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Quantum cloning of orthogonal qubits

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An optimal universal cloning transformation is derived that produces M copies of an unknown qubit from a pair of orthogonal qubits. For $M > 6$, the corresponding cloning fidelity is higher than that of the optimal copying of a pair of identical qubits.

1 Introduction

Unlike classical information, quantum information cannot be copied. The *no-cloning* theorem [1] which states that it is impossible to prepare several exact copies (or clones) of an unknown quantum state $|\psi\rangle$ is a direct consequence of the linearity of quantum theory. Although exact cloning is forbidden, one can design various quantum cloning machines producing approximate clones. Much attention has been devoted to the optimal universal cloning machines for qubits, which prepare M identical approximate clones out of N replicas of an unknown qubit, and such that the fidelity of the clones is state-independent [2]. Cloning machines for states in a d-dimensional Hilbert space were also investigated [3], as well as continuous-variable cloning machines [4] for coherent states.

In the limit of an infinite number of clones, the optimal cloning reduces to the optimal quantum measurement. In this context, a very interesting observation has been made by Gisin and Popescu [5]: the information about a direction in space is better encoded into two orthogonal qubits than in two identical ones. If we possess a two-qubit state $|\psi, \psi_\perp\rangle$ with $\langle\psi|\psi_\perp\rangle = 0$, then we can estimate $|\psi\rangle$ with a fidelity $F_\perp = (1 + 1/\sqrt{3})/2 \approx 0.789$ [5, 6]. This slightly exceeds the fidelity of the optimal measurement on a qubit pair $|\psi, \psi\rangle$, $F_{||} = 3/4$. A similar situation occurs for continuous quantum variables. A (randomly chosen) position in phase space is better encoded into a pair of phase-conjugate coherent states $|\alpha, \alpha^*\rangle$ than into a pair of identical states $|\alpha, \alpha\rangle$ [7]. Similarly, $|\alpha, \alpha^*\rangle$ gives an advantage when cloning coherent states: for M sufficiently large, M identical approximate clones of a coherent state $|\alpha\rangle$ can be prepared with a higher fidelity from the state $|\alpha, \alpha^*\rangle$ than from $|\alpha, \alpha\rangle$ [8].

Motivated by this result, we were naturally led to tackle the following question: Can M clones of a qubit $|\psi\rangle$ be produced from an orthogonal qubit pair $|\psi, \psi_\perp\rangle$ with a higher fidelity than from an identical pair $|\psi, \psi\rangle$? In this paper we answer this question by an affirmative and we present a universal cloning machine acting on an orthogonal qubit pair that approximately implements the transformation $|\psi\rangle|\psi_\perp\rangle \to |\psi\rangle^{\otimes M}$ with the optimal fidelity [9].

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2 Optimal cloning transformation

Let us seek for a unitary transformation which *optimally* approximates the transformation $|\psi\rangle|\psi_\perp\rangle \to |\psi\rangle^{\otimes M}$. Since we seek a transformation such that the final state of the clones is left invariant by permutations amongst them, we will suppose that the clones lie in the symmetric M-qubit space. Our motivation, when making this simplifying hypothesis, is that in the case of quantum cloning, an optimal universal machine can always be chosen to be of this form [2]. Moreover, since the set of all states of the form $|\psi\rangle|\psi_\perp\rangle$ span the whole Hilbert space of two qubits, the most general transformation is of the form:

$$|i\rangle|j\rangle|R\rangle \to \sum_{k=0}^{M} |M,k\rangle|R_{ijk}\rangle, \qquad i,j=0,1, \tag{1}$$

where $|R\rangle$ and $|R_{ijk}\rangle$ respectively denote the initial and final states of the ancilla, while $|M,k\rangle$ ($k = 0,\ldots,M$) denotes a symmetric M-qubit state with k qubits in state $|0\rangle$ and $M-k$ qubits in state $|1\rangle$.

The arbitrary state of a qubit $|\psi\rangle$ can be conveniently written as $|\psi\rangle = d(\Omega)|0\rangle = \sum_i d_{i0}(\Omega)|i\rangle$, where the matrix $d(\Omega)$ is given by

$$d(\Omega) = \begin{pmatrix} \cos\frac{\vartheta}{2} & e^{-i\phi}\sin\frac{\vartheta}{2} \\ e^{i\phi}\sin\frac{\vartheta}{2} & -\cos\frac{\vartheta}{2} \end{pmatrix}, \tag{2}$$

with ϑ and ϕ denoting the usual polar and azimuthal angles pointing in direction Ω. The linearity of (1) implies that an arbitrary pair of orthogonal qubits transforms according to

$$|\psi\rangle|\psi_\perp\rangle \to |\Psi_{\rm out}(\psi)\rangle = \sum_{ijk} d_{i0}(\Omega)d_{j1}(\Omega)|M,k\rangle|R_{ijk}\rangle. \tag{3}$$

We will measure the quality of the transformation by the average single-clone fidelity $F_\perp(M)$. Denoting by $\mathrm{Tr}_{1',\rm anc}$ the partial trace over the ancilla and all the clones but the first one, we get

$$F_\perp(M) = \int d\Omega \langle\psi|\,\mathrm{Tr}_{1',\rm anc}[|\Psi_{\rm out}(\psi)\rangle\langle\Psi_{\rm out}(\psi)|]\,|\psi\rangle = \sum_{i'j'k'}\sum_{ijk}\langle R_{i'j'k'}|R_{ijk}\rangle A_{ijk}^{i'j'k'}, \tag{4}$$

where

$$A_{ijk}^{i'j'k'} = \sum_{n,n'}\langle n'|\mathrm{Tr}_{1'}[\,|M,k\rangle\langle M,k'|\,]|n\rangle \int d\Omega\, d_{n0}(\Omega)d_{n'0}^*(\Omega)d_{i0}(\Omega)d_{j1}(\Omega)d_{i'0}^*(\Omega)d_{j'1}^*(\Omega).$$

The coefficients $A_{ijk}^{i'j'k'}$ can be considered as matrix elements of an operator A acting on the space $\mathcal{H}\otimes\mathcal{K}$, where $\mathcal{H}$ denotes the Hilbert space of the two input qubits and $\mathcal{K}$ denotes the Hilbert space of symmetric states of M output qubits. Similarly, $\chi_{ijk}^{i'j'k'} = \langle R_{ijk}|R_{i'j'k'}\rangle$ define matrix elements of an operator χ also acting on $\mathcal{H}\otimes\mathcal{K}$. The formula (4) for the fidelity thus simplifies to $F_\perp(M) = \mathrm{Tr}_{\mathcal{H},\mathcal{K}}[\chi A]$. The operator χ uniquely represents the completely positive cloning map, which transforms operators supported on $\mathcal{H}$ onto operators supported on $\mathcal{K}$. By definition, the operators A and χ are Hermitian and positive semidefinite, $A \geq 0$ and $\chi \geq 0$. After a tedious but otherwise straightforward calculation we obtain

$$A = S_{00}\otimes Q_{00} + S_{11}\otimes Q_{11} + S_{01}\otimes Q_{01} + S_{10}\otimes Q_{10}, \tag{5}$$

where the operators Q_{ij} read

$$Q_{00} = \sum_{k=0}^{M}\frac{k}{M}|M,k\rangle\langle M,k|, \qquad Q_{11} = \sum_{k=0}^{M}\frac{M-k}{M}|M,k\rangle\langle M,k|,$$

$$Q_{10} = \sum_{k=1}^{M}\frac{\sqrt{k(M-k+1)}}{M}|M,k-1\rangle\langle M,k|,$$

and $Q_{01} = Q_{10}^T$. The operators S_{ij} in the basis $|00\rangle$, $|11\rangle$, $|10\rangle$, $|01\rangle$ are given by

$$S_{00} = \frac{1}{12}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix} \qquad S_{11} = \frac{1}{12}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

$$S_{10} = \frac{1}{12}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and $S_{01} = S_{10}^T$.

Of course, the transformation (1) should be unitary, which reads $\sum_k \langle R_{i'j'k}|R_{ijk}\rangle = \delta_{i'i}\delta_{j'j}$. This is equivalent to

$$\mathrm{Tr}_{\mathcal{K}}[\chi] = \mathbf{1}_{\mathcal{H}}, \tag{6}$$

where $\mathbf{1}_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. Thus, introducing a set of Lagrange multipliers $\lambda_{ij}^{i'j'}$ for these unitarity constraints, our problem amounts to extremize the quantity $W = \mathrm{Tr}_{\mathcal{H},\mathcal{K}}[(A - \Lambda)\chi]$ under the constraint $\chi \geq 0$, where $\Lambda = \lambda \otimes \mathbf{1}_{\mathcal{K}}$ and λ is the matrix of Lagrange multipliers ($\mathbf{1}_{\mathcal{K}}$ is the identity operator on $\mathcal{K}$). Varying W with respect to the eigenstates of the operator χ, we get the extremal equation

$$(A - \lambda \otimes \mathbf{1}_{\mathcal{K}})\chi = 0 \tag{7}$$

for the optimal χ. The second extremal equation can be derived from the theory of semidefinite programming [9, 10],

$$\lambda \otimes \mathbf{1}_{\mathcal{K}} - A \geq 0. \tag{8}$$

If the conditions (7) and (8) are satisfied for some positive semidefinite λ and χ then χ is the optimal CP map for which the mean fidelity $F_\perp(M)$ achieves its global maximum on the convex set of trace-preserving CP maps. To prove this, first note that it follows from the inequality (8) that

$$\mathrm{Tr}_{\mathcal{H},\mathcal{K}}[\chi\lambda \otimes \mathbf{1}_{\mathcal{K}}] = \mathrm{Tr}_{\mathcal{H}}[\lambda] \geq \mathrm{Tr}_{\mathcal{H},\mathcal{K}}[\chi A] \tag{9}$$

will hold for any trace-preserving CP map χ. We have used the unitarity (trace-preservation) constraint (6) to simplify the left-hand side of this inequality. The Lagrange multiplier λ thus provides an upper bound on the achievable fidelity: $F_\perp(M) \leq \mathrm{Tr}_{\mathcal{H}}[\lambda]$. If the Eq. (7) is satisfied then $\mathrm{Tr}_{\mathcal{H},\mathcal{K}}[\chi A] = \mathrm{Tr}_{\mathcal{H}}[\lambda]$ and the upper bound on the fidelity is reached.

Following [11], the extremal equation (7) can be further transformed into a form suitable for numerical solution via repeated applications of

$$\chi = \Lambda^{-1} A \chi A \Lambda^{-1}, \qquad \lambda = (\mathrm{Tr}_{\mathcal{K}}[A\chi A])^{1/2}. \tag{10}$$

Note that the matrix $\lambda > 0$ is determined from the unitarity constraints (6). By numerically solving Eq. (10) for $M = 2, \ldots 15$, we have guessed the general solution of Eq. (10). The transformation we obtain is:

$$|\psi, \psi_\perp\rangle \to \sum_{j=0}^{M} \alpha_{j,M} |j\psi, (M-j)\psi_\perp\rangle \otimes |j\psi_\perp, (M-j)\psi\rangle, \tag{11}$$

where

$$\alpha_{j,M} = (-1)^j \left[\frac{1}{\sqrt{2(M+1)}} + \frac{\sqrt{3}(2j - M)}{\sqrt{2M(M+1)(M+2)}}\right], \tag{12}$$

with $|j\psi, (M-j)\psi_\perp\rangle$ denoting a totally symmetric state of M qubits where j qubits are in state $|\psi\rangle$ and $M-j$ qubits are in state $|\psi_\perp\rangle$. The first M output qubits contain the clones of state $|\psi\rangle$ while the other M qubits contain the clones of $|\psi_\perp\rangle$ (or anticlones). We shall prove below that this transformation is indeed optimal. We stress here that the cloning transformation (11) is unitary and the proof can be found in Ref. [9].

Let us now calculate the fidelity of the clones. We can see from Eq. (11) that the cloning machine preserves the symmetry of the input state $|\psi, \psi_\perp\rangle$, so the clones of both states $|\psi\rangle$ and $|\psi_\perp\rangle$ have the same fidelity. This state-independent single-qubit fidelity can be obtained by summing a series,

$$F_\perp(M) = \sum_{j=0}^{M} \frac{j}{M}\, \alpha_{j,M}^2. \tag{13}$$

After some algebra, we arrive at the expression

$$F_\perp(M) = \frac{1}{2}\left(1+\sqrt{\frac{M+2}{3M}}\right). \tag{14}$$

Upon comparing this fidelity to that of the optimal cloner for a pair of identical qubits $F_{||}(M) = (3M+2)/(4M)$ [2], we see that $F_{||}(M) \geq F_\perp(M)$ for $M \leq 6$, while $F_\perp(M) > F_{||}(M)$ for $M > 6$ and the cloner (11) outperforms the standard cloner. We also note that for $M \to \infty$, the fidelity $F_\perp(M)$ tends to the optimal measurement fidelity $F_\perp$, as expected.

Let us now prove the optimality of our universal cloner. From the numerical solution of Eqs. (10) we have determined the structure of the operator λ. In basis $|00\rangle$, $|11\rangle$, $|10\rangle$, $|01\rangle$ we have

$$\lambda = \frac{F_\perp(M)}{6}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}. \tag{15}$$

The block-diagonal matrix $\lambda \otimes \mathbf{1}_\mathcal{K} - A$ is positive semidefinite and has three different eigenvalues which read $\mu_1 = \frac{1}{12}\sqrt{(M+2)/(3M)}$, $\mu_2 = \frac{1}{3}\sqrt{(M+2)/(3M)}$, and $\mu_3 = 0$. Since the upper bound $\mathrm{Tr}_\mathcal{H}[\lambda] = F_\perp(M)$ is saturated by our cloning machine, we conclude that our cloner is optimal.

In summary, we have derived a universal cloning transformation for orthogonal qubit pairs. We have shown that this transformation achieves a higher fidelity for $M > 6$ than the standard cloner and thus conclude that the advantage of orthogonal qubits over identical qubits that was discovered in the context of measurement also extends to cloning. Finally, we note that the cloning transformation can be probabilistically implemented via stimulated parametric downconversion [9].

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References

[1] W. K. Wootters and W. H. Zurek, Nature (London) **299**, 802 (1982).
D. Dieks, Phys. Lett. **92A**, 271 (1982).
[2] V. Bužek and M. Hillery, Phys. Rev. A **54**, 1844 (1996).
N. Gisin and S. Massar, Phys. Rev. Lett. **79**, 2153 (1997).
R. F. Werner, Phys. Rev. A **58**, 1827 (1998).
M. Keyl and R. F. Werner, J. Math. Phys. **40**, 3283 (1999).
D. Bruss, A. Ekert, and C. Macchiavello, Phys. Rev. Lett. **81**, 2598 (1998).

[3] V. Bužek and M. Hillery, Phys. Rev. Lett. **81**, 5003 (1998).
N. J. Cerf, J. Mod. Opt. **47**, 187 (2000).
S. L. Braunstein, V. Bužek, and M. Hillery, Phys. Rev. A **63**, 052313 (2001).
[4] N. J. Cerf, A. Ipe, and X. Rottenberg, Phys. Rev. Lett. **85**, 1754 (2000).
S. L. Braunstein et al., Phys. Rev. Lett. **86**, 4938 (2001).
J. Fiurášek, Phys. Rev. Lett. **86**, 4942 (2001).
[5] N. Gisin and S. Popescu, Phys. Rev. Lett. **83**, 432 (1999).
[6] S. Massar, Phys. Rev. A **62**, 040101(R) (2000).
[7] N. J. Cerf and S. Iblisdir, Phys. Rev. A **64**, 032307 (2001).
[8] N. J. Cerf and S. Iblisdir, Phys. Rev. Lett. **87**, 247903 (2001).
[9] J. Fiurášek, S. Iblisdir, S. Massar, and N. J. Cerf, Phys. Rev. A **65**, 040302(R) (2002).
[10] K. Audenaert and B. de Moor, Phys. Rev. A **65**, 030302(R) (2002).
[11] J. Fiurášek, Phys. Rev. A **64**, 062310 (2001).