Bell inequalities resistant to detector inefficiency

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We derive both numerically and analytically Bell inequalities and quantum measurements that present enhanced resistance to detector inefficiency. In particular, we describe several Bell inequalities which appear to be optimal with respect to inefficient detectors for small dimensionality $d=2,3,4$ and two or more measurement settings at each side. We also generalize the family of Bell inequalities described by Collins *et al.* [Phys. Rev. Lett. 88, 040404 (2002)] to take into account the inefficiency of detectors. In addition, we consider the possibility for pairs of entangled particles to be produced with probability less than 1. We show that when the pair production probability is small, one should in general use different Bell inequalities than when the pair production probability is high.

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I. INTRODUCTION

A striking feature of quantum entanglement is nonlocality. Indeed, as first shown by Bell in 1964 [1] classical local theories cannot reproduce all the correlations exhibited by entangled quantum systems. This nonlocal character of entangled states is demonstrated in Einstein-Podolsky-Rosen (EPR) type experiments through the violation of Bell inequalities. However due to experimental imperfections and technological limitations, Bell tests suffer from loopholes that allow, in principle, the experimental data to be reproduced by a local realistic description. The most famous of these loopholes are the locality loophole and the detection loophole [17]. Experiments carried on photons have closed the locality loophole [2] and recently Rowe *et al.* closed the detection loophole using trapped ions [3]. But so far, 30 years since the first experiments, both loopholes have not been closed in a *single* experiment.

The purpose of this paper is to study how one can devise new tests of nonlocality capable of lowering the detector efficiency necessary to reject any local realistic hypothesis. This could be a way towards a loophole-free test of Bell inequalities and is important for several reasons. First, as quantum entanglement is the basic ingredient of quantum information processing, it is highly desirable to possess undisputable tests of its properties such as nonlocality. Even if one is convinced (as we almost all are) that nature is quantum mechanical, we can imagine practical situations where it would be necessary to perform loophole-free tests of Bell inequalities. For example, suppose you buy a quantum cryptographic device based on the Ekert protocol. The security of your cryptographic apparatus relies on the fact that you can violate Bell inequalities with it. But if the detectors efficiencies are not sufficiently high, the salesman can exploit it and sell to you a classical device that will mimic a quantum device but which will enable him to read all your correspondence [4]. Other reasons to study the resistance of quantum tests to detector inefficiencies are connected to the classification of entanglement. Indeed an important classification of entanglement is related to quantum nonlocality. One proposed criterion to gauge how much nonlocality is exhibited by quantum correlations is the resistance to noise. This is what motivated the series of works [5,6] that led to the generalization of the CHSH inequality [12] to higher-dimensional systems [7]. The resistance to inefficient detectors is a second and different criterion that we analyze in this paper. It is closely related to the amount of classical communication required to simulate quantum correlations [8].

The idea behind the detection loophole is that in the presence of unperfect detectors, local hidden variable theories can "mask" results in contradiction with quantum mechanics by telling the detectors not to fire. This is at the origin of several local hidden variable models able to reproduce particular quantum correlations if the detector efficiencies are below some threshold value η_* (see Refs. [9–11], for example).

In this paper, we introduce two parameters that determine whether a detector will fire or not: η, the efficiency of the detector, and λ, the probability that the pair of particles is produced by the source of entangled systems. This last parameter may be important, for instance, for sources involving parametric down-conversion, where λ is typically less than 10%. So far, discussions on the detection loophole were concentrating on η, overlooking λ. However we will show below that both quantities play a role in the detection loophole and clarify the relation between these two parameters. In particular, we will introduce two different detector thresholds: η_*^{λ}, the value above which quantum correlations exhibit nonlocality for given λ, and $\eta_*^{\forall\lambda}$, the value above which quantum correlations exibit nonlocality for any λ.

We have written a numerical algorithm to determine these two thresholds for given quantum state and quantum measurements. We concentrated on the two extreme cases $\eta_*^{\lambda=1}$ and $\eta_*^{\forall\lambda}$. We searched for optimal measurements such that $\eta_*^{\lambda=1}$ and $\eta_*^{\forall\lambda}$ acquire the lowest possible value.

In the case of bipartite two dimensional systems the most important test of nonlocality is the CHSH inequality [12]. Quantum mechanics violates it if the detector efficiency η is above $=2/(\sqrt{2}+1)\approx 0.8284$ for the maximally entangled state of two qubits. In the limit of large-dimensional systems and large number of settings, it is shown in Ref. [8] that the

TABLE I. Optimal threshold detector efficiency for varying dimension d and number of settings ($N_a \times N_b$) for the detectors. $\eta_*^{\lambda=1}$ is the threshold efficiency for a source such that the pair production probability $\lambda=1$, while $\eta_*^{\forall\lambda}$ is the threshold efficiency independent of λ. The column p gives the amount of white noise p that can be added to the entangled state so that it still violates locality (we use for p the same definition as that given in Refs. [5,6]). The last column refers to the Bell inequality that reproduces the detection threshold. Except for the case $d=\infty$, these thresholds are the result of a numerical optimization carried out over the set of multiport beam-splitter measurements.

d	$N_a \times N_b$	$\eta_*^{\lambda=1}$	$\eta_*^{\forall\lambda}$	p	Bell inequality
2	2×2	0.8284	0.8284	0.2929	CHSH
2	3×3	0.8165		0.2000	Present paper (see also Refs. [1,19])
2	3×3		0.8217	0.2859	Present paper
2	3×4		0.8216	0.2862	Present paper
2	4×4		0.8214	0.2863	Present paper
3	2×2	0.8209	0.8209	0.3038	Based on Ref. [7]
3	2×3	0.8182	0.8182	0.2500	Present paper (related to Ref. [18])
3	3×3	0.8079		0.2101	Present paper
3	3×3		0.8146	0.2971	Present paper
4	2×2	0.8170	0.8170	0.3095	Based on Ref. [7]
4	2×3		0.8093	0.2756	Present paper
4	3×3		0.7939	0.2625	Present paper
5	2×2	0.8146	0.8146	0.3128	Based on Ref. [7]
6	2×2	0.8130	0.8130	0.3151	Based on Ref. [7]
7	2×2	0.8119	0.8119	0.3167	Based on Ref. [7]
∞	2×2	0.8049	0.8049	0.3266	Based on Ref. [7]

efficiency threshold can be arbitrarily lowered. This suggests that the way to devise optimal tests with respect to the resistance to detector inefficiencies is to increase the dimension of the quantum systems and the number of differents measurements performed by each party on these systems. (This argument is presented in more details in Ref. [11]). We have thus performed numerical searches for increasing dimensions and number of settings starting from the two-qubit, two-settings situation of the CHSH inequality. Our results concern a specific kind of measurement, namely "multiport beam-splitter measurements" [13], performed on maximally entangled states. They are summarized in Table I. Part of these results are accounted for by existing Bell inequalities, the other part led us to introduce new Bell inequalities.

The main conclusions that can be drawn from this work are as follows:

(1) Even in two dimensions, one can improve the resistance to inefficient detectors by increasing the number of settings.

(2) One can further increase the resistance to detection inefficiencies by increasing the dimension.

(3) There are different optimal measurements settings and Bell inequalities for a source that produces entangled particles with high probability ($\lambda\approx1$) and one that produces them extremely rarely ($\lambda\to0$). Bell inequalities associated with this last situation provide a detection threshold that does not depend on the value of the pair production probability.

(4) For the measurement scenarios numerically accessible, only small improvements in threshold detector efficiency are achieved. For instance the maximum change in threshold detector efficiency we found is $\approx4\%$

The paper is organized as follows. First, we review briefly the principle of an EPR experiment in Sec. II A and under which condition such an experiment admits a local-realistic description in Sec. II B. In Sec. II C we clarify the role played by η and λ in the detection loophole. We then present the technique we used to perform the numerical searches in Sec. II D and to construct the Bell inequalities presented in this paper in Sec. II E. Section III contains our results. In particular in, Sec. III A we generalize the family of inequalities introduced in Ref. [7] to take into account detection inefficiencies and in Sec. III C we present the two different Bell inequalities associated with the two-dimensional 3×3 settings measurement scenario. In the Appendix, we collect all the measurement settings and Bell inequalities we have obtained.

II. GENERAL FORMALISM

A. Quantum correlations

Let us review the principle of an a EPR experiment: two parties, Alice and Bob, share an entangled state ρ_{AB}. We take each particle to belong to a d-dimensional Hilbert space. The parties carry out measurements on their particles. Alice can choose between N_a different von Neumann measurements A_i ($i=1,\ldots,N_a$) and Bob can choose between N_b von Neumann measurements B_j ($j=1,\ldots,N_b$). Let k and l be Alice's and Bob's outcomes. We suppose that the number of possible outcomes is the same for each party and that the values of k and l belong to $\{0,\ldots,d-1\}$. To each measurement A_i is thus associated a complete set of d orthogonal projectors $A_i^k=|A_i^k\rangle\langle A_i^k|$ and similarly for B_j. Quantum mechanics predicts the following probabilities for the outcomes:

$$P_{kl}^{QM}(A_i,B_j)=\mathrm{Tr}[(A_i^k\otimes B_j^l)\rho_{ab}],$$
$$P_l^{QM}(B_j)=\mathrm{Tr}[(\mathbb{1}_A\otimes B_j^l)\rho_{ab}],$$
$$P_k^{QM}(A_i)=\mathrm{Tr}[(A_i^k\otimes\mathbb{1}_B)\rho_{ab}]. \tag{1}$$

In a real experiment, it can happen that the measurement gives no outcome due to detector inefficiencies, losses, or because the pair of entangled states has not been produced. To take into account these cases in the most general way, we enlarge the space of possible outcomes and add a new outcome, the "no-result outcome," which we label $\emptyset$. Quantum mechanics now predicts a modified set of correlations:

$$P_{\lambda\eta}^{QM}(A_i=k,B_j=l)=\lambda\eta^2P_{kl}^{QM}(A_i,B_j),\quad k,l\neq\emptyset,$$
$$P_{\lambda\eta}^{QM}(A_i=\emptyset,B_j=l)=\lambda\eta(1-\eta)P_l^{QM}(B_j),\quad l\neq\emptyset,$$
$$P_{\lambda\eta}^{QM}(A_i=k,B_j=\emptyset)=\lambda\eta(1-\eta)P_k^{QM}(A_i),\quad k\neq\emptyset,$$
$$P_{\lambda\eta}^{QM}(A_i=\emptyset,B_j=\emptyset)=1-\lambda+\lambda(1-\eta)^2, \tag{2}$$

where η is the detector efficiency and λ is the probability that a pair of particles is produced by the source of entangled systems. By detection efficiency η we mean the probability that the detector gives a result if a particle was produced, i.e., η includes not only the "true" efficiency of the detector but also all possible losses of the particle on the path from the source to the detectors.

B. Local hidden variable theories and Bell inequalities

Let us now define when the results (2) of an EPR experiment can be explained by a local hidden variable (LHV) theory. In a LHV theory, the outcome of Alice's measurement is determined by the setting A_i of Alice's measurement apparatus and by a random variable shared by both particles. This result should not depend on the setting of Bob's measurement apparatus if the measurements are carried out at spatially separated locations. The situation is similar for Bob's outcome. We can describe without loss of generality such a local variable theory by a set of $(d+1)^{N_a+N_b}$ probabilities $p_{K_1\cdots K_{N_a}L_1\cdots L_{N_b}}$, where Alice's local variables $K_i \in \{0,\ldots,d-1,\emptyset\}$ specify the result of the measurement A_i and Bob's variables $L_j\in\{0,\ldots,d-1,\emptyset\}$ specify the result of measurement B_j. The correlations $P(A_i=K,B_k=L)$ are obtained from these joint probabilities as marginals. The quantum predictions can then be reproduced by the LHV theory if and only if the following $N_aN_b(d+1)^2$ equations are obeyed:

$$\sum_{\mathbf{KL}} p_{\mathbf{KL}}\delta_{K_i,K}\delta_{L_j,L}=P^{QM}_{\lambda\eta}(A_i=K,B_j=L), \tag{3}$$

with the conditions

$$\sum_{\mathbf{KL}} p_{\mathbf{KL}}=1, \tag{4}$$

$$p_{\mathbf{KL}}\geqslant 0, \tag{5}$$

where we have introduced the notation $\mathbf{K}=K_1\ldots K_{N_a}$ and $\mathbf{L}=L_1\ldots L_{N_b}$. Note that Eqs. (3) are not all independent since quantum and classical probabilities share additional constraints such as the normalization conditions

$$\sum_{K,L} P(A_i=K,B_j=L)=1, \tag{6}$$

or the no-signalling conditions

$$P(A_i=K)=\sum_L P(A_i=K,B_j=L) \quad \forall j, \tag{7}$$

and similarly for B_j.

An essential result is that the necessary and sufficient conditions for a given probability distribution P^{QM} to be reproducible by a LHV theory can be expressed, alternatively to the Eqs. (3), as a set of linear inequalities for P^{QM}, the Bell inequalities [22]. They can be written as

$$I=I_{rr}+I_{\emptyset r}+I_{r\emptyset}+I_{\emptyset\emptyset}\leqslant c, \tag{8}$$

where

$$I_{rr}=\sum_{i,j}\sum_{k,l\neq\emptyset} c^{kl}_{ij}P(A_i=k,B_j=l),$$

$$I_{\emptyset r}=\sum_{i,j}\sum_{l\neq\emptyset} c^{\emptyset l}_{ij}P(A_i=\emptyset,B_j=l),$$

$$I_{r\emptyset}=\sum_{i,j}\sum_{k\neq\emptyset} c^{k\emptyset}_{ij}P(A_i=k,B_j=\emptyset),$$

$$I_{\emptyset\emptyset}=\sum_{i,j} c^{\emptyset\emptyset}_{ij}P(A_i=\emptyset,B_j=\emptyset). \tag{9}$$

For certain values of η and λ, quantum mechanics can violate one of the Bell inequalities (8) of the set. Such a violation is the signal for experimental demonstration of quantum nonlocality.

C. Detector efficiency and pair production probability

For a given quantum-mechanical probability distribution P^{QM} and given pair production probability λ, the maximum value of the detector efficiency η for which there exists a LHV variable model will be denoted $\eta^{\lambda}_{*}(P^{QM})$. It has been argued [9,14] that η_* should not depend on λ. The idea behind this argument is that the outcomes $(\emptyset,\emptyset)$ obtained when the pair of particles is not created are trivial and hence it seems safe to discard them. A more practical reason is that the pair production rate is rarely measurable in experiments. Whatever the case, the logical possibility exists that the LHV theory can exploit the pair production rate. Indeed, we will show below that this is the case when the number of settings of the measurement apparatus is larger than two. This motivates our definition of threshold detection efficiency valid for all values of λ,

$$\eta^{\forall\lambda}_{*}=\max_{\lambda\neq 0}(\eta^{\lambda}_{*})=\lim_{\lambda\to 0}\eta^{\lambda}_{*}. \tag{10}$$

The second equality follows from the fact that if a LHV model exists for a given value of λ it trivially also exists for a lower value of λ.

Let us study now the structure of the Bell expression $I(QM)$ given by quantum mechanics. This will allow us to derive an expression for $\eta^{\forall\lambda}_{*}$. Inserting the quantum probabilities (2) into the Bell expression of Eq. (8), we obtain

$$I(QM)=\lambda\eta^2 I^{QM}_{rr}+\lambda\eta(1-\eta)I^{QM}_{\emptyset r}+\lambda\eta(1-\eta)I^{QM}_{r\emptyset}+[1+\lambda(\eta^2-2\eta)]\sum_{i,j}c^{\emptyset\emptyset}_{ij}, \tag{11}$$

where I^{QM}_{rr} is obtained by replacing $P(A_i=k,B_j=l)$ with $P^{QM}_{kl}(A_i,B_j)$ in I_{rr} and $I^{QM}_{\emptyset r}$ by replacing $P(A_i=\emptyset,B_j=l)$ with $P^{QM}_{l}(B_j)$ in $I_{\emptyset r}$ and similarly for $I^{QM}_{r\emptyset}$.

For $\eta=0$, we know there exists a trivial LHV model and so the Bell inequalities cannot be violated. Replacing η by 0 in Eq. (11) we therefore deduce that

$$\sum_{i,j} c_{ij}^{\emptyset\emptyset} \leqslant c. \tag{12}$$

This divides the set of Bell inequalities into two groups: those such that $\Sigma_{i,j}c_{ij}^{\emptyset\emptyset}<c$ and those for which $\Sigma_{i,j}c_{ij}^{\emptyset\emptyset}=c$. Let us consider the first group. For small λ, these inequalities will cease to be violated. Indeed, take $\eta=1$ (which is the maximum possible value of the detector efficiency), then Eq. (11) reads

$$I(QM)=\lambda I_{rr}^{QM}+(1-\lambda)\sum_{i,j} c_{ij}^{\emptyset\emptyset}. \tag{13}$$

The condition for violation of the Bell inequality is $I(QM)>c$. But since $\Sigma_{i,j}c_{ij}^{\emptyset\emptyset}<c$, for sufficiently small λ we will have $I(QM)<c$ and the inequality will not be violated. These inequalities can therefore not be used to derive a threshold $\eta_*^{\forall\lambda}$ that does not depend on λ, but they are still interesting and will provide a threshold η_*^{λ} depending on λ. Let us now consider the inequalities such that $\Sigma_{i,j}c_{ij}^{\emptyset\emptyset}=c$. Then λ cancels in Eq. (11) and the condition for violation of the Bell inequality is that η must be greater than

$$\eta_*^{\forall\lambda}(P^{QM})=\frac{2c-I_{\emptyset r}^{QM}-I_{r\emptyset}^{QM}}{c+I_{rr}^{QM}-I_{\emptyset r}^{QM}-I_{r\emptyset}^{QM}} \tag{14}$$

independently of λ

It is interesting to note that if quantum mechanics violates a Bell inequality for perfect sources $\lambda=1$ and perfect detectors $\eta=1$, then there exists a Bell inequality that will be violated for $\eta<1$ and $\lambda\to 0$. That is there necessarily exists a Bell inequality that is insensitive to the pair production probability. Indeed the violation of a Bell inequality in the case $\lambda=1$, $\eta=1$ implies that there exists a Bell expression I_{rr} such that $I_{rr}(QM)>c$ with c the maximum value of I_{rr} allowed by LHV theories. Then let us build the following inequality:

$$I=I_{rr}+I_{r\emptyset}+I_{\emptyset r}+\sum_{i,j} c_{ij}^{\emptyset\emptyset}P(A_i=\emptyset,B_j=\emptyset)\leqslant c, \tag{15}$$

where $\Sigma_{i,j}c_{ij}^{\emptyset\emptyset}=c$ and we take in $I_{r\emptyset}$ and $I_{\emptyset r}$ sufficiently negative terms to ensure that $I\leqslant c$. For this inequality, $\eta_*^{\forall\lambda}=(2c-I_{\emptyset r}^{QM}-I_{r\emptyset}^{QM})/(c+I_{rr}^{QM}-I_{\emptyset r}^{QM}-I_{r\emptyset}^{QM})<1$, which shows that Bell inequalities valid $\forall\lambda$ always exist. One can, in principle, optimize this inequality by taking $I_{r\emptyset}$ and $I_{\emptyset r}$ as large as possible while ensuring that Eq. (15) is obeyed.

From the experimentalist's point of view, Bell tests involving inequalities that depend on λ need all events to be taken into account, including $(\emptyset,\emptyset)$ outcomes, while in tests involving inequalities insensitive to the pair production probability, it is sufficient to take into account events where at least one of the parties produces a result, i.e., double nondetection events $(\emptyset,\emptyset)$ can be discarded. Indeed, first note that one can always use the normalization conditions (6) to rewrite a Bell inequality such as Eq. (8) in a form where the term $I_{\emptyset\emptyset}$ does not appear. Second, when $\Sigma_{i,j}c_{ij}^{\emptyset\emptyset}=c$, this yields an inequality of the form $I_{rr}+I_{\emptyset r}+I_{r\emptyset}\leqslant 0$ which we can rewrite as $(I_{rr}+I_{\emptyset r}+I_{r\emptyset})/[\lambda(1-(1-\eta)^2)]\leqslant 0$ where $\lambda(1-(1-\eta)^2)=P(A_i\neq\emptyset \text{ or } B_j\neq\emptyset)$ is the probability that at least one detector clicks. Thus we obtain a new inequality expressed in term of the ratios $[P(A_i=k,B_j=l)]/[P(A_i\neq\emptyset \text{ or } B_j\neq\emptyset)]$, so that to check it one need only consider events where at least one detector fires.

D. Numerical search

We have carried numerical searches to find measurements such that the thresholds $\eta_*^{\lambda=1}$ and $\eta_*^{\forall\lambda}$ acquire the lowest possible value. This search is carried out in two steps. First of all, for given quantum-mechanical probabilities, we have determined the maximum value of η for which there exists a local hidden variable theory. Second we have searched over the set of multiport beam-splitter measurements to find the minimum values of η_*.

In order to carry out the first step, we have used the fact that the question of whether there are classical joint probabilities that satisfy Eq. (3) with the conditions (4),(5) is a typical linear optimization problem for which there exist efficient algorithms [15]. We have written a program that, given λ, η and a set of quantum measurements, determines whether Eq. (3) admits a solution or not. η_*^{λ} is then determined by performing a dichotomic search on the maximal value of η, so that the set of constraints is satisfied.

However when searching for $\eta_*^{\forall\lambda}$ it is possible to dispense with the dichotomic search by using the following trick. First of all because all the equations in Eq. (3) are not independent, we can remove the constraints that involve on the right-hand side the probabilities $P(A_i=\emptyset,B_j=\emptyset)$. Second we define rescaled variables $\lambda[1-(1-\eta)^2]\tilde{p}_{\mathbf{KL}}=p_{\mathbf{KL}}$. Inserting the quantum probabilities, Eq. (2), we obtain the set of equations

$$\sum_{\mathbf{KL}} \tilde{p}_{\mathbf{KL}}\delta_{K_i,k}\delta_{L_j,l}=\alpha P_{kl}^{QM}(A_i,B_j),\quad k,l\neq\emptyset,$$

$$\sum_{\mathbf{KL}} \tilde{p}_{\mathbf{KL}}\delta_{K_i,\emptyset}\delta_{L_j,l}=\left(1-\frac{\alpha}{2}\right)P_{l}^{QM}(B_j),\quad l\neq\emptyset,$$

$$\sum_{\mathbf{KL}} \tilde{p}_{\mathbf{KL}}\delta_{K_i,k}\delta_{L_j,\emptyset}=\left(1-\frac{\alpha}{2}\right)P_{k}^{QM}(A_i),\quad k\neq\emptyset, \tag{16}$$

with the normalization

$$\sum_{\mathbf{KL}} \tilde{p}_{\mathbf{KL}}=\frac{1}{\lambda}\,\frac{1}{1-(1-\eta)^2}, \tag{17}$$

where $\alpha=\eta^2/[1-(1-\eta)^2]$. Note that λ only appears in the last equation. We want to find the maximum α such that these equations are obeyed for all λ. Since $0<\lambda\leqslant 1$ [21] we can replace the last equation by the condition

$$\sum_{\mathbf{KL}} \tilde{p}_{\mathbf{KL}} \geq 1. \tag{18}$$

We thus are led to search for the maximum α such that Eqs. (16) are satisfied and that the $\tilde{p}_{\mathbf{KL}}$ are positive and obey condition (18). In this form the search for $\eta_*^{\forall\lambda}$ has become a linear optimization problem and can be efficiently solved numerically.

Given the two algorithms that compute $\eta_*^{\lambda=1}$ and $\eta_*^{\forall\lambda}$ for given settings, the last part of the program is to find the optimal measurements. In our search over the space of quantum strategies we considered the maximally entangled state $\Psi=\Sigma_{m=0}^{d-1}|m\rangle_a|m\rangle_b$ in dimension d. The possible measurements A_i and B_j we considered are the multiport beam-splitter measurements described in Ref. [13] and which have in previous numerical searches yielded highly nonlocal quantum correlations [5,6]. These measurements are parametrized by d phases $(\phi_{A_i}^1,\ldots,\phi_{A_i}^d)$ and $(\phi_{B_j}^1,\ldots,\phi_{B_j}^d)$ and involve the following steps: first each party acts with the phase $\phi_{A_i}(m)$ or $-\phi_{B_j}(m)$ on the state $|m\rangle$, they both then carry out a discrete Fourier transform. This brings the state Ψ to

$$\Psi=\frac{1}{d^{3/2}}\sum_{k,l,m=0}^{d-1}\exp\left[i\left(\phi_{A_i}(m)-\phi_{B_j}(m)+\frac{2\pi}{d}m(k-l)\right)\right]|k\rangle_a|l\rangle_b. \tag{19}$$

Alice then measures $|k\rangle_a$ and Bob $|l\rangle_b$. The quantum probabilities (1) thus take the form

$$P_{kl}^{QM}(A_i,B_j)=\frac{1}{d^3}\left|\sum_{m=0}^{d-1}\exp\left[i\left(\phi_{A_i}(m)-\phi_{B_j}(m)+\frac{2\pi m}{d}(k-l)\right)\right]\right|^2,$$

$$P_k^{QM}(A_i)=1/d,\quad P_l^{QM}(B_j)=1/d. \tag{20}$$

The search for minimal $\eta_*^{\lambda=1}$ and $\eta_*^{\forall\lambda}$ then reduces to a nonlinear optimization problem over Alice's and Bob's phases. For this, we used the "amoeba" search procedure with its starting point fixed by the result of a randomized search algorithm. The amoeba procedure [16] finds the extremum of a nonlinear function F of N variables by constructing a simplex of $N+1$ vertices. At each iteration, the method evaluates F at one or more trial point. The purpose of each iteration is to create a new simplex in which the previous worst vertex has been replaced. The simplex is altered by reflection, expansion, or contraction, depending on whether F is improving. This is repeated until the diameter of the simplex is less than the specified tolerance.

Note that these searches are time consuming. Indeed, the first part of the computation, the solution to the linear problem, involves the optimization of $(d+1)^{N_a+N_b}$ parameters, the classical probabilities $p_{\mathbf{KL}}$ (the situation is even worse for η_*^{λ}, since the linear problem has to be solved several times while performing a dichotomic search for η_*^{λ}). Then when searching for the optimal measurements, the first part of the algorithm has to be performed for each phase setting. This results in a rapid exponential growth of the time needed to solve the entire problem with the dimension and the number of settings involved. A second factor that complicates the search for optimal measurements is that due to the relatively large number of parameters that the algorithm has to optimize, it can fail to find the global minimum and converge to a local minimum. This is one of the reasons why we restricted our searches to multiport beam-splitter measurements, since the number of parameters needed to describe them is much lesser than that for general von Neumann measurements.

Our results for setups our computers could handle in reasonable time are summarized in Table I. In two dimensions, we also performed more general searches using von Neuman measurements, but the results we obtained were the same as for the multiport beam splitters described above.

E. Optimal Bell inequalities

Upon finding the optimal quantum measurements and the corresponding values of η_*, we have tried to find the Bell inequalities which yield these threshold detector efficiencies. This is essential to confirm analytically these numerical results and also in order for them to have practical significance, i.e., to be possible to implement them in an experiment.

To find these inequalities, we have used the approach developed in Ref. [7]. The first idea of this approach is to make use of the symmetries of the quantum probabilities and to search for Bell inequalitites which have the same symmetry. Thus, for instance, if $P(A_i=k,B_j=l)=P(A_i=k+m \pmod d, B_j=l+m \pmod d)$ for all $m\in\{0,\ldots,d-1\}$, then it is useful to introduce the probabilities

$$P(A_i=B_j+n)=\sum_{m=0}^{d-1}P(A_i=m,B_j=n+m \pmod d),$$

$$P(A_i\neq B_j+n)=\sum_{\substack{m=0\\ l\neq n}}^{d-1}P(A_i=m,B_j=l+m \pmod d) \tag{21}$$

and to search for Bell inequalities written as linear combinations of $P(A_i=B_j+n)$. This reduces considerably the number of Bell inequalities among which one must search in order to find the optimal one. The second idea is to search for the logical contradictions which force the Bell inequality to take a small value in the case of LHV theories. Thus the Bell inequality will contain terms with different weights, positive and negative, but the LHV theory cannot satisfy all the relations with large positive weights. Once we had identified a candidate Bell inequality, we ran a computer program that enumerated all the deterministic classical strategies and computed the maximum value of the Bell inequality. The deterministic classical strategies are those for which the probabilities $p_{K_1 \cdots K_{N_a} L_1 \cdots L_{N_b}}$ are equal either to 0 or to 1. In order to find the maximum classical value of a Bell expression, it suffices to consider them, since the other strategies are obtained as convex combinations of the deterministic ones [22].

However when the number of settings, N_a and N_b, and the dimensionality d increase, it becomes more and more difficult to find the optimal Bell inequalities using the above analytical approach. We therefore developed an alternative method based on the numerical algorithm which is used to find the threshold detection efficiency.

The idea of this numerical approach is based on the fact that the probabilities for which there exists a solution $p_{\mathbf{KL}}$ to Eqs. (3)–(5) form a convex polytope whose vertices are the deterministic strategies. The facets of this polytope are hyperplanes of dimension $D-1$, where D is the dimension of the space in which lies the polytope [D is lower than the dimension $(d+1)^{N_a+N_b}$ of the total space of probabilities due to constraints such as the normalizations conditions (4) and the no-signalling conditions (7)]. These hyperplanes of dimension $D-1$ correspond to Bell inequalities.

At the threshold η_*, the quantum probability $P^{QM}_{\lambda\eta_*}$ belongs to the boundary, i.e., to one of the faces, of the polytope determined by Eqs. (3)–(5). The solution $p^*_{\mathbf{KL}}$ to these equations at the threshold is computed by our algorithm and it corresponds to the convex combinations of deterministic strategies that reproduce the quantum correlations. From this solution it is then possible to construct a Bell inequality. Indeed, the face F to which $P^{QM}_{\lambda\eta_*}$ belongs is the plane passing through the deterministic strategies involved in the convex combination $p^*_{\mathbf{KL}}$. Either this face F is a facet, i.e., a hyperplane of dimension $D-1$, or F is of dimension lower than $D-1$. In the first case, the hyperplane F corresponds to the Bell inequality we are looking for . In the second case, there is an infinity of hyperplanes of dimension $D-1$ passing through F, indeed every vector $\vec{v}$ belonging to the space orthogonal to the face F determines such an hyperplane. To select one of these hyperplanes lying outside the polytope, and thus corresponding effectively to a Bell inequality, we took as vector $\vec{v}$ the component normal to F of the vector that connects the center of the polytope (that is, the vector which is an equal sum of all the deterministric strategies) and the quantum probabilities when $\eta=1$: $P^{QM}_{\lambda,\eta=1}$. Though this choice of $\vec{v}$ is arbitrary, it yields Bell inequalities which preserve the symmetry of the probabilities P^{QM}.

As in the analytical method given above, we have verified by enumeration of the deterministic strategies that this hyperplane is indeed a Bell inequality (i.e., that it lies on one side of the polytope) and that it yields the threshold detection efficiency η_*.

III. RESULTS

Our results are summarized in Table I. We now describe them in more detail.

A. Arbitrary dimension, two settings on each side ($N_a = N_b = 2$)

For dimensions up to seven, we found numerically that $\eta_*^{\lambda=1} = \eta_*^{\forall\lambda}$. The optimal measurements we found are identical to those maximizing the generalization of the CHSH inequality to higher-dimensional systems [7], thus confirming their optimality not only for the resistance to noise but also for the resistance to inefficient detectors. Our values of η_* are identical to those given in Ref. [6], where $\eta_*^{\lambda=1}$ has been calculated for these particular settings for $2 \leq d \leq 16$.

We now derive a Bell inequality that reproduces analytically these numerical results (which has also been derived by Gisin [14]). Our Bell inequality is based on the generalization of the CHSH inequality obained in Ref. [7]. We recall the form of the Bell expression used in this inequality:

$$
\begin{aligned}
I_{rr}^{d,2\times 2} = \sum_{k=0}^{[d/2]-1} \left(1-\frac{2k}{d-1}\right) ([& P(A_1=B_1+k) \\
& + P(B_1=A_2+k+1) + P(A_2=B_2+k) \\
& + P(B_2=A_1+k)] - [P(A_1=B_1-k-1) \\
& + P(B_1=A_2-k) + P(A_2=B_2-k-1) \\
& + P(B_2=A_1-k-1)]).
\end{aligned} \tag{22}
$$

For local theories, $I_{rr}^{d,2\times 2} \leq 2$ as shown in Ref. [7] where the value of $I_{rr}^{d,2\times 2}(QM)$ given by the optimal quantum measurements is also described. In order to take into account no-result outcomes we introduce the following inequalities:

$$I^{d,2\times 2} = I_{rr}^{d,2\times 2} + \frac{1}{2}\sum_{i,j} P(A_i=\emptyset, B_j=\emptyset) \leq 2. \tag{23}$$

Let us prove that the maximal allowed value of $I^{d,2\times 2}$ for local theories is 2. To this end it suffices to enumerate all the deterministic strategies. First, if all the local variables correspond to a "result" outcome, then $I_{rr}^{d,2\times 2} \leq 2$ and $I_{\emptyset\emptyset}^{d,2\times 2} = \frac{1}{2}\sum_{i,j} P(A_i=\emptyset, B_j=\emptyset) = 0$ so that $I^{d,2\times 2} \leq 2$; if one of the local variables is equal to $\emptyset$ then again $I_{rr}^{d,2\times 2} \leq 2$ (since the maximal weight of a probability in $I_{rr}^{d,2\times 2}$ is 1 and they are only two such probabilities different from zero) and $I_{\emptyset\emptyset}^{d,2\times 2} = 0$; if there are two $\emptyset$ outcomes, then $I_{rr}^{d,2\times 2} \leq 1$ and $I_{\emptyset\emptyset}^{d,2\times 2} \leq 1$; while if there are three or four $\emptyset$ then $I_{rr}^{d,2\times 2} = 0$ and $I_{\emptyset\emptyset}^{d,2\times 2} \leq 2$.

Note that the inequality (23) obeys the condition $\sum_{i,j} c_{ij}^{\emptyset\emptyset} = c$, hence it will provide a bound on $\eta_*^{\forall\lambda}$. Using Eq. (14), we obtain the value of $\eta_*^{\forall\lambda}$:

$$\eta_*^{\forall\lambda} = \frac{4}{I_{rr}^{d,2\times 2}(QM)+2}. \tag{24}$$

Inserting the optimal values of $I_{rr}^{d,2\times 2}(QM)$ given in Ref. [7] this reproduces our numerical results and those of Ref. [6]. As an example, for dimension 3, $I_{rr}^{3,2\times 2}(QM) = 2.873$ so that $\eta_*^{\forall\lambda} = 0.8209$. When $d \to \infty$, Eq. (24) gives the limit $\eta_*^{\forall\lambda} = 0.8049$.

B. Three dimensions, 2×3 settings

For three-dimensional systems, we found that adding one setting to one of the parties decreases both $\eta_*^{\lambda=1}$ and $\eta_*^{\forall\lambda}$

from 0.8209 to 0.8182 (In the case of $d=2$, it is necessary to take three settings on each side to get an improvement). The optimal settings involved are $\phi_{A_1}=(0,0,0)$, $\phi_{A_2}=(0,2\pi/3,0)$, $\phi_{B_1}=(0,\pi/3,0)$, $\phi_{B_2}=(0,2\pi/3,-\pi/3)$, $\phi_{B_3}=(0,-\pi/3,-\pi/3)$.

We have derived a Bell expression associated with these measurements:

$$\begin{aligned} I_{rr}^{3,2\times 3}= & +[P(A_1=B_1)+P(A_1=B_2)+P(A_1=B_3) \\ & +P(A_2=B_1+1)+P(A_2=B_2+2)+P(A_2=B_3)] \\ & -[P(A_1\neq B_1)+P(A_1\neq B_2)+P(A_1\neq B_3) \\ & +P(A_2\neq B_1+1)+P(A_2\neq B_2+2)+P(A_2\neq B_3)]. \end{aligned} \tag{25}$$

The maximal value of $I_{rr}^{3,2\times 3}$ for classical theories is two since for any choice of local variables four relations with a $+$ can be satisfied, but then two with a $-$ are also satisfied. For example, we can satisfy the first four relations but this implies $A_2=B_2+1$ and $A_2=B_3+1$, which gives two $-$ terms. The maximal value of $I_{rr}^{3,2\times 3}$ for quantum mechanics is given for the settings described above and is equal to $I_{rr}^{3,2\times 3}(QM)=10/3$. To take into account detection inefficiencies, consider the following inequality:

$$I^{3,2\times 3}=I_{rr}^{3,2\times 3}+I_{\emptyset r}^{3,2\times 3}+I_{\emptyset\emptyset}^{3,2\times 3}\leqslant 2, \tag{26}$$

where

$$I_{\emptyset r}^{3,2\times 3}=-\frac{1}{3}\sum_{i,j} P(A_i=\emptyset,B_j\neq\emptyset) \tag{27}$$

and

$$I_{\emptyset\emptyset}^{3,2\times 3}=\frac{1}{3}\sum_{i,j} P(A_i=\emptyset,B_j=\emptyset) \tag{28}$$

($I_{r\emptyset}$ is taken equal to zero). The principle used to show that $I^{3,2\times 3}\leqslant 2$ is the same as that used to prove that $I^{d,2\times 2}\leqslant 2$. For example, if $A_1=\emptyset$, then $I_{rr}^{3,2\times 3}\leqslant 3$, $I_{\emptyset r}^{3,2\times 3}=-1$, and $I_{\emptyset\emptyset}^{3,2\times 3}=0$, so that $I^{3,2\times 3}\leqslant 3-1=2$. From Eq. (26) and the joint probabilities (20) for the optimal quantum measurements, we deduce

$$\eta_*^{\forall\lambda}=\frac{6}{\frac{10}{3}+4}=\frac{9}{11}\simeq 0.8182, \tag{29}$$

in agreement with our numerical result.

Note that in Ref. [18], an inequality formally idendical to Eq. (25) has been introduced. However, the measurements scenario involves two measurements on Alice's side and nine binary measurements on Bob's side. By grouping appropriately the outcomes, this measurements scenario can be associated to an inequality formally identical to Eq. (25) for which the violation reaches $2\sqrt{3}$. According to Eq. (29), this result in a detection efficiency threshold $\eta_*^{\forall\lambda}$ of $6/(2\sqrt{3}+4)\approx 0.8038$.

C. Three settings for both parties

For three settings per party, things become more surprising. We have found measurements that lower $\eta_*^{\lambda=1}$ and $\eta_*^{\forall\lambda}$ with respect to 2×2 or 2×3 settings. But contrary to the previous situations, $\eta_*^{\lambda=1}$ is not equal to $\eta_*^{\forall\lambda}$, and the two optimal values are obtained for two different sets of measurements. We present in this section the two Bell inequalities associated to each of these situations for the qubit case. Let us first begin with the inequality for $\eta_*^{\lambda=1}$:

$$\begin{aligned} I_{rr}^{2,3\times 3,\lambda}= & E(A_1,B_2)+E(A_1,B_3)+E(A_2,B_1)+E(A_3,B_1) \\ & -E(A_2,B_3)-E(A_3,B_2)-\frac{4}{3}P(A_1\neq B_1) \\ & -\frac{4}{3}P(A_2\neq B_2)-\frac{4}{3}P(A_3\neq B_3)\leqslant 2, \end{aligned} \tag{30}$$

where $E(A_i,B_j)=P(A_i=B_j)-P(A_i\neq B_j)$. As usual, the fact that $I_{rr}^{2,3\times 3}\leqslant 2$ follows from considering all deterministic classical strategies. The maximal quantum-mechanical violation for this inequality is 3 and is obtained by performing the same measurements on both sides $A_1=B_1$, $A_2=B_2$, $A_3=B_3$ defined by the following phases: $\phi_{A_1}=(0,0)$, $\phi_{A_2}=(0,\pi/3)$, $\phi_{A_3}=(0,-\pi/3)$. It is interesting to note that this inequality and these settings are related to those considered by Bell [1] and Wigner [19] in the first works on quantum nonlocality. But whereas in these works it was necessary to suppose that A_i and B_j are perfectly (anti) correlated when $i=j$ in order to derive a contradiction with LHV theories, here imperfect correlations $P(A_i\neq B_i)>0$ can also lead to a contradiction since they are included in the Bell inequality.

If we now consider no-result outcomes, we can use $I_{rr}^{2,3\times 3,\lambda}$ without adding extra terms, and the quantum correlations obtained from the optimal measurements violate the inequality if

$$\lambda\,\eta^2>\frac{2}{I_{rr}^{2,3\times 3,\lambda}(QM)}=\frac{2}{3}. \tag{31}$$

Taking $\lambda=1$, we obtain $\eta_*^{\lambda=1}=\sqrt{2/3}\simeq 0.8165$. For smaller values of λ, η_*^{λ} increases until $\eta_*^{\lambda}=16/19$ is reached for $\lambda\simeq 0.9401$. At that point the contradiction with local theories ceases to depend on the production rate λ and one should switch to the following inequality:

$$\begin{aligned} I_{rr}^{2,3\times 3,\forall\lambda}= & \frac{2}{3}E(A_1,B_2)+\frac{4}{3}E(A_1,B_3)+\frac{4}{3}E(A_2,B_1) \\ & +\frac{2}{3}E(A_3,B_1)-\frac{4}{3}E(A_2,B_3)-\frac{2}{3}E(A_3,B_2) \\ & -\frac{4}{3}P(A_1\neq B_1)-\frac{4}{3}P(A_2\neq B_2)-\frac{4}{3}P(A_3\neq B_3) \\ & \leqslant 2. \end{aligned} \tag{32}$$

This inequality is similar to the former one, Eq. (30), but the symmetry between the $E(A_i,B_j)$ terms has been broken: half of the terms have an additional weight of 1/3 and the others of $-1/3$. The total inequality involving no-result outcomes is

$$I^{2,3\times3,\forall\lambda}=I_{rr}^{2,3\times3,\forall\lambda}+I_{\emptyset r}^{2,3\times3,\forall\lambda}+I_{r\emptyset}^{2,3\times3,\forall\lambda}+I_{\emptyset\emptyset}^{2,3\times3,\forall\lambda}\leqslant 2. \tag{33}$$

The particular form of the terms $I_{\emptyset r}^{2,3\times3,\forall\lambda}$, $I_{r\emptyset}^{2,3\times3,\forall\lambda}$, and $I_{\emptyset\emptyset}^{2,3\times3,\forall\lambda}$ is given in the Appendix. The important point is that $\Sigma_{i,j,k}(c_{ij}^{k\emptyset}+c_{i,j}^{\emptyset k})=-8/3$ and $\Sigma_{i,j}c_{ij}^{\emptyset\emptyset}=2$. From Eqs. (14), (9), and (20), we thus deduce

$$\eta_*^{\forall\lambda}=\frac{4+\frac{4}{3}}{I_{rr}^{2,3\times3,}(QM)^{\forall\lambda}+2+\frac{4}{3}} \tag{34}$$

The measurements that optimize the former inequality (30) give the threshold $\eta_*^{\forall\lambda}=16/19$. However these measurements are not the optimal ones for Eq. (32). The optimal phase settings are given in the Appendix. Using these settings it follows that $I_{rr}^{2,3\times3,\forall\lambda(QM)}=3.157$ and $\eta_*^{\forall\lambda}\simeq0.8217$.

D. More settings and more dimensions

Our numerical algorithm has also yielded further improvements when the number of settings increases or the dimension increases. These results are summarized in Table I. For more details, see the Appendix.

IV. CONCLUSION

In summary, we have obtained using both numerical and analytical techniques, a large number of Bell inequalities and optimal quantum measurements that exhibit an enhanced resistance to detector inefficiency. This should be contrasted with the work (reported in Refs. [5,6]) devoted to searching for Bell inequalities and measurements with increased resistance to noise. In this case only a single family has been found involving two settings on each side despite extensive numerical searches (mainly unpublished, but see Ref. [15]). Thus the structure of Bell inequalities resistant to inefficient detectors seems much richer. It would be interesting to understand the reason for such additional structure and clarify the origin of these inequalities.

It should be noted that for the Bell inequalities that we have found, the amount by which the theshold detector efficiency η_* decreases is very small, of the order of 4%. This is tantalizing because we know that for sufficiently large dimension and sufficiently large number of settings, the detector efficiency threshold decreases exponentially. To increase further the resistance to an inefficent detector, it would perhaps be necessary to consider more general measurements than the one we considered in this work or use nonmaximally entangled states (for instance, Eberhard has shown that for two-dimensional systems, the efficiency threshold η_* can be lowered to 2/3 using nonmaximally entangled states [20]). There may thus be a Bell inequality of real practical importance for closing the detection loophole just behind the corner.

Note added in proof. Recently, we found numerically that for three dimensions $(d=3)$ and three measurement settings at each site $(N_a\times N_b=3\times3)$, the lowest value of $\eta_*^{\forall\lambda}$ is attained for general von Neumann measurements, and not for multiport beam-splitter measurements. The lowest value found numerically in this case is $\eta_*^{\forall\lambda}=0.7951$. This shows that multiport beam-splitter measurements are not always the best measurements for exhibiting nonlocality.

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APPENDIX

For completeness, we present here in detail all the Bell inequalities and optimal phase settings we have found. This includes also the results of Table I which have not been discussed in the text.

1. $N_A=2$, $N_B=2$, $\forall\lambda$

Bell inequality,

$$\begin{aligned}I^{d,2\times2}=\sum_{k=0}^{[d/2]-1}\left(1-\frac{2k}{d-1}\right)\{&+[P(A_1=B_1+k)\\&+P(B_1=A_2+k+1)+P(A_2=B_2+k)\\&+P(B_2=A_1+k)]-[P(A_1=B_1-k-1)\\&+P(B_1=A_2-k)+P(A_2=B_2-k-1)\\&+P(B_2=A_1-k-1)]\}\\&+\frac{1}{2}\sum_{i,j=1}^{2}P(A_i=\emptyset,B_j=\emptyset)\leqslant 2.\end{aligned}$$

Optimal phase settings,

$$\phi_{A_1}(j)=0,\quad \phi_{A_2}(j)=\frac{\pi}{d}j,$$

$$\phi_{B_1}(j)=\frac{\pi}{2d}j,\qquad \phi_{B_2}(j)=-\frac{\pi}{2d}j.$$

Maximal violation,

$$I^{d,2\times2}(QM)=4d\sum_{k=0}^{[d/2]-1}\left(1-\frac{2k}{d-1}\right)(q_k-q_{-k-1}),$$

where $q_k=1/(2d^3\sin^2[\pi(k+1/4)/d])$.

Detection threshold,

$$\eta_*^{\forall\lambda}=\frac{4}{I^{d,2\times 2}(QM)}+2.$$

2. $d=2$, $N_A=3$, $N_B=3$, λ

Bell inequality,

$$I^{2,3\times 3,\lambda}=E(A_1,B_2)+E(A_1,B_3)+E(A_2,B_1)+E(A_3,B_1)$$
$$-E(A_2,B_3)-E(A_3,B_2)-\frac{4}{3}P(A_1\neq B_1)$$
$$-\frac{4}{3}P(A_2\neq B_2)-\frac{4}{3}P(A_3\neq B_3)\leqslant 2$$

where $E(A_i,B_j)=P(A_i=B_j)-P(A_i\neq B_j)$.

Optimal phase settings,

$$\phi_{A_1}=(0,0),\quad \phi_{A_2}=(0,\pi/3),\quad \phi_{A_3}=(0,-\pi/3),$$
$$\phi_{B_1}=(0,0),\quad \phi_{B_2}=(0,\pi/3),\quad \phi_{B_3}=(0,-\pi/3).$$

Maximal violation, $I^{2,3\times 3,\lambda}(QM)=3$.
Detection threshold, $\eta_*^{\lambda}=\sqrt{2/3\lambda}$.

3. $d=2$, $N_A=3$, $N_B=3$, $\forall\ \lambda$

Bell inequality,

$$I^{2,3\times 3,\forall\lambda}=\frac{2}{3}E(A_1,B_2)+\frac{4}{3}E(A_1,B_3)+\frac{4}{3}E(A_2,B_1)$$
$$+\frac{2}{3}E(A_3,B_1)-\frac{4}{3}E(A_2,B_3)-\frac{2}{3}E(A_3,B_2)$$
$$-\frac{4}{3}P(A_1\neq B_1)-\frac{4}{3}P(A_2\neq B_2)-\frac{4}{3}P(A_3\neq B_3)$$
$$-\frac{2}{3}F_\emptyset(A_1,B_2)-\frac{4}{3}F_\emptyset(A_2,B_3)-\frac{2}{3}F_\emptyset(A_3,B_1)$$
$$+\frac{2}{3}F_\emptyset(A_3,B_2)+\frac{4}{3}P(A_2=\emptyset,B_1\neq\emptyset)$$
$$+\frac{4}{3}P(A_1\neq\emptyset,B_3=\emptyset)+\frac{4}{3}P(A_1=\emptyset,B_1=\emptyset)$$
$$+\frac{4}{3}P(A_2=\emptyset,B_1=\emptyset)+\frac{4}{3}P(A_1=\emptyset,B_3=\emptyset)\leqslant 2,$$

where $E(A_i,B_j)=P(A_i=B_j)-P(A_i\neq B_j)$ and $F_\emptyset(A_i,B_j)=P(A_i=\emptyset,B_j\neq\emptyset)+P(A_i\neq\emptyset,B_j=\emptyset)+P(A_i=\emptyset,B_j=\emptyset)$.

Optimal phase settings,

$$\phi_{A_1}=(0,0),\quad \phi_{A_2}=(0,1.3934),$$
$$\phi_{A_3}=(0,-0.7558),$$
$$\phi_{B_1}=(0,0.5525)\,\phi_{B_2}=(0,1.3083)$$
$$\phi_{B_3}=(0,-0.8410).$$

Maximal violation, $I^{2,3\times 3,\forall\lambda}(QM)=3.157$
Detection threshold, $\eta_*^{\forall\lambda}=0.8217$.

4. $d=2$, $N_A=3$, $N_B=4$, $\forall\ \lambda$

Bell inequality,

$$I^{2,3\times 4,\forall\lambda}=-P(A_1\neq B_2)-P(A_1\neq B_3)-P(A_1\neq B_4)$$
$$+P(A_2=B_1)+P(A_2=B_2)-P(A_2\neq B_3)$$
$$+P(A_2\neq B_4)-P(A_3=B_1)+P(A_3=B_2)$$
$$-P(A_3\neq B_2)+P(A_3\neq B_3)-P(A_3=B_4)$$
$$+P(A_1\neq\emptyset,B_1=\emptyset)+P(A_2=\emptyset,B_1\neq\emptyset)$$
$$-P(A_3\neq\emptyset,B_1=\emptyset)-P(A_1=\emptyset,B_2\neq\emptyset)$$
$$+P(A_1=\emptyset,B_1=\emptyset)+P(A_2=\emptyset,B_2=\emptyset)\leqslant 2.$$

Optimal phase settings,

$$\phi_{A_1}=(0,0),\quad \phi_{A_2}=(0,0.7388),$$
$$\phi_{A_3}=(0,2.1334),$$
$$\phi_{B_1}=(0,-0.1347),\quad \phi_{B_2}=(0,1.2938),$$
$$\phi_{B_3}=(0,-0.0757),\quad \phi_{B_4}=(0,-1.0891).$$

Maximal violation, $I^{2,3\times 4}(QM)=2.8683$.
Detection threshold, $\eta_*^{\forall\lambda}=0.8216$.

5. $d=2$, $N_A=4$, $N_B=4$, $\forall\ \lambda$

Bell inequality,

$$I^{2,4\times 4,\forall\lambda}=-P(A_1=B_1)+P(A_1\neq B_3)-P(A_2=B_1)$$
$$-P(A_2=B_2)+P(A_2\neq B_4)+P(A_3\neq B_1)$$
$$-P(A_3\neq B_2)-P(A_3\neq B_3)-P(A_4\neq B_1)$$
$$-P(A_4=B_2)-P(A_4=B_3)+P(A_4\neq B_4)$$
$$+P(A_1\neq\emptyset,B_4=\emptyset)-P(A_4\neq\emptyset,B_1=\emptyset)$$
$$+P(A_1=\emptyset,B_1=\emptyset)+P(A_1=\emptyset,B_4=\emptyset)\leqslant 2.$$

Optimal phase settings,

$$\phi_{A_1}=(0,0),\quad \phi_{A_2}=(0,0.0958),$$
$$\phi_{A_3}=(0,2.1856),\quad \phi_{A_4}=(0,4.5944),$$
$$\phi_{B_1}=(0,4.0339),\quad \phi_{B_2}=(0,3.3011),$$
$$\phi_{B_3}=(0,2.2493),\quad \phi_{B_4}=(0,2.3454).$$

Maximal violation, $I^{2,4\times 4}(QM)=2.8697$.
Detection threshold, $\eta_*^{\forall\lambda}=0.8214$.

6. $d=3$, $N_A=2$, $N_B=3$, $\forall$ λ

Bell inequality,

$$\begin{aligned}I^{3,2\times 3,\lambda}=&+[P(A_1=B_1)+P(A_1=B_2)+P(A_1=B_3)\\&+P(A_2=B_1+1)+P(A_2=B_2+2)+P(A_2=B_3)]\\&-[P(A_1\neq B_1)+P(A_1\neq B_2)+P(A_1\neq B_3)\\&+P(A_2\neq B_1+1)+P(A_2\neq B_2+2)+P(A_2\neq B_3)]\\&-\frac{1}{3}\sum_{i,j}P(A_i=\emptyset,B_j\neq\emptyset)+\frac{1}{3}\sum_{i,j}P(A_i=\emptyset,B_j=\emptyset)\\\leqslant 2.\end{aligned}$$

Optimal phase settings,

$$\phi_{A_1}=(0,0,0),\quad \phi_{A_2}=(0,2\pi/3,0),$$

$$\phi_{B_1}=(0,\pi/3,0),\quad \phi_{B_2}=(0,2\pi/3,-\pi/3),$$

$$\phi_{B_3}=(0,-\pi/3,-\pi/3).$$

Maximal violation, $I^{3,2\times 3}(QM)=\frac{10}{3}$.
Detection threshold, $\eta_*^{\forall\lambda}=\frac{9}{11}\simeq 0.8182$.

7. $d=3$, $N_A=3$, $N_B=3$, λ

Bell inequality,

$$\begin{aligned}I^{3,3\times 3,\lambda}=&E_1(A_1,B_2)+E_2(A_1,B_3)+E_2(A_2,B_1)\\&-E_2(A2,B_3)+E_1(A_3,B_1)-E_1(A_3,B_2)\\&-P(A_1\neq B_1)-P(A_2\neq B_2)-P(A_3\neq B_3)\leqslant 2.\end{aligned}$$

Optimal phase settings,

$$\phi_{A_1}=(0,0,0),\quad \phi_{A_2}=(0,2\pi/9,4\pi/9),$$

$$\phi_{A_3}=(0,-2\pi/9,-4\pi/9),$$

$$\phi_{B_1}=(0,0,0),\quad \phi_{B_2}=(0,2\pi/9,4\pi/9),$$

$$\phi_{B_3}=(0,-2\pi/9,-4\pi/9).$$

Maximal violation, $I^{3,3\times 3}(QM)=3.0642$.
Detection threshold, $\eta_*^{\lambda}=2/3.0642\lambda$

8. $d=3$, $N_A=3$, $N_B=3$, $\forall$ λ

Bell inequality,

$$\begin{aligned}I^{3,3\times 3,\forall\ \lambda}=&-\frac{5}{3}P(A_1=B_1)-\frac{4}{3}P(A_1=B_1+2)\\&+P(A_1=B_2)+\frac{5}{3}P(A_1=B_2+1)\\&-\frac{5}{3}P(A_1=B_3)-P(A_1=B_3+2)\\&+\frac{5}{3}P(A_2=B_1)-2P(A_2=B_1+1)\\&-\frac{5}{3}P(A_2=B_2)+2P(A_2=B_2+1)\\&-P(A_2=B_3+1)-\frac{5}{3}P(A_2=B_3+2)\\&-\frac{11}{3}P(A_3=B_1)-2P(A_3=B_1+2)\\&+\frac{2}{3}P(A_3=B_2)+2P(A_3=B_2+1)\\&+\frac{5}{3}P(A_3=B_3)+P(A_3=B_3+2)\\&+\frac{5}{3}P(A_1\neq\emptyset,B_1=\emptyset)-\frac{5}{3}P(A_2\neq\emptyset,B_1=\emptyset)\\&-2P(A_3\neq\emptyset,B_1=\emptyset)+2P(A_1\neq\emptyset,B_2=\emptyset)\\&+\frac{5}{3}P(A_1=\emptyset,B_1=\emptyset)+2P(A_1=\emptyset,B_2=\emptyset)\leqslant 11/3.\end{aligned}$$

Optimal phase settings,

$$\phi_{A_1}=(0,0,0),\quad \phi_{A_2}=(0,1.4376,2.8753),$$

$$\phi_{A_3}=(0,0.5063,1.0125),$$

$$\phi_{B_1}=(0,2.0452,4.0904),\quad \phi_{B_2}=(0,2.9758,-0.3315),$$

$$\phi_{B_3}=(0,1.3839,2.7678).$$

Maximal violation, $I^{3,3\times 3}(QM)=5.3358$.
Detection threshold, $\eta_*^{\forall\lambda}=0.8146$.

9. $d=4$, $N_A=2$, $N_B=3$, $\forall$ λ

Bell inequality,

$$\begin{aligned}I^{4,2\times 3,\forall\lambda}=&P(A_1=B_1+1)+2P(A_1=B_1+2)\\&+2P(A_1=B_2)+P(A_1=B_2+1)+2P(A_1=B_3)\\&+2P(A_2=B_1+1)+P(A_2=B_1+2)\\&+P(A_2=B_2)+2P(A_2=B_2+1)\end{aligned}$$

$$+2P(A_2=B_3+2)+\frac{4}{3}\sum_i P(A_i=\emptyset,B_1\neq\emptyset)$$
$$+\frac{1}{3}\sum_i P(A_i=\emptyset,B_2\neq\emptyset)$$
$$+\frac{1}{3}\sum_i P(A_i=\emptyset,B_3\neq\emptyset)$$
$$+\frac{5}{3}\sum_i P(A_1\neq\emptyset,B_1=\emptyset)$$
$$+\frac{1}{3}\sum_i P(A_2\neq\emptyset,B_1=\emptyset)+\frac{8}{3}P(A_1=\emptyset,B_1=\emptyset)$$
$$+\frac{5}{3}P(A_1=\emptyset,B_2=\emptyset)+\frac{5}{3}P(A_1=\emptyset,B_3=\emptyset)s$$
$$+\frac{4}{3}P(A_2=\emptyset,B_1=\emptyset)+\frac{1}{3}P(A_2=\emptyset,B_2=\emptyset)$$
$$+\frac{1}{3}P(A_2=\emptyset,B_3=\emptyset)\leqslant 8.$$

Optimal phase settings,

$$\phi_{A_1}=(0,0,0,0),$$
$$\phi_{A_2}=(0,-1.1397,2.0019,3.1416),$$
$$\phi_{B_1}=(0,1.7863,-0.5698,2.3562),$$
$$\phi_{B_2}=(0,0.2155,5.7133,0.7854),$$
$$\phi_{B_3}=(0,1.0009,1.0009,0).$$

Maximal violation, $I^{4,2\times 3}(QM)=9.4142$.
Detection threshold, $\eta_*^{\forall\lambda}=0.8093$.

10. $d=4$, $N_A=3$, $N_B=3$, $\forall\ \lambda$

Bell inequality,

$$I^{4,3\times 3,\forall\lambda}=-P(A_1=B_1+2)+P(A_1=B_1+3)+2P(A_1=B_2+1)-P(A_1=B_2+2)-P(A_1=B_3)-3P(A_1=B_3+1)-2P(A_1=B_3+2)-P(A_2=B_1)+P(A_2=B_1+1)-P(A_2=B_2+1)+P(A_2=B_2+2)$$
$$+2P(A_2=B_3+3)+2P(A_3=B_1+1)$$
$$+P(A_3=B_2)$$
$$-2P(A_3=B_2+2)-P(A_3=B_2+3)$$
$$+2P(A_3=B_3)+P(A_3=B_3+2)$$
$$+\sum_i P(A_i=\emptyset,B_1\neq\emptyset)+P(A_1\neq\emptyset,B_1=\emptyset)$$
$$+P(A_1\neq\emptyset,B_2=\emptyset)-P(A_1=\emptyset,B_3\neq\emptyset)$$
$$+P(A_3=\emptyset,B_3\neq\emptyset)+P(A_3\neq\emptyset,B_3=\emptyset)$$
$$+2P(A_1=\emptyset,B_1=\emptyset)+P(A_1=\emptyset,B_2=\emptyset)$$
$$+P(A_2=\emptyset,B_1=\emptyset)+P(A_3=\emptyset,B_1=\emptyset)$$
$$+P(A_3=\emptyset,B_3=\emptyset)\leqslant 6.$$

Optimal phase settings,

$$\phi_{A_1}=(0,0,0,0),$$
$$\phi_{A_2}=(0,-1.2238,-1.1546,3.9048),$$
$$\phi_{A_3}=(0,3.1572,3.8330,0.7070),$$
$$\phi_{B_1}=(0,-0.9042,1.7066,0.8025),$$
$$\phi_{B_2}=(0,2.5844,3.6937,-0.0051),$$
$$\phi_{B_3}=(0,4.1396,3.0022,7.1419).$$

Maximal violation, $I^{4,3\times 3}(QM)=7.5576$.
Detection threshold, $\eta_*^{\forall\lambda}=0.7939$.

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[21] Actually Eq. (18) corresponds to $0<\lambda\leqslant 1/[1-(1-\eta)^2]$ so that λ can be greater than 1. But as stated earlier, if a LHV model exists for a given value of λ it is trivial to extend it to a LHV model for a lower value of λ. The maximum of η_*^λ over the set $\lambda\in\,]0,1/(1-(1-\eta)^2)]$ will thus be equal to the maximum over the set $\lambda\in\,]0,1]$.

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