

## Greenberger-Horne-Zeilinger Paradoxes for Many Qudits

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We construct Greenberger-Horne-Zeilinger (GHZ) contradictions for three or more parties sharing an entangled state, the dimension of each subsystem being an even integer  $d$ . The simplest example that goes beyond the standard GHZ paradox (three qubits) involves five ququats ( $d = 4$ ). We then examine the criteria that a GHZ paradox must satisfy in order to be genuinely  $M$  partite and  $d$  dimensional.

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The entanglement of bipartite quantum systems of dimension greater than two as well as the entanglement of multipartite quantum systems are far from being completely understood today, and they motivate much of the current work in quantum information theory. One of the most important insights into multipartite (actually tripartite) entanglement is provided by the Greenberger-Horne-Zeilinger (GHZ) argument [1]. In its formulation given by Mermin [2], the GHZ argument is both an intrinsic contradiction arising when dealing with noncontextual variables [a Kochen-Specker (KS) theorem] and a Bell-EPR theorem that rules out local hidden-variable models. Furthermore, the GHZ argument is an important primitive for building quantum information-theoretic protocols that decrease the communication complexity [3], and it plays a central role in the theory of entanglement since the GHZ state is the maximally entangled state of three qubits [4].

In this Letter, we show how to construct GHZ contradictions for three or more quantum systems of dimension  $d$  greater than 2 (qudits). In particular, we define several families of GHZ contradictions involving  $M$  qudits that are based on operator relations, similar to the standard GHZ paradox. We also give precise conditions that every GHZ paradox must fulfill in order to be genuinely  $M$  partite and  $d$  dimensional. This is of interest for the classification of entanglement of multipartite and multidimensional systems: since GHZ paradoxes provide an all-or-nothing refutation of local realism by quantum mechanics, one expects that GHZ states are in some sense maximally entangled states. Several extensions on the original work by GHZ and Mermin have been proposed previously, as, for example, GHZ contradictions involving more than three qubits [5]. More recently, it has also been shown how to carry out a set of measurements on a multipartite multidimensional system in a generalized GHZ state such that the correlation functions between the measurement outcomes exhibit a contradiction with local hidden-variable theories of the GHZ type [6]. However, the results of [6] are not based on relations between a set of operators. Instead, our work more closely parallels Mermin's original formulation of the GHZ argument, being based on an algebra of operators. In particular, this

implies, as in [2], that each GHZ paradox presented in this paper is associated with a state-independent KS theorem as well as a basis of GHZ states. Our work is also related to multidimensional quantum error-correcting codes (the connection between quantum codes and GHZ contradictions was discussed for qubits in [7]).

Let us consider a  $d$ -dimensional Hilbert space in which we define the *unitary* operators

$$X = \sum_{k=0}^{d-1} |(k+1) \bmod d\rangle \langle k|, \quad (1)$$

$$Y = e^{i\pi p/d} \sum_{k=0}^{d-1} e^{2\pi i k/d} |(k-1) \bmod d\rangle \langle k|, \quad (2)$$

$$Z = \sum_{k=0}^{d-1} e^{2\pi i k/d} |k\rangle \langle k|, \quad (3)$$

which satisfy  $XY = e^{i\pi p/d}Z$ , where  $p = 0$  for  $d$  odd and  $p = 1$  for  $d$  even. These operators are (up to a phase) the error operators that are used in multidimensional quantum error-correcting codes [8]. For qubits ( $d = 2$ ), they correspond to the Pauli matrices  $X = \sigma_x$ ,  $Y = \sigma_y$ , and  $Z = \sigma_z$ . The overall phases in Eqs. (1)–(3) are chosen so that these error operators satisfy

$$X^d = Z^d = Y^d = \mathbb{1}. \quad (4)$$

These operators also obey the commutation relations

$$Y^b X^a = e^{2\pi i ab/d} X^a Y^b, \quad Z^b X^a = e^{2\pi i ab/d} X^a Z^b \quad (5)$$

for all integers  $a, b$ .

A simple example of a GHZ contradiction for five parties each having a ququat (i.e.,  $d = 4$ ) is based on the following six product operators:

$$\begin{aligned}
 V_0 &= X_1 \otimes X_2 \otimes X_3 \otimes X_4 \otimes X_5, \\
 V_1 &= (X_1)^3 \otimes Y_2 \otimes Y_3 \otimes Y_4 \otimes Y_5, \\
 V_2 &= Y_1 \otimes (X_2)^3 \otimes Y_3 \otimes Y_4 \otimes Y_5, \\
 V_3 &= Y_1 \otimes Y_2 \otimes (X_3)^3 \otimes Y_4 \otimes Y_5, \\
 V_4 &= Y_1 \otimes Y_2 \otimes Y_3 \otimes (X_4)^3 \otimes Y_5, \\
 V_5 &= Y_1 \otimes Y_2 \otimes Y_3 \otimes Y_4 \otimes (X_5)^3.
 \end{aligned} \tag{6}$$

One easily checks that the operators  $V_i$  commute since  $YX = iXY$ . They can thus all be simultaneously diagonalized. The eigenvalues of each  $V_i$  are the 4th roots of the identity since  $V_i^4 = \mathbb{1}$ . Furthermore, the product  $V_0 V_1 V_2 V_3 V_4 V_5 = -\mathbb{1}$ , which implies that the product of the eigenvalues of the six operators  $V_i$  must be equal to  $-1$ . For instance, a common eigenstate of the above operators with eigenvalues  $V_0 = +1, V_1 = V_2 = \dots = V_5 = -1$  is the generalized GHZ state  $|\Psi\rangle = \frac{1}{\sqrt{4}} \sum_{k=0}^3 |k\rangle \otimes |k\rangle \otimes |k\rangle \otimes |k\rangle \otimes |k\rangle$ .

Before presenting the KS and Bell-EPR forms of the GHZ argument associated with these operators, let us note that we can always associate an observable to a unitary operator  $U = \sum_k e^{i\phi_k |u_k\rangle\langle u_k|}$ , where  $e^{i\phi_k}$  and  $|u_k\rangle$  are the eigenvalues and eigenvectors of  $U$ . Indeed, there is a simple correspondence between  $U$  and the Hermitian operator  $H = -i \log U = \sum_k \phi_k |u_k\rangle\langle u_k|$ . By measuring  $H$  and exponentiating the result, one can associate to  $U$  a  $c$ -number (of unit norm) which will be one of its eigenvalues. We call this the result of the measurement of  $U$ . (Note that for qubits the Pauli matrices are both Hermitian and unitary.)

Let us now derive the KS form of the GHZ contradiction (6), following the arguments of [2]. Namely, we will show that it is impossible to ascribe a definite value  $v(V_k)$  to each of the operators  $V_k$ . Indeed, these operators are constrained by the relation  $V_0 V_1 V_2 V_3 V_4 V_5 = -\mathbb{1}$ . Since they com-

mute, the same relation must hold for their values:

$$v(V_0)v(V_1)v(V_2)v(V_3)v(V_4)v(V_5) = -1. \tag{7}$$

Invoking noncontextuality, we can assign to the operator  $V_k$  the product of the values of the five one-party operators that appear in the tensor product defining it. For instance, we have  $v(V_1) = v(X_1)^3 v(Y_2) v(Y_3) v(Y_4) v(Y_5)$ . Inserting this in (7) gives

$$v(X_1)^4 v(Y_1)^4 \dots v(X_5)^4 v(Y_5)^4 = -1. \tag{8}$$

Now the value associated with an operator must be one of its eigenvalues. Equation (4) therefore implies that each  $v(X)$  or  $v(Y)$  must be a 4th root of unity. Therefore the product on the left-hand side of Eq. (8) is  $+1$ , although the right-hand side is  $-1$ , so the assignment of values is impossible. This is the content of the KS theorem.

The Bell-EPR form of the GHZ contradiction proceeds along the same line as the KS form but with the noteworthy difference that the assignment of values to each of the operators  $X_j, Y_j$  is now justified by the weaker assumption of local realism. Suppose the five parties are separated from each other and share a quantum state which is a simultaneous eigenstate of the six operators  $V_0, \dots, V_5$ , say, the state  $|\Psi\rangle$  defined above. In principle, one can learn the result of the measurement of  $X_j$  or  $Y_j$  by party  $j$  by adequate measurements on the other four parties since the product of the results must be one of the eigenvalues of  $\Psi$ :  $V_0 = 1, V_1 = V_2 = V_3 = V_4 = V_5 = -1$ . Therefore, according to the EPR criterion of local realism, one must assign to each party  $j$  a value  $v(X_j)$  and  $v(Y_j)$  for both the operators  $X_j$  and  $Y_j$ , which is one of the 4th roots of the identity. Reasoning as above, one gets the same contradiction. In this way the GHZ argument provides a very simple way to rule out local realism.

Let us now generalize the above GHZ contradiction to any odd number  $M (\geq 3)$  of parties, each having a qudit of dimension  $d = M - 1$ . The corresponding GHZ operators can be written as

$$\left. \begin{array}{ccccc}
 X & X & X & \dots & X \\
 X^{d-1} & Y & Y & \dots & Y \\
 Y & X^{d-1} & Y & \dots & Y \\
 Y & Y & X^{d-1} & \dots & Y \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 Y & Y & Y & \dots & X^{d-1}
 \end{array} \right\} M + 1 = d + 2 \text{ operators,} \tag{9}$$

$M = d + 1$  parties

where the columns correspond to the  $M$  different parties and the lines to the  $M + 1$  different operators. We note that the generalized GHZ state  $|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k\rangle^{\otimes M}$  is again a common eigenstate of the  $M + 1$  operators, giving rise to the same kind of contradiction. This example can be further generalized by considering the  $M + 1$  operators  $W_0, W_1, \dots, W_M$ :

$$W_0 = \underbrace{X^a \dots X^a}_{M \text{ terms}}, \quad W_1 = \underbrace{X^b \dots X^b}_n \underbrace{Y^c \dots Y^c}_p \underbrace{\mathbb{1} \dots \mathbb{1}}_q \underbrace{Y^c \dots Y^c}_p, \quad W_k = \text{cyclic permutations of } W_1 (1 < k \leq M),$$

(10)

where

$$2p = M - n - q \tag{11}$$

( $M - n - q$  is thus even). In order to have a GHZ paradox we require that (i) the operators  $W_j$  commute; (ii) if one assigns a classical value to the operators  $X_j$  and  $Y_j$  ( $j = 1, \dots, M$ ), then the product  $v(W_0) \dots v(W_M) = +1$ ; (iii) the product of

operators  $W_0 W_1 \dots W_M \neq +1$ . The first condition is already satisfied for  $j \neq 0$  because of the cyclic permutations in the construction. The requirement that  $W_0$  also commutes with  $W_j$  ( $j \neq 0$ ) imposes the additional constraint  $(e^{2\pi i ac/d})^{2p} = 1$ , or

$$2pac = kd, \tag{12}$$

where  $k > 0$  is an arbitrary integer. The second condition is satisfied if, in each column, the number of  $X$ 's and  $Y$ 's is a multiple of  $d$ . This implies that

$$2pc = k'd \tag{13}$$

and

$$nb + a = k''d, \tag{14}$$

with  $k', k'' > 0$  being arbitrary integers. [Note that Eq. (13) implies Eq. (12).] The product of the  $M + 1$  operators  $W_j$  is  $W_0 W_1 \dots W_M = e^{2\pi i [bcnp(M-n+1)/d]} \mathbb{1}$  so that, using (13), the third condition yields

$$bk'n(M - n + 1) = 2l + 1, \tag{15}$$

where  $l > 0$  is an arbitrary integer. Thus  $b, k', n$  and  $(M - n + 1)$  must be odd integers. This implies that the number of parties  $M$  must be odd and, given Eq. (13), that the dimension  $d$  must be even regardless of  $c$ . From Eq. (14), we also have that  $a$  must be odd, while Eq. (11) implies that  $q$  is even.

As an illustration, let us consider the special case  $c = 1, q = 0$ , and  $k = 1$ . Thus, for any even dimension  $d$  and any odd  $n$ , there is a GHZ contradiction for  $M = d + n$  parties, with the exponents  $a$  and  $b$  given by Eq. (14). The operators given in Eq. (9) are just the subclass  $a = 1, b = d - 1$ , and  $n = 1$ . Another example is that of five qubits ( $d = 2, M = 5, n = 3$ ):

$$\left. \begin{array}{ccccc} X & X & X & X & X \\ X & X & X & Y & Y \\ Y & X & X & X & Y \\ Y & Y & X & X & X \\ X & Y & Y & X & X \\ X & X & Y & Y & X \end{array} \right\} \begin{array}{l} 6 \text{ operators.} \\ \\ \\ \\ \\ \\ \end{array} \tag{16}$$

5 parties

The GHZ state  $|\Psi\rangle = (|00000\rangle + |11111\rangle)/\sqrt{2}$  is a common eigenstate of these operators and gives rise to a paradox. Other families of GHZ contradictions are also possible. For instance, replacing  $n = 3$  and  $q = 0$  in the above example by  $n = 1$  and  $q = 2$  yields

$$\left. \begin{array}{ccccc} X & X & X & X & X \\ X & Y & \mathbb{1} & \mathbb{1} & Y \\ Y & X & Y & \mathbb{1} & \mathbb{1} \\ \mathbb{1} & Y & X & Y & \mathbb{1} \\ \mathbb{1} & \mathbb{1} & Y & X & Y \\ Y & \mathbb{1} & \mathbb{1} & Y & X \end{array} \right\} \begin{array}{l} 6 \text{ operators,} \\ \\ \\ \\ \\ \\ \end{array} \tag{17}$$

5 parties

which is the paradox obtained from the five-qubit error-correcting code [7]. Here, the logical states  $|0_L\rangle$  and  $|1_L\rangle$  of the five-qubit code give rise to the paradox.

Although  $M$  was restricted to odd numbers in what precedes, it is also possible to build GHZ contradictions with an even number of parties. In [5], a contradiction for qubits shared between four parties was given which can be generalized to an even number  $M = d + 2$  of qudits as follows:

$$\left. \begin{array}{ccccc} X & Y^{d-1} & Y^{d-1} & \dots & Y^{d-1} \\ X^{d-1} & Y & Y & \dots & Y \\ Y & X^{d-1} & Y & \dots & Y \\ Y & Y & X^{d-1} & \dots & Y \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Y & Y & Y & \dots & X^{d-1} \\ Y^{d-1} & X & X & \dots & X \end{array} \right\} \begin{array}{l} M + 2 \text{ operators.} \\ \\ \\ \\ \\ \\ \\ \end{array}$$

M = d + 2 parties

(18)

A common eigenstate of these operators is the state  $|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{-i\pi k(k+2)/d} |k\rangle^{\otimes(d+2)}$ .

The above examples thus illustrate that it is possible to construct several families of GHZ contradictions involving many parties, each having a high-dimensional system. We now examine what should be the precise meaning of a *multipartite and multidimensional GHZ paradox*.

*Multipartite GHZ paradox*—A GHZ paradox is genuinely  $M$  partite if one cannot reduce the number of parties and still have a paradox. This is best illustrated by an example. In [5], a GHZ paradox with five qubits was defined by the following operators:

$$\left. \begin{array}{ccccc} X & X & X & X & X \\ X & Y & Y & X & X \\ Y & X & Y & Y & Y \\ Y & Y & X & Y & Y \end{array} \right\} \begin{array}{l} 4 \text{ operators.} \\ \\ \\ \\ \end{array} \tag{19}$$

5 parties

This paradox is not genuinely 5 partite according to our criterion. Indeed, these operators, restricted to the first three parties, constitute a GHZ contradiction (in fact this is the original paradox as formulated by Mermin). Moreover, these operators, restricted to the last two parties, commute. As a consequence, the eigenstates of these four operators can be written as tensor products of states belonging to the first three parties times states belonging to the last two parties. For instance, the state  $(|000\rangle + |111\rangle) \otimes (|00\rangle + |11\rangle)/2$  is a common eigenstate of these four operators. As a consequence, this state does not exhibit 5-partite entanglement.

*Multidimensional GHZ paradox*—A GHZ paradox is genuinely  $d$  dimensional if one cannot reduce the dimensionality of the Hilbert space of each of the parties to less than  $d$  and still have a paradox.

More precisely, consider a GHZ paradox defined by the  $M$ -partite operators  $W_k$  [e.g., those introduced in Eq. (10)]. Suppose that there exist  $M$  projectors  $\Pi_l$  of rank less than  $d$ , each acting on the space of the  $l$ th party, such that the operators  $\tilde{W}_k = \Pi_1 \otimes \cdots \otimes \Pi_M W_k \Pi_1 \otimes \cdots \otimes \Pi_M$  define a lower-dimensional GHZ paradox. Then, the original paradox defined by these operators  $W_k$  is not genuinely  $d$  dimensional. Let us illustrate this by a GHZ paradox in which three parties have a ququat (four-dimensional system), defined by the operators:

$$\left. \begin{array}{ccc} X & X & X \\ X^3 & Y^2 & Y^2 \\ Y^2 & X^3 & Y^2 \\ Y^2 & Y^2 & X^3 \end{array} \right\} \begin{array}{l} 4 \text{ operators.} \\ \\ \\ \end{array} \quad (20)$$

3 parties

On the basis of the commutation relations (5), one could expect that this is a genuinely four-dimensional contradiction. Indeed, the relation  $YX = XYe^{i2\pi/d}$  can be realized only in a Hilbert space whose dimension is at least  $d$ . (To prove this, suppose  $X$  is diagonal:  $X|k\rangle = e^{i\phi}|k\rangle$ . Then the commutation relation implies that the states  $Y^p|k\rangle$  are also eigenstates of  $X$  with eigenvalue  $e^{i\phi - i2\pi p/d}$ . Taking  $p = 1, \dots, d$  yields  $d$  distinct eigenvalues.) However, in the example (20), the operator  $Y$  appears only to the power 2. Hence the only commutators that are relevant to the paradox are  $XY^2 = -Y^2X$  and  $X^3Y^2 = -Y^2X^3$  which can be realized in a two-dimensional space. Using the representations (1) and (2), one sees that if each party projects onto the subspace spanned by the two vectors  $|0\rangle + |2\rangle$  and  $|1\rangle + |3\rangle$ , one still has a paradox. Thus the paradox (20) is not genuinely four dimensional, but only two dimensional. Another example is provided by [9], where seemingly multidimensional GHZ paradoxes are in fact based on anticommuting operators, and, hence, according to our criteria, are only two dimensional.

All the multipartite multidimensional GHZ contradictions that are exhibited in this paper are constructed from tensor products of operators  $X$  and  $Y$  raised to different powers (with commutation relation  $Y^a X^b = X^b Y^a e^{i2\pi ab/d}$ ). Such a paradox is genuinely  $d$  dimensional if, in each column (i.e., for each party), the algebra generated by  $X$  and  $Y$  raised to the powers which appear in that column can only be represented in a Hilbert space of dimension at least  $d$ . (This was not the case in the last example since the algebra of the operators  $\{X, X^3, Y^2\}$  could be represented in a two-dimensional space.)

The above criteria guaranteeing that a GHZ paradox is genuinely multipartite and genuinely  $d$  dimensional are satisfied by the examples given in Eqs. (9) and (16)–(18). These criteria can also be applied to the general case [Eq. (10)]. One would then obtain additional conditions on the parameters  $a$ ,  $b$ , and  $c$ . For instance, the operators that appear in each column of Eq. (10) are  $\{X^a, X^b, Y^c\}$ . The algebra generated by these operators will be realized

in a space of dimension at least  $d$  so that the paradoxes will be genuinely  $d$  dimensional if  $c$  and  $d$  are relatively prime (i.e., their greatest common divisor is 1), and if  $a$  or  $b$  is relatively prime with  $d$ . To ensure that the first condition is satisfied, we can take  $c = 1$ . This is not restrictive since, if  $c$  and  $d$  are relatively prime, there is a unitary operation that maps  $\{X^a, X^b, Y^c\}$  to  $\{X^{a'}, X^{b'}, Y\}$ , so that the algebra generated by the new set of operators is identical to the one generated by the original set. Let us now examine the conditions that are necessary for the paradoxes in Eq. (10) to be genuinely multipartite. Removing any number of columns (i.e., any parties), there are always two line  $W_k$  and  $W_l$  such that  $W_k W_l = e^{2\pi i bc/d} W_l W_k$ . Since  $e^{2\pi i bc/d} \neq 1$  because  $c$  and  $d$  are relatively prime and  $b = 1, \dots, d-1$ , the condition that all the operators  $W_j$  must commute is not satisfied, so that the remaining parties do not make a paradox. The generalization (10) is thus genuinely multipartite provided it is already genuinely  $d$  dimensional.

In summary, we have shown how to generalize the GHZ argument to multipartite higher-dimensional systems. Our method is related to the techniques used to construct error-correcting codes for arbitrary dimension. Interestingly, in all the GHZ-type paradoxes we have constructed, the dimension is even and is strictly less than the number of parties. We do not know whether this is necessarily the case, or if it is due to the restricted set of constructions we have considered. We also stressed that all the paradoxes one naively expects to be multipartite and multidimensional are not necessarily so. In some cases it is possible to reexpress the paradox in a lower-dimensional space, and in other cases the GHZ state associated with the paradox can be represented as a product of states belonging to different subsets of parties. We discussed criteria that ensure that a GHZ paradox is truly  $M$  partite and  $d$  dimensional. An interesting extension of this work would be to construct Mermin-like inequalities for qudits from these paradoxes, which would lay the grounds for an experimental testing of multipartite multidimensional nonlocality.

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