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# Cloning a qutrit

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**Abstract.** Several classes of state-dependent quantum cloners for three-level systems are investigated. These cloners optimally duplicate some of the four maximally-conjugate bases with an equal fidelity, thereby extending the phase-covariant qubit cloner to qutrits. Three distinct classes of qutrit cloners can be distinguished, depending on whether two, three, or four maximally-conjugate bases are cloned as well (the latter case simply corresponds to the universal qutrit cloner). These results apply to symmetric as well as asymmetric cloners, so that the balance between the fidelity of the two clones can also be analysed.

## 1. Introduction

Since its inception, quantum information theory has traditionally been concerned with informational processes involving two-level quantum systems, known as qubits. For example, quantum teleportation, quantum cryptography, quantum computation, or quantum cloning were all developed using qubits as fundamental units of quantum information [1]. Over the last few years, however, there has been a growing interest in quantum informational processes based on multi-level or even continuous-spectrum systems. There are several reasons for this. First, higher-dimensional quantum informational processes seem to be more efficient in certain situations. For example, multi-level quantum cryptographic schemes can be shown to be more secure against eavesdropping than their qubit-based counterparts [2]. Second, the present experimental context makes it reasonable to consider the manipulation of more-than-two-level quantum information carriers. For example, the time-bin implementation of qubits can be relatively straightforwardly extended to three or more time bins [3]. Quantum computation over

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continuous variables also seems to be a promising avenue, as it can be carried out by manipulating squeezed states of light with only linear optics elements [4].

In this paper, the concept of quantum cloning is extended from qubits to qutrits (quantum three-level systems). In spite of this apparently simple incremental step of just one more dimension, the question of cloning turns out to be already significantly more complex for qutrits, suggesting that the state-dependent cloning of multi-level systems is a rich field. Quantum cloning is a concept that was first introduced in a seminal paper by Buzek and Hillery [5], where a universal (or state-independent) and symmetric $1 \rightarrow 2$ cloning transformation was introduced for qubits. This transformation was later extended to higher dimensions by Buzek and Hillery [6] and by Werner [7], but for the special case of universal (state-independent) cloning. In contrast, we will rather focus here on non-universal (or state-dependent) cloning. Our starting point will be a general characterization of asymmetric and state-dependent $1 \rightarrow 2$ cloning transformations for $N$-level systems, as described in [8, 9]. Let us first review this formalism before analysing the special case of $N = 3$ in details.

Consider an arbitrary state $|\psi\rangle$ in a $N$-dimensional Hilbert space of which we wish to produce two (approximate) clones. The class of cloning transformations to be analysed is such that, if the input state is $|\psi\rangle$, then the two output clones (called $A$ and $B$) are produced in a mixture of the states $|\psi_{m,n}\rangle = U_{m,n}|\psi\rangle$:

$$\rho_A = \sum_{m,n=0}^{N-1} p_{m,n} |\psi_{m,n}\rangle\langle\psi_{m,n}|$$

$$\rho_B = \sum_{m,n=0}^{N-1} q_{m,n} |\psi_{m,n}\rangle\langle\psi_{m,n}| \tag{1}$$

where the unitary operators

$$U_{m,n} = \sum_{k=0}^{N-1} \exp\left[2\pi \mathrm{i}(kn/N)\right] |k + m \bmod N\rangle\langle k| \tag{2}$$

correspond to *error* operators: $U_{m,n}$ shifts the state by $m$ units (modulo $N$) in the computational basis, and multiplies it by a phase so as to shift its Fourier transform by $n$ units (modulo $N$). Of course, $U_{0,0} = I$, which corresponds to no error. In the special case of a qubit ($N = 2$), we have $U_{1,0} = \sigma_x$, $U_{0,1} = \sigma_z$, and $U_{1,1} = -\mathrm{i}\sigma_y$, and the corresponding class of so-called Pauli cloners can be investigated exhaustively [10, 11].

From equation (1), it is clear that the clones $A$ and $B$ are characterized in general by the weight functions $p_{m,n}$ and $q_{m,n}$, respectively. As will be seen below, the class of cloners we will restrict attention to are defined by a particular relation between these weight functions [8, 9]. More specifically, the focus will be on cloners satisfying

$$p_{m,n} = |a_{m,n}|^2$$

$$q_{m,n} = |b_{m,n}|^2 \tag{3}$$

where $a_{m,n}$ and $b_{m,n}$ are two (complex) amplitude functions that are dual under a Fourier transform:

$$b_{m,n} = \frac{1}{N}\sum_{x,y=0}^{N-1} \exp\left[2\pi \mathrm{i}(nx - my)\right]/N a_{x,y}. \tag{4}$$

Of course, these amplitudes are normalized: $\sum_{m,n}|a_{m,n}|^2 = \sum_{m,n}|b_{m,n}|^2 = 1$. Interestingly enough, the cloners obeying equations (3) and (4) form a fairly general class, which contains most cloners discovered so far.

The Fourier transform that underlies the relation between the $p_{m,n}$ and the $q_{m,n}$ is responsible for the *complementarity* between the quality of the two clones: if clone $A$ is very good ($a_{m,n}$ is a 'peaked' function), then clone $B$ is very bad ($b_{m,n}$ is a rather 'flat' function, so that many error operators $U_{m,n}$ act on $|\psi\rangle$ with significant probabilities). Note that the relation between these dual amplitude functions $a_{m,n}$ and $b_{m,n}$ can be re-expressed in a simple way by associating them with Fourier transformed amplitude functions $a^F_{m,n}$ and $b^F_{m,n}$ defined as

$$c^F_{m,n} = \frac{1}{\sqrt{N}}\sum_{n'=0}^{N} \exp\left[-2\pi \mathrm{i}(nn'/N)\right]c_{m,n'} \tag{5}$$

or, conversely,

$$c_{m,n} = \frac{1}{\sqrt{N}}\sum_{n'=0}^{N} \exp\left[2\pi \mathrm{i}(nn'/N)\right]c^F_{m,n'} \tag{6}$$

where $c_{m,n}$ stands for $a_{m,n}$ or $b_{m,n}$. We have

$$\begin{aligned} b^F_{m,n} &= \frac{1}{\sqrt{N}}\sum_{n'=0}^{N-1} \exp\left[-2\pi \mathrm{i}(nn'/N)\right]b_{m,n'} \\ &= \frac{1}{\sqrt{N}}\sum_{n'=0}^{N-1} \exp\left[-2\pi \mathrm{i}(nn'/N)\right]\frac{1}{N}\sum_{x,y=0}^{N-1} \exp\left[2\pi i(n'x - my)\right]/N a_{x,y} \\ &= \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} \exp\left[-2\pi \mathrm{i}(my/N)\right]a_{n,y} \\ &= a^F_{n,m}. \end{aligned} \tag{7}$$

Therefore, the Fourier transformed amplitudes $a^F_{m,n}$ and $b^F_{m,n}$ of the two clones are simply *transposes* of each other, which will be helpful in the following. The balance between the quality of clones $A$ and $B$ can be alternatively expressed by an entropic no-cloning uncertainty relation that relates the probability distributions $p_{m,n}$ and $q_{m,n}$ [9]:

$$H[p_{m,n}] + H[q_{m,n}] \geqslant \log_2(N^2) \tag{8}$$

where $H[p]$ denote the Shannon entropy of the probability distribution $p$. This inequality is actually a special case of an information-theoretic no-cloning uncertainty relation involving the *losses* of the channels yielding the two clones [11]. Also, more refined uncertainty relations can be found that express the fact that the index $m$ of output $A$ is dual to the index $n$ of output $B$, and conversely [9].

Let us now describe the class of cloning transformations that actually produce the clones characterized by equation (1). For this, we need first to define the set of $N^2$ generalized Bell states for a pair of N-dimensional systems:

$$|B_{m,n}\rangle = N^{-1/2}\sum_{k=0}^{N-1}\exp\,[2\pi i(kn/N)]|k\rangle|k+m\rangle \tag{9}$$

with $m$ and $n$ ($0 \leqslant m, n \leqslant N-1$) labelling these Bell states. We characterize the cloning transformation by assuming that the cloner input is prepared in the joint state $|B_{0,0}\rangle$ together with a ($N$-dimensional) system called a *reference* system and denoted $R$. We consider a unitary cloning transformation $\mathcal{U}_{\rm cl}$ acting on this input system together with two additional $N$-dimensional systems prepared each in an initial state $|0\rangle$: a blank copy and the cloning machine itself. After transformation, the input system and the blank copy become respectively the clones $A$ and $B$, while the cloning machine denoted as $C$ can be traced over. We thus are interested in the joint state (after cloning) of the reference $R$, the two clones ($A$ and $B$), and the cloning machine $C$, that is

$$(I_R \otimes \mathcal{U}_{\rm cl}) = |B_{0,0}\rangle|0\rangle|0\rangle = |\Psi\rangle_{RABC}. \tag{10}$$

More specifically, we consider only joint states which can be written as

$$|\Psi\rangle_{RABC} = \sum_{m,n=0}^{N-1} a_{m,n}|B_{m,n}\rangle_{R,A}|B_{m,-n}\rangle_{B,C} = \sum_{m,n=0}^{N-1} b_{m,n}|B_{m,n}\rangle_{R,B}|B_{m,-n}\rangle_{A,C} \tag{11}$$

with $a_{m,n}$ and $b_{m,n}$ being related by equation (4). This construction is very useful because one can easily express the output state resulting from cloning an arbitrary input state $|\psi\rangle$ simply by projecting the reference system onto an appropriate state. Indeed, before cloning, projecting $R$ onto state $|\psi^*\rangle$ amounts to project the input system onto $|\psi\rangle$ since these two systems are in state $|B_{0,0}\rangle$. Therefore, as this projection of $R$ onto $|\psi^*\rangle$ can as well be performed *after* cloning, it is easy to write the resulting joint state of the two clones and the cloning machine when the input state is $|\psi\rangle$. Using $|B_{m,n}\rangle = (I \otimes U_{m,n})|B_{0,0}\rangle$, we get

$$|\psi\rangle \rightarrow \sum_{m,n=0}^{N-1} a_{m,n}\, U_{m,n}|\psi\rangle_A|B_{m,-n}\rangle_{B,C} = \sum_{m,n=0}^{N-1} b_{m,n} U_{m,n}|\psi\rangle_B|B_{m,-n}\rangle_{A,C}. \tag{12}$$

Now, it is easy to check that tracing over systems $B$ and $C$ (or $A$ and $C$) yields the expected final states of clone $A$ (or clone $B$), in accordance with (1). Thus, the $N^2$ amplitudes $a_{m,n}$ (or $b_{m,n}$) completely define the state after cloning, equation (12), so they completely characterize the class of cloning transformations of interest here.

Finally, let us see how the cloning fidelity can be calculated based on these amplitude matrices $a_{m,n}$ or $b_{m,n}$. The fidelity of the first clone when copying a state $|\psi\rangle$ can be written, in general, as

$$F_A = \langle\psi|\rho_A|\psi\rangle = \sum_{m,n=0}^{N-1} |a_{m,n}|^2|\langle\psi|\psi_{m,n}\rangle|^2. \tag{13}$$

(Of course, the same relation can be used for the second clone by replacing $a_{m,n}$ by $b_{m,n}$.) For example, for any state $|k\rangle$ ($k = 0, \ldots N-1$) in the computation basis, the fidelity of the first clone is equal to

$$F_A = \sum_{n=0}^{N-1} |a_{0,n}|^2. \tag{14}$$

As we will see later on for $N = 3$, the cloning fidelity for other bases can also be written as a sum of three squared terms of the $a_{m,n}$ matrix. This will make it possible to express constraints on the state-dependent cloners of interest.

In the rest of this paper, we use this general characterization of cloning in order to investigate the state-dependent cloning of a qutrit. Four maximally-conjugate bases can be defined in a three-dimensional space: these bases are such that any basis state in one basis has equal squared amplitudes when expressed in any other basis. We analyse transformations that optimally clone a subset of these four maximally-conjugate bases for a qutrit. Three interesting situations occur depending on whether we consider a subset of two or three of these bases, or all four bases. The case of a qubit is also treated in the Appendix for completeness.†

## 2. Cloning a three-level system

In a three-dimensional Hilbert space, one can define four maximally-conjugate (or mutually unbiased) bases [12]. Conventionally, one chooses the first basis to be simply the computation basis $\{|0\rangle, |1\rangle, |2\rangle\}$. The second basis is defined as

$$\begin{aligned} |0'\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle), \\ |1'\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + \gamma|1\rangle + \gamma^2|2\rangle), \\ |2'\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + \gamma^2|1\rangle + \gamma|2\rangle), \end{aligned} \tag{15}$$

where $\gamma = \exp(2\pi i/3)$. Similarly, the third basis is defined as

$$\begin{aligned} |0''\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + \gamma|2\rangle), \\ |1''\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + \gamma|1\rangle + |2\rangle), \\ |2''\rangle &= \frac{1}{\sqrt{3}}(|0\rangle\gamma + |1\rangle + |2\rangle), \end{aligned} \tag{16}$$

while the fourth basis is defined as

$$\begin{aligned} |0'''\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + \gamma^2|2\rangle), \\ |1'''\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + \gamma^2|1\rangle + |2\rangle), \\ |2'''\rangle &= \frac{1}{\sqrt{3}}(\gamma^2|0\rangle + |1\rangle + |2\rangle). \end{aligned} \tag{17}$$

† The reader may want to read this appendix first as an introduction to the concept of state-dependent cloning. The case of a qubit is indeed much simpler to treat than that of a qutrit.

It is easy to check that the scalar product between any two basis states belonging to two distinct bases is $1/\sqrt{3}$, as expected. We can also check that the first and second bases are connected by a discrete Fourier transform. The same relation holds for the third and fourth bases.

Let us start by calculating the action of the nine error operators $U_{m,n}$ on these basis states. Within each of these four bases, it can be shown that applying $U_{m,n}$ to one basis state yields either $|k\rangle$ or $|k+1 \bmod 3\rangle$ or $|k+2 \bmod 3\rangle$ up to a phase $\gamma$ or $\gamma^2$. Using this property, we can express the fidelity of the first (or second) clone in each basis. The fidelity of one of the clones when copying a state $|\psi\rangle$ is defined as

$$F = \langle\psi|\rho|\psi\rangle \tag{18}$$

where $\rho$ is defined in equation (1). Note that, unlike the situation for a qubit, there are *two* possible errors when copying the basis state $|k\rangle$ (in a given basis) for a qutrit depending on it being transformed into $|k+1 \bmod 3\rangle$ or $|k+2 \bmod 3\rangle$. Therefore, we define *two* disturbances $D_1$ and $D_2$ corresponding to these two errors. Remembering that the state of the first clone is completely characterized by the matrix

$$(p_{m,n}) = \begin{pmatrix} p_{0,0} & p_{0,1} & p_{0,2} \\ p_{1,0} & p_{1,1} & p_{1,2} \\ p_{2,0} & p_{2,1} & p_{2,2} \end{pmatrix} \tag{19}$$

we can calculate the fidelity and the two disturbances when cloning any basis state in any basis. For example, for the first maximally-conjugate basis, we have

$$F = p_{0,0} + p_{0,1} + p_{0,2}, \tag{20}$$

$$D_1 = p_{1,0} + p_{1,1} + p_{1,2}, \tag{21}$$

$$D_2 = p_{2,0} + p_{2,1} + p_{2,2}. \tag{22}$$

The cloning of the three last maximally-conjugate bases can be treated together by considering the state

$$|\psi_0\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \exp(\mathrm{i}\alpha)|1\rangle + \exp(\mathrm{i}\beta)|2\rangle) \tag{23}$$

with arbitrary $\alpha$ and $\beta$. By direct computation, we get the fidelity

$$\begin{aligned} F = \langle\psi_0|\rho|\psi_0\rangle &= p_{0,0} + \tfrac{1}{3}(p_{1,0} + p_{2,0} + p_{1,2} + p_{2,1} + p_{1,1} + p_{2,2}) \\ &+ \tfrac{2}{9}(p_{1,0} + p_{2,0})[\cos(\alpha+\beta) + \cos(\alpha-2\beta) + \cos(\beta-2\alpha)] \\ &+ \tfrac{2}{9}(p_{1,2} + p_{2,1})[\cos(\alpha+\beta+2\pi/3) + \cos(\alpha-2\beta+2\pi/3) \\ &+ \cos(\beta-2\alpha+2\pi/3)] \\ &+ \tfrac{2}{9}(p_{1,1} + p_{2,2})[\cos(\alpha+\beta-2\pi/3) + \cos(\alpha-2\beta-2\pi/3) \\ &+ \cos(\beta-2\alpha-2\pi/3)]. \end{aligned} \tag{24}$$

Before calculating the disturbances, we need first to define the states

$$|\psi_1\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \gamma\ \exp(\mathrm{i}\alpha)|1\rangle + \gamma^2\ \exp(\mathrm{i}\beta)|2\rangle),$$
$$|\psi_2\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \gamma^2 \exp(\mathrm{i}\alpha)|1\rangle + \gamma\ \exp(\mathrm{i}\beta)|2\rangle), \tag{25}$$

which, together with $|\psi_0\rangle$, form an orthonormal basis. We can easily rewrite the second, third, and fourth maximally-conjugate bases in the form $\{|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle\}$ for well-chosen values of $\alpha$ and $\beta$. This is of course not true for the first (computational) basis. We can now calculate the disturbances,

$$\begin{aligned} D_1 = \langle\psi_1|\rho|\psi_1\rangle &= p_{0,1} + \tfrac{1}{3}(p_{1,1} + p_{2,1} + p_{1,0} + p_{2,2} + p_{1,2} + p_{2,0}) \\ &+ \tfrac{2}{9}(p_{1,1} + p_{2,1})[\cos(\alpha+\beta) + \cos(\alpha - 2\beta) + \cos(\beta - 2\alpha)] \\ &+ \tfrac{2}{9}(p_{1,0} + p_{2,2})[\cos(\alpha+\beta+2\pi/3) + \cos(\alpha - 2\beta + 2\pi/3) \\ &+ \cos(\beta - 2\alpha + 2\pi/3)] \\ &+ \tfrac{2}{9}(p_{1,2} + p_{2,0})[\cos(\alpha+\beta-2\pi/3) + \cos(\alpha - 2\beta - 2\pi/3) \\ &+ \cos(\beta - 2\alpha - 2\pi/3)] \end{aligned} \tag{26}$$

and

$$\begin{aligned} D_2 = \langle\psi_2|\rho|\psi_2\rangle &= p_{0,2} + \tfrac{1}{3}(p_{1,2} + p_{2,2} + p_{1,1} + p_{2,0} + p_{1,0} + p_{2,1}) \\ &+ \tfrac{2}{9}(p_{1,2} + p_{2,2})[\cos(\alpha+\beta) + \cos(\alpha - 2\beta) + \cos(\beta - 2\alpha)] \\ &+ \tfrac{2}{9}(p_{1,1} + p_{2,0})[\cos(\alpha+\beta+2\pi/3) + \cos(\alpha - 2\beta + 2\pi/3) \\ &+ \cos(\beta - 2\alpha + 2\pi/3)] \\ &+ \tfrac{2}{9}(p_{1,0} + p_{2,1})[\cos(\alpha+\beta-2\pi/3) + \cos(\alpha - 2\beta - 2\pi/3) \\ &+ \cos(\beta - 2\alpha - 2\pi/3)]. \end{aligned} \tag{27}$$

It can be easily checked that $F$, $D_1$, and $D_2$ are invariant if we replace $(\alpha, \beta)$ by $(\alpha \pm 2\pi/3, \beta \pm 4\pi/3)$, and that these phase shifts simply permute cyclically the states $|\psi_0\rangle$, $|\psi_1\rangle$, and $|\psi_2\rangle$. Therefore, the values of $F$, $D_1$ and $D_2$ are invariant under a cyclic permutation of the states of the maximally-conjugate bases. Note that all the states of the second maximally-conjugate basis fulfil $\alpha + \beta = \alpha - 2\beta = \beta - 2\alpha = 0$. Similarly, in the third and fourth maximally-conjugate bases, we have $\alpha + \beta = \alpha - 2\beta = \beta - 2\alpha = 2\pi/3$ and $-2\pi/3$, respectively. For those states, equations (24), (26) and (27) can be simplified. For instance, the fidelity and disturbances when cloning any basis state of the second maximally-conjugate basis are given by

$$\begin{aligned} F' &= p_{0,0} + p_{1,0} + p_{2,0,} \\ D'_1 &= p_{0,1} + p_{1,1} + p_{2,1}, \\ D'_2 &= p_{0,2} + p_{1,2} + p_{2,2}. \end{aligned} \tag{28}$$

For the third basis, we have

$$F'' = p_{0,0} + p_{1,1} + p_{2,2},$$
$$D_1'' = p_{0,1} + p_{1,2} + p_{2,0},$$
$$D_2'' = p_{0,2} + p_{1,0} + p_{2,1}, \tag{29}$$

while the fourth basis yields

$$F''' = p_{0,0} + p_{1,2} = p_{2,1},$$
$$D_1''' = p_{0,1} + p_{1,0} + p_{2,2},$$
$$D_2''' = p_{0,2} + p_{1,1} + p_{2,0}. \tag{30}$$

In the following, we will be interested in extending to a three-dimensional space the so-called phase-covariant qubit cloner described in the Appendix. Two extensions can be considered, depending on whether two or three of the maximally-conjugate bases are copied equally well. The cloner that copies all four bases with an equal fidelity is simply the universal cloner, as discussed in the final section.

## 3. Optimal cloner of two maximally-conjugate bases

Here, we consider a state-dependent cloner that clones equally well the third and fourth maximally-conjugate bases. This imposes the conditions

$$p_{1,1} + p_{2,2} = p_{1,2} + p_{2,1},$$
$$p_{1,2} + p_{2,0} = p_{1,0} + p_{2,2},$$
$$p_{1,0} + p_{2,1} = p_{1,1} + p_{2,0}. \tag{31}$$

It is easy to deduce from these constraints together with equation (24) that the cloning fidelity for an arbitrary state $|\psi_0\rangle$ is given by

$$F = p_{0,0} + \tfrac{1}{3}(p_{1,0} + p_{2,0} + 2(p_{1,2} + p_{2,1})) + \tfrac{2}{9}[p_{1,0} + p_{2,0} - (p_{1,2} + p_{2,1})][\cos(\alpha+\beta) + \cos(\alpha - 2\beta) + \cos(\beta - 2\alpha)]. \tag{32}$$

The function $\cos(\alpha+\beta) + \cos(\alpha - 2\beta) + \cos(\beta - 2\alpha)$ reaches its extremal value $-3/2$ when $\alpha + \beta = \alpha - 2\beta = \beta - 2\alpha = 2\pi/3$ or $-2\pi/3$, that is, when $|\psi_0\rangle$ belongs to the third or fourth basis. Therefore, when the second basis is not cloned as well as the third and fourth maximally-conjugate bases, i.e. when $p_{1,0} + p_{2,0} < p_{1,1} + p_{2,2}$, there exists no state of the form $\frac{1}{\sqrt{3}}(|0\rangle + \exp(i\alpha)|1\rangle + \exp(i\beta)|2\rangle)$ outside the third and fourth bases that is equally well cloned. Similarly, we expect that when the first and second bases are not cloned as well as the third and fourth maximally-conjugate bases, i.e. when $p_{0,1} + p_{0,2} < p_{1,1} + p_{2,2}$ and $p_{1,0} + p_{2,0} < p_{1,1} + p_{2,2}$, then there exists no state at all outside the third and fourth bases that is equally well cloned.

Let us now consider a state-dependent qutrit cloner that is characterized by the amplitude matrix

$$(a_{m,n}) = \begin{pmatrix} v & y & y \\ y & x & x \\ y & x & x \end{pmatrix} \tag{33}$$

where $v$, $x$, and $y$ are real parameters obeying the normalization condition $v^2 + 4x^2 + 4y^2 = 1$. This matrix corresponds to the probability matrix $p_{m,n} = a^2_{m,n}$. It is easy to check that this cloner results in a same fidelity (and same disturbances: $D_1 = D_2$) for all the basis states of the two last bases $\{|0''\rangle, |1''\rangle, |2''\rangle\}$ and $\{|0'''\rangle, |1'''\rangle, |2'''\rangle\}$:

$$F'' = F''' = v^2 + 2x^2,$$

$$D''_{1,2} = D'''_{1,2} = x^2 + 2y^2. \tag{34}$$

Of course, we have $F + D_1 + D_2 = 1$. Using equations (5), (6), and (7), we get

$$(a^F_{m,n}) = \frac{1}{\sqrt{3}} \begin{pmatrix} v+2y & v-y & v-y \\ y+2x & y-x & y-x \\ y+2x & y-x & y-x \end{pmatrix}, \tag{35}$$

$$(b^F_{m,n}) = \frac{1}{\sqrt{3}} \begin{pmatrix} v+2y & y+2x & y+2x \\ v-y & y-x & y-x \\ v-y & y-x & y-x \end{pmatrix}, \tag{36}$$

$$(b_{m,n}) = \frac{1}{3} \begin{pmatrix} v+4x+4y & v-2x+y & v-2x+y \\ v-2x+y & v+x-2y & v+x-2y \\ v-2x+y & v+x-2y & v+x-2y \end{pmatrix}, \tag{37}$$

so that the matrix $b_{m,n}$ characterizing the second clone has the same form as $a_{m,n}$ with the substitution:

$$v \to (v + 4x + 4y)/3, \tag{38}$$

$$x \to (v + x - 2y)/3, \tag{39}$$

$$y \to (v - 2x + y)/3. \tag{40}$$

Consequently, the states of the last two bases are again copied with the same fidelity (and same disturbances) onto the second clone:

$$\tilde{F} = (v^2 + 8x^2 + 6y^2 + 4vy + 8xy)/3, \tag{41}$$

$$\tilde{D}_{1,2} = (v^2 + 2x^2 + 3y^2 - 2vy - 4xy)/3. \tag{42}$$

We will now be interested in finding the optimal cloner, that is the cloner that maximizes the fidelity of the second clone for a given fidelity of the first clone. Maximizing $\tilde{F}$ with the constraint that $F$ is given and using the normalization condition yields the solution

$$v = F, \tag{43}$$

$$x = \sqrt{F(1-F)/2}, \tag{44}$$

$$y = (1-F)/2. \tag{45}$$

Hence, the fidelity of the second clone can be written as a function of the fidelity of the first clone

$$\tilde{F} = \frac{2-F}{3} + \frac{2\sqrt{2}}{3}\sqrt{F(1-F)} \qquad (46)$$

which expresses the complementarity between the clones. As expected, $F=1$ implies $\tilde{F}=1/3$, and conversely. An interesting special case is the symmetric cloner, which yields two clones of equal fidelity

$$F = \tilde{F} = \frac{1}{2} + \frac{1}{\sqrt{12}} \simeq 0.789. \qquad (47)$$

It should be noted that equations (46) and (47) hold regardless of which bases are optimally cloned, provided that two of them are equally cloned. The two remaining ones are then copied with a lower fidelity $v^2 + 2y^2 = \frac{1}{2} + \frac{1}{2\sqrt{12}} \simeq 0.644$. Note also that a more general cloner could be constructed for which these two remaining bases are not cloned with an equal fidelity, but this will not be considered here.

## 4. Optimal cloner of three maximally-conjugate bases

Now, we consider a state-dependent cloner that clones equally well the final three maximally-conjugate bases and for which $D_1 = D_2$. Again, our result will actually be independent of which three bases are optimally cloned, so we consider only the final three for simplicity. This imposes the conditions

$$\begin{aligned} p_{0,1} + p_{1,1} + p_{2,1} &= p_{0,2} + p_{1,2} + p_{2,2} \\ p_{1,0} + p_{2,0} &= p_{1,1} + p_{2,2} = p_{1,2} + p_{2,1} \\ p_{1,1} + p_{2,1} &= p_{1,2} + p_{2,0} = p_{1,0} + p_{2,2} \\ p_{1,2} + p_{2,2} &= p_{1,0} + p_{2,1} = p_{1,1} + p_{2,0}. \end{aligned} \qquad (48)$$

It is easy to deduce from these constraints together with equations (24), (26) and (27) that the cloning fidelity for an arbitrary state $|\psi_0\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \exp(\mathrm{i}\alpha)|1\rangle + \exp(\mathrm{i}\beta)|2\rangle)$ is simply given by

$$F = p_{0,0} + p_{1,0} + p_{2,0}, \qquad (49)$$

that is, it coincides with the cloning fidelity of the elements of the three last bases. This simplification occurs because of cyclical compensations in equation (24) which originate from the fact that the number of bases that are equally well cloned here is equal to the dimension of the Hilbert space (3 in the present case). This situation generalizes the one encountered with the phase-covariant cloner for a qubit (see the Appendix). In that case, the cloner that clones equally well two maximally-conjugate bases in a Hilbert space of dimension 2 can be shown to clone equally well all the states of an equator of the Bloch sphere. In the present case, the cloner that clones equally well two plus one maximally-conjugate bases clones equally well the generalized equator, i.e. a 1 + 1-dimensional variety that contains all the states of the form $|\psi_0\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \exp(\mathrm{i}\alpha|1\rangle + \exp(\mathrm{i}\beta)|2\rangle$.

It can be shown that the general solution of equation (48) is a probability matrix $p_{m,n}$ of the form

$$(p_{m,n}) = \begin{pmatrix} v^2 & x^2 & x^2 \\ y^2 & y^2 & y^2 \\ z^2 & z^2 & z^2 \end{pmatrix}. \tag{50}$$

For instance, we have that

$$p_{1,0} + p_{2,0} - (p_{1,1} + p_{2,1}) = (p_{1,1} + p_{2,2} + p_{1,2} + p_{2,1})/2 - (p_{1,2} + p_{2,0} + p_{1,0} + p_{2,2})/2$$
$$= (p_{1,1} + p_{2,1} - (p_{2,0} + p_{1,0}))/2,$$

so that $p_{1,0} + p_{2,0} = p_{1,1} + p_{2,1}$. But we have $p_{0,0} + p_{1,0} + p_{2,0} = p_{0,0} + p_{1,1} + p_{2,2}$, so that $p_{2,1} = p_{2,2}$. We deduce in a similar way that one must have $p_{2,0} = p_{2,1} = p_{2,2}$, $p_{1,0} = p_{1,1} = p_{1,2}$, and $p_{0,1} = p_{0,2}$. It is easy to check that these conditions are also sufficient conditions.

### 4.1. *Symmetric cloner*

Let us now consider the symmetric state-dependent cloner that clones equally well the final three bases and is characterized by the amplitude matrix

$$(a_{m,n}) = \begin{pmatrix} x+y+z & x+\alpha y+\alpha^2 z & x+\alpha^2 y+\alpha z \\ y & y & y \\ z & z & z \end{pmatrix} \tag{51}$$

where $x$, $y$, and $z$ are real parameters and with the normalization condition $3x^2 + 6x^2 + 6z^2 = 1$. This matrix corresponds to the probability matrix $p_{m,n} = a_{m,n}^2$. It is easy to check that this cloner results in the same fidelity (and same disturbance) for all basis states of the final three bases $\{|0'\rangle, |1'\rangle, |2'\rangle\}$, $\{|0''\rangle, |1''\rangle, |2''\rangle\}$ and $\{|0'''\rangle, |1'''\rangle, |2'''\rangle\}$:

$$F' = F'' = F''' = x^2 + 2y^2 + 2z^2 + 2xy + 2yz + 2xz, \tag{52}$$

$$D'_{1,2} = D''_{1,2} = D'''_{1,2} = x^2 + 2y^2 + 2z^2 - xy - yz - xz. \tag{53}$$

Of course we have $F + D_1 + D_2 = 1$. By equations (5), (7), and (6):

$$(a^F_{m,n}) = \frac{1}{\sqrt{3}} \begin{pmatrix} 3x & 3y & 3z \\ 3y & 0 & 0 \\ 3z & 0 & 0 \end{pmatrix}, \tag{54}$$

$$(b^F_{m,n}) = (a^F_{m,n}), \tag{55}$$

$$(b_{m,n}) = (a_{m,n}), \tag{56}$$

which shows that this cloner is symmetric. The cloner is optimal when the fidelity $x^2 + 2y^2 + 2z^2 + 2xy + 2yz + 2xz$ is maximal under the constraint that $x^2 + 2y^2 + 2z^2 = 1/3$. By the method of Lagrange, we obtain that the fidelity is extremal when the following equations are satisfied:

$$y + z = \lambda x$$

$$x + z = 2\lambda y$$

$$x + y = 2\lambda z, \tag{57}$$

where $\lambda$ is a Lagrange multiplier. From the last two equations, we deduce that either $\lambda = -\frac{1}{2}$ or $y = z$. If $\lambda = -\frac{1}{2}$, then $x = 0$, $y = -z$ and $F = \frac{1}{6}$ which is a minimum. If $y = z$, then $\lambda = \frac{1 \pm \sqrt{17}}{4}$ and $F = \frac{5 \pm \sqrt{17}}{12}$. The maximal fidelity is thus equal to

$$F_{\max} = \frac{5 + \sqrt{17}}{12} \simeq 0.760. \tag{58}$$

This corresponds to an amplitude matrix

$$(a_{m,n}) = \begin{pmatrix} x + 2y & x - y & x - y \\ y & y & y \\ y & y & y \end{pmatrix} \tag{59}$$

with $x = \sqrt{\frac{17 - \sqrt{17}}{102}}$ and $y = \sqrt{\frac{17 + \sqrt{17}}{408}}$. It should be noted that this symmetric cloner exactly coincides with the so-called double-phase covariant qutrit cloner that was independently derived in [14].

### 4.2. *Asymmetric cloner*

Let us now consider the asymmetric state-dependent cloner that clones equally well the final three maximally-conjugate bases, and is characterized by the amplitude matrix

$$(a_{m,n}) = \begin{pmatrix} v & y & y \\ x & x & x \\ x & x & x \end{pmatrix} \tag{60}$$

where $v$, $x$, and $y$ are real parameters and with the normalization condition $v^2 + 6x^2 + 2y^2 = 1$. This matrix corresponds to the probability matrix $p_{m,n} = a_{m,n}^2$. It is easy to check that this cloner results in the same fidelity (and same disturbances) for all basis states of the final three bases $\{|0'\rangle, |1'\rangle, |2'\rangle\}, \{|0''\rangle, |1''\rangle, |2''\rangle\}$ and $\{|0'''\rangle, |1'''\rangle, |2'''\rangle\}$:

$$F' = F'' = F''' = v^2 + 2x^2, \tag{61}$$

$$D'_{1,2} = D''_{1,2} = D'''_{1,2} = 2x^2 + y^2. \tag{62}$$

Of course, we have again $F + D_1 + D_2 = 1$. By use of equations (5), (6) and (7) we get

$$(a^F_{m,n}) = \frac{1}{\sqrt{3}} \begin{pmatrix} v+2y & v-y & v-y \\ 3x & 0 & 0 \\ 3x & 0 & 0 \end{pmatrix}, \tag{63}$$

$$(b^F_{m,n}) = \frac{1}{\sqrt{3}} \begin{pmatrix} v+2y & 3x & 3x \\ v-y & 0 & 0 \\ v-y & 0 & 0 \end{pmatrix}, \tag{64}$$

$$(b_{m,n}) = \frac{1}{3} \begin{pmatrix} v+6x+2y & v-3x+2y & v-3x+2y \\ v-y & v-y & v-y \\ v-y & v-y & v-y \end{pmatrix}. \tag{65}$$

Hence, for the second clone, the matrix $b_{m,n}$ has again the same form as $a_{m,n}$ with the substitution

$$v \to (v+6x+2y)/3, \tag{66}$$

$$x \to (v-y)/3, \tag{67}$$

$$y \to (v-3x+2y)/3, \tag{68}$$

so that the states of the final three bases are all copied onto the second clone with the same fidelity (and same disturbances):

$$\tilde{F} = (v^2 + 12x^2 + 2y^2 + 4vx + 8xy)/3, \tag{69}$$

$$\tilde{D}_{1,2} = (v^2 + 3x^2 + 2y^2 - 2vx - 4xy)/3. \tag{70}$$

For the optimal cloner, we need to maximize $\tilde{F}$ for a given value of $F$ using the normalization condition, just as before. However, in contrast with the case of the asymmetric cloner for two maximally-conjugate bases, we have found no simple analytical solution for this problem. A numerical solution and its connections with quantum cryptography will be discussed elsewhere. Note that an asymmetric state-dependent cloner could be constructed for which the final three bases are all copied equally well but with a more general matrix $a_{m,n}$ than in equation (60). It can be shown however, that the *optimal* such cloner must necessarily obey equation (60) so that this possibility will not be considered here.

## 5. Optimal cloner of all the maximally-conjugate bases

Let us finally consider an asymmetric cloner that copies equally well all four maximally-conjugate bases and for which $D_1 = D_2$. We have already shown that the constraints (48) must be obeyed to clone equally well the final three maximally-conjugate bases. In order to clone the fourth basis equally well, we must impose the additional constraints:

$$
\begin{aligned}
p_{0,1} + p_{0,2} &= p_{1,0} + p_{2,0} \\
p_{1,0} + p_{1,2} &= p_{0,1} + p_{2,1} \\
p_{2,0} + p_{2,1} &= p_{0,2} + p_{1,2}.
\end{aligned} \tag{71}
$$

Equivalently, using equation (50), we have

$$
\begin{aligned}
2x^2 &= y^2 + z^2 \\
2y^2 &= x^2 + z^2 \\
2z^2 &= x^2 + y^2.
\end{aligned} \tag{72}
$$

Hence, $x^2 = y^2 = z^2$, and $p_{m,n}$ must be of the form

$$
(p_{m,n}) = \begin{pmatrix} v^2 & x^2 & x^2 \\ x^2 & x^2 & x^2 \\ x^2 & x^2 & x^2 \end{pmatrix}. \tag{73}
$$

It is thus natural to consider the following amplitude matrix:

$$
(a_{m,n}) = \begin{pmatrix} v & x & x \\ x & x & x \\ x & x & x \end{pmatrix} \tag{74}
$$

where $v$ and $x$ are real parameters that satisfy the normalization condition $v^2 + 8x^2 = 1$. This matrix corresponds to the probability matrix $p_{m,n} = a_{m,n}^2$. By use of equations (5), (6) and (7), we have

$$
(a^F_{m,n}) = \frac{1}{\sqrt{3}} \begin{pmatrix} v + 2x & v - x & v - x \\ 3x & 0 & 0 \\ 3x & 0 & 0 \end{pmatrix}, \tag{75}
$$

$$
(b^F_{m,n}) = \frac{1}{\sqrt{3}} \begin{pmatrix} v + 2x & 3x & 3x \\ v - x & 0 & 0 \\ v - x & 0 & 0 \end{pmatrix}, \tag{76}
$$

$$
(b_{m,n}) = \frac{1}{3} \begin{pmatrix} v + 8x & v - x & v - x \\ v - x & v - x & v - x \\ v - x & v - x & v - x \end{pmatrix}, \tag{77}
$$

so that, for the second clone, the matrix $b_{m,n}$ has the same form as $a_{m,n}$ with the substitution:

$$v \to \frac{v+8x}{3}, \tag{78}$$

$$x \to \frac{v-x}{3}. \tag{79}$$

It is convenient here to change the variables $v$ and $x$ into $\alpha$ and $\beta$ according to

$$v = \frac{\alpha+\beta}{3}$$

$$x = \frac{\beta}{3} \tag{80}$$

so that we have

$$(a_{m,n}) = \begin{pmatrix} \alpha+\frac{\beta}{3} & \frac{\beta}{3} & \frac{\beta}{3} \\ \frac{\beta}{3} & \frac{\beta}{3} & \frac{\beta}{3} \\ \frac{\beta}{3} & \frac{\beta}{3} & \frac{\beta}{3} \end{pmatrix}, \tag{81}$$

$$(b_{m,n}) = \begin{pmatrix} \beta+\frac{\alpha}{3} & \frac{\alpha}{3} & \frac{\alpha}{3} \\ \frac{\alpha}{3} & \frac{\alpha}{3} & \frac{\alpha}{3} \\ \frac{\alpha}{3} & \frac{\alpha}{3} & \frac{\alpha}{3} \end{pmatrix}. \tag{82}$$

It is easy to check that this cloner results in the same fidelity (and same disturbance) for any qutrit state:

$$F = \alpha^2 + 2\frac{\alpha\beta}{3} + \frac{\beta^2}{3}, \tag{83}$$

$$D_{1,2} = \frac{\beta^2}{3}. \tag{84}$$

Of course we have $F + D_1 + D_2 = 1$. This is the special case of a state-independent (or universal) N-dimensional cloner [8, 9], which can be obtained simply by letting

$$a_{m,n} = \alpha\delta_{m,0}\delta_{n,0} + \beta/N, \tag{85}$$

$$b_{m,n} = \beta\delta_{m,0}\delta_{n,0} + \alpha/N. \tag{86}$$

This is consistent with equation (4) since the constant function $1/N$ is the Fourier transform of $\delta_{m,0}\,\delta_{n,0}$. Thus, $\alpha = 1(\beta = 0)$ is the case where the first clone is perfect, whereas $\beta = 1(\alpha = 0)$ is the case where the second clone is perfect. The normalization relation implies that

$$|\alpha|^2 + \frac{2}{N}\mathrm{Re}\,(\alpha\beta^*) + |\beta|^2 = 1, \tag{87}$$

which characterizes the balance between the quality of the two clones. In particular, the *symmetric* universal $N$-dimensional cloner corresponds to the case where

$$\alpha^2 = \beta^2 = \frac{N}{2(1+N)}. \tag{88}$$

Using equation (14) for the cloning fidelity in the computational basis (since all states are copies with the same fidelity), we recover the standard formula for the universal cloner [6–9]

$$F = \left(\alpha + \frac{\beta}{N}\right)^2 + (N-1)\left(\frac{\beta}{N}\right)^2 = \frac{3+N}{2(1+N)}. \tag{89}$$

In particular, the symmetric universal qutrit cloner ($N = 3$) is characterized by a fidelity of 3/4.

## 6. Conclusion

We have investigated several categories of $1 \to 2$ cloning transformations for a three-dimensional system (a qutrit). First, we have analysed cloners that optimally copy the states of any two of the four maximally-conjugate bases. The symmetric cloner of this class has a cloning fidelity of $\frac{1}{2} + \frac{1}{\sqrt{12}} \simeq 0.789$. Second, we studied cloners that copy equally well and with the highest fidelity three maximally-conjugate bases. These cloners can be shown to copy all states of the form $\frac{1}{\sqrt{3}}(|0\rangle + \exp\,(\mathrm{i}\alpha)|1\rangle + \exp\,(\mathrm{i}\beta)|2\rangle)$ with the same fidelity for any $\alpha$ and $\beta$, so they are the natural extension of the phase-covariant qubit cloners. The symmetric cloner of this class copies all these states with a fidelity $\frac{5+\sqrt{17}}{12} \simeq 0.760$, and coincides with the so-called double-phase covariant qutrit cloner analysed independently in [14]. Finally, cloners that optimally copy all four maximally-conjugate bases can be shown to copy all states of a qutrit equally well, so they simply correspond to universal qutrit cloners. The symmetric universal qutrit cloner has a fidelity of 3/4, in accordance with [6, 7]. We conclude thus that, quite naturally, the cloning fidelity decreases when we put a stronger requirement on the cloner (namely two, three or four bases must be copied optimally). The study also suggests that there is still much room for further investigation of multi-level non-universal quantum cloning.

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**Appendix. Phase-covariant cloner for a qubit**

In this Appendix, we show that the phase-covariant qubit cloner [13] can be obtained in just a few lines by using the general characterization of Pauli cloners [8–11]. The phase-covariant qubit cloner is defined as a transformation that optimally copies all states of the form $\frac{1}{\sqrt{2}}(|0\rangle + \exp(\mathrm{i}\alpha|1\rangle)$ for any $\alpha$. Here, we rather look for a qubit cloner that copies any two maximally-conjugate bases. In the Hilbert space of a qubit, there are three maximally-conjugate bases, which correspond to the eigenstates of the three Pauli matrices:

$$|0\rangle, \quad |1\rangle, \tag{A1}$$

$$|0'\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |1'\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \tag{A2}$$

$$|0''\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \mathrm{i}|1\rangle), \qquad |1''\rangle = \frac{1}{\sqrt{2}}(\mathrm{i}|0\rangle + |1\rangle). \tag{A3}$$

The universal qubit cloning machine [5] copies the states of each of these three bases with the same fidelity. In contrast, the cloner we are interested in here is required to optimally copy only the first two bases with the same (and maximum) fidelity. This is equivalent to requiring that the states on the 'equatorial' plane $x - z$ of the Bloch sphere are all shrunk by the same factor. (Note that, conventionally, the phase-covariant qubit cloner is rather required to optimally copy the final two bases, or, by extension, all states of the equatorial plane $x - y$ [13].) The cloning fidelity can be higher than that of the universal cloner, but this is at the expense of cloning fidelity for the third basis, which must be lower.

Let us calculate the fidelity of this phase-covariant cloner. Consider the effect of the error operators on the elements of the two first bases. We have

$$\begin{array}{llll} & |0\rangle \to |0\rangle & & |0\rangle \to |0\rangle \\ U_{0,0} & & U_{0,1} & \\ & |1\rangle \to |1\rangle & & |1\rangle \to -|1\rangle \\ & |0\rangle \to |0\rangle & & |0\rangle \to |0\rangle \\ U_{1,0} & & U_{1,1} & \\ & |1\rangle \to |0\rangle & & |1\rangle \to -|0\rangle \end{array} \tag{A4}$$

so the elements of the first basis are left unchanged (up to a sign) by the error operators $U_{0,0}$ and $U_{0,1}$. Similarly, we have

$$\begin{array}{llll} & |0'\rangle \to |0'\rangle & & |0'\rangle \to |1'\rangle \\ U_{0,0} & & U_{0,1} & \\ & |1'\rangle \to |1'\rangle & & |1'\rangle \to |0'\rangle \\ & |0'\rangle \to |0'\rangle & & |0'\rangle \to -|0'\rangle \\ U_{1,0} & & U_{1,1} & \\ & |1'\rangle \to -|1'\rangle & & |1'\rangle \to -|0'\rangle \end{array} \tag{A5}$$

so the elements of the second basis are left unchanged (up to a sign) under $U_{0,0}$ and $U_{1,0}$. Now, using the general formula for the cloning fidelity equation (13), we find that the elements $|0\rangle$ and $|1\rangle$ of the first basis are cloned with the fidelity

$$F = p_{0,0} + p_{0,1}, \tag{A6}$$

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while the elements $|0'\rangle$ and $|1'\rangle$ of the second basis are cloned with the fidelity

$$F' = p_{0,0} + p_{1,0}. \tag{A7}$$

The requirement of having a phase covariant cloner $(F + F')$ can thus be written simply as $p_{0,1} = p_{1,0}$. Consequently, we consider a cloner characterized by the amplitude matrix

$$(a_{m,n}) = \begin{pmatrix} v & x \\ x & y \end{pmatrix} \tag{A8}$$

where $x$, $y$ and $v$ are real and positive, and with the normalization condition $v^2 + 2x^2 + y^2 = 1$. The fidelity $F$ (and disturbance $D = 1 - F$) of the first clone are thus given in both bases by

$$F = F' = v^2 + x^2, \tag{A9}$$

$$D = D' = x^2 + y^2. \tag{A10}$$

For the second clone, equation (4) (or, equivalently, equation (7)), implies that the matrix $b_{m,n}$ has the same form as $a_{m,n}$ with the substitution

$$v \to (v + 2x + y)/2, \tag{A11}$$

$$x \to (v - y)/2, \tag{A12}$$

$$y \to (v - 2x + y)/2, \tag{A13}$$

so that the states of the two conjugate bases are again copied all with the same fidelity (and the same disturbance):

$$\tilde{F} = (v^2 + 2x^2 + y^2 + 2vx + 2xy)/2 = 1/2 + vx + xy, \tag{A14}$$

$$\tilde{D} = (v^2 + 2x^2 + y^2 - 2vx - 2xy)/2 = 1/2 - vx - xy. \tag{A15}$$

We are now interested in finding the cloner that maximizes the fidelity of the second clone $\tilde{F}$ for a given fidelity of the first clone $F$. A simple constrained maximization calculation yields the solution

$$v = F, \tag{A16}$$

$$x = \sqrt{F(1-F)}, \tag{A17}$$

$$y = 1 - F, \tag{A18}$$

so that the maximum fidelity of the second clone can be written as a function of the fidelity of the first clone:

$$\tilde{F} = \tfrac{1}{2} + \sqrt{F(1-F)}. \tag{A19}$$

This expresses the balance between the quality of the two clones in the case of a phase-covariant qubit cloner. As expected, $F = 1$ yields $\tilde{F} = 1/2$, and conversely. The symmetric phase-covariant cloner yields two clones of equal fidelity:

$$F = \tilde{F} = \frac{1}{2} + \frac{1}{\sqrt{8}} \simeq 0.854, \tag{A20}$$

in agreement with [13]. As expected, this fidelity is slightly higher than the fidelity of the universal qubit cloner, namely $F = 5/6$. In contrast, the third basis is now copied with a fidelity equal to 3/4, that is, lower than the fidelity of the universal cloner.

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