{ip\hat{x}} = \int_{-\infty}^{\infty} dx'\, e^{ipx'} |x' + x\rangle\langle x'|, \tag{3}$$

which form a continuous Heisenberg group. Physically, $\hat{D}(x,p)$ denotes a momentum shift of p followed by a position shift of x. If $\hat{D}(x,p)$ acts on, say, variable 2, then

$$\mathbb{1}_1 \otimes \hat{D}_2(x,p)\,|\psi(0,0)\rangle = |\psi(x,p)\rangle. \tag{4}$$

This will be useful to specify the errors induced by the continuous cloning machines considered later on. Assume that the input variable of a cloner is initially entangled with another (so-called reference) variable, so that their joint state is $|\psi(0,0)\rangle$. If cloning induces, say, a position-shift error of x on the copy, then the joint state of the reference and copy variables will be $|\psi(x,0)\rangle$ as a result of Eq. (4). Similarly, a momentum-shift error of p will result in $|\psi(0,p)\rangle$. More generally, if these x and p errors occur according to the probability distribution $P(x,p)$, then the joint state will be the mixture

$$\iint_{-\infty}^{\infty} dx\, dp\, P(x,p)\, |\psi(x,p)\rangle\langle\psi(x,p)|. \tag{5}$$

Consider now a cloning machine defined as the unitary transformation $\hat{\mathcal{U}}$ acting on three continuous variables: the input variable (variable 2) supplemented with two auxiliary variables, the blank copy (variable 3), and an ancilla (variable 4). A reference variable (variable 1) maximally entangled with the cloner input is also introduced in order to simplify the analysis. We assume that the reference and input variables are in the joint state $|\psi(0,0)\rangle_{1,2}$, while the auxiliary variables 3 and 4 are initially prepared in the state

$$|\chi\rangle_{3,4} = \iint_{-\infty}^{\infty} dx\, dp\, f(x,p)\, |\psi(x,-p)\rangle_{3,4}, \tag{6}$$

where $f(x,p)$ is an (arbitrary) complex amplitude function. After applying $\hat{\mathcal{U}}$, variables 2 and 3 are taken as the two outputs of the cloner, while variable 4 (the ancilla) is simply traced over. The goal will be to find a transformation $\hat{\mathcal{U}}$ such that the joint state of the reference, the two copies, and the ancilla after cloning, i.e., $|\Phi\rangle_{1,2,3,4} = \mathbb{1}_1 \otimes \hat{\mathcal{U}}_{2,3,4}\, |\psi(0,0)\rangle_{1,2}\, |\chi\rangle_{3,4}$, is given by

$$|\Phi\rangle = \iint_{-\infty}^{\infty} dx\, dp\, f(x,p)\, |\psi(x,p)\rangle_{1,2} |\psi(x,-p)\rangle_{3,4}. \tag{7}$$

This is a very peculiar state in that it can be reexpressed in a similar form by exchanging variables 2 and 3, namely

$$|\Phi\rangle = \iint_{-\infty}^{\infty} dx\, dp\, g(x,p)\, |\psi(x,p)\rangle_{1,3} |\psi(x,-p)\rangle_{2,4} \tag{8}$$

with

$$g(x,p) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} dx' dp' e^{i(px' - xp')} f(x', p'). \tag{9}$$

Thus, interchanging the two cloner outputs amounts to substitute the function f with its two-dimensional Fourier transform g [10]. This property is crucial as it ensures that the two copies suffer from *complementary* position and momentum errors. Indeed, using Eq. (7) and tracing over variables 3 and 4, we see that the joint state of the reference and the first output is given by Eq. (5), with $|f|^2$ playing the role of P. Hence, the first copy (called copy a later on) is imperfect in that the input variable gets a random position- and momentum-shift error drawn from the probability distribution $P_a(x,p) = |f(x,p)|^2$. Similarly, tracing the state (8) over variables 2 and 4 implies that the second copy (or copy b) is affected by a position- and momentum-shift error distributed as $P_b(x,p) = |g(x,p)|^2$. The trade-off between the quality of the two copies originates from the Fourier relation between the amplitude functions f and g, in close analogy with discrete quantum cloners [11].

Interestingly, it turns out that the cloning transformation that effects Eq. (7) can be written as

$$\hat{\mathcal{U}}_{2,3,4} = e^{-i(\hat{x}_4 - \hat{x}_3)\hat{p}_2} e^{-i\hat{x}_2(\hat{p}_3 + \hat{p}_4)}, \tag{10}$$

where $\hat{x}_k$ ($\hat{p}_k$) is the position (momentum) operator for variable k. This can be interpreted as a sequence of four continuous controlled-NOT (CNOT) gates, defined as the unitary transformation $e^{-i\hat{x}_k\hat{p}_l}$ with k (l) referring to the control (target) variable [2]. Remarkably, Eq. (10) coincides with the discrete CNOT gate sequence that achieves the qubit cloning transformation [8], up to a sign ambiguity originating from the fact that a continuous CNOT gate is not equal to its inverse [2]. This discrete-to-continuous correspondence thus suggests that the above cloning transformation is truly "universal."

Now, let us apply the cloning transformation $\hat{\mathcal{U}}$ on an input *position* state $|x_0\rangle$. We simply need to project the reference variable onto state $|x_0\rangle$. Indeed, applied to the initial joint state of the reference and the input $|\psi(0,0)\rangle_{1,2}$, the projection operator $|x_0\rangle\langle x_0| \otimes \mathbb{1}$ yields $|x_0\rangle_1|x_0\rangle_2$ up to a normalization, so the input is projected onto the desired state. Applying this projector to the state $|\Phi\rangle$ as given by Eq. (7) results in the state

$$\iint_{-\infty}^{\infty} dx\, dp\, f(x,p) e^{ipx_0} |x_0 + x\rangle_2\, |\psi(x,-p)\rangle_{3,4} \tag{11}$$

for the remaining variables 2, 3, and 4. The state of copy a (or variable 2) is then obtained by tracing over variables 3 and 4,

$$\rho_a = \int_{-\infty}^{\infty} dx\, P_a(x)\, |x_0 + x\rangle\langle x_0 + x|, \qquad (12)$$

where $P_a(x) = \int_{-\infty}^{\infty} dp\, P_a(x,p)$ is the position-error (marginal) distribution affecting copy a. Hence, the first copy undergoes a position error distributed as $P_a(x)$. Similarly, applying the projector to the alternate expression for $|\Phi\rangle$, Eq. (8), and tracing over variables 2 and 4 results in a state of the second copy ρ_b akin to Eq. (12) with $P_b(x) = \int_{-\infty}^{\infty} dp\, P_b(x,p)$. The result of cloning an input *momentum* state $|p_0\rangle$ can also be easily determined by projecting the reference variable onto $|-p_0\rangle$, so that the initial joint state of the reference and the input is projected on $|-p_0\rangle_1 |p_0\rangle_2$. Using Eqs. (7) and (8), we obtain the analogous expressions for the state of copies a and b,

$$\rho_{a(b)} = \int_{-\infty}^{\infty} dp\, P_{a(b)}(p)\, |p_0 + p\rangle\langle p_0 + p|, \qquad (13)$$

where $P_{a(b)}(p) = \int_{-\infty}^{\infty} dx\, P_{a(b)}(x,p)$. Consequently, the two copies undergo a momentum error distributed as $P_{a(b)}(p)$. The tradeoff between the quality of the copies can be expressed by relating the variances of $P_a(x,p)$ and $P_b(x,p)$.

Let us analyze this no-cloning complementary by applying the Heisenberg uncertainty relation to the state

$$|\zeta\rangle_{1,2} = \iint_{-\infty}^{\infty} dx\, dp\, f(x,p)\, |x\rangle_1 |p\rangle_2 = \iint_{-\infty}^{\infty} dx\, dp\, g(-x,-p)\, |p\rangle_1 |x\rangle_2 . \qquad (14)$$

The two pairs of canonically conjugate operators $(\hat{x}_1, \hat{p}_1)$ and $(\hat{p}_2, \hat{x}_2)$ give rise, respectively, to the no-cloning uncertainty relations

$$(\Delta x_a)^2 (\Delta p_b)^2 \geq 1/4, \quad (\Delta x_b)^2 (\Delta p_a)^2 \geq 1/4, \qquad (15)$$

where $(\Delta x_a)^2$ and $(\Delta x_b)^2$ denote the variance of $P_a(x)$ and $P_b(x)$, respectively, while the same notation holds for the momentum-shift variances. Consequently, if the cloning process induces a small position (momentum) error on the first copy, then the second copy is necessarily affected by a large momentum (position) error.

We now turn to a particular symmetric cloner that is *rotation covariant* in phase space and saturates the above no-cloning uncertainty relations. We restrict ourselves to solutions of the form $f(x,p) = q(x)Q(-p)$ where $Q(p) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} dx\, e^{-ipx} q(x)$ is the Fourier transform of $q(x)$. This choice satisfies the symmetry requirement $|g(x,p)|^2 = |f(x,p)|^2$ (even though it is not unique). For the cloner to act equally on position and momentum states, $q(x)$ must be equal (in magnitude) to its Fourier transform, so that we choose $f(x,p) = e^{-(x^2+p^2)/2}/\sqrt{\pi}$. Hence, $P_{a(b)}(x,p) = e^{-(x^2+p^2)}/\pi$ is simply a bivariate Gaussian of variance $1/2$ on x and p axis. The two auxiliary variables must then be prepared in the state

$$|\chi\rangle = \frac{1}{\sqrt{\pi}} \iint_{-\infty}^{\infty} dy\, dz\, e^{-(y^2+z^2)/2} |y\rangle |y + z\rangle. \qquad (16)$$

The resulting transformation effected by this Gaussian cloner on an input position state $|x\rangle$ is given by

$$|x\rangle|\chi\rangle \to \frac{1}{\sqrt{\pi}} \iint_{-\infty}^{\infty} dy\, dz\, e^{-(y^2+z^2)/2} \times |x + y\rangle |x + z\rangle |x + y + z\rangle, \qquad (17)$$

where the three variables denote the two copies and the ancilla, respectively. It is easy to check that Eq. (17) implies Eq. (12) and its counterpart for copy b, with $P_a(x) = P_b(x) = \exp(-x^2)/\sqrt{\pi}$, so that both copies are affected by a Gaussian-distributed position error of variance $1/2$. The choice $(\Delta x_a)^2 = (\Delta x_b)^2 = (\Delta p_a)^2 = (\Delta p_b)^2$ ensures that position *and* momentum states are copied with the same error variance, while the value $1/2$ implies that the cloner is optimal among the class considered here, in view of Eq. (15). Furthermore, the rotation invariance of $|f(x,p)|^2$ implies that this cloner copies the eigenstates of any operator of the form $c\hat{x} + d\hat{p}$ with the same error distribution, as we will show.

Let us first determine the operation of this Gaussian cloner on an arbitrary state $|\xi\rangle$ expressed in position basis as $\int_{-\infty}^{\infty} dx\, \xi(x)\, |x\rangle$. For this, we project the reference variable onto state $|\xi^*\rangle$, i.e., the state obtained by changing $\xi(x)$ into its complex conjugate. Applying $|\xi^*\rangle\langle\xi^*| \otimes \mathbb{1}$ on the initial state $|\psi(0,0)\rangle_{1,2}$ yields the state $|\xi^*\rangle_1 |\xi\rangle_2$, up to a normalization, so the input is indeed projected onto $|\xi\rangle$. Now, applying this projector to the state $|\Phi\rangle$ after cloning implies that the remaining three variables are left in the state

$$\iint_{-\infty}^{\infty} dx\, dp\, f(x,p)\, |\xi(x,p)\rangle_2 |\psi(x,-p)\rangle_{3,4}, \qquad (18)$$

where $|\xi(x,p)\rangle = \hat{D}(x,p)|\xi\rangle = \int_{-\infty}^{\infty} dx' \xi(x') e^{ipx'} |x' + x\rangle$ is the input state $|\xi\rangle$ affected by a momentum shift of p followed by a position shift of x. This yields

$$\rho_{a(b)} = \iint_{-\infty}^{\infty} dx\, dp\, P_{a(b)}(x,p)\, |\xi(x,p)\rangle\langle\xi(x,p)| \qquad (19)$$

so the two outputs are mixtures of shifted states $|\xi(x,p)\rangle$, with x and p distributed according to $P_{a(b)}(x,p)$. Expressed in terms of Wigner distributions, Eq. (19) implies that $W_{\text{out}}(x,p) = W_{\text{in}}(x,p) \circ P(x,p)$ with $\circ$ denoting convolution. In particular, the Gaussian cloner simply effects a spreading out of the input Wigner function by a bivariate Gaussian of variance $1/2$.

These considerations can be easily generalized to any pair of canonically conjugate variables in a rotated phase space. First note that, using the Baker-Hausdorff formula and $[\hat{x}, \hat{p}] = i$, the displacement operator can be rewritten as $\hat{D}(x,p) = e^{-ixp/2} e^{i(p\hat{x} - x\hat{p})}$. Consider now any pair of observables $\hat{u}$ and $\hat{v}$ satisfying the commutation rule $[\hat{u},\hat{v}] = i$. Let $\hat{u} = c\hat{x} + d\hat{p}$ and $\hat{v} = -d\hat{x} + c\hat{p}$, where c and d are real and satisfy $c^2 + d^2 = 1$. It is easy to check that $v\hat{u} - u\hat{v} = p\hat{x} - x\hat{p}$, where the variables u and v are defined just as $\hat{u}$ and $\hat{v}$, so that $\hat{D}$ takes a similar form in terms of $\hat{u}$ and $\hat{v}$ (up to an irrelevant phase). Therefore, as a consequence of the rotation invariance of the Gaussian $|f(x,p)|^2$, the eigenstates $|u\rangle$ of the observable $\hat{u}$ undergo a random shift of u that is distributed as $\exp(-u^2)/\sqrt{\pi}$. (The position and momentum states are just two special cases of this.) We can also treat the cloning of coherent states (Gaussian wave packets) by considering the complex rotation that defines the annihilation and creation operators $\hat{a} = (\hat{x} + i\hat{p})/\sqrt{2}$ and $\hat{a}^\dagger = (\hat{x} - i\hat{p})/\sqrt{2}$. The displacement operator can then be written (up to an irrelevant phase) in the usual form $\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$, where $\alpha = (x + ip)/\sqrt{2}$ is a c number that characterizes the position and momentum shift. In the coherent state representation, where $|\alpha_0\rangle$ denotes the eigenstate of $\hat{a}$ with eigenvalue α_0, this operator effects the transformation $\hat{D}(\alpha)\,|\alpha_0\rangle = e^{i\theta}|\alpha_0 + \alpha\rangle$, with $\theta = \mathrm{Im}(\alpha\,\alpha_0^*)$. Thus, if the input of the cloner is a coherent state $|\alpha_0\rangle$, its two outputs are a mixture of coherent states characterized by

$$\rho = \int d^2\alpha\, G(\alpha)\,|\alpha_0 + \alpha\rangle\langle\alpha_0 + \alpha|\,, \qquad (20)$$

where the integral is over the complex plane, and $G(\alpha) = 2\exp(-2|\alpha|^2)/\pi$ is a Gaussian distribution in α space. Then, using $|\langle\alpha|\alpha'\rangle|^2 = \exp(-|\alpha - \alpha'|^2)$, it is easy to calculate the fidelity of the Gaussian cloner:

$$f = \langle\alpha_0|\rho|\alpha_0\rangle = \frac{2}{\pi}\int d^2\alpha\, e^{-3|\alpha|^2} = \frac{2}{3}\,. \qquad (21)$$

This fidelity does not depend on α_0, so it is *invariant* for all coherent states.

Finally, consider the cloning of quadrature squeezed states, defined as the eigenstates of $\hat{b} = (\hat{x}/\sigma + i\sigma\hat{p})/\sqrt{2}$, where σ is a real parameter. These states can be denoted as $|\beta\rangle$, where $\beta = (x/\sigma + i\sigma p)/\sqrt{2}$ is a c number. We have again $\hat{D}(\beta) = e^{\beta\hat{b}^\dagger - \beta^*\hat{b}}$, so that $\hat{D}(\beta)|\beta_0\rangle = |\beta_0 + \beta\rangle$ up to a phase. In order to keep the fidelity maximum, however, we must use here a modified cloner defined by

$$f(x,p) = \frac{1}{\sqrt{\pi}}\, e^{-(x^2/\sigma^2 + \sigma^2 p^2)/2}. \qquad (22)$$

Both copies yielded by this cloner are affected by an x error of variance $\sigma^2/2$ and a p error of variance $1/(2\sigma^2)$, which implies that the output density operator has the same form as Eq. (20) with $G(\beta) = 2\exp(-2|\beta|^2)/\pi$. As a consequence, there exists a specific σ-dependent cloning machine that copies all squeezed states corresponding to each value of σ with a fidelity of $2/3$. In contrast, cloning these states using the rotation-covariant cloner above gives a fidelity that decreases as squeezing increases.

We have shown that a Gaussian cloning machine for continuous quantum variables can be defined that transforms position (momentum) states into a Gaussian-distributed mixture of position (momentum) states with an error variance of $1/2$. It is translation and rotation covariant in phase space, as the eigenstates of any linear combination of $\hat{x}$ and $\hat{p}$ are copied with the same error distribution. In particular, it duplicates all coherent states with a fidelity of $2/3$. We conjecture that this cloning fidelity is optimal. An experimental realization of this cloner could be envisaged based on the manipulation of modes of the electromagnetic field. The cloning transformation $\hat{\mathcal{U}}$ would then couple two auxiliary modes to an input mode. Since $\hat{\mathcal{U}}$ amounts to a sequence of continuous CNOT gates, it could be implemented by pairwise quantum nondemolition (QND) coupling between these three modes [2]. As a final remark, it is worth noting that the two auxiliary modes must be prepared in state (16), which is simply the product vacuum state $|0\rangle_3|0\rangle_4$ processed by a CNOT gate $e^{-i\hat{x}_3\hat{p}_4}$. This suggests that the noise that inevitably arises when cloning the input mode is intrinsically linked to the vacuum fluctuations of the auxiliary modes. It also corroborates the suggestion that the physical mechanism that prevents perfect cloning is spontaneous emission [12].

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