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Linear optics and projective measurements alone suffice to create large-photon-number path entanglement

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We propose a method for preparing maximal path entanglement with a definite photon-number N, larger than two, using projective measurements. In contrast with the previously known schemes, our method uses only linear optics. Specifically, we exhibit a way of generating four-photon, path-entangled states of the form $|4,0\rangle+|0,4\rangle$, using only four beam splitters and two detectors. These states are of major interest as a resource for quantum interferometric sensors as well as for optical quantum lithography and quantum holography.

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Quantum entanglement plays a central role in quantum communication and computation. It also provides a significant improvement in frequency standards as well as in the performance of interferometric sensors [1,2]. In this context, it has been shown that the Heisenberg limit for phase sensitivity of a Mach-Zehnder interferometer can be reached by using maximally entangled states with a definite number of photons N, that is, $|N,0\rangle_{A,B}+|0,N\rangle_{A,B}$. Here, A and B denote the two arms of the interferometer. These states, also called path-entangled photon-number states, allow a phase sensitivity of order $1/N$, whereas coherent light yields the shot-noise limit of $1/\sqrt{\bar{n}}$, with mean photon-number $\bar{n}$ [3]. The use of quantum entanglement can also be applied to optical lithography. It has been shown recently that the Rayleigh diffraction limit in optical lithography can be beaten by the use of path-entangled photon-number states [4]. In order to obtain an N-fold resolution enhancement, with quantum interferometric optical lithography, one again needs to create the N-photon path-entangled state given above. Due to interference of the two paths, one obtains an intensity pattern at the lithographic surface which is proportional to $1+\cos N\varphi$, where φ parametrizes the position on the surface. A superposition of these states with varying N and suitable phase shifts then yields a Fourier series of the desired pattern, up to a constant [5].

In view of these potential applications, finding methods for generating path-entangled states has been a long-standing endeavor in quantum optics. Unfortunately, with the notable exception of $N=2$, the optical generation of these states seemed to require single-photon quantum logic gates that involve a large nonlinear interaction, namely, a Kerr element with $\chi^{(3)}$ on the order of unity. Typically, $\chi^{(3)}$ is of the order 10^{-16} cm^2 s^{-1} V^{-2} [6]. This makes a physical implementation with previously known techniques very difficult [7–9]. Recently, however, several methods for the realization of probabilistic single-photon quantum logic gates have been proposed, which make use solely of linear optics and projective measurements (PMs) [10–12]. PMs are performed by measuring some part of the system while the rest of it is projected onto a desired state (state reduction). Since the state obtained is conditioned on a measurement outcome, this method only works probabilistically. Such a protocol has been employed experimentally, by the group of Zeilinger, to postselect on four-photon *polarization* entanglement [13].

In this paper, we devise a technique for generating maximally path-entangled photon-number states based on this paradigm. In particular, our method circumvents the use of $\chi^{(3)}$ nonlinearities in a Fredkin gate approach, for example [9]. We suggest several linear optical schemes, based on projective measurements, for the preparation of a four-photon, path-entangled state. We also discuss the feasibility of these schemes, by investigating the consequence of inefficient detectors on the state preparation process.

It is well known that two-photon, path-entangled states can be created using a Hong-Ou-Mandel (HOM) interferometer, where a photon pair from a parametric down converter impinges onto a 50:50 beam splitter [14]. The beam splitter yields the path-entangled state $|2,0\rangle_{A'B'}+|0,2\rangle_{A'B'}$ from the product state $|1,1\rangle_{AB}$. In other words, the probability amplitude for having $|1,1\rangle_{A'B'}$ at the output of the beam splitter vanishes. This can be understood by a simple diagrammatic analysis (see Fig. 1).

In our convention, the reflected mode acquires a phase π while the transmitted mode acquires a phase of $\pi/2$, consistent with the reciprocity requirement, so that the two possible ways of producing a state $|1,1\rangle$ interfere destructively [15]. However, a beam splitter is not sufficient any more if the goal is to produce path-entangled states with a photon number larger than two [16]. Consequently, it is commonly assumed that $\chi^{(3)}$ nonlinear optical components are needed for $N>2$. By contrast, we show here that the recourse to such nonlinearity can be avoided if single-photon detectors are

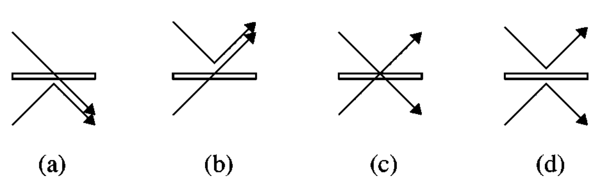

FIG. 1. Four possibilities when sending a $|1,1\rangle$ state through a beam splitter. The diagrams (c) and (d) lead to the same final state, but interfere destructively; (c) transmission-transmission $(i)(i)=-1$; (d) refection-reflection $(-1)(-1)=1$.

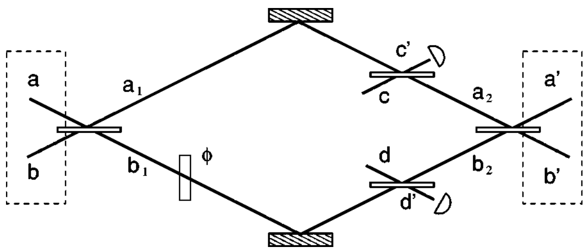

FIG. 2. Mach-Zehnder interferometer with two additional beam splitters in the lower and upper arms, which direct the reflected beams to photodetectors. A one-photon count at both detectors allows the projective generation of the states $|2,0\rangle+|0,2\rangle$ or $|4,0\rangle+|0,4\rangle$, depending on the input state.

added to the scheme. The desired path-entangled states are then obtained, conditioned on the measurement outcome.

Before considering the interesting case of $N=4$, it is instructive to first exhibit the generation of the state $|2,0\rangle_{A'B'}+|0,2\rangle_{A'B'}$ using projective measurements, instead of a simple beam splitter. Let us consider a Mach-Zehnder interferometer with two additional beam splitters, each of them being followed by a detector (see Fig. 2). In such a configuration, with all paths balanced, one can select the desired state via state reduction, conditionally on both detectors clicking. Formally, we are dealing with a four-port optical device, which may be characterized by expressing the output bosonic mode operators $\hat{a}'$, $\hat{b}'$, $\hat{c}'$, and $\hat{d}'$ as a function of the input mode operators $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ [2]. For the transformation effected by a single beam splitter (say, the first one in Fig. 2), we use the convention $\hat{a}_1=(-\hat{a}+i\hat{b})/\sqrt{2}$, $\hat{b}_1=(i\hat{a}-\hat{b})/\sqrt{2}$.

Combining the transformations of the first, the last, and the two intermediate beam splitters in the lower and upper arms, we get the overall transformation

$$\hat{a}'=\hat{b}/\sqrt{2}+(\hat{c}-i\hat{d})/2,$$

$$\hat{b}'=\hat{a}/\sqrt{2}+(\hat{d}-i\hat{c})/2,$$

$$\hat{c}'=(\hat{a}-i\hat{b})/2+i\hat{c}/\sqrt{2},$$

$$\hat{d}'=(\hat{b}-i\hat{a})/2+i\hat{d}/\sqrt{2}. \qquad (1)$$

Note that we neglect here the phase induced by the mirrors and that accumulated along the optical path, since they cancel for a suitably balanced interferometer. For a given input state, one obtains the output state simply by expressing the input modes in terms of the output modes, that is, by inverting Eqs. (1). Suppose the input state is $|2,2\rangle_{AB}=\frac{1}{2}(\hat{a}^{\dagger})^2(\hat{b}^{\dagger})^2|0\rangle$. Then, the term of order $\hat{c}'\dagger\,\hat{d}'\dagger$ in the expansion of $(\hat{a}^{\dagger})^2(\hat{b}^{\dagger})^2$ can be shown to be $-i/4[(\hat{a}'^{\dagger})^2+(\hat{b}'^{\dagger})^2]$. If we call $|\psi_{\rm am}\rangle$ the state before the projective measurement (*ante* measurement), we have the output state (*post* measurement) $|\psi_{\rm pm}\rangle=\langle 1,1|\psi_{\rm am}\rangle\propto|2,0\rangle+|0,2\rangle$. Thus, if one and only one photon is measured at each detector, one obtains the envisioned two-photon path-entangled output state. The probability of this event is 1/16.

The way this projective method works can be understood very simply. After passing through the first beam splitter, the product state $|2,2\rangle$ becomes a linear superposition of $|4,0\rangle$, $|2,2\rangle$, and $|0,4\rangle$. Again, the states $|3,1\rangle$ and $|1,3\rangle$ do not appear for the same reason as the vanishing of the HOM output state $|1,1\rangle$, when the input is $|1,1\rangle$ (see Fig 3). Since the detection of one photon at each detector requires at least one photon in both the upper and lower arms of the interferometer, the $|4,0\rangle$ and $|0,4\rangle$ states cannot contribute to such events. Consequently, only the $|2,2\rangle$ term is left, which then becomes $|1,1\rangle$ if one photon is detected in each arm. This $|1,1\rangle$ state is thus found at the input of the last beam splitter, which results in the expected state $|2,0\rangle+|0,2\rangle$.

We can now use this approach to proceed to generate the $|4,0\rangle+|0,4\rangle$ state. The key reason why projective measurement is useful in the above scheme is that it enables us to conditionally suppress the extreme components $|4,0\rangle$ and $|0,4\rangle$, while leaving the middle component $|2,2\rangle$ unchanged. More generally, the generation of path-entangled states with $N>2$ requires eliminating the extreme components with respect to the middle terms. Suppose we want to produce the state $|4,0\rangle+|0,4\rangle$. Then, a simple matrix inversion shows that the state we need at the input of the last beam splitter is generated an operator of the form $(\hat{a}^{\dagger})^4-6(\hat{a}^{\dagger})^2(\hat{b}^{\dagger})^2+(\hat{b}^{\dagger})^4$. Similarly, to produce the output state $|4,0\rangle-|0,4\rangle$, the required input operator is of the form $(\hat{a}^{\dagger})^3(\hat{b}^{\dagger})-(\hat{a}^{\dagger})(\hat{b}^{\dagger})^3$. Since the latter operator has fewer terms, we will focus for the moment on producing $|4,0\rangle-|0,4\rangle$.

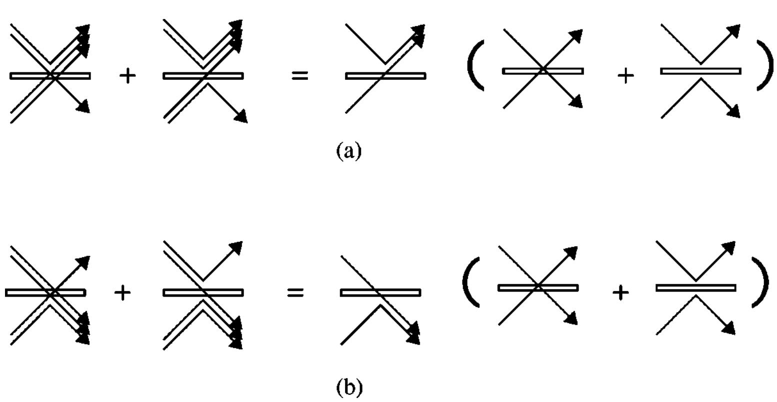

FIG. 3. Two possible ways of making (a) $|3,1\rangle$ or (b) $|1,3\rangle$ from an input state $|2,2\rangle$ passing through a beam splitter. The two diagrams interfere destructively just as in Fig. 1.

Let us show how to produce this state taking $|3,3\rangle$ as the input state and using the same interferometric setup as in Fig. 2. The first beam splitter transforms $|3,3\rangle=\frac{1}{6}(\hat{a}^{\dagger})^3(\hat{b}^{\dagger})^3|0\rangle$ into a linear superposition of $|6,0\rangle$, $|4,2\rangle$, $|2,4\rangle$, and $|0,6\rangle$ generated by

$$(\hat{a}^{\dagger})^6+3(\hat{a}^{\dagger})^4(\hat{b}^{\dagger})^2+3(\hat{a}^{\dagger})^2(\hat{b}^{\dagger})^4+(\hat{b}^{\dagger})^6. \tag{2}$$

After passing through the two intermediate beam splitters, and if one and only one photon is counted at each detector, the state is then projected onto an equal superposition of $|3,1\rangle$ and $|1,3\rangle$. Indeed, the states $|6,0\rangle$ or $|0,6\rangle$ are again eliminated by this projective measurement, since they cannot yield a click at *both* detectors. The $|4,2\rangle$ and $|2,4\rangle$ states, on the other hand, lose one photon in each arm of the interferometer and are therefore reduced to $|3,1\rangle$ and $|1,3\rangle$, respectively. Thus, just before the last beam splitter, we have $|3,1\rangle+|1,3\rangle$. Finally, we need to add a $\pi/2$-phase shifter in the lower arm of the interferometer (see Fig. 2) in order to get the relative phase π that is needed between the two terms. This transforms Eq. (2) into

$$(\hat{a}^{\dagger})^6-3(\hat{a}^{\dagger})^4(\hat{b}^{\dagger})^2+3(\hat{a}^{\dagger})^2(\hat{b}^{\dagger})^4-(\hat{b}^{\dagger})^6, \tag{3}$$

so that the state after the projective measurement is reduced to $|3,1\rangle-|1,3\rangle$. Consequently, after the last beam splitter, we get the desired state $|4,0\rangle-|0,4\rangle$. Of course, the state $|4,0\rangle+|0,4\rangle$ can simply be obtained by putting an extra $\pi/4$-phase shifter at the end of one path. A straightforward calculation shows that if the input state is $\frac{1}{6}(\hat{a}^{\dagger})^3(\hat{b}^{\dagger})^3|0\rangle$, then, as before, the output state reads $|\psi_{\rm pm}\rangle=\langle 1,1|\psi_{\rm am}\rangle\propto|4,0\rangle+|0,4\rangle$. A proper normalization shows that the probability to yield the desired state $|4,0\rangle+|0,4\rangle$ is 3/64. Note that any $|2N+1,2N+1\rangle$ input state may be used in this configuration to yield $|4,0\rangle+|0,4\rangle$ by detecting $2N-1$ photons at each detector, but with a smaller yield as N increases.

An alternative way of producing $|4,0\rangle+|0,4\rangle$ was found that requires the ability of preparing the input states $|2,2\rangle$ and $|1,1\rangle$, instead of $|3,3\rangle$. The idea is to feed the previously unused input ports of the two intermediate beam splitters (modes c and d in Fig. 2) with the state $|2,0\rangle+|0,2\rangle$. This state is obtained by sending $|1,1\rangle$ through a HOM beam splitter. Suppose we have an input state $|2,2\rangle$, which after the first beam splitter gives a superposition of $|4,0\rangle$, $|2,2\rangle$, and $|0,4\rangle$, as explained above. Consider, first, the middle term $|2,2\rangle_{A_1B_1}$, which gives

$$|2,2\rangle_{A_1B_1}(|2,0\rangle_{CD}+|0,2\rangle_{CD}) \\ =|2,2\rangle_{A_1C}|2,0\rangle_{B_1D}+|2,0\rangle_{A_1C}|2,2\rangle_{B_1D}, \tag{4}$$

so that either the beam splitter in the upper arm or that in the lower arm is fed again with $|2,2\rangle$. As shown in Fig. 3, this leads to the measurement of zero or two photons at the corresponding detector, but cannot give one count. Consequently, the middle term cannot contribute to $|1,1\rangle_{C'D'}$. Take now the first term $|4,0\rangle_{A_1B_1}$, which gives

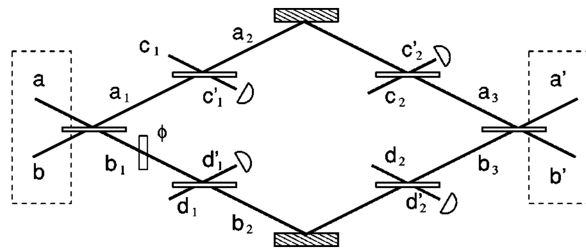

FIG. 4. Four-detector scheme with a Mach-Zehnder interferometer. When we feed the c_1,d_1 modes with $|N-1,0\rangle_{C_1D_1}+|0,N-1\rangle_{C_1D_1}$, we have the output state $|N,0\rangle_{A'B'}-|0,N\rangle_{A'B'}$ ($N\in\{2,3,4\}$) conditioned upon one photon detection at each detector. Here the input state is $|5,0\rangle_{AB}$ and $\phi=\pi/2$.

$$|4,0\rangle_{A_1B_1}(|2,0\rangle_{CD}+|0,2\rangle_{CD}) \\ =|4,2\rangle_{A_1C}|0,0\rangle_{B_1D}+|4,0\rangle_{A_1C}|0,2\rangle_{B_1D}. \tag{5}$$

Clearly, the first term in the latter expression cannot give a click at the lower detector. In contrast, the second term can give a click at both detectors, which results in the state $|3,1\rangle_{A_2C'}|1,1\rangle_{B_2D'}$, after the intermediate beam splitters. Thus, postselecting on one count at each detector yields $|3,1\rangle_{A_2B_2}$. Similarly, for the third term $|0,4\rangle_{A_1B_1}$, we get the state $|1,3\rangle_{A_2B_2}$ after postselection. Consequently, we only now need to adjust the relative phase between the $|4,0\rangle_{A_1B_1}$ and $|0,4\rangle_{A_1B_1}$ states in order to get $|3,1\rangle-|1,3\rangle$ before the last beam splitter. This can be done by inserting a $\pi/4$-phase shifter in the lower arm of the interferometer. Then the desired state $|4,0\rangle-|0,4\rangle$ is produced after the last beam splitter. A simple calculation shows that an input state $\frac{1}{4}(\hat{a}^{\dagger})^2(\hat{b}^{\dagger})^2[(\hat{c}^{\dagger})^2+(\hat{d}^{\dagger})^2]|0\rangle$ yields the same output as before up to an irrelevant global phase, so that one-photon detection at each detector projects the output state onto $|4,0\rangle-|0,4\rangle$ with probability 3/64. The yield is thus equal to that of the previous scheme. Note again, that with this configuration, any $|2N,2N\rangle(|2,0\rangle-|0,2\rangle)$ input state yields, conditionally on the detection of $2N-1$ photons at each detector, the same output state $|4,0\rangle-|0,4\rangle$. However, the probabilities decrease as N increases.

The schemes we have shown so far relied on symmetric product states $|N,N\rangle$ as inputs. States of this form are typically produced in optical parametric oscillators and down converters [17,18]. We have also devised schemes, which start from the state $|5,0\rangle$ instead, and from which we generate states of the form $|N,0\rangle+|0,N\rangle$, for $N\in\{2,3,4\}$ (see Fig. 4). Such input states as $|N,0\rangle$ can be produced by manipulating states of the form $|N,N\rangle$, or from N-photon sources, now under development [18,19].

Finally, let us discuss the consequence of using realistic detectors in our schemes. We can model the detector efficiency η^2 with an ideal detector preceded by a beam splitter with transmissivity η. The photons deflected from the detector represent the loss. When two photons enter the inefficient detector, one of them might be lost, thus yielding an incorrect detector outcome. This is particularly important here, since we condition the outgoing state on single-photon detection events. The projective measurement associated with a single-photon detection can be modeled by the projector $\Sigma_{n=1}^{\infty} n\eta^2(1-\eta^2)^{n-1}|n\rangle\langle n|$. Applying this to the first proposed scheme for generating $|4,0\rangle_{A'B'}-|0,4\rangle_{A'B'}$ (see Fig. 2), we obtain a state $\rho_{A'B'}\propto\Sigma_{n,m=1}^{6} nm\eta^4(1-\eta^2)^{n+m-2}\rho_{A'B'}^{(n,m)}$, where n,m are the number of photons lost in modes C' and D', and $\rho_{A'B'}^{(n,m)}\propto{}_{C'D'}\langle n,m|\rho_{A'B'C'D'}|n,m\rangle_{C'D'}$. These density matrices $\rho_{A'B'}^{(n,m)}$, which arise due to imperfect detections, also correspond to N-photon path-entangled states, but with $N<4$. (See Table I.)

Thus, the output state is a mixture of path-entangled states with different values of N. For a realistic, single-photon resolution, photodetector with efficiency $\eta^2=0.88$ [19], the fidelity of the outgoing state with respect to the envisioned state $|\Psi\rangle=|4,0\rangle+|0,4\rangle$ is $F=\langle\Psi|\rho|\Psi\rangle=0.64$, conditioned on a single-photon detector coincidence. Even though these imperfect detections lead to a degraded fidelity, this might be exploited in order to create incoherent superpositions of path-entangled states, which may be useful for the pseudo-Fourier method in quantum lithography [5].

In conclusion, we have shown that conditioning the output of a linear optical setup on single-photon detection events makes it possible to generate path-entangled photon number states with more than two photons. The price of eliminating nonlinear components is the relatively low yield of the projective process, which is only about 5% for the state $|4,0\rangle+|0,4\rangle$. Of course, the optical schemes we have found so far are not necessarily the most efficient ones, so finding the optimal protocols remains an interesting open problem. In particular, employing the teleportation "fix" used by Knill, Laflamme, and Milburn [10], in future work we plan to devise schemes where the yield scales more efficiently with N.

TABLE I. The outgoing states $\rho^{(n,m)}$ of the interferometer of Fig. 2 (only the ket parts are given since these states are pure). The left column lists the photon-number coincidence in the two detectors, while the right column gives the corresponding outgoing state. When the detector outcomes are interchanged, i.e., $(n,m)\rightarrow(m,n)$, the corresponding state picks up a relative minus sign.

(n,m)	$\rho^{(n,m)}$
(1,2)	$\|3,0\rangle+\|0,3\rangle$
(2,2)	$\|2,0\rangle+\|0,2\rangle$
(3,1)	$\|2,0\rangle+\|0,2\rangle$
(3,2)	$\|1,0\rangle+\|0,1\rangle$
(4,1)	$\|1,0\rangle+\|0,1\rangle$

Another inherent difficulty is that our proposed schemes require detectors that are able to resolve one or more photons. This problem may, however, not be critical in applications where incoherent superpositions of path-entangled photon number states are needed anyway, such as in quantum lithography. The projective generation method also requires the availability of photon-number sources, which clearly is another challenge [20]. The technique presented here can be extended to generating path-entangled states with arbitrary N, which is presented in a subsequent report [21].

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