# Optimal Cloning of Coherent States with a Linear Amplifier and Beam Splitters

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A transformation achieving the optimal symmetric $N \rightarrow M$ cloning of coherent states is presented. Its implementation requires only a phase-insensitive linear amplifier and a network of beam splitters. An experimental demonstration of this continuous-variable cloner should therefore be in the scope of current technology. The link between optimal quantum cloning and optimal amplification of quantum states is also pointed out.

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Quantum systems cannot be cloned *exactly* [1], but only approximately. Finding the optimal *approximate* quantum cloning transformation has been a fundamental issue in quantum information theory for the last five years. In quantum cryptography, for instance, this problem happens to be strongly related to the assessment of security [2]. Cloning has been extensively studied to date for discrete quantum variables, such as quantum bits [3–9] or $d$-level systems [10–12], since quantum information theory was initially developed for these kinds of systems. Recent progress has shown, however, that continuous spectrum systems might be experimentally simpler to manipulate than their discrete counterparts in order to process quantum information (see, e.g., [13,14]).

Stimulated by this progress, we investigate in this Letter the possibility of achieving the cloning of continuous-variable quantum information. Commonly, a distinction is made between *universal* cloning, if the set of input states contains all possible states for a given Hilbert space dimension, and *state-dependent* cloning, if the input states are restricted to a certain set which does not contain all possible states. For any Hilbert space dimension, the optimal universal cloner [10–12] that clones all possible input states equally well can be constructed from a single family of quantum circuits [15]. This universal cloner reduces to a classical probability distributor in the continuous limit. Besides the universal cloner, quantum cloning of continuous-variable systems has been considered first in a state-dependent context. In Ref. [16], the duplication of coherent states was investigated, and an explicit transformation that is covariant under displacement and rotation in phase space was derived. This transformation therefore clones all coherent states with the same fidelity ($F = 2/3$), although it is not universal, strictly speaking, as its cloning fidelity is lower for other classes of states such as squeezed states. The optimality of this continuous-variable cloning transformation was then proven in Ref. [17]. More generally, it was shown that, if one attempts to produce $M$ clones from $N$ original replicas of a coherent state $|\alpha\rangle$ ($M \geq N$) with an equal fidelity for all $\alpha$'s, the so-called $N$-to-$M$ cloning transformation must result in an additional noise on both quadratures of each of the $M$ outputs which has a minimum variance

$$\overline{\sigma}^2_{N,M} = \left(\frac{2}{N} - \frac{2}{M}\right)\Delta x^2_{\text{vac}}\,, \tag{1}$$

where the vacuum noise on a quadrature is denoted as $\Delta x^2_{\text{vac}} = 1/2$ ($\hbar = 1$). The corresponding maximum $N \rightarrow M$ cloning fidelity is $F_{N,M} = MN/(MN + M - N)$. However, finding the optimal $N \rightarrow M$ cloning transformation and proving that it actually achieves this maximum fidelity was an open problem.

The present Letter resolves this question. We use the Heisenberg picture in order to derive explicitly an $N \rightarrow M$ symmetric cloning transformation that attains Eq. (1). Remarkably, it appears that implementing this transformation requires only a phase-insensitive linear amplifier and a set of beam splitters. Let $|\Psi\rangle = |\alpha\rangle^{\otimes N} \otimes |0\rangle^{\otimes M-N} \otimes |0\rangle_z$ denote the initial joint state of the $N$ input modes to be cloned (prepared in the coherent state $|\alpha\rangle$), the additional $M - N$ blank modes, and an ancillary mode $z$. The blank modes and the ancilla are assumed to be initially in the vacuum state $|0\rangle$. Let $\{x_k, p_k\}$ denote the pair of quadrature operators associated with each mode $k$ involved in the cloning transformation: $k = 0, \ldots, N - 1$ refers to the $N$ original input modes, and $k = N, \ldots, M - 1$ refers to the additional blank modes. Cloning can be thought of as some unitary transformation $U$ acting on $|\Psi\rangle$, and resulting in a state $|\Psi''\rangle = U|\Psi\rangle$ such that the $M$ modes are left in the same (mixed) state which is maximally close to $|\alpha\rangle$. Alternatively, in the Heisenberg picture, this transformation can be described by a canonical transformation acting on the operators $\{x_k, p_k\}$:

$$x''_k = U^\dagger x_k U, \qquad p''_k = U^\dagger p_k U\,, \tag{2}$$

while leaving the state $|\Psi\rangle$ invariant. We will work in the Heisenberg picture and use the above notation throughout this paper, with $x''_k$ denoting the clones (i.e., the output modes of the cloning circuit except the ancilla $z$), because cloning turns out to be much simpler to describe from that point of view. We will now impose several requirements on transformation (2) that translate the expected properties for an optimal cloning transformation. First, we require that the $M$ output modes have the desired mean values:

$$\langle x_k'' \rangle = \langle \alpha | x_0 | \alpha \rangle, \qquad \langle p_k'' \rangle = \langle \alpha | p_0 | \alpha \rangle, \tag{3}$$

for $k = 0, \ldots, M - 1$. Roughly speaking, this means that the state of the clones is centered on the original coherent state. Our second requirement is covariance with respect to rotation in phase space. Coherent states have the property that the quadrature variances are left invariant by complex rotations in phase space. So, for any input mode $k$ of the cloning transformation and for any operator $v_k = c x_k + d p_k$ (where $c$ and $d$ are complex numbers satisfying $|c|^2 + |d|^2 = 1$), the error variance $\sigma_{v_k}^2$ is the same:

$$\sigma_{v_k}^2 = \langle (v_k)^2 \rangle - \langle v_k \rangle^2 = \Delta x_{\mathrm{vac}}^2 = \frac{1}{2}. \tag{4}$$

We impose this property to be conserved through the cloning process. Taking optimality into account, Eq. (1), rotational covariance yields

$$\sigma_{v_k''}^2 = \left(1 + \frac{2}{N} - \frac{2}{M}\right) \Delta x_{\mathrm{vac}}^2, \tag{5}$$

where $v_k'' = c x_k'' + d p_k''$. Our third requirement is, of course, the unitarity of the cloning transformation (2). In the Heisenberg picture, this is equivalent to demanding that the commutation relations are preserved through the evolution,

$$[x_j'', x_k''] = [p_j'', p_k''] = 0, \qquad [x_j'', p_k''] = i \delta_{jk}, \tag{6}$$

for $j, k = 0, \ldots, M - 1$ and for the ancilla.

Let us first focus on the continuous-variable duplication ($N = 1$, $M = 2$). A simple transformation obeying the three conditions mentioned above is given by

$$\begin{aligned} x_0'' &= x_0 + \frac{x_1}{\sqrt{2}} + \frac{x_z}{\sqrt{2}}, \qquad p_0'' = p_0 + \frac{p_1}{\sqrt{2}} - \frac{p_z}{\sqrt{2}}, \\ x_1'' &= x_0 - \frac{x_1}{\sqrt{2}} + \frac{x_z}{\sqrt{2}}, \qquad p_1'' = p_0 - \frac{p_1}{\sqrt{2}} - \frac{p_z}{\sqrt{2}}, \\ x_z' &= x_0 + \sqrt{2}\, x_z, \qquad p_z' = -p_0 + \sqrt{2}\, p_z. \end{aligned} \tag{7}$$

This transformation clearly conserves the commutation rules, and yields the expected mean values $(\langle x_0 \rangle, \langle p_0 \rangle)$ for the two clones (modes $0''$ and $1''$). Also, one can check that the quadrature variances of both clones are equal to $2\Delta x_{\mathrm{vac}}^2$, in accordance with Eq. (5). This transformation actually coincides with the Gaussian cloning machine introduced in Ref. [16]. Interestingly, we note here that the state in which the ancilla $z$ is left after cloning is centered on $(\langle x_0 \rangle, -\langle p_0 \rangle)$, that is, the *phase-conjugated* state $|\alpha^*\rangle$. This means that, in analogy with the universal qubit cloning machine [4], the continuous-variable cloner generates an "anticlone" (or time-reversed state) together with the two clones.

Now, let us show how this duplicator can be implemented in practice. Equation (7) can be interpreted as a sequence of two canonical transformations:

$$\begin{aligned} a_0' &= \sqrt{2}\, a_0 + a_z^\dagger, \qquad a_z' = a_0^\dagger + \sqrt{2}\, a_z, \\ a_0'' &= \frac{1}{\sqrt{2}} (a_0' + a_1), \qquad a_1'' = \frac{1}{\sqrt{2}} (a_0' - a_1), \end{aligned} \tag{8}$$

where $a_k = (x_k + i p_k)/\sqrt{2}$ and $a_k^\dagger = (x_k - i p_k)/\sqrt{2}$ denote the annihilation and creation operators for mode $k$. As shown in Fig. 1, the interpretation of this transformation becomes then straightforward: the first step (which transforms $a_0$ and $a_z$ into $a_0'$ and $a_z'$) is a phase-insensitive amplifier whose (power) gain $G$ is equal to 2, while the second step (which transforms $a_0'$ and $a_1$ into $a_0''$ and $a_1''$) is a phase-free 50:50 beam splitter [18]. Clearly, rotational covariance is guaranteed here by the use of a *phase-insensitive* amplifier. As discussed in Ref. [19], the ancilla $z$ involved in linear amplification can always be chosen such that $\langle a_z \rangle = 0$, so that we have $\langle a_0'' \rangle = \langle a_1'' \rangle = \langle a_0 \rangle$ as required. Finally, the optimality of our cloner can be confirmed from known results on linear amplifiers. For an amplifier of (power) gain $G$, each quadrature's excess noise variance is bounded by [19]

$$\sigma_{LA}^2 \geq (G - 1)/2. \tag{9}$$

Hence, the optimal amplifier of gain $G = 2$ yields $\sigma_{LA}^2 = 1/2$, so that our cloning transformation is optimal.

Let us now consider the $N \to M$ cloning transformation. In order to achieve cloning, energy has to be brought to the $M - N$ blank modes in order to drive them from the vacuum state into a state which has the desired mean value. We will again achieve this operation with the help of a linear amplifier. From Eq. (9), we see that the cloning induced noise essentially originates from the amplification process, and grows with the amplifier gain. So, we should preferably amplify as little as possible. Loosely speaking, the cloning procedure should then be as follows: (i) symmetrically amplifying the $N$ input modes by *concentrating* them into one single mode, which is then

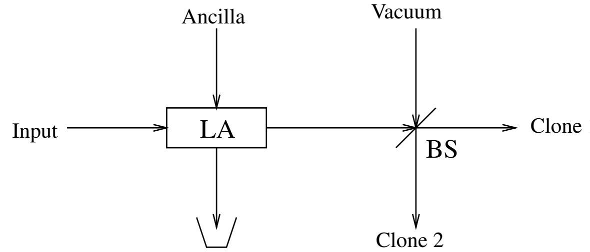


FIG. 1. Implementation of a $1 \to 2$ continuous-variable cloning machine. LA stands for linear amplifier, and BS represents a phase-free 50:50 beam splitter.

amplified; (ii) symmetrically *distributing* the output of this amplifier among the $M$ output modes. As we will see, a convenient way to achieve these concentration and distribution processes is provided by the discrete Fourier transform (DFT). Cloning is then achieved by the following three-step procedure (see Fig. 2). First step: a DFT (acting on $N$ modes),

$$a'_k = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} \exp(ikl2\pi/N) a_l \,, \tag{10}$$

with $k = 0, \dots, N-1$. This operation concentrates the energy of the $N$ input modes into one single mode (renamed $a_0$) and leaves the remaining $N-1$ modes $(a'_1, \dots, a'_{N-1})$ in the vacuum state. Second step: the mode $a_0$ is amplified with a linear amplifier of gain $G = M/N$. This results in

$$\begin{aligned} a'_0 &= \sqrt{\frac{M}{N}}\, a_0 + \sqrt{\frac{M}{N} - 1}\, a_z^\dagger, \\ a'_z &= \sqrt{\frac{M}{N} - 1}\, a_0^\dagger + \sqrt{\frac{M}{N}}\, a_z \,. \end{aligned} \tag{11}$$

Third step: amplitude distribution by performing a DFT (acting on $M$ modes) between the mode $a'_0$ and $M-1$ modes in the vacuum state:

$$a''_k = \frac{1}{\sqrt{M}} \sum_{l=0}^{M-1} \exp(ikl2\pi/M) a'_l \,, \tag{12}$$

with $k = 0, \dots, M-1$, and $a'_i = a_i$ for $i = N, \dots, M-1$. The DFT now distributes the energy contained in the output of the amplifier among the $M$ output clones.

It is readily checked that this procedure meets our three requirements, and is optimal provided that the amplifier is optimal, that is $\sigma_{LA}^2 = [(M/N) - 1]/2$. The quadrature variances of the $M$ output modes coincide with Eq. (5). As in the case of duplication, the quality of cloning decreases as $\sigma_{LA}^2$ increases, that is cloning and amplifying coherent states are two equivalent problems. For $1 \to 2$ cloning, we have seen that the final amplitude distribution among the output clones is achieved with a single beam splitter. In fact, any unitary matrix such as the DFT used here can be realized with a sequence of beam splitters (and phase shifters) [20]. This means that the $N \to M$ cloning transformation can be implemented using only passive elements except for a single linear amplifier.

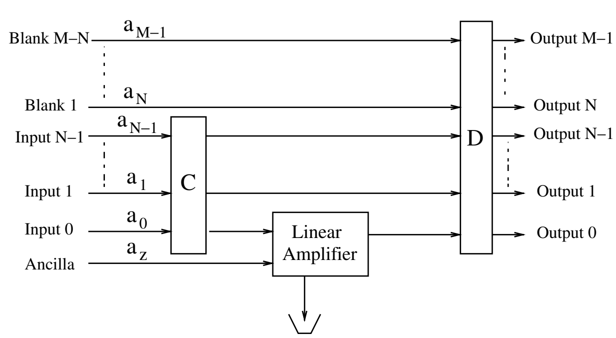


FIG. 2. Implementation of an $N \to M$ continuous-variable cloning machine. C stands for the amplitude concentration operation, while D refers to the amplitude distribution. Both can be achieved by using a DFT, or, alternatively, an inverse "$N$-splitter" and an "$M$-splitter" (in which case we shift the indices by one in the text, i.e., $k = 1, \dots, M$).

We will now explicitly give the *simplest* beam splitter combination that enables the above transformation. For convenience, let us now use the indices $k = 1, \dots, N$ for the $N$ original input modes $a_k$, and $k = N+1, \dots, M$ for the additional blank modes $a_k$. With an ideal (phase-free) beam splitter operation acting on two modes $c_k$ and $c_l$,

$$\begin{pmatrix} c'_k \\ c'_l \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} c_k \\ c_l \end{pmatrix}, \tag{13}$$

we define a matrix $B_{kl}(\theta)$ which is an $M$-dimensional identity matrix with the entries $I_{kk}$, $I_{kl}$, $I_{lk}$, and $I_{ll}$ replaced by the corresponding entries of the above beam splitter matrix. Now we can define a sequence of beam splitters acting on $M$ modes ("$M$-splitter" [14]) as

$$\begin{aligned} \mathcal{U}(M) \equiv{}& B_{M-1M}\Big(\sin^{-1}\frac{1}{\sqrt{2}}\Big) B_{M-2M-1}\Big(\sin^{-1}\frac{1}{\sqrt{3}}\Big) \\ &\times \cdots \times B_{12}\Big(\sin^{-1}\frac{1}{\sqrt{M}}\Big). \end{aligned} \tag{14}$$

The individual beam splitters in Eq. (14) depend only on their reflectivity/transmittance parameter $\theta$. In order to concentrate the $N$ identical inputs, we send them now through an inverse $N$-splitter,

$$(a'_1\ a'_2 \cdots a'_N)^T = \mathcal{U}^\dagger(N)\,(a_1\ a_2 \cdots a_N)^T. \tag{15}$$

Again, we end up with one mode (renamed $a_1$) having nonzero mean value and $N-1$ modes $(a'_2, \dots, a'_N)$ in the vacuum state. After amplifying mode $a_1$, $a'_1 = \sqrt{M/N}\, a_1 + \sqrt{M/N - 1}\, a_z^\dagger$, etc., a final $M$-splitter operation yields the output clones

$$(a''_1\ a''_2 \cdots a''_M)^T = \mathcal{U}(M)\,(a'_1\ a'_2 \cdots a'_M)^T, \tag{16}$$

with $a'_i = a_i$ for $i = N+1, \dots, M$.

Since the amplification produces extra noise, our cloning circuits used as little amplification as possible. However, rather surprisingly, by first amplifying each input copy $k = 1, \dots, N$ individually,

$$\begin{aligned} a'_k &= \sqrt{\frac{M}{N}}\, a_k + \sqrt{\frac{M}{N} - 1}\, a_{z,k}^\dagger \,, \\ a'_{z,k} &= \sqrt{\frac{M}{N} - 1}\, a_k^\dagger + \sqrt{\frac{M}{N}}\, a_{z,k} \,, \end{aligned} \tag{17}$$

a circuit can also be constructed that yields optimum fidelities. In the next step, the amplified modes are *each* sent together with $M-1$ vacuum modes $b_{k,1}, b_{k,2}, \dots, b_{k,M-1}$ through an $M$-splitter

$$(a'_{k,1}\ a'_{k,2} \cdots a'_{k,M})^T = \mathcal{U}(M)\,(a'_k\ b_{k,1} \cdots b_{k,M-1})^T. \tag{18}$$

The $NM$ output modes after this operation can be written as

$$a'_{k,l} = \frac{1}{\sqrt{N}}\,a_k + \sqrt{\frac{M-N}{MN}}\,a^{\dagger}_{z,k} + d_{k,l}\,, \tag{19}$$

where $l = 1, \ldots, M$. The noise in each $M$-splitter output coming from the $M - 1$ vacuum inputs is represented by mode $d_{k,l}$ having zero mean value and quadrature variances of $(M - 1)/2M$. The final step now consists of $M$ inverse $N$-splitters acting on all modes with the same index $l$, i.e., the $N$ modes $a'_{k,1}$, and the $N$ modes $a'_{k,2}$, etc. The output modes at each $N$-splitter,

$$(a''_l\ e_{1,l} \cdots e_{N-1,l})^T = \mathcal{U}^{\dagger}(N)\,(a'_{1,l}\ a'_{2,l} \cdots a'_{N,l})^T, \tag{20}$$

contain only noise except for one mode,

$$a''_l = \sum_{k=1}^{N}\left(\frac{1}{N}\,a_k + \sqrt{\frac{M-N}{MN^2}}\,a^{\dagger}_{z,k} + \frac{1}{\sqrt{N}}\,d_{k,l}\right). \tag{21}$$

Again, all $M$ clones are optimal, although additional noise has been introduced at the intermediate steps which results in $M(N - 1)$ "waste" output modes. However, this particular circuit points out that $N \rightarrow M$ cloning of coherent states is effectively a "classical plumbing" procedure distributing classical amplitudes.

Finally, we note that for squeezed-state inputs rather than coherent states, the transformations and circuits presented require all auxiliary vacuum modes (the blank modes and the ancillary mode $z$) be correspondingly squeezed in order to maintain optimum cloning fidelities. This means, in particular, that the amplifier mode $z$ needs to be controlled which requires a device different from a simple phase-insensitive amplifier, namely a two-mode parametric amplifier. One can say that the cloning machine capable of optimal cloning of all squeezed states with *fixed* and *known* squeezing then operates in a nonuniversal fashion with respect to all possible squeezed states at the input [16,17].

In summary, an optimal $N$-to-$M$ continuous-variable cloning transformation for coherent states has been derived, which attains the maximum cloning fidelity $F_{N,M} = MN/(MN + M - N)$. A possible experimental implementation of this cloner has been proposed. We trust that this implementation should be achievable with current technology since it requires only a single linear amplifier and $N + M - 2$ beam splitters. In Ref. [21], an alternative one-to-two cloning scheme has been proposed based on three nondegenerate optical parametric amplifiers, and its experimental realization is currently underway at Roma University. Finally, we pointed out the link between the quality of the best cloner and the minimum noise induced by the amplification of a quantum state, emphasizing that spontaneous emission is here again the mechanism that prevents the perfect cloning of quantum states of light [3,9].

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