Quantum search by local adiabatic evolution

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The adiabatic theorem has been recently used to design quantum algorithms of a new kind, where the quantum computer evolves slowly enough so that it remains near its instantaneous ground state, which tends to the solution. We apply this time-dependent Hamiltonian approach to Grover's problem, i.e., searching a marked item in an unstructured database. We find that by adjusting the evolution rate of the Hamiltonian so as to keep the evolution adiabatic on each infinitesimal time interval, the total running time is of order $\sqrt{N}$, where N is the number of items in the database. We thus recover the advantage of Grover's standard algorithm as compared to a classical search, scaling as N. This is in contrast with the constant-rate adiabatic approach of Farhi *et al.* (e-print quant-ph/0001106), where the requirement of adiabaticity is expressed only globally, resulting in a time of order N.

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INTRODUCTION

Although quantum computation is still mostly a theoretical concept today, many proposals for experimental quantum computing have appeared over the last few years, which have focused on different implementations of elementary quantum gates as well as on the potential scalability of these implementations (see [1] for a recent review). On the theoretical side, several quantum algorithms have been designed and shown to outperform all known classical algorithms, thereby giving a strong motivation for the development of quantum computers. Probably the most spectacular result is Shor's algorithm [2], which can factor a large number with a computation time polynomial in the size of the number, whereas all known classical algorithms require a (sub-) exponential time. Another remarkable algorithm, due to Grover, concerns the problem of searching in an unsorted database [3]. Suppose we have a database of N items, one of which is marked. The goal is to find this unknown marked item by accessing the database a minimum number of times. Classically, $N/2$ items must be tested, on average, before finding the right one. Grover's quantum algorithm performs the same task with a complexity of order $\sqrt{N}$, giving rise to a quadratic speed up.

While Grover's algorithm was originally presented within the standard paradigm for quantum computation, that is, using a discrete sequence of unitary logic gates, we will turn here to another type of quantum computation where the state of the quantum register evolves continuously under the influence of some driving Hamiltonian. This concept of quantum computation viewed as a continuous time evolution was pioneered by Farhi and Gutmann in [4], where they proposed an ''analog'' version of Grover's algorithm based on a time-independent Hamiltonian applied for a time T. Their algorithm required T to be of order $\sqrt{N}$, which thus gives the same complexity as Grover's algorithm. In a more recent paper with co-workers, they considered an alternative class of continuous-time quantum algorithms based on a time-dependent Hamiltonian that effects an *adiabatic* evolution of the quantum register [5] (the concept of adiabatic quantum computing is also explored in [6]). If the evolution of a quantum system is governed by a Hamiltonian that varies slowly enough, this system will stay near its instantaneous ground state. This adiabatic evolution can then be used to switch gradually from an initial Hamiltonian, whose ground state is known, to a final Hamiltonian, whose ground state encodes the unknown solution. The time required for this switching to remain globally adiabatic determines the computation time. Farhi *et al.* have solved Grover's search problem using this adiabatic evolution approach, but this unfortunately resulted in a complexity of order N, which is no better than a classical algorithm that simply checks all possible solutions [5].

In the present article, we show that one can recover the quadratic speed-up of Grover's original algorithm by continuously adjusting the rate with which the initial Hamiltonian is switched to the final Hamiltonian so as to fulfill the condition of adiabaticity *locally*, i.e. at each time. Interestingly, this local adiabatic-evolution approach makes it possible to improve the scaling law of the complexity of the quantum-search algorithm simply by varying the speed of this adiabatic sweep. This offers the perspective of speeding up more sophisticated adiabatic-evolution algorithms, such as those applied to NP-complete problems [7]. It might also be used to design an adiabatic-evolution version of the nested quantum-search technique proposed in [8] for solving structured problems.

ADIABATIC THEOREM

Consider a quantum system in a state $|\psi(t)\rangle$, which evolves according to the Schrödinger equation

$$i\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle, \tag{1}$$

where $H(t)$ is the Hamiltonian of the system (we let $\hbar = 1$). If this Hamiltonian is time independent and the system is initially in its ground state, then it will remain in this state. The adiabatic theorem (see, e.g., [9]) states that if the Hamiltonian varies slowly enough, roughly speaking, the state of

the system will stay close to the instantaneous ground state of the Hamiltonian at each time t. More specifically, let $|E_k;t\rangle$ be the eigenstates of $H(t)$ satisfying

$$H(t)|E_k;t\rangle = E_k(t)|E_k;t\rangle, \tag{2}$$

where $E_k(t)$ are the corresponding eigenvalues and k labels the eigenstates ($k=0$ labels the ground state). We define the minimum gap between the lowest two eigenvalues as

$$g_{\min} = \min_{0\leqslant t\leqslant T} [E_1(t) - E_0(t)] \tag{3}$$

and the maximum value of the matrix element of dH/dt between the two corresponding eigenstates as

$$D_{\max} = \max_{0\leqslant t\leqslant T} \left| \left\langle \frac{dH}{dt} \right\rangle_{1,0} \right| \tag{4}$$

with $\langle dH/dt\rangle_{1,0} = \langle E_1;t|dH/dt|E_0;t\rangle$. The adiabatic theorem states that if we prepare the system at time $t=0$ in its ground state $|E_0;0\rangle$ and let it evolve under the Hamiltonian $H(t)$, then

$$|\langle E_0;T|\psi(T)\rangle|^2 \geqslant 1-\varepsilon^2 \tag{5}$$

provided that

$$\frac{D_{\max}}{g_{\min}^2} \leqslant \varepsilon, \tag{6}$$

where $\varepsilon \ll 1$. In particular, this implies that the minimum gap cannot be lower than a certain value if we require the state at time t to differ from the instantaneous ground state by a negligible amount (a smaller gap implies a higher transition probability to the first excited state). This result can be used to design a new type of quantum algorithm based on a time-dependent Hamiltonian [5]. Assume we can build a Hamiltonian for which we know that the ground state encodes the solution of a problem. Then, it suffices to prepare the system in the ground state of another Hamiltonian, easy to build, and change progressively this Hamiltonian into the other one in order to get, after measurement, the sought solution with a large probability. The adiabatic theorem imposes the minimum time it takes for this switching to be adiabatic, and this time can be thought of as the algorithm complexity.

GLOBAL VS LOCAL ADIABATIC EVOLUTION FOR SOLVING THE QUANTUM-SEARCH PROBLEM

We now apply this adiabatic-evolution method to the problem of finding an item in an unsorted database. Before describing our local adiabatic-search algorithm, we first summarize the method of [5], based on a global adiabatic evolution. Consider a set of N items among which one is marked, the goal being to find it in minimum time. We use n qubits to label the items, so that the Hilbert space is of dimension $N=2^n$. In this space, the basis states are written as $|i\rangle$, with $i=0,\ldots,N-1$, while the marked state is denoted by $|m\rangle$. As we do not know $|m\rangle$ *a priori*, we use as an initial state an equal superposition of all basis states

$$|\psi_0\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N-1} |i\rangle. \tag{7}$$

More precisely, the Hamiltonian of the system is initially chosen as

$$H_0 = I - |\psi_0\rangle\langle\psi_0|, \tag{8}$$

whose ground state is $|\psi_0\rangle$ with energy zero. Let us suppose we are also able to apply to our system the Hamiltonian

$$H_m = I - |m\rangle\langle m|, \tag{9}$$

whose ground state, the marked state $|m\rangle$, is unknown. The fact that H_m can be applied without explicitly knowing m is equivalent to the assumption, in the standard description of Grover's algorithm, that a quantum ''oracle'' is available (evolving the system with H_m during a certain time interval is roughly equivalent to applying the quantum oracle; see [10] for a detailed discussion of Grover's algorithm as an oracle-based algorithm). The time-dependent Hamiltonian underlying the algorithm is a linear interpolation between these two Hamiltonians, that is,

$$H(t) = (1-t/T)H_0 + t/T H_m, \tag{10}$$

or, with $s=t/T$,

$$\tilde{H}(s) = (1-s)H_0 + sH_m. \tag{11}$$

The algorithm consists in preparing the system in the state $|\psi(0)\rangle = |\psi_0\rangle$ and then applying the Hamiltonian $H(t)$ during a time T.

Now that our Hamiltonian $H(t)$ is well defined, we can solve the eigenproblem (2) to find the eigenvalues $E_k(t)$ and eigenstates $|E_k;t\rangle$, and then to evaluate the gap (3) and the matrix element in Eq. (4). The eigenvalues of $\tilde{H}(s) = H(sT)$ are plotted as a function of s in Fig. 1. The highest eigenvalue $E_2=1$ is $(N-2)$ times degenerated, while the two lowest ones E_0 and E_1 have a degeneracy 1 and are separated by a gap

$$g(s) = \sqrt{1 - 4\left(1 - \frac{1}{N}\right)s(1-s)}. \tag{12}$$

We note that the minimum gap $g_{\min} = 1/\sqrt{N}$ is attained for $s=1/2$.

The matrix element in Eq. (4) can be reexpressed by using

$$\left\langle \frac{dH}{dt} \right\rangle_{1,0} = \frac{ds}{dt} \left\langle \frac{d\tilde{H}}{ds} \right\rangle_{1,0} = \frac{1}{T} \left\langle \frac{d\tilde{H}}{ds} \right\rangle_{1,0} \tag{13}$$

with $d\tilde{H}/ds = H_m - H_0$. Bounding the matrix element from above using

$$\left| \left\langle \frac{d\tilde{H}}{ds} \right\rangle_{1,0} \right| \leqslant 1, \tag{14}$$

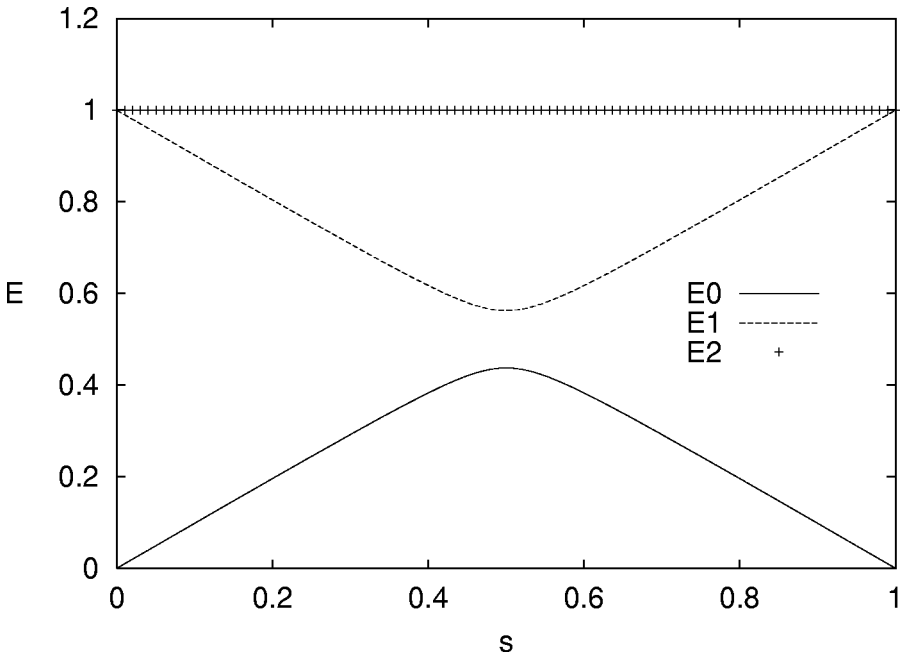

FIG. 1. Eigenvalues of the time-dependent Hamiltonian $\tilde{H}(s)$ as a function of the reduced time s for $N=64$. (With the convention $\hbar=1$, the energy as well as the reduced time are dimensionless quantities.)

we conclude that the adiabatic condition (6) is verified provided that

$$T \geqslant \frac{N}{\varepsilon}. \tag{15}$$

Thus, the computation time is of order N, and there is no advantage of this method compared to a classical search.

Now, let us show how to improve on this adiabatic-evolution method. One should note that by applying Eq. (6) globally, i.e., to the entire time interval T, we impose a limit on the evolution rate during the whole computation while this limit is only severe around $s=1/2$, where the gap g is minimum. Thus, by dividing T into infinitesimal time intervals dt and applying the adiabaticity condition locally to each of these intervals, we can vary the evolution rate continuously in time, thereby speeding up the computation. In other words, we do not use a linear evolution function $s(t)$ any more, but we adapt the evolution rate ds/dt to the local adiabaticity condition (in a very different context, namely, nuclear magnetic resonance, a similar method known as adiabatic fast passage has been widely used [11]). Let us find the optimum $s(t)$ with the boundary conditions $s(0)=0$ and $s(T)=1$. Applying Eq. (6) to each infinitesimal time interval, we get the new condition

$$\left|\frac{ds}{dt}\right| \leqslant \varepsilon \frac{g^2(s)}{\left|\left\langle \frac{d\tilde{H}}{ds} \right\rangle_{1,0}\right|} \tag{16}$$

for all times t. Using Eqs. (12) and (14), we choose to make the Hamiltonian evolve at a rate that is a solution of

$$\frac{ds}{dt} = \varepsilon g^2(s) = \varepsilon \left[1 - 4\frac{N-1}{N} s(1-s) \right], \tag{17}$$

where $\varepsilon \ll 1$. After integration, we find

$$t = \frac{1}{2\varepsilon} \frac{N}{\sqrt{N-1}} [\arctan\{\sqrt{N-1}(2s-1)\} + \arctan\sqrt{N-1}]. \tag{18}$$

By inverting this function, we obtain $s(t)$ as plotted in Fig. 2, which shows the gradual change in the switching between H_0 and H_m.

We see that $H(t)$ is changing faster when the gap $g(s)$ is large while it evolves slower when s is close to 1/2, that is, where the gap is minimum. We may now evaluate the computation time of our new algorithm by taking $s=1$, which gives

$$T = \frac{1}{\varepsilon} \frac{N}{\sqrt{N-1}} \arctan\sqrt{N-1}. \tag{19}$$

With the approximation $N \gg 1$, we get

$$T \simeq \frac{\pi}{2\varepsilon}\sqrt{N}, \tag{20}$$

improving upon Eq. (15). As a consequence, we obtain a quadratic speed-up with respect to a classical search, so that this algorithm can be viewed as the adiabatic-evolution version of Grover's algorithm. One can actually perform a more accurate calculation that takes into account the exact expression for $\langle d\tilde{H}/ds \rangle_{1,0}$ as a function of s, but this simply changes the prefactor in Eq. (20) to $\pi/(4\varepsilon)$. We have checked numerically that T scales as $\sqrt{N}$ for a given error probability ε^2 by solving the time-dependent Schrödinger equation up to $N=10^5$. It is worth noting that this scaling is robust against a change in the evolution function $s(t)$. In particular, if one adds a small component $\varepsilon g_{\min}^2$ in Eq. (17), the scaling remains unchanged while the prefactor in Eq.

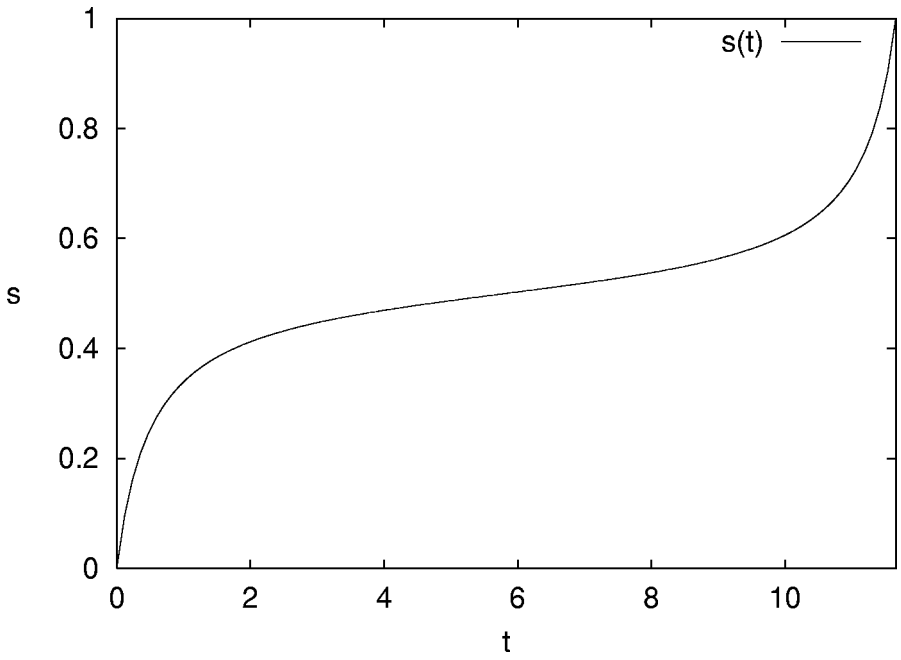

FIG. 2. Dynamic evolution of the Hamiltonian that drives the initial ground state to the solution state: plot of the evolution function $s(t)$ for $N=64$. The global adiabatic-evolution method of [5] would appear here as a straight line between $s(0)=0$ and $s(1)=1$.

(20) is increased. In the Appendix, we show that our algorithm is optimal, that is, the computation time cannot be shorter than $O(\sqrt{N})$ using any other evolution function $s(t)$.

Finally, let us prove that our algorithm can be extended to the case where there is more than one solution (see Fig. 3). Let M be the number of solutions. Equation (9) then becomes

$$H_m = I - \sum_{m \in \mathcal{M}} |m\rangle\langle m|, \tag{21}$$

where $\mathcal{M}$ is the ensemble of solutions (of size M). The Hamiltonian $\tilde{H}(s)$ remains defined as in Eq. (11) but it has a new eigenvalue E_3, which is $(M-1)$ times degenerated, while $E_2=1$ is now $(N-M-1)$ times degenerated. The eigenvalues E_0 and E_1 still have degeneracy 1, but they are now separated by the gap

$$g(s) = \sqrt{1-4\left(1-\frac{M}{N}\right)s(1-s)}. \tag{22}$$

When applying the adiabatic theorem to this case, one does not have to consider the eigenstates corresponding to E_2 and E_3 because they are totally uncoupled from the ground and first excited states. This is due to the fact that the Hamiltonians H_0 and H_m are symmetric under the permutation of any two solution states (or two nonsolution states), and that the eigenstates corresponding to E_0 and E_1 are the only ones

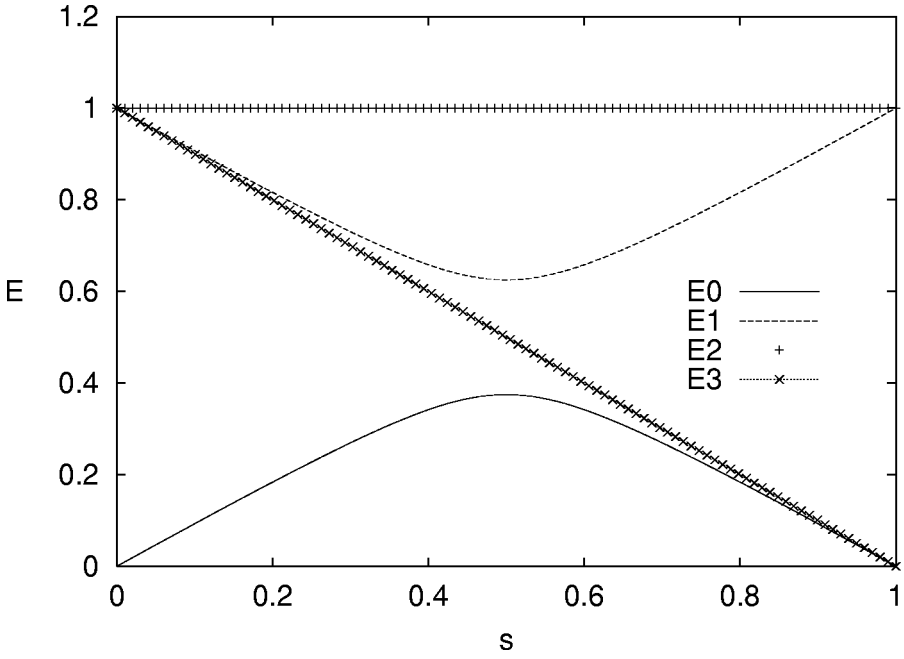

FIG. 3. Eigenvalues of the time-dependent Hamiltonian $\tilde{H}(s)$ as a function of the reduced time s in the case where there is more than one solution, for $N=64$ and $M=4$.

to have this symmetry. Therefore, we see that our previous reasoning can be applied to this case simply by replacing N by N/M. The minimum gap is now $g_{\min}=\sqrt{M/N}$. By applying the adiabatic condition (6) globally, we find the following bound for the computation time:

$$T \geqslant \frac{N}{\varepsilon M}, \tag{23}$$

while a local adiabatic evolution reduces this bound to

$$T \simeq \frac{\pi}{2\varepsilon}\sqrt{\frac{N}{M}}. \tag{24}$$

CONCLUSION

In this article, we have applied the adiabatic-evolution technique of [5] to design a quantum algorithm for solving Grover's problem, i.e., the search for a marked item in an unstructured database. We have shown that applying the adiabatic theorem globally (as in [5]) imposes a running time of order N, where N is the number of items in the database, whereas adjusting the evolution rate of the Hamiltonian continuously in time so as to fulfill the adiabaticity condition locally results in a time of order $\sqrt{N}$. We, therefore, recover the advantage of Grover's usual algorithm compared to a classical search [3]. This result also extends to the case of a problem with M solutions, for which we obtain a time of order $\sqrt{N/M}$. We should notice that this speed up was achieved by switching the Hamiltonian according to Eq. (18), which is only possible because the gap $g(s)$ can be derived analytically here, and does not depend on the solution $|m\rangle$ of the problem. As long as these conditions are satisfied, such a local adiabatic-evolution method could be applied to more complicated—and more realistic—problems such as NP-complete problems, treated using either a quantum adiabatic-evolution algorithm [7] or a nested version of Grover's algorithm exploiting the problem structure [8].

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APPENDIX: PROOF OF OPTIMALITY

In this appendix, we show that using our algorithm, no other choice of the evolution function $s(t)$ could lead to a better complexity than Eq. (20). This proof follows closely the lines of the optimality proof of the ''analog'' Grover's algorithm based on a time-independent Hamiltonian [4].

Let $|\psi_m,t\rangle$ be the state of our quantum register during the computation when the solution state is $|m\rangle$. After the computation time T, the states corresponding to different solutions (m and m') must be sufficiently different,

$$1-|\langle\psi_m,T|\psi_{m'},T\rangle|^2 \geqslant \varepsilon \quad \forall \quad m\neq m'. \tag{A1}$$

Let us decompose the Hamiltonian $\tilde{H}(s)$ into two parts

$$\tilde{H}(s)=\tilde{H}_1(s)+\tilde{H}_{2m}(s), \tag{A2}$$

where

$$\tilde{H}_1(s)=I-(1-s)|\psi_0\rangle\langle\psi_0|, \tag{A3}$$

$$\tilde{H}_{2m}(s)=-s|m\rangle\langle m|. \tag{A4}$$

$|\psi_m\rangle$ and $|\psi_{m'}\rangle$ are solutions of the Schrödinger equations

$$i\frac{d}{dt}|\psi_m,t\rangle=(H_1+H_{2m})|\psi_m,t\rangle, \tag{A5}$$

$$i\frac{d}{dt}|\psi_{m'},t\rangle=(H_1+H_{2m'})|\psi_{m'},t\rangle \tag{A6}$$

with initial conditions

$$|\psi_m,0\rangle=|\psi_{m'},0\rangle=|\psi_0\rangle. \tag{A7}$$

We will now derive a necessary condition on T for Eq. (A1) to be satisfied. Using Eqs. (A5) and (A6), we have

$$\frac{d}{dt}[1-|\langle\psi_m,t|\psi_{m'},t\rangle|^2] \tag{A8}$$

$$=2\,\mathrm{Im}[\langle\psi_m,t|H_{2m}-H_{2m'}|\psi_{m'},t\rangle\langle\psi_{m'},t|\psi_m,t\rangle] \tag{A9}$$

$$\leqslant 2|\langle\psi_m,t|H_{2m}-H_{2m'}|\psi_{m'},t\rangle||\langle\psi_{m'},t|\psi_m,t\rangle| \tag{A10}$$

$$\leqslant 2[|\langle\psi_m,t|H_{2m}|\psi_{m'},t\rangle|+|\langle\psi_m,t|H_{2m'}|\psi_{m'},t\rangle|]. \tag{A11}$$

Summing over m and m', we get

$$\frac{d}{dt}\sum_{m,m'}[1-|\langle\psi_m,t|\psi_{m'},t\rangle|^2] \tag{A12}$$

$$\leqslant 4\sum_{m,m'}|\langle\psi_m,t|H_{2m}|\psi_{m'},t\rangle| \tag{A13}$$

$$\leqslant 4\sum_{m,m'}\|H_{2m}|\psi_m,t\rangle\|\|\psi_{m'},t\rangle\| \tag{A14}$$

$$\leqslant 4N\sum_{m}\|H_{2m}|\psi_m,t\rangle\|, \tag{A15}$$

where we have used the Cauchy-Schwartz inequality along with the fact that $|\psi_{m'},t\rangle$ is normalized. The property

$$\sum_m \|H_{2m}|\psi,t\rangle\|^2=s^2 \Rightarrow \sum_m \|H_{2m}|\psi,t\rangle\| \leqslant \sqrt{N}s \tag{A16}$$

then leads to

$$\frac{d}{dt}\sum_{m,m'}[1-|\langle\psi_m,t|\psi_{m'},t\rangle|^2]\leqslant 4N\sqrt{N}s. \qquad \text{(A17)}$$

We may now integrate this inequality using the initial conditions (A7),

$$\sum_{m,m'}[1-|\langle\psi_m,t|\psi_{m'},t\rangle|^2]\leqslant 4N\sqrt{N}\int_0^T s(t)dt. \qquad \text{(A18)}$$

Finally, using condition (A1) and $0\leqslant s(t)\leqslant 1$, we find

$$T\geqslant\frac{\varepsilon}{4}\frac{N-1}{\sqrt{N}} \qquad \text{(A19)}$$

or, with $N\gg 1$,

$$T\geqslant\frac{\varepsilon}{4}\sqrt{N}. \qquad \text{(A20)}$$

We conclude that in order to be able to distinguish between states corresponding to different solutions ($\varepsilon>0$), the computation must last a minimum time of order $\sqrt{N}$, which is what we found in Eq. (20). Our choice of $s(t)$ is thus optimal.

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