# Cloning a real *d*-dimensional quantum state on the edge of the no-signaling condition

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We investigate the class of quantum cloning machines that equally duplicate all real states in a Hilbert space of arbitrary dimension. By using the no-signaling condition, namely, that cloning cannot make superluminal communication possible, we derive an upper bound on the fidelity of this class of quantum cloning machines. Then, for each dimension *d*, we construct an optimal symmetric cloner whose fidelity saturates this bound. Similar calculations can also be performed in order to recover the fidelity of the optimal universal cloner in *d* dimensions.

## I. INTRODUCTION

The intimate connection between the impossibility of making a perfect quantum cloning machine (QCM) and the no-signaling condition, which prevents any superluminal communication, has been realized since the seminal papers of Dieks [1] and of Wootters and Zurek [2]. More recently, Gisin has shown that this connection can actually be exploited in order to recover the fidelity 5/6 of the Buzek-Hillery universal QCM for qubits [3]. Any cloning machine that would duplicate a qubit with a fidelity exceeding 5/6 would necessarily open a way to superluminal communication. In short, the no-signaling condition is taken into account by expressing that two statistical ensembles realizing the same input density matrix (e.g., an equal mixture of $|0\rangle$ and $|1\rangle$ or an equal mixture of $|0\rangle+|1\rangle$ and $|0\rangle-|1\rangle$) must result in indistinguishable output density matrices for the clones. Since then, this no-signaling constraint has also been used to recover the fidelity of other classes of cloners, namely, the asymmetric universal and phase-covariant qubit cloners [4,5].

In this paper, we exploit this no-signaling condition in order to derive an upper bound on the fidelity of a class of QCMs that have not been considered in the literature. We analyze symmetric QCMs that duplicate any *d*-dimensional *real* state with an equal fidelity. These are the counterpart of the well-known universal QCMs but within the realm of the so-called "real" quantum mechanics. We also find a constructive method to build QCMs that saturate this upper bound, and therefore are optimal. In particular, using this method for $d=2$, we find a cloner unitarily equivalent to the phase-covariant qubit cloner [6,7] which clones all states $a|0\rangle+b|1\rangle$ ($a$, $b$ being real and satisfying $a^2+b^2=1$) with a fidelity $(1+1/\sqrt{2})/2$. For an arbitrary dimension *d*, we use techniques from tensor calculus in order to derive the no-signaling bound and the explicit cloner. A specific application of this notion of real QCMs arises in four dimensions, when these cloners are equivalent, up to a unitary transformation, to the universal cloners over the set of maximally entangled qubit pairs [8].

In general, the no-signaling requirement does not provide a tight upper bound on the optimal cloning fidelity. The linearity and trace preserving properties (which, combined, imply the no-signaling condition) need to be supplemented with

the complete positivity property in order to determine the best possible cloning transformation [9]. However, for the real QCMs of interest, it is sufficient to combine the no-signaling requirement together with positivity (and trace preservation) in order to find a tight bound, just as in Gisin's original paper [3]. The impossibility of signaling is crucial to derive this bound: would signaling be possible, then no constraint could be put on cloning in this way. Furthermore, we show that a similar reasoning can also be applied in order to find the optimal fidelity of the universal QCM in *d* dimensions [10–12]. Thus, the requirement of no-signaling allows us to recover more simply and straightforwardly some standard results on cloning.

A reason for which the no-signaling upper bound on the fidelity is saturated is that the set of states we are cloning is somehow "large." The set of *d*-dimensional real states is realized by means of the SO(*d*) group representation, while the whole set of *d*-dimensional complex states is realized by means of the usual SU(*d*) representation. If we impose that the QCM acts equally on all the input states defined by one of these representations, then the number of arbitrary independent parameters characterizing the cloning transformation is considerably reduced. For the real QCM, we will show that the density matrix can be reexpressed under the form of a covariant real tensor. This simplification allows us to diagonalize the resulting density matrix and easily express no-signaling and positivity. Then, the initial optimization problem is turned into a simpler one involving only seven independent parameters, which can be solved analytically.

## II. NO-SIGNALING UPPER BOUND ON THE CLONING FIDELITY

The real input state to be cloned is defined in the computational basis $\{|i\rangle\}$ as

$$|\psi\rangle = \sum_{i=0}^{d-1} n_i|i\rangle, \qquad (1)$$

where the amplitudes $n_i$ are real and normalized as $\sum_{i=0}^{d-1} n_i^2 = 1$. The two-clone output density matrix corresponding to this input state $\mathbf{n} = (n_0, \ldots, n_{d-1})$ is defined as

TABLE I. Eigenvector decomposition of the rank-4 $d^2 \times d^2$ tensor $r_{ij,kl}$ characterizing the two-clone density matrix of the real QCM in $d$ dimensions when the input state is $n_i$.

| Eigenvalue | Eigenvectors | Degeneracy |
|---|---|---|
| $\lambda_A$ | $V_{ij,\mu\nu} = \cos\alpha\sin(2\phi)\left[\dfrac{1}{\sqrt{d-1}}n_i n_j + \dfrac{\cot\phi}{d-1}(n_i n_j - \delta_{ij})\right]\delta_{\mu\nu}$ | $\dfrac{d(d-1)}{2}$ |
| | $+\dfrac{e^{i\alpha}}{2}(m_k^\mu m_l^\nu + m_k^\nu m_l^\mu)$ | |
| $\lambda_B$ | $V_{ij} = \cos\phi\, n_i n_j - \dfrac{\sin\phi}{\sqrt{d-1}}(n_i n_j - \delta_{ij})$ | 1 |
| $\lambda_C$ | $V_{ij,\mu} = \cos\theta\, n_i m_j^\mu + \sin\theta\, m_i^\mu n_j$ | $d-1$ |
| $\lambda_D$ | $V'_{ij,\mu} = -\sin\theta\, n_i m_j^\mu + \cos\theta\, m_i^\mu n_j$ | $d-1$ |
| $\lambda_E$ | $V_{ij,\mu_1,\dots,\mu_{d-3}} = \displaystyle\sum_{\mu_{d-2},\mu_{d-1}=1}^{d-1}\dfrac{1}{\sqrt{2}}\epsilon_{\mu_1,\dots,\mu_{d-1}}m_i^{\mu_{d-1}}m_j^{\mu_{d-2}}$ | $\dfrac{(d-1)(d-2)}{2}$ |
| Total | | $d^2$ |

$$\rho_{out}(\mathbf{n}) = \sum_{i,j,k,l=0}^{d-1} r_{ij,kl}(\mathbf{n})|i\rangle\langle k| \otimes |j\rangle\langle l|. \tag{2}$$

We require that the QCM cloner act similarly on all real input states, that is,

$$\rho_{out}(\mathbf{n}') = U \otimes U\, \rho_{out}(\mathbf{n})U^\dagger \otimes U^\dagger, \tag{3}$$

where $\mathbf{n}' = (n'_0, \dots, n'_{d-1})$ with $n'_i = R_{ij}n_j$ and $U = \Sigma_{i,j}R_{ij}|i\rangle\langle j|$ is an arbitrary real rotation in the $d$-dimensional space satisfying $R_{ij}R_{kj} = \delta_{ij}$ (the summation symbol will be omitted from now on when dealing with tensors). This covariance property implies that $r_{ij,kl}(\mathbf{n})$ is a tensor of rank 4, i.e., it satisfies

$$r_{ij,kl}(\mathbf{n}') = R_{ii'}R_{jj'}R_{kk'}R_{ll'}r_{i'j',k'l'}(\mathbf{n}). \tag{4}$$

Since we seek a symmetric cloner, the output density matrix must be invariant under the interchange of the two clones, i.e., under the permutations $i \leftrightarrow j$ and $k \leftrightarrow l$. The covariance and the permutation symmetry of the tensor impose the following general form [13]:

$$r_{ij,kl}(\mathbf{n}) = \kappa_1\delta_{ik}\delta_{jl} + \kappa_2\delta_{il}\delta_{jk} + \kappa_3\delta_{ij}\delta_{kl} + \kappa_4(n_i n_k \delta_{jl}$$
$$+ n_j n_l \delta_{ik}) + \kappa_5(n_i n_l \delta_{jk} + n_j n_k \delta_{il}) + \kappa_6(n_i n_j \delta_{kl}$$
$$+ n_k n_l \delta_{ij}) + \kappa_7 n_i n_j n_k n_l, \tag{5}$$

where the $\kappa_\alpha$ are seven independent real parameters. Note that if $\kappa_7 = 1$ and all other parameters vanish, the two clones are perfect. The main result below is that the no-signaling condition imposes that $\kappa_7 = 0$, so perfect cloning is precluded.

It is convenient, in what follows, to diagonalize this tensor, Eq. (5), and use its eigenvalues (along with a few other coefficients) as independent parameters that characterize the tensor. The optimization will then be made over these parameters. The diagonalization of Eq. (5) results in

$$r_{ij,kl}(\mathbf{n}) = \lambda_A\sum_{\mu,\nu=1}^{d-1}V_{ij,\mu\nu}V^*_{kl,\mu\nu} + \lambda_B V_{ij}V^*_{kl}$$
$$+ \lambda_C\sum_{\mu=1}^{d-1}V_{ij,\mu}V^*_{kl,\mu} + \lambda_D\sum_{\mu=1}^{d-1}V'_{ij,\mu}V'^*_{kl,\mu}$$
$$+ \lambda_E\sum_{\substack{\mu_1,\dots,\mu_{d-3}=1 \\ \mu_1 \neq \cdots \neq \mu_{d-3}}}V_{ij,\mu_1,\dots,\mu_{d-3}}V^*_{kl,\mu_1,\dots,\mu_{d-3}}, \tag{6}$$

where the complete set of orthonormal eigenvectors is defined in Table I.

Note that all the eigenvectors are normalized to unity except for the off-diagonal eigenvectors of the symmetric subset $V_{ij,\mu\nu}$ which are normalized to $1/2$, i.e., $\Sigma_{i,j}V_{ij,\mu\nu}V^*_{ij,\mu'\nu'} = (\delta_{\mu\mu'}\delta_{\nu\nu'} + \delta_{\mu\nu'}\delta_{\nu\mu'})/2$. Here, the coefficients $m_i^\mu$ denote an arbitrary set of basis vectors ($1 \leq \mu \leq d-1$) of the subspace orthogonal to $\mathbf{n}$. The notation $\epsilon_{\mu_1,\dots,\mu_{d-1}}$ stands for the unit antisymmetric tensor of rank $d-1$, which is equal to 1 if $(\mu_1,\dots,\mu_{d-1})$ is an even permutation of $(1,\dots,d-1)$, to $-1$ if $(\mu_1,\dots,\mu_{d-1})$ is an odd permutation of $(1,\dots,d-1)$, and to 0 if any index is repeated. The permutation symmetry between the two clones imposes that $(\lambda_C - \lambda_D)\cos(2\theta) = 0$, so that either $\lambda_C = \lambda_D$ or $\cos\theta = \pm 1/\sqrt{2}$. This constraint reduces to seven the number of independent parameters among the eight parameters $\lambda_I$ ($I = A, B, C, D, E$), $\alpha$, $\phi$, and $\theta$. A straightforward identification between expressions (5) and (6) allows us to unambiguously express the seven independent parameters $\kappa_\alpha$ in terms of the new ones.

Let us now consider the density matrix of each of the two clones and their fidelity with respect to the input state. The two clones are in the same mixture due to permutation symmetry, and the covariance imposes that the density matrix is given by a rank-2 tensor of the form

$$\mathrm{Tr}_1\rho_{out}(\mathbf{n}) = \mathrm{Tr}_2\rho_{out}(\mathbf{n})$$

$$= \sum_{i,j=0}^{d-1} \frac{1}{d-1}[(dF-1)n_i n_j$$

$$+ (1-F)\delta_{ij}]|i\rangle\langle j|$$

$$= F|\psi\rangle\langle\psi| + \frac{1-F}{d-1}(1-|\psi\rangle\langle\psi|), \qquad (7)$$

where $F$ is the fidelity

$$F = \mathrm{Tr}_1[|\psi\rangle\langle\psi|\rho_{out}(\mathbf{n})] = n_i n_k r_{ij,kj}(\mathbf{n}). \qquad (8)$$

Using Eq. (6), we can express the fidelity in terms of the eigenvalues and eigenvector parameters,

$$F = \lambda_A \cos^2\alpha \sin^2(2\phi) + \lambda_B \cos^2\phi + (\lambda_C \cos^2\theta + \lambda_D \sin^2\theta)$$

$$\times (d-1). \qquad (9)$$

When maximizing $F$, we will have to take into account the three following constraints.

(i) *Positivity.* $\rho_{out} \geq 0$. This gives $\lambda_I \geq 0$ with $I = A, B, C, D, E$.

(ii) *Trace preservation.* $\mathrm{Tr}(\rho_{out}) = 1$. This gives

$$\frac{d(d-1)}{2}\lambda_A + \lambda_B + (d-1)(\lambda_C + \lambda_D) + \frac{(d-1)(d-2)}{2}\lambda_E = 1. \qquad (10)$$

(iii) *No-signaling condition.* This implies that the uniform mixtures of any two basis sets $\mathbf{n}^\mu$ and $\mathbf{n}^{\mu\prime}$ (which thus both realize the same input density matrix, namely, the identity) result in two equal output density matrices. Otherwise, the density matrices would be distinguishable and measuring the clones would give information on which mixture was used, which means signaling. Thus,

$$\sum_{\mu=0}^{d-1} \rho_{out}(\mathbf{n}^\mu) = \sum_{\mu=0}^{d-1} \rho_{out}(\mathbf{n}'^\mu) \qquad (11)$$

can be used as a necessary (but possibly not sufficient) condition to respect the no-signaling constraint. Using Eq. (5) and the completion relation $\sum_{\mu=0}^{d-1} n_i^\mu n_j^\mu = \sum_{\mu=0}^{d-1} n_i^{\mu\prime} n_j^{\mu\prime} = \delta_{ij}$, the only way of satisfying Eq. (11) is to forbid quartic term in Eq. (5), i.e., to impose $\kappa_7 = 0$. As mentioned earlier, this means that the ''perfect cloning'' term in Eq. (5) is forbidden. Indeed, if Eq. (11) was not required, then by maximizing $F$ we would obtain a perfect cloner described in terms of the only eigenvector $V^{ij}$ setting $\phi = 0$. Thus, we observe that the no-signaling condition is sufficient to impose Eq. (11), that is, to exclude perfect cloning. In terms of the eigenvalues and eigenvector parameters, this no-signaling condition becomes

$$\lambda_A t_A + \lambda_B t_B = \lambda_C(1 + \sin 2\theta) + \lambda_D(1 - \sin 2\theta), \qquad (12)$$

where we have defined the positive coefficients

$$t_A = \cos^2\alpha\left(\frac{d-2}{d-1}\sin^2(2\phi) + \frac{\sin(4\phi)}{\sqrt{d-1}}\right) + 1 \geq 0, \qquad (13)$$

$$t_B = \left(\cos\phi - \frac{\sin\phi}{\sqrt{d-1}}\right)^2 \geq 0. \qquad (14)$$

Now, the constrained optimization problem can be solved analytically in order to upper bound the cloning fidelity. First, we observe that when $\lambda_E \neq 0$, we can always increase the fidelity by substituting $\lambda_I(I = A, B, C, D)$ with $\lambda_I/(1 - (d-1)(d-2)\lambda_E/2)$ and $\lambda_E$ with 0. This substitution increases the fidelity while keeping the constraints satisfied. Therefore, the requirement $\lambda_E = 0$ always gives an optimal fidelity. Second, remember that the permutation symmetry imposes either (a) $\cos\theta = \pm 1/\sqrt{2}$ or (b) $\lambda_C = \lambda_D$. We will consider these two possibilities.

*Case a.* Let us examine the case $\cos\theta = 1/\sqrt{2}$. (The case $\cos\theta = -1/\sqrt{2}$ is treated similarly.) We eliminate the variable $\lambda_C$ between Eqs. (10) and (12), resulting in

$$\frac{d-1}{2}(d+t_A)\lambda_A + \left(1 + \frac{d-1}{2}t_B\right)\lambda_B + (d-1)\lambda_D = 1. \qquad (15)$$

Similarly, combining Eqs. (9) and (12) gives

$$F = \left(\cos^2\alpha \sin^2(2\phi) + \frac{d-1}{4}t_A\right)\lambda_A + \left(\cos^2\phi + \frac{d-1}{4}t_B\right)\lambda_B$$

$$+ \frac{d-1}{2}\lambda_D. \qquad (16)$$

The coefficients in front of the eigenvalues $\lambda_A$, $\lambda_B$, and $\lambda_D$ are all semipositive in Eqs. (15) and (16), so that only one of these eigenvalues is nonzero in the optimum. For each nonzero eigenvalue, Eqs. (15) and (16) give a value for the fidelity, and the maximum fidelity is simply chosen as the best of these three possibilities. We find that the fidelity is upper bounded by

$$\max\left\{\frac{\cos^2\alpha \sin^2(2\phi) + \frac{d-1}{4}t_A}{\frac{d-1}{2}(d+t_A)}, \frac{\cos^2\phi + \frac{d-1}{4}t_B}{1 + \frac{d-1}{2}t_B}, \frac{1}{2}\right\}. \qquad (17)$$

The first term in the maximum, Eq. (17), must be greater than 1/2 to be of interest. This condition is fulfilled only if $\cos^2\alpha\sin^2(2\phi) \geq d(d-1)/4$ and this can be the case only when $d=2$. But, for $d=2$, we notice that the first term is maximized by choosing $\cos(\alpha) = 1$, since the optimum always lies within the range $\pi \leq 4\phi \leq 3\pi/2$. Moreover, if we substitute $2\phi$ with $-\phi$, we recover the second term of Eq. (17). Thus, optimizing the first term for $d=2$ amounts to optimizing the second term. As a consequence, we are left with maximizing the second term of Eq. (17) for any dimension, which only depends on $\phi$. The maximum is found for

$$\tan\phi = \frac{d+4-\sqrt{d^2+4d+20}}{2\sqrt{d-1}}. \tag{18}$$

Consequently, the cloning fidelity of the real QCM in $d$ dimensions cannot exceed the following upper bound:

$$F \leq F_{max} = \frac{1}{2} + \frac{\sqrt{d^2+4d+20}-d+2}{4(d+2)} \tag{19}$$

in order to make signaling via cloning impossible. This is the main result of this section.

*Case b.* In order to be complete, let us consider the second case $\lambda_C = \lambda_D$ and show that the upper bound cannot be improved. Similarly to the first case, we eliminate the variable $\lambda_C$ from Eqs. (10) and (12), and obtain equations similar to Eqs. (15) and (16), namely,

$$\frac{d-1}{2}(d+2t_A)\lambda_A + (1+(d-1)t_B)\lambda_B = 1 \tag{20}$$

and

$$F = \left(\cos^2\alpha\,\sin^2(2\phi) + \frac{d-1}{2}t_A\right)\lambda_A + \left(\cos^2\phi + \frac{d-1}{2}t_B\right)\lambda_B. \tag{21}$$

We then obtain an upper bound on $F$ given by

$$\max\left\{ \frac{\cos^2\alpha\,\sin^2(2\phi) + \frac{d-1}{2}t_A}{\frac{d-1}{2}(d+2t_A)}, \frac{\cos^2\phi + \frac{d-1}{2}t_B}{1+(d-1)t_B}\right\}. \tag{22}$$

From Eq. (22), we note that for the fidelity to be greater than 1/2, then either $\cos^2\alpha\,\sin^2(2\phi) > 1/2$ or $\cos^2\phi > 1/2$. But if one of these conditions is satisfied, then each term in Eq. (22) is lower than the corresponding one in Eq. (17). Therefore, we conclude that the no-signaling upper bound is indeed given by Eq. (19).

### III. REAL QCM SATURATING THE NO-SIGNALING BOUND

We will now explicitly construct a real QCM and observe that it saturates the no-signaling upper bound Eq. (19). Hence, we will have found an *optimal* real QCM in $d$ dimensions. The fact that Eq. (19) gives a tight bound on the fidelity shows that, in this case, Eq. (11) is actually a necessary and sufficient condition for no-signaling, so that the no-cloning and no-signaling constraints coincide (when supplemented with positivity, trace preservation, and covariance).

We will follow here the constructive method described in Ref. [12], which consists in considering the cloning of an input system that is maximally entangled with a reference system denoted as $R$, i.e., $\sum_{i=1}^{d}|i\rangle|i\rangle$. In this case, the joint state $|\Psi\rangle_{R,1,2,A}$ of the reference, the two output clones, and the ancilla completely characterizes the cloning transformation. (The reference and ancilla systems are assumed to be-

long to a space of dimension $d$, just as the input and the two clones.) We consider here the most general state

$$|\Psi\rangle_{R,1,2,A} = \sum_{i,j,k,l=0}^{d-1} u_{ijkl}|l\rangle_R|i\rangle_1|j\rangle_2|k\rangle_A, \tag{23}$$

where the indexes 1, 2, $R$, and $A$ refer, respectively, to the two clones, the reference, and the ancilla. Note that this state does not depend on **n**, but it can be easily used in order to define the cloning transformation applied on state $|\psi\rangle$: projecting the reference system of $|\Psi\rangle_{R,1,2,A}$ onto $|\psi\rangle$ [15] amounts to defining the cloning transformation as

$$|\psi\rangle \to \sum_{i,j,k,l=0}^{d-1} u_{ijkl}n_l|i\rangle_1|j\rangle_2|k\rangle_A. \tag{24}$$

We require that this state obeys the following covariance principle:

$$|\Psi\rangle_{R,1,2,A} = U^*\otimes U\otimes U\otimes U^*|\Psi\rangle_{R,1,2,A} \tag{25}$$

for all real unitary rotations $U$ as those used in Eq. (3). This strong requirement allows to recover property (3), while the converse is not necessarily true. Condition (24) physically means that applying a rotation $U$ on the input (or rotating the reference by $U^*$) is equivalent to rotating the two clones by $U$ and the ancilla by $U^*$. This covariance principle implies that $u_{ijkl}$ is a tensor of rank 4, that is, it satisfies

$$u_{ijkl} = R_{ii'}R_{jj'}R_{kk'}R_{ll'}u_{i'j'k'l'}. \tag{26}$$

The most general tensor obeying Eq. (26) can be written as [13]

$$u_{ijkl} = A\,\delta_{il}\delta_{jk} + B\,\delta_{jl}\delta_{ik} + C\,\delta_{kl}\delta_{ij}. \tag{27}$$

The symmetry permutation between the two clones imposes that $A = B$. The fidelity $F$ can be obtained from Eq. (24) by tracing over one of the clones and the ancilla, resulting in

$$F = (d+3)|A|^2 + |C|^2 + 2(AC^* + CA^*). \tag{28}$$

This expression has to be maximized under the normalization constraint

$$2(d+1)|A|^2 + d|C|^2 + 2(AC^* + CA^*) = 1. \tag{29}$$

It can be checked that this maximization procedure exactly gives the right-hand side of Eq. (19), so that the cloner we have constructed saturates the no-signaling bound. The corresponding optimal coefficients are given by

$$A = \left[\frac{d\sqrt{d^2+4d+20}+2d+d^2-8}{4(d-1)(d+2)\sqrt{d^2+4d+20}}\right]^{1/2}, \tag{30}$$

$$C = \frac{\sqrt{d^2+4d+20}-d-2}{4}A. \tag{31}$$

## IV. CASE OF THE $d$-DIMENSIONAL UNIVERSAL CLONER

Consider now a universal cloner, that is, a QCM such that any pure state $|\psi\rangle = \Sigma_{i=0}^{d-1} c_i |i\rangle$ (with $c_i$ being complex amplitudes) is cloned with the same fidelity. To obey the covariance properties (3) for *any* unitary transformation $U$, the output density matrix must have the more restricted form

$$r_{ij,kl}(c_i) = \kappa_1 \delta_{ik}\delta_{jl} + \kappa_2 \delta_{il}\delta_{jk} + \kappa_4 (c_i c_k^* \delta_{jl} + c_j c_l^* \delta_{ik})$$
$$+ \kappa_5 (c_i c_l^* \delta_{jk} + c_j c_k^* \delta_{il}) + \kappa_7 c_i c_j c_k^* c_l^* . \quad (32)$$

In comparison with Eq. (5), the covariance condition imposes that $\kappa_3 = \kappa_6 = 0$ and, consequently, that $\phi = 0$. As a result, the second term in Eq. (17) gives a smaller upper bound so that we find

$$F \le \frac{1}{2} + \frac{1}{d+1}. \quad (33)$$

The universal $d$-dimensional cloner saturating this bound has been discussed in Refs. [10–12], so we see that the no-signaling condition again gives a tight bound. We can recover this cloner by following Sec. III. The covariance condition implies that $C = 0$ and, as a consequence, $A = 1/\sqrt{2(d+1)}$.

## V. DISCUSSION AND CONCLUSION

We have found a class of QCMs that duplicate any $d$-dimensional real state with an equal fidelity

$$F = \frac{1}{2} + \frac{\sqrt{d^2 + 4d + 20} - d + 2}{4(d+2)}. \quad (34)$$

Furthermore, for these universal cloners over real states in $d$ dimensions, we have demonstrated that the no-signaling requirement provides a sufficient constraint to unambiguously determine the optimal performance of the cloners. Hence, we have found the optimal real QCMs.

In the special case of $d = 2$, we recover the phase-covariant qubit cloner of fidelity

$$F_{d=2} = \frac{1 + 1/\sqrt{2}}{2} \simeq 0.854 \quad (35)$$

as derived in Ref. [6] (see also the Appendix of [7]). For qutrits ($d = 3$), we get a cloner of fidelity

$$F_{d=3} = \frac{9 + \sqrt{41}}{20} \simeq 0.770. \quad (36)$$

This result is distinct from the fidelity of the three known QCMs for qutrits [7]: $F = 3/4$ for the universal qutrit cloner,
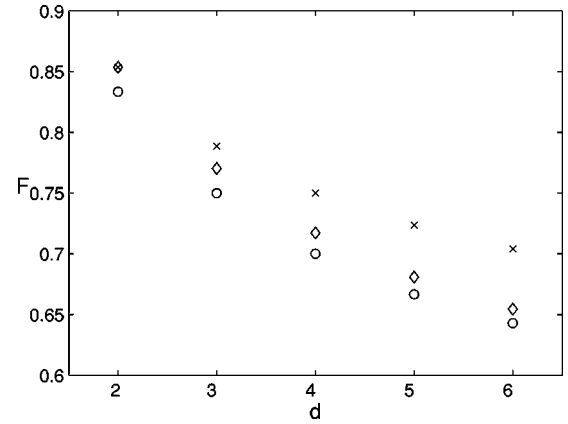


FIG. 1. Fidelity $F$ as a function of the dimension $d$ for the universal cloner (o) [10–12], the real cloner derived in the present paper ($\diamond$), and the cloner of two mutually unbiased bases ($\times$) [14].

$F = (5 + \sqrt{17})/12 \simeq 0.760$ for the two-phase covariant qutrit cloner, and $F = 1/2 + 1/\sqrt{12} \simeq 0.789$ for the qutrit cloner of two mutually unbiased bases. This suggests that these real QCMs form a distinct class of QCMs. Note, finally, that when the dimension $d$ tends to infinity, the cloning fidelity $F$ tends to $1/2 + O(1/d)$. In Fig. 1, we have plotted, for comparison, the fidelity as a function of the dimension $d$ for the universal cloner, the real cloner derived here, and the optimal cloner of two mutually unbiased bases obtained in Ref. [14]. As expected, we observe that the real QCM has a higher fidelity than the universal QCM since it clones the restricted class of real states. However, the real QCM performs less well than the cloner of two mutually unbiased bases (except when $d = 2$ where they coincide).

An interesting issue of this work is the potential generalization of this method exploiting the no-signaling constraint to other classes of cloners. It is likely, however, that for a more restrictive set of states to be cloned equally, the no-signaling constraint may only give a nontight upper bound on the fidelity. A typical example may be cloning the set of two mutually unbiased bases [14]: this smaller set might impose a weaker covariance constraint to the cloning transformation, so the maximum fidelity consistent with no-signaling might correspond to a cloner that is not allowed by quantum mechanics. This will be further investigated.

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