Experimental Purification of Single Qubits

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We report the experimental realization of the purification protocol for single qubits sent through a depolarizing channel. The qubits are associated with polarization states of single photons and the protocol is achieved by means of passive linear optical elements. The present approach may represent a convenient alternative to the distillation and error correction protocols of quantum information.

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Modern quantum data processing using realistic (imperfect) quantum gates and long-distance quantum communication in the presence of a noisy environment requires a large supply of qubits with a high degree of *purity*. Indeed, the fidelity of most quantum information (QI) protocols critically depends on the preservation of the purity of the QI carriers. It is therefore crucial to develop techniques that protect quantum states from the unavoidable losses and decoherence processes accompanying the transmission. One of these techniques is the quantum error correction [1,2], which works by encoding the quantum state into a higher-dimensional Hilbert space. One can also distribute several copies of an entangled state and extract fewer states by means of entanglement distillation [3–7] in order to be able to subsequently transmit an arbitrary state with high fidelity by quantum teleportation [8–11]. An interesting alternative to the error correction and entanglement purification protocols is to transmit several copies of the state over the noisy channel and then purify the resulting mixed states at the receiver's station [12–14].

The present work focuses on the purification procedure that was theoretically proposed by Cirac *et al.* in 1999 [12]. It addresses the issue of the purification of N equally prepared qubits in the mixed state $\rho = \xi|\phi\rangle\langle\phi| + \frac{1}{2}\times(1-\xi)\mathbb{I}$, where $0 \le \xi \le 1$. This purification procedure allows us to distill from a set of mixed states a subset of states with a higher degree of purity; i.e., it probabilistically increases the purity by filtering out some of the noise. The procedure is based on a set of projections onto the symmetric subspace of the N qubits (i.e., the subspace spanned by all the states that are invariant under any permutation of the N qubits) and onto orthogonal subspaces that contain symmetric subspaces for subsets of the initial N qubits. This procedure is designed to be optimal and universal; i.e., it acts with the same fidelity for all input states. Since it is optimal, the purity cannot be further increased by any means. The purification via symmetrization can find a wide variety of applications in QI processing; e.g., it has been proposed to stabilize quantum computation in the presence of noise and decoherence [15]. It was also shown that this method can be used to improve the precision of frequency standards based on laser cooled ions in the presence of decoherence [16] and to improve the fidelity of quantum teleportation with mixed entangled states [14]. Furthermore, the purification helps to increase the fidelity of estimation of states transmitted through a depolarizing channel [17].

In this Letter, we consider the case of two qubits: i.e., $N = 2$, for which the purification procedure reduces to a projection of the two-qubit state onto the symmetric subspace. The procedure works as follows. Consider two independent qubits, a and b, both originally in the state $|\phi\rangle$, that are transmitted over a noisy channel from which they emerge in a mixed state represented by the density matrix $\rho_a = \xi|\phi\rangle\langle\phi| + \frac{1}{2}\times(1-\xi)\mathbb{I} = \frac{1+\xi}{2}|\phi\rangle\langle\phi| + \frac{1-\xi}{2}|\phi^{\perp}\rangle\langle\phi^{\perp}|$, where $|\phi^{\perp}\rangle$ is a state orthogonal to $|\phi\rangle$. Our goal is then to purify the transmitted qubits in order to obtain two qubits that are as close as possible to the original state $|\phi\rangle$. The overall two-qubit input state $\rho_{ab}^{\rm in} = \rho_a^{\rm in} \otimes \rho_b^{\rm in}$ is expressed in the basis $\{|\phi\rangle_a|\phi\rangle_b, |\phi\rangle_a|\phi^{\perp}\rangle_b, |\phi^{\perp}\rangle_a|\phi\rangle_b, |\phi^{\perp}\rangle_a|\phi^{\perp}\rangle_b\}$ by the matrix

$$\rho_{ab}^{\rm in} = \frac{1}{4}\begin{pmatrix} (1+\xi)^2 & 0 & 0 & 0 \\ 0 & 1-\xi^2 & 0 & 0 \\ 0 & 0 & 1-\xi^2 & 0 \\ 0 & 0 & 0 & (1-\xi)^2 \end{pmatrix}. \quad (1)$$

As mentioned above, the purification protocol consists of the projection of the two-qubit state onto the symmetric subspace: if the projection is successful, we obtain two equal output qubits that are the optimal "purified" ones, otherwise we discard the output states. We note that this protocol can be implemented, for every qubit encoding, by a quantum circuit requiring an ancilla qubit and a Toffoli gate [18]. After a successful projection, the output qubits are in the state

$$\rho_{ab}^{\text{out}} = \frac{\Pi \rho_{ab}^{\text{in}} \Pi^\dagger}{Tr[\Pi \rho_{ab}^{\text{in}} \Pi^\dagger]} = \frac{1}{3+\xi^2}\begin{pmatrix} (1+\xi)^2 & 0 & 0 & 0 \\ 0 & \frac{1-\xi^2}{2} & \frac{1-\xi^2}{2} & 0 \\ 0 & \frac{1-\xi^2}{2} & \frac{1-\xi^2}{2} & 0 \\ 0 & 0 & 0 & (1-\xi)^2 \end{pmatrix}, \quad (2)$$

where $\Pi = \mathbb{I}_{ab} - |\Psi_{ab}^-\rangle\langle\Psi_{ab}^-|$ is the projector onto the symmetric subspace and $|\Psi_{ab}^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ is the singlet state of two qubits. The success probability of the procedure is $p = \text{Tr}[\Pi \rho_{ab}^{\text{in}} \Pi^\dagger] = \frac{3+\xi^2}{4}$. Since ρ_{ab}^{out} belongs to the symmetric subspace, the reduced density matrices of the resulting single qubits, expressed in the basis $\{|\phi\rangle, |\phi^\perp\rangle\}$, are found to be identical,

$$\rho_a^{\text{out}} = \rho_b^{\text{out}} = \frac{1}{2}\begin{pmatrix} 1+\xi_P & 0 \\ 0 & 1-\xi_P \end{pmatrix}, \quad (3)$$

where $\xi_P = \frac{4}{3+\xi^2}\xi \geq \xi$ and the purification gain factor is $\eta = \frac{4}{3+\xi^2}$. Note that p and η are related by the equation $\eta p = 1$ so a higher purification gain factor is necessarily accompanied by a lower probability of success.

We report the implementation of the above protocol for qubits encoded in the polarization of single photons (see Fig. 1). The qubit to be purified is $\frac{1+\xi}{2}|\phi\rangle\langle\phi| + \frac{1-\xi}{2}|\phi^\perp\rangle\langle\phi^\perp|$, where $|\phi\rangle = \alpha|H\rangle + \beta|V\rangle$ and $|H\rangle$, $|V\rangle$ respectively correspond to the horizontal and vertical linear polarizations. In the present experiment, pairs of photons with wavelength $\lambda = 532$ nm and coherence time $\tau_{\text{coh}} = 80$ fs, were generated in a Type I, beta barium borate (BBO) crystal slab in the *product state* $|H\rangle_a|H\rangle_b$ by spontaneous parametric down conversion (SPDC) process excited by a cw fourth-harmonic-generation laser (Coherent Verdi +MBD-266). The output

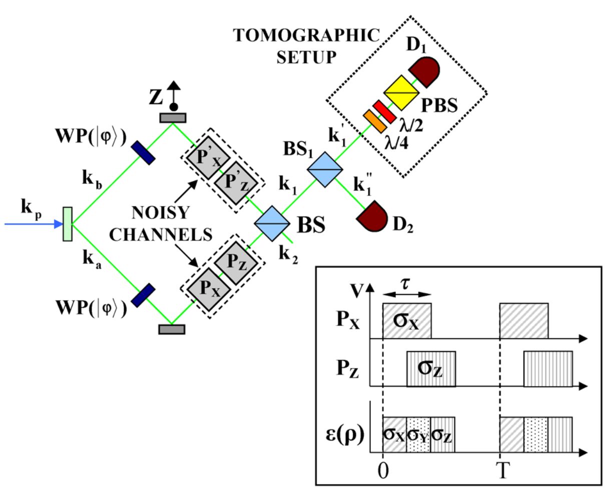

FIG. 1 (color online). Experimental setup for the optical implementation of the purification procedure. Inset shows the realization of the depolarizing channel employing two Pockels cells.

state was first encoded in the state $|\phi\rangle_a|\phi\rangle_b$ by means of two *equal* wave plates (WP) WP($|\phi\rangle$) and then each photon, injected into a noisy channel P, emerged in the mixed state: $\rho_a = \rho_b = \xi|\phi\rangle\langle\phi| + (1-\xi)\frac{\mathbb{I}}{2}$. The two mixed qubits, associated with the two modes k_a and k_b, were linearly superimposed at beam-splitter (BS) with a mutual time delay Δt micrometrically adjustable by a translation stage with position settings $Z = 2\Delta tc$, with c denoting the velocity of light. The value $Z = 0$ was assumed to correspond to the full overlapping of the photon pulses injected into BS, i.e., to the maximum photon interference leading to the simultaneous detection of two photons on either output modes k_1 or k_2 of BS [19,20]. Recently it has been shown that the projection of the overall state in the symmetric subspace, precisely the one implying the present purification procedure, is unambiguously identified by the maximum interference condition: $Z = 0$ [20,21].

Let us give more details about the realization of the two equal depolarizing channels P and P' operating on the BS input modes k_a, k_b, respectively. Each channel implemented the quantum map $\mathcal{E}(\rho) = \xi\rho + (1-\xi)\mathcal{E}_{DEP}(\rho)$ where $\mathcal{E}_{DEP}(\rho)$ maps any unknown input state ρ into a *fully mixed* one. This transformation can be achieved by stochastically applying the full set of Pauli operators $\{\mathbb{I}, \sigma_x, \sigma_y, \sigma_z\}$ with the same statistical weight; that is, $\mathcal{E}_{DEP}(\rho) = \frac{1}{4}(\mathbb{I}\rho\mathbb{I} + \sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z)$. Let us consider here only one of the noisy channels, say P, the one operating on the mode k_a. The $\mathcal{E}(\rho)$ map was realized by means of a pair of equal electro-optic (EO) $LiNbO_3$ Pockels cells (P cell), P_X, and P_Z, carefully aligned with a 45° mutual spatial orientation of the optical axes of the EO crystals (see Fig. 1). All P cells were Shanghai Institute of Ceramics devices with a $\frac{\lambda}{2}$ voltage: $V_{\lambda/2} = 390$ V. Each P cell of the pair was driven by a cw periodic square-wave electric field with maximum $V = V_{\lambda/2}$, *fixed* frequency $f = T^{-1}$ and *variable* pulse duration τ corresponding to a *duty-cycle* $\nu = \tau/T$ adjustable in the range: $0 < \upsilon < 1/2$. The excitation pulses feeding the two P cells were mutually delayed by a time equal to $\tau/2$ (see inset of Fig. 1). Consider a single excitation cycle. In the time intervals $\Delta\tau = \tau/2$ in which only one P cell was active, either the σ_x or the σ_z transformation was implemented depending on the corresponding crystal orientation. In the interval $\Delta\tau = \tau/2$ in which both P cells were simultaneously active, the σ_y transformation was realized. In summary, each operator $\mathbb{I}$, σ_x, σ_y, σ_z was applied to the input state over a time $\Delta\tau = \tau/2$ and the total depolarizing process lasted a time 2τ over each period T, thus achieving an average depolarizing fraction $(1-\xi) = 2\nu$. In order to avoid any correlation between the two qubits to be purified, ρ_a and ρ_b, the two channels P andP' were driven by different frequencies: $f = 6$ KHz and $f' = 1.7 \times f$. Correspondingly, different values of τ were adopted for the two channels in order to realize, within each experimental run, equal values of ξ for the two input qubits.

In the analysis, we have assumed an identical preparation of the two input qubits, while the output ones are described by the same density matrix $\rho = \rho_a = \rho_b$. With this assumption, carefully checked over each channel, the verification of the purification procedure lies on the tomography of the density matrix of one of the input and one of the output qubits. For the sake of simplicity, we only analyzed the measurements performed on the BS output mode k_1 (see Fig. 1), selecting counts in coincidence between the detectors $[D_1, D_2]$ to trigger the realization of the projection of ρ onto the symmetric subspace. The detectors D_1, D_2 were coupled to mode k_1 by a 50:50 beam splitter BS_1. D_1 provided the measured outcomes of a simple tomographic setup consisting of a $\lambda/2$ WP, a $\lambda/4$ WP, and a polarizing beam-splitter (*P*BS).

Consider first the projector switched off, by setting $Z \gg c\tau_{coh}$, i.e., by spoiling any interference on the photons impinging on BS. A tomographic reconstruction of the qubit in the mode k_1' based on the measurement of the corresponding Stokes parameters by four different settings of the WP's $\frac{\lambda}{2}, \frac{\lambda}{4}$ was undertaken. It is easy to see that this qubit, corresponding to the qubit to be purified, is expressed by the density matrix $\rho_1 = \frac{\rho_a^{in} + \rho_b^{in}}{2}$ [22]. By turning on the projector, i.e., by restoring the BS interference setting $Z = 0$, the mode k_1 contains the two photons described by the density matrix ρ_{ab}^{out}. In this case, we measured on the mode k_1' the purified qubit $\rho_1' = \rho_a^{out}$. From the density matrices reconstructed in absence and in presence of interference, we obtain ξ, ξ_p and thus the purification factor $\eta = \xi_p/\xi$. In addition, from the coincidence rates determined for $Z = 0$ and for $Z \gg c\tau_{coh}$, we inferred the success probability p of the purification protocol. We may check that an increase of the purification gain factor for any qubit pair, i.e., a larger η, corresponds to a lower success probability of the overall protocol, as expected. In Fig. 2, we plotted the experimental values of η and p obtained for different ξ's, i.e., for different experimental values of $\nu = (1 - \xi)/2$, for three input states: $|H\rangle$, $|L\rangle = 2^{-\frac{1}{2}}(|H\rangle + |V\rangle)$, and $|E\rangle = [\cos(3\pi/16)|H\rangle + i\sin(3\pi/16)|V\rangle]$ corresponding, respectively, to horizontal, 45°-diagonal, and a very general elliptical polarization of the input qubits. The mutual agreement of the data for different input states demonstrates the universality of the purification procedure. The deviations of the experimental data from the theoretical values were mainly due to the imperfections of the optical components. In particular, the nonideal properties of the main BS were found to be highly critical. We believe they are responsible for the better approximation to theory of the data taken for η than the ones for p, in Fig. 2. In order to achieve the projection onto the symmetric subspace, the BS transmittances T_H and T_V for the H and V polarization modes should be equal, with a high level of precision, and any difference between T_H and T_V partially spoils the purification. Notice, however, that deviations of $T_H = T_V$ from 50% only decrease the success probability p but do not alter the purification gain factor η.

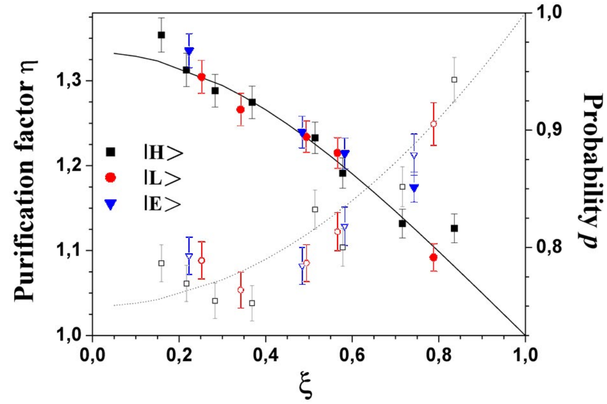

FIG. 2 (color online). Experimental results of the purification procedure for different input qubits corresponding to the encoded polarizations: $|H\rangle$, $|L\rangle = 2^{-1/2}(|H\rangle + |V\rangle)$, and $|E\rangle = [\cos(\frac{\theta}{2})|H\rangle + i\sin(\frac{\theta}{2})|V\rangle]$ with $\theta = \frac{3}{8}\pi$. Filled markers denote the experimental purification factor η data while open markers denote the experimental data of the procedure probability p.

We may generalize the above method by accounting for any possible asymmetry of the preparation of the input qubits. Allowing the input qubits to have different degree of mixedness, i.e., $\rho_a^{in} = \zeta|\phi\rangle\langle\phi| + (1 - \zeta)\frac{I}{2}$, $\rho_b^{in} = \kappa|\phi\rangle\langle\phi| + (1 - \kappa)\frac{I}{2}$, the output qubits are still in the state given by Eq. (3) with $\xi_P = 2(\zeta + \kappa)/(3 + \kappa\zeta)$. The purification factor is $\eta = \xi_P/\xi = \frac{4}{3+\kappa\zeta} = 1/p$, where $\xi \equiv \frac{1}{2}(\zeta + \kappa)$ is the average input mixedness factor. This process may be investigated by recourse to the quantum "relative entropy" that measures the closeness of any output state σ with respect to a corresponding input pure state ρ, e.g., after propagation through a noisy communication channel: $S(\rho \parallel \sigma) \equiv \mathrm{Tr}(\rho \log\rho) - Tr(\rho\log\sigma)$ [23]. Suppose that two qubits are equally prepared in the *pure* state $\rho = |\phi\rangle_a\langle\phi|_a \otimes |\phi\rangle_b\langle\phi|_b$, $S(\rho) = 0$. After corruption by noise, the entropy is: $S(\rho \parallel \rho^{in}) = \log[\frac{1}{2}(1 + \zeta)] + \log[\frac{1}{2}(1 + \kappa)]$. If the qubits are further purified by symmetrization, the following result is obtained: $S(\rho \parallel \rho^{out}) = \log[\frac{1}{2}(1 + \zeta)] + \log[\frac{1}{2} \times (1 + \kappa)] - \log\eta$. Then the symmetrization leads to the positive *information gain* $\Delta S = S(\rho \parallel \rho^{in}) - S(\rho \parallel \rho^{out}) = \log\eta$ at the expense of a reduced rate p of success: $\Delta S = -\log p$, $\frac{3}{4} \leq p \leq 1$.

An interesting case is represented by the purification of a *fully mixed state* by a *pure* state, e.g., by the initial conditions $\zeta = 1$ and $\kappa = 0$. This precisely corresponds to the probabilistic *quantum cloning* process recently realized by our Laboratory in Rome by a symmetrization procedure [18,20,21]. There $\xi = \frac{1}{2}$, and a purification gain factor $\eta = 4/3$ was attained with a success probability: $p = 3/4$.

In conclusion, we have experimentally demonstrated the optimal purification of two depolarized qubits using the interference of two photons at a beam splitter, conditionally effecting symmetrization. The experimentally observed purification gain factors are in very good agreement with the theoretical estimates. We therefore envision that single qubit purification may become a viable procedure for protecting quantum states against noise. Note also that the projection on the symmetric subspace of more than two qubits can be carried out with a sequence of beam splitters, which is in the reach of present optical technology. Furthermore note that the purification via symmetrization, which is well suited for photons because of their bosonic nature, may be in principle applicable also to other physical systems as trapped ions or NMR. However, in these cases, one would have to implement the projection onto the symmetric subspace by using an ancilla qubit that would first interact with the two qubits to be purified and then be measured.

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