Unitary-gate synthesis for continuous-variable systems

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We investigate the synthesis of continuous-variable two-mode unitary gates in the setting where two modes A and B are coupled by a fixed quadratic Hamiltonian H. The gate synthesis consists of a sequence of evolutions governed by Hamiltonian H, which are interspaced by local phase shifts applied to A and B. We concentrate on protocols that require the minimum number of necessary steps and we show how to implement the beam splitter and the two-mode squeezer in just three steps. Particular attention is paid to Hamiltonian $x_A p_B$ that describes the effective off-resonant interaction of light with the collective atomic spin.

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I. INTRODUCTION

One of the central problems of the quantum information theory is to establish what resources are sufficient for universal quantum computation. In this context, the question whether a given Hamiltonian H can *simulate* another one has attracted considerable attention recently [1–6]. In its simplest form, this problem may be formulated as follows. Consider two parties, traditionally referred to as Alice and Bob, possessing a single qubit each. The interaction between those two qubits is governed by a fixed Hamiltonian H, which is determined by the physical properties of the systems that represent the qubits. In addition to interaction H, Alice and Bob may attach (local) ancillas to their qubits and perform arbitrary local unitary operations on their subsystems. It is usually assumed that these local operations are very fast compared to the evolution induced by Hamiltonian H. The task for Alice and Bob is to simulate the evolution due to a different Hamiltonian H'. Two kinds of simulations should be distinguished. The infinitesimal time simulation [1–6] consists of simulating the action of Hamiltonian H' for an infinitesimally short interval Δt. The *gate synthesis* [7–12] requires the implementation of the unitary transformation $U'=\exp(-iH't)$ for some *finite* time t.

It turns out that in the two-party setting, all nonlocal Hamiltonians H are qualitatively equivalent. Given enough time τ, Alice and Bob can, with the help of local ancillas, simulate the evolution $\exp(-iH't)$ for any H' [2]. The central question, then, is what is the optimal simulation. The latter may be defined as a simulation that requires the shortest time. For the two-qubit case, this problem has been completely solved and the optimal protocols for Hamiltonian [2,6] and unitary-gate [8] simulations have been determined. The situation becomes much more complicated for higher-dimensional systems and for higher number of involved parties. Simulation protocols suggested for these generic settings are rather involved and it is not known which protocols are optimal.

It should be stressed that most of the work focused on discrete variable systems: qubits or, more generally, qudits. Recently, however, Kraus *et al.* extended the notion of Hamiltonian simulation to continuous variable systems [13]. They assumed that Alice and Bob possess a single-mode system each and these two modes are coupled via quadratic Hamiltonian (we assume $\hbar=1$ throughout this paper):

$$H=c_{11}x_Ax_B+c_{12}x_Ap_B+c_{21}p_Ax_B+c_{22}p_Ap_B, \tag{1}$$

where x_j and p_j are two conjugate quadratures of the jth mode. Kraus *et al.* showed that almost every Hamiltonian (1) is capable of simulating any other Hamiltonian of form (1), provided that Alice and Bob can apply fast local phase shifts described by single-mode Hamiltonians $H_A=x_A^2+p_A^2$ and $H_B=x_B^2+p_B^2$.

These results are interesting from both theoretical and experimental points of view. In particular, the off-resonant interaction of light with the collective atomic spin [14–20] can be described by the effective unitary transformation

$$U=\exp(-itH_{AL}), \tag{2}$$

where the Hamiltonian

$$H_{AL}=\kappa x_Ap_B \tag{3}$$

is a special instance of Eq. (1). The typical geometry of the experiments is such that a light beam with strong coherent field polarized along the x axis propagates along the z axis through the atomic sample, whose spin is also polarized along the x axis. The x and p quadratures are defined as the properly normalized y and z components of the collective spin operators describing the polarization state of light and atomic ensemble, respectively [14,16,17]. In recent beautiful experiments it was demonstrated that interaction (2) can be employed to squeeze the atomic spin [15], entangle two distant atomic ensembles [18], and transfer the quantum state of light into the atoms [19]. Schemes for teleportation and swapping of the quantum state of collective atomic spin have been suggested [16,17]. These experiments and proposals in fact rely on the quantum nondemolition (QND) measurement of the atomic quadrature, possibly accompanied by a suitable feedback.

As showed by Kraus *et al.* [13], Hamiltonian (3) can simulate any Hamiltonian (1). In particular, H_{AL} can be used to implement a beam splitter and a two-mode squeezer. This is very appealing because it suggests that, for instance, the storage of the quantum state of light in atoms and the subse-

quent readout of the quantum memory, i.e., the transfer of quantum state of atoms onto light, can be implemented in a unitary way if Hamiltonian (3) is used to simulate a beam splitter.

However, currently there are technical difficulties that will complicate the actual practical realization of this procedure. The effective unitary transformation (2) describes the modification of the polarization state of the light pulse after the passage through the atomic sample and the corresponding backaction of the light pulse on the atoms. This means that the Hamiltonian simulation requires several passages of the light pulse through the atomic sample (cf. the detailed description of the simulation protocol in Sec. II). In currently envisaged experiments, the pulse width must be at least 1 μs [21], which corresponds to a length of 300 m. This long pulse would have to be stored somewhere (e.g., in an optical fiber) until its tail leaves the atomic sample. Only then can the pulse with properly applied phase shifts be fed to the atomic sample again. It may be helpful to place the atoms inside a (bad) optical cavity that could increase the coupling strength between light and atoms, which would allow one to reduce the pulse length. The interaction of light with atomic ensembles inside a cavity is currently an area of active research. For instance, a polarization squeezed light has been generated recently with such a setup [22]. However, the QND-type interaction (3) has not yet been demonstrated in the system involving an atomic ensemble inside a cavity.

The above practical considerations imply that the approach relying on the infinitesimal time simulation is not very convenient from the experimental point of view. It is possible to simulate a gate by concatenating a large sequence of short-time Hamiltonian simulations but this would require a large number of manipulations and passages of the light pulse through the sample. Since, in practice, every round of the gate synthesis procedure is necessarily accompanied by some losses and other errors, the accumulation of the errors would negatively influence the simulation.

In this paper, we show how to simulate several important two-mode interactions, such that the number of the applications of Hamiltonian (1) is minimized. We demonstrate that only three sequences of evolution governed by Hamiltonian H, interspaced by (fast) local phase shifts on both subsystems, suffice to implement a two-mode squeezer and a beam splitter. For the specific Hamiltonian (3), we also provide an analytical prescription for a single-mode squeezing gate, which involves four evolution steps, and we show that three-step implementations can be found numerically. These results illustrate that several important quantum information processing tasks, such as entangling the light and collective atomic spin, or a transfer of the quantum state of light into atomic clouds and vice versa, can be carried out with a small number of repeated passages of the light pulse through the atomic sample.

This paper is structured as follows. In Sec. II we introduce the notation, the canonical form of the interaction Hamiltonian (1) and we describe the gate simulation protocol. In Sec. III we consider the simple interaction Hamiltonian (3) and we show how to implement the two-mode squeezing operation, beam splitter transformation, and also single-mode squeezing as a sequence of three (or four) intervals of evolution governed by Hamiltonian (3) combined with local phase-shift operations. In Sec. IV we extend this analysis to the generic interaction Hamiltonians (1). Finally, the conclusions are drawn in Sec. V.

II. DESCRIPTION OF THE SIMULATION PROTOCOL

In this section we describe the simulation protocol. The gate synthesis consists of a sequence of N intervals of evolution governed by Hamiltonian H, followed by local unitary phase-shift transformations. The resulting unitary gate G is given by

$$G = V_{N+1} e^{-iHt_N} V_N e^{-iHt_{N-1}} \cdots V_3 e^{-iHt_2} V_2 e^{-iHt_1} V_1 . \tag{4}$$

The local phase-shift operation applied to modes A and B reads

$$V_j = e^{-i\phi_{Aj} a^\dagger a} \otimes e^{-i\phi_{Bj} b^\dagger b}, \tag{5}$$

where a and b are the annihilation operators of modes A and B, respectively. We can further decompose transformations V_j as follows:

$$V_1 = \tilde{V}_1, \quad V_j = \tilde{V}_j \tilde{V}_{j-1}^\dagger, \quad j = 2, \ldots, N+1, \tag{6}$$

where $\tilde{V}_j = \exp(-i\varphi_{Aj} a^\dagger a - i\varphi_{Bj} b^\dagger b)$ and $\varphi_{A1} = \phi_{A1}$, $\varphi_{Aj} = \phi_{Aj} + \varphi_{A,j-1}$, $j = 2, \ldots, N+1$ and similar formulas hold also for φ_{Bj}. On inserting decompositions (6) into Eq. (4) and making use of the identity

$$U^\dagger \exp(-iHt) U = \exp(-iU^\dagger H U t), \tag{7}$$

we obtain

$$G = \tilde{V}_{N+1} e^{-iH_N t_N} \cdots e^{-iH_2 t_2} e^{-iH_1 t_1}, \tag{8}$$

where $H_j = \tilde{V}_j^\dagger H \tilde{V}_j$.

Hamiltonian (1) is characterized by four parameters. However, by means of local rotations, we can always transform this Hamiltonian to a simpler form:

$$H_c = c_1 x_A p_B + c_2 p_A x_B , \tag{9}$$

where $c_1 = \sigma_1$ and $c_2 = \sigma_2 \det[C]/|\det[C]|$, and σ_1 and σ_2 are the singular values of matrix C defined as $(C)_{ij} = c_{ij}$ [13]. In close analogy to the qubit case [2], we may refer to H_c as the *canonical form* of H. Mathematically, we have

$$\exp(-iH_c t) = W^\dagger \exp(-iHt) W, \tag{10}$$

where W is a local rotation (5). This shows that without loss of generality, we may assume that H has the canonical form (9). In particular, it follows that H is able to simulate an arbitrary H' (1) if and only if H_c is able to simulate an arbitrary canonical Hamiltonian (9).

Generally, phase shifts φ_{Aj} and φ_{Bj} may be arbitrary. In what follows, we focus on the phase shifts that preserve the

canonical form of H, so that all H_j have form (9). There are four inequivalent possibilities:

(a) $\varphi_A=0,\ \varphi_B=0,$

$$H_1=c_1x_Ap_B+c_2p_Ax_B. \tag{11}$$

(b) $\varphi_A=\pi/2,\ \varphi_B=3\pi/2,$

$$H_2=c_2x_Ap_B+c_1p_Ax_B. \tag{12}$$

(c) $\varphi_A=\pi,\ \varphi_B=0,$

$$H_3=-c_1x_Ap_B-c_2p_Ax_B. \tag{13}$$

(d) $\varphi_A=\pi/2,\ \varphi_B=\pi/2,$

$$H_4=-c_2x_Ap_B-c_1p_Ax_B. \tag{14}$$

From the structure of these Hamiltonians we can deduce that two different noncommuting canonical Hamiltonians H_1 and H_2 are available. Furthermore, we can see that $H_3=-H_1$ and $H_4=-H_2$, hence we can implement any transformation $\exp(-iH_1t)$ and $\exp(-iH_2t)$, where t is an *arbitrary* real number, positive or negative. The two specific cases $c_1=c_2$ and $c_1=-c_2$, when $H_1=\pm H_2$ and the simulation based on the protocol (4) is not possible, correspond to the Hamiltonians of a two-mode squeezer and a beam splitter, respectively.

III. XP COUPLING

Having established the notation and described the simulation protocol, we may proceed to the unitary-gate synthesis. Namely, we would like to decompose the unitary transformation G, which we want to simulate, into a sequence of unitary evolutions governed by Hamiltonians H_1 and H_2, which were defined in the preceding section:

$$G=e^{-iH_2t_N}e^{-iH_1t_{N-1}}\cdots e^{-iH_2t_2}e^{-iH_1t_1}. \tag{15}$$

We are particularly interested in the simulations that involve the lowest possible number of steps N, because such simulations require low number of local manipulations in the eventual experimental implementation.

We note here that Eq. (15) is an example of a decomposition of a group element into a product of N other group elements. In the present case, the underlying group is the symplectic group $\mathrm{Sp}(4,\mathbb{R})$ of all linear canonical transformations of the quadratures of the two modes A and B [23,24]. It is worth mentioning here that the related problem of the decomposition of the symplectic transformation into a sequence of simple evolutions associated with the common passive and active linear optical elements has been studied recently. Braunstein has shown that any N-mode symplectic transformation can be implemented as a sequence of an N-mode passive linear interferometer, followed by N single-mode squeezers and another passive interferometer, the so-called Bloch-Messiah decomposition [25]. The decompositions of this kind have also been applied to investigate the properties of nonlinear optical couplers [26,27].

In this section, we shall consider the simplest and also the experimentally relevant coupling between the two systems described by the interaction Hamiltonian (3). Without loss of generality, we may assume that the coupling constant is equal to unity, hence the two relevant Hamiltonians read

$$H_1=x_Ap_B,\quad H_2=p_Ax_B. \tag{16}$$

All the canonical Hamiltonians (9) have the important property that the x and p quadratures are not mutually coupled when we write down the Heisenberg equations of motion for x_j and p_j. This means that the evolution of operators $\boldsymbol{x}=(x_A,x_B)^T$ and $\boldsymbol{p}=(p_A,p_B)^T$ is governed by the following linear canonical transformations:

$$\boldsymbol{x}_{\mathrm{out}}=\boldsymbol{S}\boldsymbol{x}_{\mathrm{in}},\quad \boldsymbol{p}_{\mathrm{out}}=\boldsymbol{R}\boldsymbol{p}_{\mathrm{in}}. \tag{17}$$

This decoupling of x and p quadratures greatly simplifies the analysis. Transformation (17) must preserve the canonical commutation relations $[x_j,p_k]=i\delta_{jk}$. From these conditions we can express matrix $\boldsymbol{R}$ in terms of $\boldsymbol{S}$,

$$\boldsymbol{R}=(\boldsymbol{S}^T)^{-1}, \tag{18}$$

hence the evolution of the p quadratures is uniquely determined by the evolution of the x quadratures.

Our task is to construct the two-mode unitary gates (symplectic transformations) as a sequence of a small number of unitary transformations generated by Hamiltonians (16). Matrices $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ associated with the unitary evolutions $U_1=\exp(-iH_1t)$ and $U_2=\exp(-iH_2t)$, respectively, read

$$\boldsymbol{S}_1(t)=\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix},\quad \boldsymbol{S}_2(t)=\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}. \tag{19}$$

Factorization (15) can be rewritten in terms of matrices S_j as follows:

$$\boldsymbol{S}=\boldsymbol{S}_2(t_N)\boldsymbol{S}_1(t_{N-1})\cdots\boldsymbol{S}_2(t_2)\boldsymbol{S}_1(t_1), \tag{20}$$

where $\boldsymbol{S}$ is the matrix associated with gate G. Since $\det\boldsymbol{S}_1=\det\boldsymbol{S}_2=1$, we are restricted to a three-parametric subgroup of transformations $\boldsymbol{S}$ such that $\det\boldsymbol{S}=1$. In what follows, we will discuss the implementation of three important gates: a beam splitter, a two-mode squeezer, and a single-mode squeezer.

A. Beam splitter

The beam splitter operation is described by the matrix

$$\boldsymbol{S}_{BS}(\theta)=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}. \tag{21}$$

It is proved in the Appendix that it is impossible to simulate $\boldsymbol{S}_{BS}$ via a two-step protocol, so we must consider three-step protocols. We show that $\boldsymbol{S}_{BS}$ can be implemented as a sequence of three evolutions (19):

$$\boldsymbol{S}_{BS}(\theta)=\boldsymbol{S}_1(\gamma)\boldsymbol{S}_2(\beta)\boldsymbol{S}_1(\alpha). \tag{22}$$

The explicit multiplication yields

$$\boldsymbol{S}_{BS}(\theta)=\begin{pmatrix} 1+\alpha\beta & \beta \\ \alpha+\gamma(1+\alpha\beta) & 1+\gamma\beta \end{pmatrix}. \tag{23}$$

If we compare the elements of the matrices on left- and right-hand sides of Eq. (23), we obtain a set of equations for parameters α, β, and γ, whose solution yields

$$\alpha=-\tan\frac{\theta}{2}, \quad \beta=\sin\theta, \quad \gamma=\alpha. \tag{24}$$

The interaction times α and β can be negative but this is not an obstacle, as explained in the preceding section, because we can change the sign of Hamiltonian H_1 or H_2 by π rotation of one of the systems. Two cases of particular importance are (i) the balanced beam splitter ($\theta=\pi/4$), which requires $\alpha=1-\sqrt{2}$ and $\beta=\sqrt{2}/2$ and (ii) the swap ($\theta=\pi/2$) that exchanges the quantum states of the two systems, $\alpha=-1$ and $\beta=1$.

The swap gate is closely related to the two-step protocol for mapping the state of collective atomic spin on light, which was suggested by Kuzmich and Polzik [20]. In fact, their two-step protocol can be obtained by simply removing the last step of the present three-step swap gate. The removal of the third step means that the mapping adds some noise and the procedure is, thus, only approximate. A possible way of improving its performance is to use squeezing. For details, see Ref. [20].

B. Two-mode squeezer

Let us now turn our attention to the two-mode squeezer, described by the following matrix:

$$\boldsymbol{S}_{\mathrm{TMS}}(r)=\begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix}. \tag{25}$$

As in the case of the beam splitter, we attempt to implement this transformation as a sequence of three evolutions, cf. Eq. (22). By comparison of the right-hand side of Eq. (23) with matrix (25), we again obtain a system of nonlinear equations for parameters α, β, and γ having the solution

$$\alpha=\tanh\frac{r}{2}, \quad \beta=\sinh r, \quad \gamma=\alpha. \tag{26}$$

The parameters are finite for any finite r. However, β grows exponentially with r and for large r we have $\beta\propto e^r$. On the other hand, for small r we get $\beta\approx r$. This implies that we may reduce the synthesis time if we implement the two-mode squeezing transformation as a sequence of n two-mode squeezers with $r'=r/n$. The reduction of the time is achieved at the expense of a higher number of steps of the gate synthesis protocol. For modest values of squeezing r, the sequence of three evolutions is advantageous because it involves the minimum number of necessary manipulations of systems A and B.

C. Single-mode squeezer

After dealing with two-mode gates, let us now focus on the single-mode gate, namely, the single-mode squeezer. Since $\det\boldsymbol{S}=1$, the squeezing of quadrature x_A will necessarily be accompanied by antisqueezing of x_B, and vice versa:

$$\boldsymbol{S}_{\mathrm{SMS}}(r)=\begin{pmatrix} e^r & 0 \\ 0 & e^{-r} \end{pmatrix}. \tag{27}$$

It turns out that in this case, a sequence of three transformations (19) is insufficient and we must consider a sequence of four evolutions:

$$\boldsymbol{S}_{\mathrm{SMS}}(r)=\boldsymbol{S}_2(\delta)\boldsymbol{S}_1(\gamma)\boldsymbol{S}_2(\beta)\boldsymbol{S}_1(\alpha). \tag{28}$$

We proceed as before and derive equations for the four parameters appearing in Eq. (28):

$$e^r=1+\alpha\beta+\delta(\alpha+\gamma+\alpha\beta\gamma),$$

$$e^{-r}=1+\gamma\beta,$$

$$0=\beta+\delta(1+\gamma\beta),$$

$$0=\alpha+\gamma(1+\alpha\beta).$$

This system of equations has a one-parametric class of solutions given by

$$\beta=\frac{e^r-1}{\alpha}, \quad \gamma=-\alpha e^{-r}, \quad \delta=\frac{e^r(1-e^r)}{\alpha}, \tag{29}$$

and α is arbitrary but nonzero. We may choose the optimal value of α, which minimizes the total interaction time T:

$$T=|\alpha|+|\beta|+|\gamma|+|\delta|. \tag{30}$$

Assuming that $r>0$, we obtain, by solving $dT/d\alpha=0$, the optimal value

$$\alpha=\sqrt{\frac{e^{2r}-1}{1+e^{-r}}}. \tag{31}$$

In the limit of small r, all the four parameters α, β, γ, and δ are proportional to $\sqrt{r}$. This stems from the fact that the single-mode squeezing Hamiltonian $H_{\mathrm{SMS}}=x_Ap_A$ cannot be obtained as a linear combination of the two-mode Hamiltonians H_1 and H_2 and only the terms of the order of $O(t^2)$, or higher, in Eq. (15) may give rise to the contribution proportional to H_{SMS}.

D. General protocols and numerical results

We have seen that the synthesis of the single-mode squeezing gate requires a sequence of four steps if we restrict ourselves to protocols (15). However, it is not *a priori* clear whether the four-step simulation (28) is the optimal way of designing single-mode squeezing gate. It may be the case that the number of steps and/or the required interaction time

TABLE I. Parameters of a three-step protocol that accomplishes a single-mode squeezing of mode A with squeezing constant $r=1$.

j	ϕ_{Aj}	ϕ_{Bj}	t_j
1	−2.78869	−0.65328	1.78008
2	−0.00040	4.33771	1.41767
3	3.14193	1.13816	1.82622
4	−1.21861	2.64601	

can be reduced by using the general protocol (4). The general *three-step* procedure (4) is characterized by 11 parameters: eight phase-shifts ϕ_{Aj} and ϕ_{Bj}, and three interaction times t_j. Since any two-mode symplectic transformation is fully specified by ten parameters, one may expect that the single-mode squeezing can be achieved by means of some general three-step procedure.

As before, it is convenient to work in the Heisenberg picture, where vector of the quadratures $\boldsymbol{v}=(x_A,p_A,x_B,p_B)^T$ transforms according to

$$\boldsymbol{v}_{\rm out}=\boldsymbol{M}_G\boldsymbol{v}_{\rm in}, \tag{32}$$

where $\boldsymbol{M}_G$ is a 4×4 symplectic matrix associated with G. Transformation (32) should result in single-mode squeezing of mode A, which means that the first two rows of $\boldsymbol{M}_G$ should read $M_{G,ij}=M_{G,ii}\delta_{ij}$, $i=1,2$, $j=1,\ldots,4$ and $M_{G,11}=e^r$ and $M_{G,22}=e^{-r}$. The elements of matrix $\boldsymbol{M}_G$ can be expressed as functions of ϕ_{Aj}, ϕ_{Bj}, and t_j. The resulting system of nonlinear equations for these parameters is rather complicated and was solved numerically. Note also that here we are interested only in the transformation of mode A and we do not specify the squeezing of mode B. Thus, we consider a much broader class of protocols than in Sec. III C.

The numerical calculations reveal that it is indeed possible to squeeze mode A in just three steps. Moreover, we have found that these three-step protocols may be more efficient than the four-step protocol given in Sec. III C, also in terms of the required interaction time. An explicit example is given in Table I, which shows the parameters of the protocol that accomplishes a single-mode squeezing of mode A with the squeezing constant $r=1$. The total interaction time $T=\Sigma_{j=1}^{3}|t_j|\approx5.02$ is significantly smaller than time $T=5.91$ required for the best four-step protocol (28).

The main complication when dealing with the general protocol (4) is that the equations for the relevant parameters are very complicated and can be solved only numerically. Moreover, there exist many different solutions to these equations, i.e., there are many different three-step protocols for each single-mode squeezing gate. It would be interesting to find the optimal protocol among them, which minimizes the total interaction time T for a fixed squeezing r. However, this is a highly nontrivial optimization problem that is beyond the scope of the present paper.

A related problem is whether the three-step simulations of the beam splitter and two-mode squeezing transformations discussed in Secs. III A and III B are time optimal, or whether some general three-step protocol (4) can exhibit better performance. Again we have investigated this issue with the help of the numerical calculations. As before, we have found that there are many different general three-step protocols for gates (21) and (25) and that these protocols differ in the total interaction time. Interestingly, the numerics provide some evidence that the protocols derived in Secs. III A and III B may be time optimal because, in contrast to the the case of the single-mode squeezer, we have not been able to find a three-step protocol that would require less time (for a given value of θ or r) than the protocols given in Secs. III A and III B.

IV. GENERIC QUADRATIC COUPLING

In this section, we shall assume that the interaction Hamiltonian has the generic canonical form. Although the mathematical analysis will be more involved, we shall still be able to derive analytical formulas for the interaction times characterizing the gate synthesis. Without loss of generality, we may assume that $c_1=1$ in Eq. (9). We have to distinguish the two classes of Hamiltonians giving rise to qualitatively different evolutions of the quadratures in the Heisenberg picture. For $c_2>0$ the dynamics resembles an amplifier, while for $c_2<0$ we obtain oscillatory dynamics reminiscent of a beam splitter. We shall discuss these two cases separately. As proved in the Appendix, it is not possible to simulate the beam splitter and the two-mode squeezer in only two steps with Hamiltonians (1). Therefore, we shall focus on the three-step protocols.

A. Amplifierlike Hamiltonians

We suppose first that $c_2>0$ and introduce a more convenient notation $c_2=s^2$, $s>0$, hence

$$H_1=x_Ap_B+s^2p_Ax_B,\quad H_2=s^2x_Ap_B+p_Ax_B. \tag{33}$$

It is an easy exercise to derive matrices $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ corresponding to the unitary evolutions governed by H_1 and H_2, respectively,

$$\boldsymbol{S}_1^{+}(t)=\begin{pmatrix}\cosh(st) & s\sinh(st)\\ \dfrac{1}{s}\sinh(st) & \cosh(st)\end{pmatrix}, \tag{34}$$

$$\boldsymbol{S}_2^{+}(t)=\begin{pmatrix}\cosh(st) & \dfrac{1}{s}\sinh(st)\\ s\sinh(st) & \cosh(st)\end{pmatrix}. \tag{35}$$

In what follows, we will focus on the beam splitter and two-mode squeezer transformations. We have seen in the preceding section that these transformations could be implemented as a sequence of three basic evolutions. Moreover, there was an inherent symmetry in this gate synthesis; we have found that $\gamma=\alpha$. It turns out that these basic symmetry properties remain valid also for the generic Hamiltonians (33) and we can thus decompose the two-mode squeezing transformation as

$$\boldsymbol{S}_{\mathrm{TMS}}(r)=\boldsymbol{S}_1^{+}(\alpha/s)\boldsymbol{S}_2^{+}(\beta/s)\boldsymbol{S}_1^{+}(\alpha/s). \tag{36}$$

Here parameters $\alpha=st_1$ and $\beta=st_2$ are the rescaled interaction times. The nonlinear equations for parameters α and β are much more complicated than before. Nevertheless, analytical results can be obtained. From condition $S_{12}=S_{21}$, we get

$$\tanh\beta=\frac{2\sinh\alpha\cosh\alpha}{\cosh^2\alpha-(s^{-2}+1+s^2)\sinh^2\alpha}. \tag{37}$$

Condition $S_{11}=S_{22}$ is satisfied due to symmetry ($\gamma=\alpha$), and parameter α can be determined from the last independent equation $S_{12}/S_{11}=\tanh r$, which yields

$$\frac{2y(s+s^{-1})}{1+y^2(s^{-2}+1+s^2)}=\tanh r,$$

where $y=\tanh\alpha$. This is a quadratic equation for y, whose solution reads

$$\tanh\alpha=\frac{s+s^{-1}-\sqrt{1+(s^{-2}+1+s^2)(\cosh r)^{-2}}}{(s^{-2}+1+s^2)\tanh r}. \tag{38}$$

We have selected the root that yields the correct limit $\tanh\alpha=0$ when $r\to 0$. In the opposite limit $r\to\infty$, we obtain

$$\tanh\alpha_\infty=\frac{1}{s^{-1}+1+s}. \tag{39}$$

On inserting this back into Eq. (37), we find that

$$\lim_{r\to\infty}\tanh\beta=1. \tag{40}$$

It is easy to check that the equations for α and β have finite solutions for any finite r. In the limit $r\to\infty$, α approaches a finite asymptotic value, cf. Eq. (39), while β grows to infinity.

Beam splitter (21) can also be simulated by the symmetric sequence (36). The calculations of parameters α and β parallel those for the two-mode squeezer. From condition $S_{12}=-S_{21}$, we express β in terms of α:

$$\tanh\beta=\frac{-2\sinh\alpha\cosh\alpha}{\cosh^2\alpha+(s^{-2}-1+s^2)\sinh^2\alpha}. \tag{41}$$

Since $(s^{-2}-1+s^2)\geq 1$, it follows that $|\tanh\beta|\leq|\tanh(2\alpha)|$. From condition $S_{12}/S_{11}=\tan\theta$, we obtain quadratic equation for $\tanh\alpha$, leading to

$$\tanh\alpha=\frac{(s^{-1}-s)-\mathrm{sgn}(s^{-1}-s)\sqrt{\dfrac{s^{-2}-1+s^2}{\cos^2\theta}-1}}{(s^{-2}-1+s^2)\tan\theta}. \tag{42}$$

The sgn function in the above formula selects the root that yields the correct limit $\alpha=0$ when $\theta\to 0$. Formula (42) is applicable in interval $\theta\in[0,\pi/2]$. In the limiting case $\theta=\pi/2$, we have

$$\tanh\alpha_{\pi/2}=\frac{\mathrm{sgn}(s-s^{-1})}{\sqrt{s^{-2}-1+s^2}}, \tag{43}$$

which implies that $|\tanh\alpha_{\pi/2}|<1$ iff $s\neq 1$. Furthermore, it can be shown that α is a monotonic function of θ in interval $[0,\pi/2]$. We can thus conclude that with the interaction Hamiltonian of the amplifier type (33), we can implement any beam splitter transformation (21) with the mixing angles in interval $[0,\pi/2]$, which includes the two important cases of a balanced beam splitter ($\theta=\pi/4$) and the swap ($\theta=\pi/2$). It follows from expression (43) that the simulation becomes more and more time consuming for Hamiltonians close to the two-mode squeezing Hamiltonian $H_{\mathrm{TMS}}=x_Ap_B+p_Ax_B$, i.e., when $s\to 1$.

B. Beam splitter–like Hamiltonians

Having derived the gate synthesis parameters for Hamiltonians (33), we proceed to the interaction Hamiltonians leading to oscillatory dynamics:

$$H_1=x_Ap_B-s^2p_Ax_B,\quad H_2=-s^2x_Ap_B+p_Ax_B. \tag{44}$$

The **S** matrices associated with these Hamiltonians read

$$\boldsymbol{S}_1^{-}(t)=\begin{pmatrix}\cos(st) & -s\sin(st)\\ \dfrac{1}{s}\sin(st) & \cos(st)\end{pmatrix}, \tag{45}$$

$$\boldsymbol{S}_2^{-}(t)=\begin{pmatrix}\cos(st) & \dfrac{1}{s}\sin(st)\\ -s\sin(st) & \cos(st)\end{pmatrix}. \tag{46}$$

We shall not repeat the details of the derivations of parameters α and β that fully characterize the gate synthesis protocol (36) and we only summarize the results here. The beam splitter transformation can be accomplished by the following choice:

$$\tan\beta=\frac{-2\sin\alpha\cos\alpha}{\cos^2\alpha+(s^{-2}+1+s^2)\sin^2\alpha} \tag{47}$$

and

$$\tan\alpha=\frac{s^{-1}+s-\sqrt{(s^{-2}+1+s^2)(\cos\theta)^{-2}+1}}{(s^{-2}+1+s^2)\tan\theta}. \tag{48}$$

This reveals that simulation of any beam splitter with $\theta\in[0,\pi/2]$ is possible and the parameters satisfy $|\alpha|\leq\pi/2$ and $|\beta|\leq\pi/2$.

Consider now the two-mode squeezing operation (25). After some algebra, one obtains

$$\tan\beta = \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha - (s^{-2}-1+s^2)\sin^2\alpha} \tag{49}$$

and

$$\tan\alpha = \frac{s^{-1}-s-\mathrm{sgn}(s^{-1}-s)\sqrt{\dfrac{s^{-2}-1+s^2}{\cosh^2 r}-1}}{(s^{-2}-1+s^2)\tanh r}. \tag{50}$$

The function $\tan\alpha$ must be real, which implies that the term under the square root must be non-negative. This constraint, in turn, limits the amount of two-mode squeezing that can be produced via a three-step protocol (36). It holds that $r \leqslant r_{\mathrm{th}}$, where

$$\cosh r_{\mathrm{th}} = \sqrt{s^{-2}-1+s^2}. \tag{51}$$

The origin of this bound lies in the fact that the dynamics governed by Hamiltonians (44) is oscillatory and fully periodic with period $2\pi/s$. Squeezing above threshold r_{th} can be achieved only if we concatenate several three-step protocols. It thus appears that the amplifierlike Hamiltonians (33) are, in certain sense, more versatile than the beam splitter–like Hamiltonians (44), because the former allow one to implement any two-mode squeezing gate and also any beam splitter with $\theta \in [0,\pi/2]$, via a three-step protocol (36).

V. CONCLUSIONS

In this paper we have addressed the problem of gate synthesis for continuous variable systems. We have assumed that two single-mode systems A and B interact via a quadratic Hamiltonian H and we have studied how to implement a unitary symplectic gate G with the use of this Hamiltonian as a resource. The gate synthesis protocol consists of a sequence of evolutions governed by H and followed by fast local phase shifts applied to systems A and B. We have focused on the gate simulation protocols that involve the minimal number of necessary steps, because these protocols require a low number of local control operations, which is important from the experimental point of view. We have shown that a three-step protocol suffices for simulation of the two-mode squeezer as well as a beam splitter. For the specific case of Hamiltonian (3), we have also found a protocol for single-mode squeezing and we have numerically investigated the general three-step simulation protocols (4). Our results are applicable to any physical system whose two parts are coupled via the quadratic Hamiltonian. In particular, the gate synthesis protocols proposed in the present paper may find applications in the experiments where light interacts with atomic ensembles via a Kerr-like coupling [15,18,19].

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APPENDIX: TWO-STEP PROTOCOLS ARE INSUFFICIENT

Here we prove that with the interaction Hamiltonians (9), it is not possible to synthesize the beam splitter or two-mode squeezing gates by using the general two-step protocol (4) that reads

$$G = V_3 e^{-iHt_2} V_2 e^{-iHt_1} V_1. \tag{A1}$$

We present the proof for the amplifierlike Hamiltonian $H = x_A p_B + s^2 p_A x_B$. The proof for the beam splitter–like Hamiltonians (44) is analogous. Finally, the proof for Hamiltonian $H = x_A p_B$ can be obtained by taking limit $s \to 0$.

Consider a generic beam splitter transformation with mixing angle θ and arbitrary phase factors. In the Heisenberg picture, the output annihilation operators are linear combinations of the input ones:

$$\begin{aligned} a_{\mathrm{out}} &= a_{\mathrm{in}} e^{i\psi_1}\cos\theta + b_{\mathrm{in}} e^{i\psi_2}\sin\theta, \\ b_{\mathrm{out}} &= b_{\mathrm{in}} e^{i\psi_3}\cos\theta - a_{\mathrm{in}} e^{i\psi_4}\sin\theta, \end{aligned} \tag{A2}$$

where phases ψ_j must satisfy

$$\psi_1 - \psi_2 + \psi_3 - \psi_4 = 0, \tag{A3}$$

because Eq. (A2) must preserve the canonical commutation relations. Similarly, the generic two-mode squeezing transformation can be written as follows:

$$\begin{aligned} a_{\mathrm{out}} &= a_{\mathrm{in}} e^{i\psi_1}\cosh r + b_{\mathrm{in}}^{\dagger} e^{i\psi_2}\sinh r, \\ b_{\mathrm{out}} &= b_{\mathrm{in}} e^{i\psi_4}\cosh r + a_{\mathrm{in}}^{\dagger} e^{i\psi_3}\sinh r. \end{aligned} \tag{A4}$$

Assume that G is the transformation of type (A2) or (A4). Since the local unitary phase-shift operation V_j adds only phase factors to the annihilation and creation operators, a modified transformation $G' = V_5 G V_4$ would still have a form (A2) or (A4). In particular, by choosing $V_4 = V_1^{\dagger}$ and $V_5 = V_3^{\dagger}$, we obtain a simplified protocol:

$$G' = e^{-iHt_2} e^{-i\phi_A a^{\dagger}a - i\phi_B b^{\dagger}b} e^{-iHt_1}. \tag{A5}$$

As described in Sec. III D, we can associate a symplectic matrix $\boldsymbol{M}_{G'}$ with unitary G'. For any transformation (A2) and (A4), the elements of the symplectic matrix $\boldsymbol{M}$ must satisfy

$$M_{x_A p_A} = -M_{p_A x_A}, \quad M_{x_B p_B} = -M_{p_B x_B}, \tag{A6}$$

irrespective of phases ψ_j. The relevant matrix elements of $\boldsymbol{M}_{G'}$ read

$$M_{G',x_A p_A} = \cosh\alpha\cosh\beta\sin\phi_A - s^2\sinh\alpha\sinh\beta\sin\phi_B,$$

$$M_{G',p_A x_A} = \frac{1}{s^2}\sinh\alpha \sinh\beta \sin\phi_B - \cosh\alpha \cosh\beta \sin\phi_A ,$$

$$M_{G',x_B p_B} = \cosh\alpha \cosh\beta \sin\phi_B - \frac{1}{s^2}\sinh\alpha \sinh\beta \sin\phi_A ,$$

$$M_{G',p_B x_B} = s^2 \sinh\alpha \sinh\beta \sin\phi_A - \cosh\alpha \cosh\beta \sin\phi_B , \tag{A7}$$

where $\alpha = st_1$ and $\beta = st_2$. On inserting matrix elements (A7) into Eq. (A6), we obtain two equations:

$$(s^2 - s^{-2})\sinh\alpha \sinh\beta \sin\phi_B = 0,$$

$$(s^2 - s^{-2})\sinh\alpha \sinh\beta \sin\phi_A = 0. \tag{A8}$$

For $s \neq 1$, these two equations have only trivial solutions $\alpha = 0$, or $\beta = 0$ or $\sin\phi_A = \sin\phi_B = 0$. In all cases, the unitary transformation (A5) effectively reduces to $\exp(-iHt)$, which clearly differs from the beam splitter or two-mode squeezing transformation. This proves that the two-step protocols cannot be used to simulate gates (A2) and (A4).

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