Conditional generation of arbitrary multimode entangled states of light with linear optics

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We propose a universal scheme for the probabilistic generation of an arbitrary multimode entangled state of light with finite expansion in Fock basis. The suggested setup involves passive linear optics, single-photon sources, strong coherent laser beams, and photodetectors with single-photon resolution. The efficiency of this setup may be greatly enhanced if, in addition, a quantum memory is available.

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I. INTRODUCTION

The generation of nonclassical states of light is one of the primary research areas in quantum optics. In particular, the preparation of *entangled* states of light has attracted a considerable amount of attention recently, since these states have been identified as a key resource in quantum-information processing [1]. Such states would, for example, be needed in multiparty quantum-communication protocols such as quantum secret sharing [2–4]. Also, the realization of quantum error correcting codes based on qubits encoded in harmonic oscillators [5] would require highly complex light states. The preparation of entangled light states is also important for investigating the foundations of quantum mechanics. Certain two-mode entangled states have been shown to yield a strong violation of Bell-CHSH inequalities when Alice and Bob perform balanced homodyne measurements followed by an appropriate binning [6], while other light states may be necessary to exhibit Greenberger-Horne-Zeilinger paradoxes using homodyne measurements [7]. Finally, multimode entangled states of light also have applications in ultrahigh precision measurements [8–11] and quantum optical lithography [12–14].

The kinds of interactions between light fields that are experimentally accessible are rather limited, thereby restricting the class of quantum states of the optical field that can be generated in the lab. However, this class can be significantly extended if one considers probabilistic generation schemes, whose success is conditioned on the detection of a particular outcome of a measurement performed on some ancilla system. Schemes for the probabilistic preparation of Fock states [15], arbitrary superpositions of Fock states of single-mode field [16,17], or superpositions of classically distinguishable states [18,19] have been found. In addition, several schemes for the generation of two-mode N-photon path-entangled states have been suggested [20–24], which may be useful to enhance the precision of quantum interferometric setups. On the experimental side, the conditional preparation of a single-photon Fock state with negative Wigner function has recently been reported [25], suggesting that more complex conditional generation schemes may become feasible in the future.

All the above-mentioned schemes are nevertheless quite restrictive, in the sense that they are capable to prepare only single-mode states or some particular class of two-mode states (such as the N-photon states). In this paper, we design a truly *universal* scheme, which can be used to probabilistically generate an *arbitrary* multimode entangled state of light, provided that each mode does not contain more than N photons, where N is an arbitrary but finite integer. The resources required for the present scheme comprise passive linear optical elements (beam splitters and phase shifters), single-photon sources, strong coherent laser beams, and photodetectors. It is now well known that linear optics and single photons are sufficient resources for performing universal quantum computing [26–31], so our proposal provides another illustration of the surprising versatility and power of this paradigm. Our work may have some important applications in view of the numerous possible utilizations of complex multimode states of light outlined above.

Note that, very recently, Clausen *et al.* suggested a scheme for approximate conditional implementation of general single-mode or multimode operators acting on states of traveling optical field [32]. The scheme described in Ref. [32] is also based on linear optics and may be in principle used for (approximate) quantum-state preparation. We emphasize, however, that the approach proposed in the present paper conceptually differs from that of Ref. [32]. We are primarily interested in exact state preparation and our setup is specifically tailored for that purpose.

We will first explain all the essential features of the state-preparation procedure on the simplest yet nontrivial example of two-mode entangled state that is formed by a superposition of two product states,

$$|\psi\rangle_{AB}=q_f|f\rangle_A|f'\rangle_B+q_g e^{i\theta}|g\rangle_A|g'\rangle_B, \tag{1}$$

where $|f\rangle$, $|f'\rangle$, $|g\rangle$, and $|g'\rangle$ are normalized states and q_f and q_g are real. Then we will generalize the preparation procedure and design a scheme for the generation of arbitrary multimode entangled states of light.

The paper is structured as follows. In Sec. II, we propose a method to probabilistically generate the two-mode entangled state (1) with the help of passive linear optics, single-photon sources, strong coherent laser fields, and photodetectors. The performance of the scheme under realistic conditions is discussed in Sec. III, where the effects of imperfect detectors and single-photon sources are analyzed. In

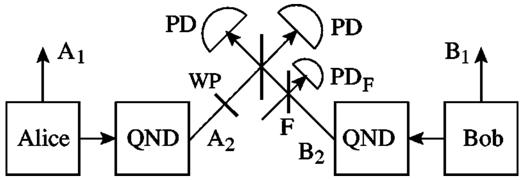

FIG. 1. Schematic of the setup for the generation of the entangled state (1) via entanglement swapping. The two modes emerging from the quantum nondemolition (QND) measurement devices are combined on a balanced beam splitter and the swapping succeeds when each photodetector (PD) detects exactly one photon. The wave plates (WP) and the filter (F) transform the projection on the entangled state (4) into the projection on the singlet state $|VH\rangle-|HV\rangle$. The filtering succeeds if PD_F does not detect any photon. The boxes denoted Alice and Bob correspond to the setup depicted in Fig. 2.

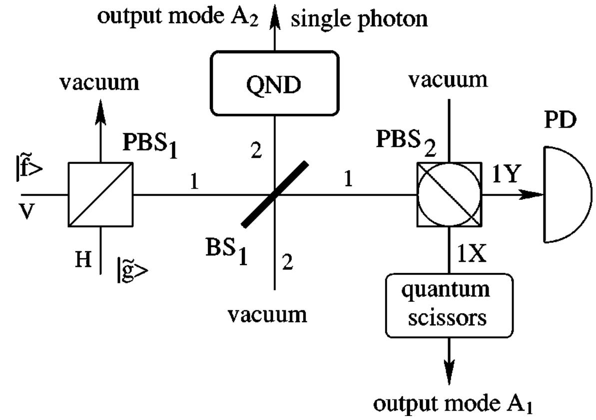

FIG. 2. Setup for the conditional generation of the entangled state (2). The polarizing beam splitter PBS_1 transmits vertically polarized photons and reflects horizontally polarized ones. The box labeled as QND performs the quantum nondemolition measurement of the photon number in the spatial mode while preserving the polarization state of the photon, see Fig. 3. The polarizing beam splitter PBS_2 partially reflects and partially transmits both vertically and horizontally polarized photons. BS_1 is a usual beam splitter and PD denotes photodetector with single-photon sensitivity.

Sec. IV we shall extend the scheme to allow for more than two modes and more than two terms in the superposition. We will thus design a universal device that enables one to conditionally prepare an arbitrary multimode state of light. Finally, the conclusions are drawn in Sec. V.

II. CONDITIONAL GENERATION OF TWO-MODE ENTANGLED STATES

Let us first note that, in order to simplify the notation, we shall omit the normalization factors in front of the quantum states in what follows.

Since the state $|\psi\rangle_{AB}$ is rather complicated, we divide its preparation into several steps. The main simplification stems from the observation that the entangled state (1) can be prepared by means of entanglement swapping [33–35] if we possess the two three-mode states

$$|\phi\rangle_A=|f\rangle_{A_1}|V\rangle_{A_2}+|g\rangle_{A_1}|H\rangle_{A_2}, \qquad (2)$$

$$|\phi'\rangle_B=|f'\rangle_{B_1}|V\rangle_{B_2}+|g'\rangle_{B_1}|H\rangle_{B_2}, \qquad (3)$$

where $|V\rangle$ and $|H\rangle$ denote the state of a single photon in spatial mode A_2 (or B_2) that is polarized vertically or horizontally, respectively. As shown in Fig. 1, we easily obtain state (1) from the state $|\phi\rangle_A|\phi'\rangle_B$ if we project the single photons in spatial modes A_2 and B_2 onto the entangled state

$$|\Phi\rangle=q_f|V\rangle|V\rangle+q_g e^{-i\theta}|H\rangle|H\rangle. \qquad (4)$$

Without loss of generality, we may assume that $q_g>q_f$. This projection may be accomplished by first applying a filter $|V\rangle\rightarrow q_f/q_g|V\rangle$, $|H\rangle\rightarrow|H\rangle$ to one of the photons, followed by projection onto the Bell state $|V\rangle|V\rangle+e^{-i\theta}|H\rangle|H\rangle$. The filter F (see Fig. 1) can be implemented by a single beam splitter that is fully transparent for horizontally polarized photons and partially reflects vertically polarized photons as described by amplitude transmittances $t_H=1$, $t_V=q_f/q_g$. The filtering succeeds if the photodetector PD_F does not detect any photon.

The entanglement swapping yields the target state (1) for arbitrary $|f\rangle$, $|g\rangle$, $|f'\rangle$, and $|g'\rangle$. The generation of state (1) thus boils down to the preparation of the entangled state (2). [State (3) can be prepared similarly.] This latter task will be accomplished with the help of a quantum nondemolition (QND) measurement of a single photon, which creates entanglement. The preparation procedure specified below works with a finite probability of success only if all the states appearing in Eq. (1) have finite expansions in Fock state basis. From now on we therefore assume that the states $|f\rangle$, $|g\rangle$, $|f'\rangle$, and $|g'\rangle$ contain no more than N photons, where N is arbitrary but finite integer, and we can write

$$|f\rangle=\sum_{n=0}^{N} f_n|n\rangle, \qquad |g\rangle=\sum_{n=0}^{N} g_n|n\rangle. \qquad (5)$$

Here $|n\rangle$ denotes the n-photon Fock state and similar expansions hold also for $|f'\rangle$ and $|g'\rangle$.

A. QND measurement

The proposed setup is depicted in Fig. 2. The input polarization modes $1V$ and $1H$ are prepared in the pure single-mode states $|\tilde{f}\rangle$ and $|\tilde{g}\rangle$ that contain no more than $N+1$ photons,

$$|\tilde{f}\rangle=\sum_{n=0}^{N+1} \tilde{f}_n|n\rangle, \qquad |\tilde{g}\rangle=\sum_{n=0}^{N+1} \tilde{g}_n|n\rangle. \qquad (6)$$

States (6) are, of course, closely related to the states $|f\rangle$ and $|g\rangle$ appearing in Eq. (2). The exact relationship between them will be specified later. Note that it was shown by Dakna *et al.* [17] that the single-mode states (6) can be probabilistically generated with the help of passive linear optics, single-photon sources, strong coherent laser pulses, and photodetectors with single-photon sensitivity.

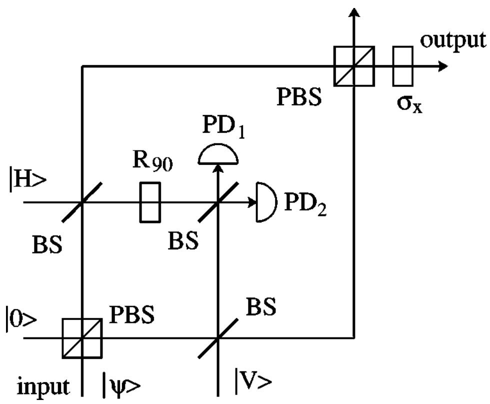

FIG. 3. Interferometric scheme for a QND measurement of a single photon [36]. BS denotes beam splitters. PBS denotes polarizing beam splitters that transmit horizontally polarized photons and reflect vertically polarized photons. R_{90} is a waveplate that rotates polarization state by 90°.

As shown in Fig. 2, the input beam containing the states $|\tilde{f}\rangle$ and $|\tilde{g}\rangle$ in the V and H polarization modes impinges on a beam splitter BS_1 with transmittance t_1 and reflectance r_1. The purpose of this beam splitter is to separate a single photon from one of the input states and thus create entanglement. Of course, BS_1 may separate more than one photon or no photon at all. Therefore we must verify that there is exactly a single photon present in the output spatial mode 2, without disturbing its polarization state. Such QND measurements of a single photon have been recently thoroughly discussed in Ref. [36] where several schemes have been proposed and analyzed.

One possible method relies on teleportation [37,38]. We have to first prepare the "event-ready" singlet state $|VH\rangle_{34} - |HV\rangle_{34}$ in the auxiliary modes 3 and 4. This can be done with the help of the conditional Controlled-NOT (CNOT) gate for photonic qubits [26] or using the schemes proposed in Refs. [39,40]. Then we teleport the polarization state of a photon in mode 2 onto the photon in mode 4 by performing a Bell measurement on modes 2 and 3. This measurement should be carried out with two detectors that are capable to resolve the number of photons. If exactly two photons are detected in coincidence at the two detectors, then we know that there was a single photon in mode 2 and its polarization state has been coherently transferred onto the state of a single photon in mode 4.

Kok *et al.* also proposed QND measurement schemes that do not require *a priori* entanglement [36]. For our purposes it suffices to employ the simple interferometric scheme depicted in Fig. 3 which performs a partial QND photon number measurement. The coincident detection of a single photon in detectors PD_1 and PD_2 indicates that there was at least one photon in the input mode. Moreover, if there was *exactly* a single photon at the input, then its polarization state is unperturbed by the measurement. The probability of successful QND measurement of a single photon provided that there *is* a single photon in the spatial mode 2 is equal to 1/8. In the preparation scheme shown in Fig. 1 the two QND measurements are followed by the entanglement swapping that succeeds only if each photodetector in Fig. 1 detects exactly one photon. In this way we select the events when there is exactly a single photon in each mode A_2 and B_2.

The state after the QND measurement can be written in the form

$$|\phi_{\mathrm{QND}}\rangle = B_1|\tilde{f}\rangle_{1V} B_0|\tilde{g}\rangle_{1H}|V\rangle_2 + B_0|\tilde{f}\rangle_{1V} B_1|\tilde{g}\rangle_{1H}|H\rangle_2, \tag{7}$$

where the nonunitary operators B_0 and B_1 describe removal of none or a single photon at BS_1, respectively,

$$B_0 = \sum_{n=0}^{\infty} t_1^n |n\rangle\langle n|,$$

$$B_1 = \sum_{n=0}^{\infty} \sqrt{n+1}\, t_1^n r_1 |n\rangle\langle n+1|. \tag{8}$$

B. Quantum erasure

The QND measurement has thus created the entanglement. Still, $B_j|\tilde{f}\rangle$ and $B_j|\tilde{g}\rangle$ are states of two different polarization modes while $|f\rangle$ and $|g\rangle$ in Eq. (2) are states of a *single* mode. We achieve this by erasing the information about the polarization. This is done with the help of a polarizing beam splitter PBS_2, see Fig. 2. The beam splitter is rotated such that it combines the V and H polarizations. If we associate the annihilation operators a_{1V} and a_{1H} with the modes $1V$ and $1H$, then the transformation carried out by PBS_2 is given by

$$a_{1X} = r_2 a_{1V} + t_2 a_{1H},$$

$$a_{1Y} = r_2 a_{1H} - t_2 a_{1V}, \tag{9}$$

where the polarization modes $1X$ and $1Y$ are *spatially separated* at the output of PBS_2 as indicated in Fig. 2. The erasing succeeds if and only if the detector PD placed on the $1Y$ mode does not detect any photon, which ensures that all photons have been transmitted to the linearly polarized output mode $1X$.

Since the input states do not contain more than $N+1$ photons each and since a single photon was subtracted on BS_1, the state after erasure of the polarization information reads

$$|\phi_{\mathrm{PBS}}\rangle = \sum_{n=0}^{2N+1} (f_n|n\rangle|V\rangle + g_n|n\rangle|H\rangle), \tag{10}$$

where the output complex amplitudes f_n and g_n are given by the following formula:

$$f_n = \sum_{k=0}^{n} \sqrt{k+1} \sqrt{\binom{n}{k}}\, t_1^n r_1 r_2^k t_2^{n-k} \tilde{f}_{k+1} \tilde{g}_{n-k},$$

$$g_n = \sum_{k=0}^{n} \sqrt{k+1} \sqrt{\binom{n}{k}}\, t_1^n r_1 r_2^{n-k} t_2^k \tilde{g}_{k+1} \tilde{f}_{n-k}. \tag{11}$$

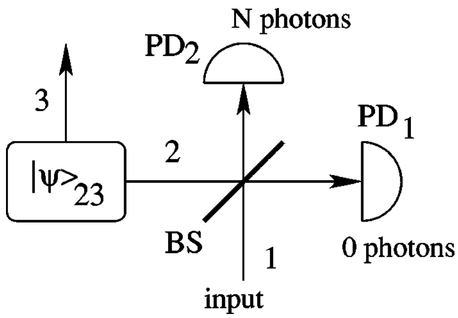

FIG. 4. Quantum scissors (see Refs. [41,42]).

It follows that we can fix the values of output coefficients f_n by properly choosing the complex amplitudes of the input states. We have the recurrence equations

$$\tilde{f}_{n+1}=\frac{f_n-F_n}{\sqrt{n+1}\,t_1^n r_1 r_2^n \tilde{g}_0},$$

$$F_n=\sum_{k=0}^{n-1}\sqrt{k+1}\sqrt{\binom{n}{k}}\,t_1^n r_1 r_2^k t_2^{n-k}\tilde{f}_{k+1}\tilde{g}_{n-k}, \tag{12}$$

and a similar formula holds for $\tilde{g}_{n+1}$. The amplitudes $\tilde{f}_{n+1}$ and $\tilde{g}_{n+1}$ are calculated as follows. First, some nonzero values for $\tilde{f}_0$ and $\tilde{g}_0$ are chosen. Then one uses repeatedly Eq. (12) to calculate $\tilde{f}_{n+1}$ and $\tilde{g}_{n+1}$ from $\tilde{f}_k$ and $\tilde{g}_k$, $0\leqslant k\leqslant n$, and the iterations stop at $n=N$.

We have thus almost obtained state (2). However, while the effective states $|f\rangle$ and $|g\rangle$ in Eq. (10) have the correct structure in the subspace spanned by the first $N+1$ Fock states $|0\rangle,|1\rangle,\ldots,|N\rangle$, they also contain an unwanted tail in the subspace spanned by the Fock states $|N+1\rangle,\ldots,|2N+1\rangle$.

C. Quantum-state truncation

The last step in our scheme is to get rid of that tail. We do so with the help of quantum scissors [41–45], which project the state in the mode 1 onto the subspace spanned by the first $N+1$ Fock states. This action is formally described by the projector $\Pi_{\mathrm{QS}}=\Sigma_{n=0}^{N}|n\rangle\langle n|$. After this truncation, we finally obtain the target entangled state (2) in the output modes A_1 and A_2 of our device.

The quantum scissors have been discussed in detail in several recent papers [41–45]. The schemes proposed there can be implemented with the use of linear optics, single-photon sources, and photodetectors. The setup that we shall briefly describe in the present paper is depicted in Fig. 4 (for more details, see Refs. [41,42]). The modes 2 and 3 are prepared in a pure N-photon two-mode entangled state

$$|\psi\rangle_{23}=\sum_{k=0}^{N}c_k|k\rangle_2|N-k\rangle_3. \tag{13}$$

It was shown recently that these states may be conditionally generated with the help of linear optics, single-photon sources, and photodetectors [20–23]. We then combine the modes 1 and 2 on a balanced beam splitter and measure the number of photons in output modes 1 and 2 by means of two photodetectors PD_1 and PD_2 with single-photon resolution. The truncation is a conditional operation that succeeds when a particular measurement outcome has been detected, say, no photons at PD_1 and N photons at PD_2, which we denote as $(0,N)$. Note that one could also consider other detection events, but the total number of photons detected by the photodetectors should be equal to N. The quantum scissors is essentially a quantum-state teleportation in the subspace of the first $N+1$ Fock states, where state (13) serves as the quantum channel, and the Bell-type measurement is carried out with the use of the beam splitter in Fig. 4 that couples the modes 1 and 2.

For the particular choice of the detection event $(0,N)$, and assuming a balanced beam splitter, one finds that

$$c_k=\sqrt{P_{\mathrm{QS}}}\frac{\sqrt{k!(N-k)!}}{2^{-N/2}\sqrt{N!}} \tag{14}$$

should hold [42], where

$$P_{\mathrm{QS}}=\left(\sum_{k=0}^{N}\frac{k!(N-k)!}{2^{-N}N!}\right)^{-1} \tag{15}$$

is the probability of the successful quantum-state truncation that projects onto the subspace spanned by $|0\rangle,|1\rangle,\ldots,|N\rangle$.

III. THE INFLUENCE OF IMPERFECTIONS

In our discussion so far we have assumed that the setup is constructed from perfect components. However, in reality the components will never be perfect, and any attempt to experimentally demonstrate the operation of the present scheme will have to deal with various errors and imperfections. It is therefore essential to investigate the behavior of the setup under more realistic conditions.

Before discussing the specific sources of possible errors that may occur, let us make some general observations. Quite generally, we can quantify the performance of a quantum-information processing device by the fidelity F. Suppose the input of the device is a pure state $|\psi_{\mathrm{in}}\rangle$. Ideally, the device should produce (with some probability) an output state $|\psi_{\mathrm{out}}\rangle$. However, due to the errors, the output (generally mixed) state ρ_{out} differs from the expected ideal outcome, and the fidelity is defined as $F=\langle\psi_{\mathrm{out}}|\rho_{\mathrm{out}}|\psi_{\mathrm{out}}\rangle$. Now the setup depicted in Figs. 1 and 2 can be viewed as a chain of several blocks, and we can associate the fidelity F_j with jth block. These blocks include, for instance, the devices for preparation of input single-mode states $|\tilde{f}\rangle$ and $|\tilde{g}\rangle$, the quantum entangler shown in Fig. 2, and the final entanglement swapping operation. The total fidelity F_{tot} of the quantum-state preparation can be then roughly estimated as the product of the fidelities of each block,

$$F_{\mathrm{tot}}\simeq F_1F_2\cdots F_k, \tag{16}$$

where k is the number of the basic building blocks.

What may reduce the fidelity? Two main sources of errors can be identified. First, the photodetectors with single-photon resolution and unit efficiency $\eta=1$ are not currently available. Second, perfect single-photon sources are not currently available. We investigate the effect of each type of imperfection in turn.

We first consider imperfect detectors. The commercially available avalanche photodiodes usually employed in the experiments exhibit single-photon sensitivity but not a single-photon resolution (see, however Refs. [46–48]). The outcome of the measurement is dichotomic: either a "click" or "no click." Moreover, the typical detection efficiencies are of the order of 50% and the highest reported efficiency of the single-photon detector is $\eta_{\max}\approx 88\%$ [48,49].

The fact that the photodiodes are not able to distinguish the number of incoming photons is not a serious limitation and this drawback can be circumvented. A series of recent experiments demonstrated that it is possible to simulate detectors with single-photon resolution by splitting the light beam into many beams that are detected by many detectors [50–53]. If the average number of photons in the beam impinging on each detector is much lower than one, then the number of observed clicks gives a very good estimate of the number of photons. In practice, the splitting is performed by means of fiber couplers and strongly unbalanced Mach-Zehnder interferometers, which allow us to divide the signal into many time bins. The performance of these measuring apparatuses is limited mainly by the efficiency of the photodetectors. The different outcomes of such a measuring device are the number n of detectors that click. To them are associated positive-operator-valued measures elements Π_n. If n is much smaller than the number of detectors, then Π_n can be approximated by

$$\Pi_n=\sum_{m\geqslant n}\binom{m}{n}\eta^n(1-\eta)^{m-n}|m\rangle\langle m|. \tag{17}$$

To give an explicit example how the fidelity is influenced by the nonunit detector efficiency, we have carried out an explicit calculation of F for the entangling device depicted in Fig. 2. We have chosen simple but nontrivial target states $|f\rangle=|0\rangle$ and $|g\rangle=|1\rangle$. Assuming that all the beam splitters in the setup are balanced ($r=t=1/\sqrt{2}$), the corresponding input single-mode states $|\tilde{f}\rangle$ and $|\tilde{g}\rangle$ read

$$|\tilde{f}\rangle=\frac{1}{\sqrt{3}}(|0\rangle+\sqrt{2}|1\rangle), \qquad |\tilde{g}\rangle=\frac{1}{\sqrt{5}}(|0\rangle+2|2\rangle). \tag{18}$$

In this case the quantum scissors can be implemented with just a single auxiliary photon split on a balanced beam splitter because $|\psi\rangle_{23}=(|01\rangle+|10\rangle)/\sqrt{2}$. We further assume that the QND measurement is carried out with the use of the simple interferometric device shown in Fig. 3. In agreement with our definition of the fidelity we assume that the input states (18) are perfect. Ideally, the output of the device in Fig. 2 should be

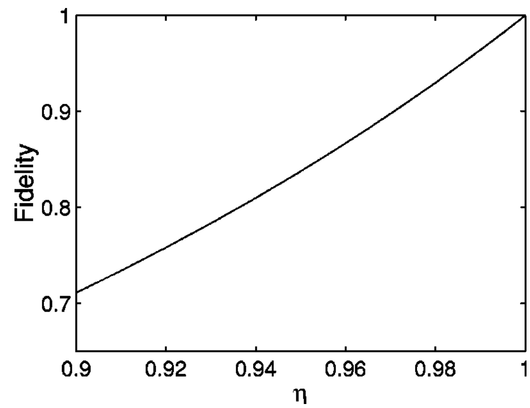

FIG. 5. The dependence of the fidelity of the entangling device shown in Fig. 2 on the efficiency of the photodetectors η. See text for more details.

$$|0\rangle_{A_1}|V\rangle_{A_2}+|1\rangle_{A_1}|H\rangle_{A_2}+|\phi\rangle_{A_1A_2}, \tag{19}$$

where the state $|\phi\rangle_{A_1A_2}$ has more than one photon in mode A_2. Note that $|\phi\rangle_{A_1A_2}$ arises because we use the partial QND measurement scheme depicted in Fig. 3. The tail $|\phi\rangle_{A_1A_2}$ is removed during the final entanglement swapping procedure, see Fig. 1 and the discussion in Sec. II A.

The setup involves altogether eight different modes and five photodetectors (two for QND, two for quantum scissors, and one for information erasure). We have numerically calculated the dependence of the fidelity on η and the results are given in Fig. 5. Although F decreases with decreasing η, the reduction of the fidelity is perhaps not as severe as one might expect, given the relatively large number (five) of detectors involved. Of course, since the total fidelity is approximately product of fidelities of the building blocks, see Eq. (16), a reliable operation of the whole scheme would require detectors with efficiencies of about 95%.

The second main source of errors is the necessity to provide auxiliary single photons on demand. The fabrication of a reliable single-photon source is one of the holy grails of quantum optics. Currently available triggered single-photon sources operating by means of fluorescence from a single molecule [54], a single quantum dot [55–57], or a nitrogen-vacancy center in a diamond [58] exhibit significantly sub-Poissonian statistics, with tenfold or even higher suppression of two-photon emission in comparison to the coherent light with the same mean photon number. Instead of those novel sources of single photons one could also utilize single photons conditionally prepared from photon pairs generated by means of spontaneous parametric down-conversion [59]. If a single idler photon is detected then the signal beam collapses to a single-photon state [25]. However, the experimentally observed correlations between signal and idler photons are never perfect. These problems are quite generic. Although the triggered single-photon source [54–58] may emit a single photon each time a button is pressed, the main difficulty is to collect the emitted photon into a single well defined spatiotemporal mode. In practice, only a fraction $R<1$ of the emitted photons is collected and detected.

We model the imperfect single-photon source as a source that emits a single photon with probability R and does not emit any photon with probability $1-R$. For the particular example, we have calculated how the fidelity of the device shown in Fig. 2 depends on R, assuming ideal photodetectors and perfect input states (18). The obtained dependence $F(R)$ was quite similar to the dependence $F(\eta)$ given in Fig. 5, which indicates that $R>95\%$ would be required for a reliable operation of the quantum-state entangler.

We can conclude that highly efficient photodetectors and single-photon sources are necessary for the reliable operation of the device proposed in this paper. This is hardly surprising because the suggested experimental setup consists of a large number of photodetectors and auxiliary single-photon sources. We have seen, however, that small errors and inefficiencies of the order of several percent can be tolerated and do not significantly degrade the functionality of the device.

IV. GENERAL SCHEME

Up to now we have considered the generation of a two-mode state that consists of a superposition of two product states. In this section we shall generalize our scheme in several ways.

A. *M*-mode entangled state

First, we will discuss the preparation of a M-mode entangled state which has a form limited to the superposition of two product states, namely,

$$|\psi\rangle=|f_1\rangle|f_2\rangle\cdots|f_M\rangle+e^{i\theta}|g_1\rangle|g_2\rangle\cdots|g_M\rangle. \tag{20}$$

To prepare this state, we first generate M states

$$|\phi_j\rangle=|f_j\rangle|V\rangle+|g_j\rangle|H\rangle \tag{21}$$

as before, and then project the M single photons onto the maximally entangled state

$$|\Psi_{GHZ}\rangle=|VV\cdots V\rangle+e^{-i\theta}|HH\cdots H\rangle. \tag{22}$$

This projection can be performed with the Greenberger-Horne-Zeilinger (GHZ) state analyzer described by Pan and Zeilinger [60]. This analyzer consists of polarizing beam splitters, half-wave plates, and photodetectors allows one to distinguish two GHZ states of M qubits.

B. Arbitrary two-mode entangled states

Our second generalization extends the preparation procedure to two-mode states that are superpositions of d product states,

$$|\psi\rangle_{AB}=\sum_{j=1}^{d} q_j e^{i\theta_j}|f_j\rangle_A|g_j\rangle_B\,. \tag{23}$$

Obviously, a natural strategy is to prepare this state via d-dimensional entanglement swapping from two states of the form

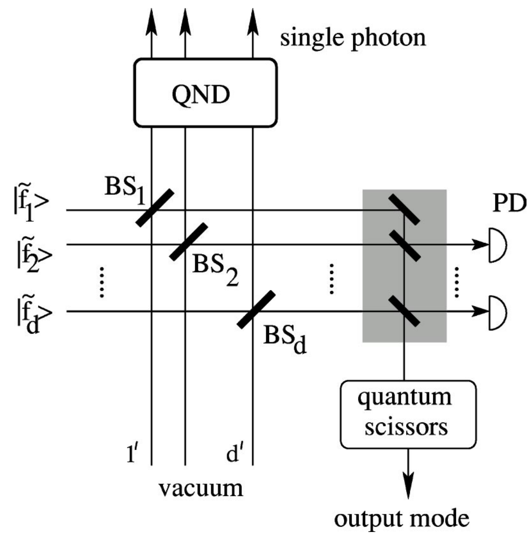

FIG. 6. Scheme for conditional generation of the entangled state (24). The erasing of the spatial information is achieved conditionally on no photon being detected at the $d-1$ photodetectors PD.

$$|\phi\rangle_{AC}=\sum_{j=1}^{d} |f_j\rangle_A|1\rangle_{C_j}, \qquad |\phi'\rangle_{BD}=\sum_{j=1}^{d} |g_j\rangle_B|1\rangle_{D_j}, \tag{24}$$

where $|1\rangle_{C_j}$ denotes a state of a single photon in the C_jth auxiliary spatial mode. The state $|\psi\rangle_{AB}$ is obtained by projecting the modes C_j and D_j onto a two-photon entangled state

$$\sum_{j=1}^{d} q_j e^{-i\theta_j}|1\rangle_{C_j}|1\rangle_{D_j}. \tag{25}$$

To prepare states (24), we proceed essentially as in Sec. II. The extended scheme is depicted in Fig. 6. Initially, d spatial modes are prepared in states $|\tilde{f}_j\rangle$. The array of d beam splitters BS$_j$ extracts, with a certain probability, a single photon. We must verify the presence of exactly one single photon in total in the modes $1',\ldots,d'$ while preserving the coherence among these modes by performing a QND measurement. This can be accomplished by d-dimensional teleportation similarly as discussed in Sec. II. However, this time we need to teleport a qudit encoded as a state of a single photon in d spatial modes. Fortunately, it was shown recently that the probabilistic gates for quantum computing with linear optics [26–30], which were introduced for qubits, can be easily extended to qudits [61]. This means that all the necessary manipulations, such as the generation of maximally entangled state of two qudits, Bell measurement, and the projection onto state (25), can be probabilistically carried out with the resources that we assume here.

After the QND measurement, we must erase the spatial information contained in the resulting state

$$|\psi_{QND}\rangle=\sum_{j=1}^{d}\left(\prod_{k=1}^{d} B_{\delta_{j,k}}|\tilde{f}_k\rangle\right)|1\rangle_j\,, \tag{26}$$

where B_j are given by Eq. (8). This is done by the second array of $d-1$ beam splitters indicated by the shaded zone in Fig. 6. The erasure succeeds when the detectors PD do not detect any photons. The preparation is finished by the application of the quantum scissors that truncate the Fock state expansion of the output state at N photons.

Suppose that the output mode after erasing is a balanced linear superposition of all d modes

$$a_{\text{out}}=\frac{1}{\sqrt{d}}\sum_{j=1}^{d} a_j . \tag{27}$$

The complex amplitudes of the states $|f_j\rangle=\Sigma_n f_{j,n}|n\rangle$ can then be expressed in terms of the complex amplitudes of the input states $\tilde{f}_{j,n}$ as follows:

$$f_{j,n}=\sqrt{n!}\;\frac{t_1^n r_1}{d^{n/2}}\sum_{\boldsymbol{n}}{}' \;\sqrt{n_j+1}\;\frac{\tilde{f}_{j,n_j+1}}{\sqrt{n_j!}}\prod_{k\neq j}\frac{\tilde{f}_{k,n_k}}{\sqrt{n_k!}}, \tag{28}$$

where the prime indicates summation over all $\boldsymbol{n}=(n_1,\ldots,n_d)$ satisfying the constraint $\Sigma_{k=1}^d n_k=n$. Note that formula (28) may be easily inverted and we can determine the complex amplitudes $\tilde{f}_{j,k}$ of the input states for any given prescribed output states $|f_j\rangle$.

C. Universal scheme

Finally, we point out that those two generalizations can be combined and we can thus generate an arbitrary multimode entangled state of light where each mode contains no more than N photons. Any M-mode state of this kind can be written as a superposition of no more than $d=(N+1)^M$ product states,

$$|\psi\rangle=\sum_{j=1}^{d} q_j e^{i\theta_j}|f_{j1}\rangle_1|f_{j2}\rangle_2\cdots|f_{jM}\rangle_M . \tag{29}$$

We can conditionally generate state (29) if we first prepare M entangled states $(k=1,\ldots,M)$

$$|\phi_k\rangle=\sum_{j=1}^{d} |f_{jk}\rangle|1\rangle_{jk}, \tag{30}$$

where $|1\rangle_{jk}$ denotes a single-photon state of the (j,k)th auxiliary spatial mode. As discussed above, states (30) can be prepared with the help of the scheme depicted in Fig. 6. Then, we carry out a joint measurement on all auxiliary photons. The state $|\psi\rangle$ is obtained if we project the auxiliary modes onto an M-photon entangled state:

$$|\Psi_{M,d}\rangle=\sum_{j=1}^{d} q_j e^{-i\theta_j}|1\rangle_{j1}|1\rangle_{j2}\cdots|1\rangle_{jM} . \tag{31}$$

If we denote $|\Phi_{M,d}\rangle=|\phi_1\rangle|\phi_2\rangle\cdots|\phi_M\rangle$ then we can write

$$|\psi\rangle=\langle\Psi_{M,d}|\Phi_{M,d}\rangle. \tag{32}$$

With the help of the techniques developed in the framework of the quantum computation with linear optics, the entangled M-photon state (31) can be mapped onto a product state of M photons which can then be detected by observing an appropriate M-photon coincidence. The factorization consists of a sequence of the controlled-SHIFT (C-SHIFT) gates for two qudits. The C-SHIFT gate is defined as follows:

$$|1\rangle_{jk}|1\rangle_{j'k'}\rightarrow|1\rangle_{jk}|1\rangle_{j'-j,k'} , \tag{33}$$

where $j'-j$ should be calculated modulo d. As shown in Ref. [61] this transformation can be (probabilistically) performed with the use of single-photon sources, passive linear optics, and photodetectors with single-photon sensitivity. If we apply the C-SHIFT gate to the qudits 1 and k, where $k=2,\ldots,M$, then state (31) is mapped onto product state of M photons

$$|\Psi_{M,d}\rangle\rightarrow\left(\sum_{j=1}^{d} q_j e^{-i\theta_j}|1\rangle_{j1}\right)|1\rangle_{d2}\cdots|1\rangle_{dM} . \tag{34}$$

D. State preparation with quantum memory

The above described procedure for the preparation of the generic M-mode entangled state clearly shows that our approach is indeed general since it allows us, in principle, to generate an arbitrary state. But the success probability of the scheme will be exponentially small in general. The preparation procedure will therefore have to be repeated exponentially many times to produce the desired state. In the rest of this section we shall argue that the number of operations required to produce the desired state can be significantly decreased if a quantum memory is available.

The advantage of the quantum memory is that it allows one to wait until successive steps in the state generation protocol have succeeded before proceeding with the next steps. For instance, in the scheme described in Fig. 6, one can first produce the states $|\tilde{f}_j\rangle$ and store them in a memory before proceeding with the production of the entangled state (24). Suppose that the state $|\tilde{f}_j\rangle$ can be prepared with the probability P_j. Without quantum memory the probability of simultaneous preparation of the d states $|\tilde{f}_j\rangle$ is $P=\Pi_{j=1}^d P_j$ and the number of required operations thus scales as $N_O\propto 1/P$. If, after a successful preparation, we store each state $|\tilde{f}_j\rangle$ in a memory, we reduce the number of required operations to $N_O\propto\Sigma_{j=1}^d 1/P_j$. The quantum memory is also essential in boosting the probability of success of the probabilistic logic gates for single photons proposed by Knill, Laflamme, and Milburn [26]. As explained in the preceding section, these gates are required in the present scheme for performing the QND measurements of photon number and for manipulations and Bell measurements on qudits represented by single photons in d spatial modes.

A more elaborate use of a quantum memory is to build the state using a *recursive procedure* based on the Schmidt decomposition. This method is similar to that analyzed in Ref. [24] where the number of operations required to produce ‘‘cat states’’ was drastically decreased by using a recursive procedure and a quantum memory. The method is best explained on an explicit example. Suppose we would like to prepare a four-mode state $|\psi\rangle_{ABCD}$. The Schmidt decomposition of this four-mode state with respect to a bipartite splitting into the AB and CD modes can be written as follows:

$$|\psi\rangle_{ABCD}=\sum_{j=1}^{d} q_j|\varphi_j\rangle_{AB}|\pi_j\rangle_{CD}, \tag{35}$$

where $|\varphi_j\rangle_{AB}$ and $|\pi_j\rangle_{CD}$ form orthogonal bases in some subspaces of the Hilbert space of modes AB and CD, respectively. The number of the terms in the Schmidt decomposition (35) is bounded by $d\leqslant(N+1)^2$ while, in general, $(N+1)^4$ terms were necessary in the general procedure of Sec. IV C. Decomposition (35) suggests that we could obtain the state $|\psi\rangle_{ABCD}$ via entanglement swapping if we first prepare two entangled states

$$|\Phi\rangle_{ABX}=\sum_{j=1}^{d} |\varphi_j\rangle_{AB}|1\rangle_{X_j},$$

$$|\Phi'\rangle_{CDY}=\sum_{j=1}^{d} |\pi_j\rangle_{CD}|1\rangle_{Y_j}. \tag{36}$$

Now the entangled states $|\Phi\rangle_{ABX}$ and $|\Phi'\rangle_{CDY}$ can be prepared using the general procedure of Sec. IV C. However, for states of the specific form (36), a simpler procedure can be devised. One first generates d states $|\tilde{\varphi}_j\rangle_{AB}$ where the tilde indicates that $|\tilde{\varphi}_j\rangle$ is related to $|\varphi_j\rangle$ via relations similar to those described by Eqs. (12). The state $|\Phi\rangle_{ABX}$ is then prepared with the help of the device shown in Fig. 6 where the mode B of $|\tilde{\varphi}_j\rangle_{AB}$ is sent to the jth input port of the interferometer. The only modification is that we must also erase the spatial information carried by the A modes of the states $|\tilde{\varphi}_j\rangle_{AB}$ and perform the quantum-state truncation after the erasing. Briefly, all d modes A_j are combined on an array of $d-1$ beam splitters and the detectors monitor the first $d-1$ output ports. If these detectors do not register any photon, then the erasing procedure succeeded and we apply the quantum scissors to the state in the dth output port.

Clearly, this recursive procedure can be extended to any number of modes. The resulting scheme resembles a tree structure involving repeated applications of the entangling scheme shown in Fig. 6 and the entanglement swapping that produces an $2m$-mode state from two m-mode states entangled with auxiliary single photons. This scheme relying on quantum memory may possibly lead to a substantial improvement of the generation probability, similarly to the case of the two-mode N-photon cat states [24].

V. CONCLUSIONS

In summary, the scheme proposed in the present paper is universal, in that arbitrary multimode entangled states of light can be probabilistically generated. The resources required are passive linear optics, single-photon sources, strong coherent states, and detectors with single-photon resolution. This result clearly illustrates the versatility and power of the approach relying only on linear optics and single photons.

It is fair to say, though, that the suggested setup is rather complicated and involves several nontrivial operations such as the quantum nondemolition measurement of a single photon, quantum scissors, and transformations of photonic qudits encoded as a state of a single photon in d spatial modes. We have shown that all these operations can be (probabilistically) implemented with the resources that we consider here. The resulting procedure may become very complex for general states. However, it should be emphasized that building a state like (29) must necessarily be complicated because of the large number of parameters [$(N+1)^M-1$ complex numbers] that characterize this state and that must be fixed by the preparation scheme. We have also analyzed in a simple case the effect of realistic experimental conditions (imperfect detectors and imperfect single-photon sources). Although they decrease the fidelity of the final state, imperfections of the order of a few percent may be acceptable. Finally, we have argued that the use of a quantum memory can decrease the number of operations required to produce the desired state.

We hope that this paper will stimulate the efforts towards experimental demonstrations of the basic building blocks of our scheme, such as the conditional generation of single-mode finite superpositions of Fock states and the preparation of two-mode entangled N-photon states required for the quantum-state truncation.

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