Quantum Cloning Machines with Phase-Conjugate Input Modes

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A quantum cloning machine is introduced that yields M identical optimal clones from N replicas of a coherent state and N' replicas of its phase conjugate. It also optimally produces $M' = M + N' - N$ phase-conjugate clones at no cost. For well chosen ratios N'/N, this machine is shown to provide better cloning fidelities than the standard $(N + N') \rightarrow M$ cloner. The special cases of the optimal balanced cloner ($N = N'$) and optimal measurement ($M = \infty$) are investigated.

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The concept of cloning plays a central role in quantum information theory. It is, for example, crucial in quantum cryptography, as the optimal duplication of a quantum state directly determines the security of a cryptosystem [1,2], or in quantum estimation theory, as cloning provides a constructive way to achieve a generalized measurement [3]. The optimal N-to-M cloning transformations, which produce M clones from N originals, have been found for both quantum bits [4] and continuous variables [5,6]. Interestingly, producing infinitely many clones from N identical replicas of a quantum state is equivalent to performing an optimal measurement, which reflects the existence of a close link between cloning and measurement theories. In the context of measurement, recent work has revealed that pairs of antiparallel qubits are intrinsically more informative than pairs of parallel qubits [7], a result that has later been extended to continuous variables: more information can be encoded in a pair of phase-conjugate coherent states $|\psi\rangle|\psi^*\rangle$ than in two identical replicas $|\psi\rangle|\psi\rangle$ [8]. This property suggests that cloning machines with antiparallel input qubits (or phase-conjugate input modes) might yield better fidelities than standard $N \rightarrow M$ cloning machines, thereby opening a new avenue in the investigation of quantum cloning.

In this Letter, we will focus on quantum information carried by continuous variables, and seek for a cloning transformation that, taking as input N replicas of a coherent state $|\psi\rangle$ and N' replicas of its complex conjugate $|\psi^*\rangle$, produces M optimal clones of $|\psi\rangle$. The resulting concept of phase-conjugate input (PCI) cloning machines will turn out to be closely connected to that of the amplification of light, just as for standard cloning [5,6,9,10]. As a matter of fact, PCI cloning can be decomposed as a sequence of beam splitters, a single nonlinear process, and another sequence of beam splitters, which is consistent with the Bloch-Messiah reduction theorem [11]. We will start by deriving the optimal canonical transformation that acts on two modes in a coherent state with respective mean values $\alpha\psi$ and $\beta\psi^*$ (where α, β are real while ψ is a complex number), and generates a mode whose mean value is $\gamma\psi$, where γ is real. Remarkably, this transformation will be shown to have a structure similar to that of a conventional phase-insensitive phase-preserving amplifier as defined in [12], where both the signal and idle modes are used as inputs. After having derived this transformation, we will apply it to the case of integer α^2, β^2, and γ^2, and see how it can be supplemented with beam splitters to provide a PCI cloning machine for continuous variables. This machine will be shown to produce $M' = M + N' - N$ additional phase-conjugate clones (or anticlones). The quality of the clones and anticlones will be discussed in the case of a balanced cloner ($N = N'$), as well as for arbitrary phase-conjugate input fractions $N'/(N + N')$. The related question of the optimal measurement ($M = \infty$) of phase-conjugate coherent states will also be treated. To our knowledge, the PCI cloner is the first example of a quantum information-theoretic process for continuous variables for which no discrete-variable analog has been found yet.

Let $\{a_i\}$ and $\{b_i\}$ $(i = 1, \ldots, 3)$ denote, respectively, the input and output mode annihilation operators of the cloning transformation. The indices $i = 1, 2$ refer to the input and phase-conjugate input modes, respectively, while $i = 3$ refers to an auxiliary mode. In full generality, we are seeking for a linear canonical transformation,

$$b_i = M_{ij} a_j + L_{ij} a_j^\dagger \qquad (i, j = 1, \ldots, 3), \tag{1}$$

that meets the three following requirements (the sum over repeated indices being implicit). First, starting with modes a_1 and a_2 with mean values $\langle a_1 \rangle = \alpha\psi$ and $\langle a_2 \rangle = \beta\psi^*$, we require $\langle b_1 \rangle = \gamma\psi$. We will consider only the case $|\gamma| \geq |\alpha|$, since, otherwise, the problem becomes trivial: one would just have to attenuate the input coherent state $|\alpha\psi\rangle$ with an unbalanced beam splitter, yielding a coherent state of amplitude $\gamma\psi$. To simplify the problem, we may assume that $\beta = 1$, which amounts to substitute ψ for $\beta\psi$. Then, we have

$$\begin{aligned} \alpha M_{11} + L_{12} &= \gamma , \\ M_{12} + \alpha L_{11} &= 0 . \end{aligned} \tag{2}$$

Second, this transformation must obey the commutation rules $[b_i, b_k] = 0$ and $[b_i, b_k^\dagger] = \delta_{ik}$ ($\hbar = 1$), that is,

$$\begin{aligned} M_{ij} L_{kj} - L_{ij} M_{kj} &= 0 , \\ M_{ij} M_{kj}^* - L_{ij} L_{kj}^* &= \delta_{ik} . \end{aligned} \tag{3}$$

Third, the noise of the output mode b_1 of this transformation should be phase insensitive and minimum.

Before sketching our calculation, let us note that a further simplification comes from the fact that the annihilation operators are defined up to an arbitrary phase, so that a transformation $a_i \to e^{i\mu_i} a_i$ and $b_1 \to e^{i\nu} b_1$ allows us to take M_{1j} and L_{1j} as real and positive. Since we focus on a phase-insensitive transformation, minimizing the noise amounts to minimizing the sole quantity $(\Delta b_1)^2 = \frac{1}{2}\langle b_1 b_1^\dagger + b_1^\dagger b_1\rangle - \langle b_1\rangle\langle b_1^\dagger\rangle$ [12]. Thus, using the fact that $(\Delta a_i)^2 = 1/2$ for a mode a_i in a coherent state, we need to minimize

$$(\Delta b_1)^2 = \tfrac{1}{2}(M_{1j}M_{1j} + L_{1j}L_{1j}), \tag{4}$$

under the constraints Eqs. (2) and (3). Rather than solving this full problem, we use here a common trick in constrained extremization problems that consists in solving a simpler problem with weaker constraints (bearing in mind that taking weaker constraints can only yield better solutions) and then checking that the solution of this simpler problem is one of the full problem. Specifically, we minimize $(\Delta b_1)^2$, taking into account the only condition $M_{1j}M_{1j} - L_{1j}L_{1j} = 1$. Taking Eq. (2) into account and introducing a Lagrange multiplier λ, we minimize the quantity $M_{11}^2 + (\gamma - \alpha M_{11})^2 + (1 + \alpha^2)L_{11}^2 + M_{13}^2 + L_{13}^2 + \lambda[M_{11}^2 - (\gamma - \alpha M_{11})^2 - (1 - \alpha^2)L_{11}^2 + M_{13}^2 - L_{13}^2 - 1]$, with respect to M_{11}, L_{11}, M_{13}, and L_{13}. Some algebra shows that this problem admits only one solution $M_{13} = L_{13} = L_{11} = M_{12} = 0$; that is, the auxiliary mode is unnecessary. The optimal transformation has then the same structure as that of a phase-insensitive amplifier of gain G. Restoring β, we get

$$\begin{aligned} b_1 &= \sqrt{G}\, a_1 + \sqrt{G-1}\, a_2^\dagger, \\ b_2 &= \sqrt{G-1}\, a_1^\dagger + \sqrt{G}\, a_2, \end{aligned} \tag{5}$$

with

$$\sqrt{G} = \frac{-\alpha\gamma + \beta\sqrt{\gamma^2 - \alpha^2 + \beta^2}}{\beta^2 - \alpha^2}, \tag{6}$$

It can easily be checked that, for $\beta = 0$ (or $\alpha = 0$), Eq. (5) reduces to a phase-insensitive phase-preserving (or phase-conjugating) amplifier as defined in [12], and can be used to carry out the $N \to M$ cloning (or phase-conjugating) transformation described in [5] (or [8]).

Let us now turn to the special case where α^2, β^2, and γ^2 are integers (which we will denote, respectively, as N, N', and M). The transformation Eq. (5) can be used as the central element of a PCI cloning machine, which is covariant for translations and rotations in phase space (see Fig. 1). Indeed, the following procedure can be used to produce M optimal clones of a coherent state $|\psi\rangle$ from $|\psi\rangle^{\otimes N}|\psi^*\rangle^{\otimes N'}$.

(i) Concentrate the N replicas of $|\psi\rangle$ stored in the N modes $\{c_l\}$ $(l = 0, \dots, N-1)$ into a single mode a_1; this results in a coherent state of amplitude $\sqrt{N}\,\psi$. This operation can be performed with a network of beam splitters achieving a N-mode discrete Fourier transform (DFT) [5], which yields the mode

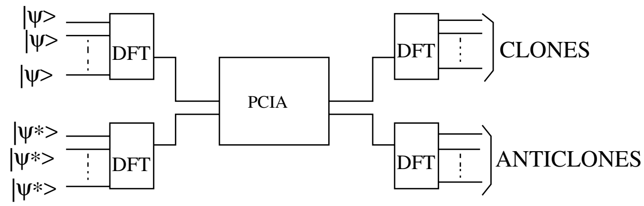

FIG. 1. PCI cloner that produces M clones and M' anticlones from N replicas of $|\psi\rangle$ and N' replicas of $|\psi^*\rangle$. Modes are concentrated and distributed by discrete Fourier transform (DFT). PCIA stands for a phase-conjugate input amplifier.

$$a_1 = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} c_l, \tag{7}$$

and $N - 1$ vacuum modes. Similarly, concentrate the N' replicas of $|\psi^*\rangle$ stored in the N' modes $\{d_l\}$ $(l = 0, \dots, N'-1)$ into a single mode a_2 in a coherent state of amplitude $\sqrt{N'}\,\psi^*$ with the help of a N'-mode DFT:

$$a_2 = \frac{1}{\sqrt{N'}} \sum_{l=0}^{N'-1} d_l. \tag{8}$$

(ii) Process the modes a_1 and a_2 into a "phase-conjugate input" amplifier (PCIA), resulting in modes b_1 and b_2 as defined in Eqs. (5) and (6).

(iii) Distribute the output b_1 into M clones $\{c'_l\}$ $(l = 0, \dots, M-1)$ with a M-mode DFT:

$$c'_l = \frac{1}{\sqrt{M}}(b_1 + e^{i2\pi kl/M} v_k), \tag{9}$$

where $\{v_k\}$ $(k = 1, \dots, M-1)$ denote $M - 1$ vacuum modes. It is readily verified that this procedure yields M clones of $|\psi\rangle$.

Interestingly, the amplitude b_2 of the other output of the PCIA has a mean value $\sqrt{M'}\,\psi^*$, with

$$N - N' = M - M'. \tag{10}$$

Therefore, it can be used to produce M' phase-conjugate clones (or anticlones) of $|\psi\rangle$, $\{d'_l\}$ $(l = 0, \dots, M'-1)$, using a M'-mode DFT:

$$d'_l = \frac{1}{\sqrt{M'}}(b_2 + e^{i2\pi kl/M} w_k), \tag{11}$$

where $\{w_k\}$ $(k = 1, \dots, M'-1)$ denote $M' - 1$ vacuum modes. Clearly, this procedure is optimal to produce M clones since its central element, the PCIA, is optimal, and the beam splitters are passive elements. In addition, the M' anticlones that are produced at no cost are also optimal. Indeed, our transformation produces M optimal clones with $M \geq N$, and is symmetric with respect to the interchange of labels 1 and 2. So, if our initial problem was to produce M' optimal anticlones with $M' \geq N'$, we would find the same solution. Noting that $M \geq N \Longleftrightarrow M' \geq N'$ [see Eq. (10)], it therefore is clear that our transformation yields both optimal clones and anticlones. Furthermore, since the PCIA is linear and phase-insensitive, the resulting PCI cloner is covariant with respect to translations and rotations of the state to be copied: all coherent states are copied equally well, and the cloning-induced noise is the same for all quadrature components.

Using Eqs. (5)–(9) and (11), the noise of the clones and anticlones can be written as

$$(\Delta c_l')^2 = \frac{1}{2} + \frac{G-1}{M}, \qquad (\Delta d_l')^2 = \frac{1}{2} + \frac{G-1}{M'}, \tag{12}$$

where the gain can be reexpressed as a function of the number of inputs and outputs,

$$\sqrt{G} = \frac{\sqrt{N'M'} - \sqrt{NM}}{N' - N}. \tag{13}$$

As expected, the variance of the output clones exceeds $1/2$, implying that the clones are not exactly in the coherent state $|\psi\rangle$. Instead, they suffer from a thermal noise with a mean number of photons given by $\langle n_{th}\rangle = (G-1)/M$. In other words, their P function [13] is a Gaussian distribution

$$P(\xi, \xi^*) = \frac{1}{\pi\langle n_{th}\rangle} e^{-|\xi-\psi|^2/\langle n_{th}\rangle}, \tag{14}$$

rather than a Dirac distribution $P(\xi,\xi^*) = \delta^{(2)}(\xi - \psi)$.

Consider now the balanced case ($N = N'$, $M = M'$), for which simple analytical expressions of the noise variances can be obtained. Taking the limit $\alpha \to \beta$ in Eq. (6) and replacing α^2 by N and γ^2 by M yields $G = (M+N)^2/4MN$, so that the error variances of the clones and anticlones are

$$(\Delta c_l')^2 = (\Delta d_l')^2 = \frac{1}{2} + \frac{(M-N)^2}{4M^2N}. \tag{15}$$

Note that this balanced cloner is optimal among all PCI cloners in the sense that it minimizes $\langle n_{th}\rangle$ for fixed $N + N'$ and $M + M'$. It is convenient to characterize the quality of cloning in terms of the fidelity $f_{\binom{N}{N}\to M} = \langle\psi|\rho_c|\psi\rangle/|\langle\psi\,|\,\psi\rangle|^2$ where ρ_c denotes the state of the clones. Using Eq. (14), we get

$$f_{\binom{N}{N}\to M} = \frac{1}{1+\langle n_{th}\rangle} = \frac{4M^2N}{4M^2N + (M-N)^2}. \tag{16}$$

Let us now compare the production of M clones from N replicas and N antireplicas to the production of M clones from $2N$ identical replicas. The variance and fidelity of the clones k_i obtained by standard cloning are given by [14]

$$(\Delta k_i')^2 = \frac{1}{2} + \left(\frac{1}{2N} - \frac{1}{M}\right), \tag{17}$$

and

$$f_{2N\to M} = \frac{2MN}{2MN + M - 2N}. \tag{18}$$

Of course, in the trivial case where $M = 2N$, standard cloning can be achieved perfectly, while the balanced PCI cloner yields an additional variance $\langle n_{th}\rangle = 1/(16N)$. However, if M is sufficiently large, the $\binom{N}{N} \to M$ balanced cloner yields a lower variance (hence a higher fidelity) than the $2N \to M$ cloning machine. The balanced PCI cloner is also better for the anticlones: more anticlones are produced at no cost, and they have a better fidelity. Indeed, a standard $2N \to M$ cloning machine produces $M - 2N$ anticlones of fidelity $2N/(2N+1)$, which actually is the fidelity of an optimal measurement of $2N$ replicas of $|\psi\rangle$. In contrast, a PCI cloner produces M anticlones with a higher fidelity, as given by Eq. (16). In particular, for $M \to \infty$, we see from Eqs. (15) and (17) that the additional noise induced by a PCI cloner is $1/4N$, that is, *one-half* of the noise induced by a standard $2N \to \infty$ cloner $(1/2N)$. Note that in this case, the output of the PCIA can be considered as classical and the underlying process appears to be equivalent to a measurement. This reflects that *more classical information can be encoded in N pairs of phase-conjugate replicas of a coherent state than in 2N identical replicas,* a result which was proven for $N = 1$ in [8]. More generally, in the unbalanced case ($N \neq N'$), it can be shown that the optimal measurement results in a noise that is equal to that obtained by measuring $(\sqrt{N} + \sqrt{N'})^2$ identical replicas of the input, in the absence of phase-conjugate inputs.

We have shown that the balanced PCI cloner results in better cloning fidelities than a standard cloner. More generally, we may ask the following question: *If we want to produce M clones of a coherent state $|\psi\rangle$ from a fixed total number n of input modes, N of which being in the coherent state $|\psi\rangle$ and N' of which being in the phase-conjugate state $|\psi^*\rangle$, what is the phase-conjugate fraction $a = N'/n$ that minimizes the error variances of the clones?*

From Eq. (5), we see that for fixed values of the total number of inputs n and number of outputs M, the gain G (and thus the thermal noise of the clones $\langle n_{th}\rangle$) depends only on a, and varies as

$$G(a) = \left(\frac{\sqrt{a}\sqrt{\frac{M}{n} + (2a-1)} - \sqrt{\frac{M}{n}}\sqrt{1-a}}{2a-1}\right)^2. \tag{19}$$

In Fig. 2, we have plotted $\sqrt{\langle n_{th}\rangle}$ as a function of a for $n = 8$ and different values of $M \geq n$. (Of course, only rational values of a are relevant here, but the whole curve has been plotted for simplicity.) In the trivial case where $M = n = 8$, the minimum additional variance is, of course, zero, and is obtained for $a = 0$. The cloning transformation is then just the identity. However, when $M \geq n + 1$, using phase-conjugate input modes yields lower variances than in standard cloning if a is correctly chosen (the lowest variance is attained for $a \neq 0$). Remarkably, the value of a achieving the minimum variance is not equal to $1/2$ for finite M; that is, the optimal input partition contains more replicas than antireplicas. In the limit of large M, however,

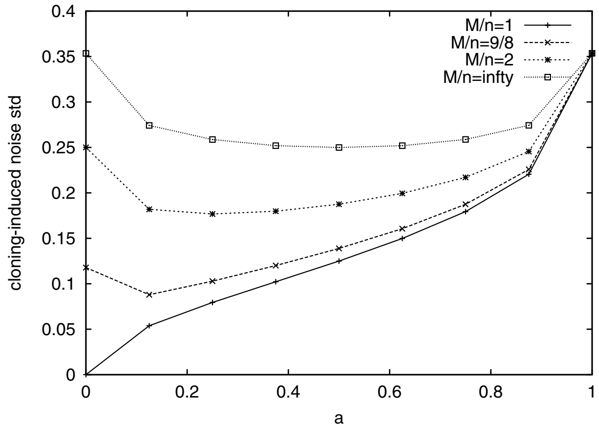

FIG. 2. Cloning-induced noise standard deviation $\sqrt{\langle n_{th}\rangle}$ as a function of the phase-conjugate fraction $a = N'/n$, for $n = 8$ and several values of M/n.

the number of antireplicas achieving the lowest variances tends to $n/2$, and the curve $G(a)$ tends to a symmetric curve around $a = 1/2$. This symmetry is not surprising, since $M = \infty$ corresponds to a measurement [4,14], and we expect that measuring the value of ψ from N replicas of $|\psi\rangle$ and N' replicas of $|\psi^*\rangle$ is equivalent to starting from N' replicas of $|\psi\rangle$ and N replicas of $|\psi^*\rangle$. So, we have found that the optimal measurement is achieved with balanced inputs ($N = N'$). Finally, in the case where $a = 1$, the transformation consists in producing M clones of $|\psi\rangle$ from n replicas of $|\psi^*\rangle$. This is just phase conjugation, for which we know that the best strategy is to perform a measurement [8]. The additional variance is therefore given by $1/n$, which does not depend on M. This explains why the curves all converge to the same point at $a = 1$.

In summary, we have derived a continuous-variable cloning transformation using phase-conjugate inputs. This transformation has been shown to be decomposable in a sequence of beam splitters, a central amplification stage, and another sequence of beam splitters. A possible way to implement this central stage would be to use four-wave mixing. Two weak fields entering the $\chi^{(3)}$ medium would then play the role of the phase-conjugate inputs, and energy would be brought to the system by two external modes in a large coherent state (see [13] for details). We have shown evidence that PCI cloning transformations outperform standard cloning transformations (taking only identical inputs) if the goal is to produce clones and anticlones of a state or to get knowledge about a state through measurement. The special case of the balanced cloner, which produces M pairs of phase-conjugate clones from N pairs of phase-conjugate replicas, has been analyzed and was shown to be optimal. As far as we know, no qubit analog of our transformation has been proposed yet, though it is very plausible that a cloning machine that produces M clones from N qubits in an arbitrary state and N' qubits in its orthogonal state can be defined. Another possible extension of this work would be to study the case where the number of anticlones is a free parameter (in the PCI cloner derived here, it is constrained by N, N', and M). Also, an interesting generalization would be to investigate fully asymmetric PCI cloning transformations, that is, transformations whose output clones have all different fidelities. Finally, our work raises the question of why a better cloning can be achieved when phase-conjugate inputs are available rather than identical inputs. It is known that, in standard cloning, the spontaneous emission occurring during the amplification stage is the physical mechanism that hinders perfect cloning. Thus, an open question left by our work is to understand why things happen as if comparatively less spontaneous emission occurred in a PCI cloning machine.

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