Noise resistance of adiabatic quantum computation using random matrix theory

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Besides the traditional circuit-based model of quantum computation, several quantum algorithms based on a continuous-time Hamiltonian evolution have recently been introduced, including for instance continuous-time quantum walk algorithms as well as adiabatic quantum algorithms. Unfortunately, very little is known today about the behavior of these Hamiltonian algorithms in the presence of noise. Here, we perform a fully analytical study of the resistance to noise of these algorithms using perturbation theory combined with a theoretical noise model based on random matrices drawn from the Gaussian orthogonal ensemble, whose elements vary in time and form a stationary random process.

I. INTRODUCTION

There has been a growing interest recently in the concept of Hamiltonian-based quantum algorithms, as opposed to the standard circuit-based paradigm of quantum computing. The Hamiltonian algorithms rely on the continuous-time evolution of a quantum register according to the Schrödinger equation and include in particular the quantum search algorithms by adiabatic evolution [1] or by continuous quantum walks [2,3]. While these algorithms may be translated into circuit-based algorithms so that they could be implemented on a "standard" quantum computer [4,5], another possibility is to consider a "continuous" quantum computer specifically designed to run this type of algorithms [6]. For a realistic implementation, it seems therefore crucial to investigate how well such a quantum computer would behave in the presence of noise. Until now, this question has only been addressed for some specific algorithms subject to some very particular noise. For instance, Childs *et al.* have considered an adiabatic quantum algorithm for solving combinatorial problems [7] affected by an error modeled by an extra term which is random but deterministically evolves in time [8]. While this study was purely numerical, later on Shenvi *et al.* [9] analytically analyzed the effect of a Markovian stochastic variable perturbing the amplitude of the oracle Hamiltonian in the specific case of the analog analogue of Grover's search algorithm [10]. In contrast, the purpose of the present paper is to derive more generic results for an Hamiltonian-based algorithm perturbed by a noise that is described by a stationary Gaussian random process. This makes it possible to carry out a fully analytical treatment of the tolerance to noise, although this is at the price of some (fairly general) assumptions on the noise model and of the use of perturbation theory.

This paper is organized as follows. In Sec. II, we describe our theoretical model of noise based on the Gaussian orthogonal ensemble. In Sec. III, we use perturbation theory to analyze the effect of noise on a *time-independent* Hamiltonian evolution and apply our results to the analog analogue of quantum search. In Sec. IV, we consider the tolerance to noise of a *time-dependent* quantum algorithm by adiabatic evolution and then focus on the quantum search by local adiabatic evolution. Finally, in Sec. V, we conclude by dis-

cussing the scaling of the noise-induced error probability with problem size as a function of the noise bandwidth.

II. NOISE MODEL

Suppose we have a Hamiltonian algorithm based on the Hamiltonian $\bar{H}(t)$,

$$i\hbar \frac{d}{dt}|\bar{\psi}(t)\rangle = \bar{H}(t)|\bar{\psi}(t)\rangle. \tag{1}$$

At the end of the computation ($t=T$), we obtain the state $|\bar{\psi}(T)\rangle$, which, after measurement, defines the output of the algorithm. Now, suppose that a perturbation $\varepsilon h(t)$ adds to this Hamiltonian:

$$H(t) = \bar{H}(t) + \varepsilon h(t). \tag{2}$$

Instead of $|\bar{\psi}(T)\rangle$, we will get at the end of the computation a different state $|\psi(T)\rangle$. The problem in the following will be to evaluate the error probability

$$p_{\mathrm{err}} = 1 - |\langle \bar{\psi}(T)|\psi(T)\rangle|^2 \tag{3}$$

induced by the perturbation.

In order to derive analytical results, we will have to make some assumptions on the noise term $\varepsilon h(t)$. First, we limit ourselves to a noise of small amplitude $\varepsilon \ll 1$, so that the use of perturbation theory is justified. Second, we assume that, in any basis $|\varphi_k\rangle$ ($k=,0\ldots,N-1$) of the N-dimensional Hilbert space where the computation takes place, the matrix elements of $h(t)$ are normal random variables:

$$h_{kl}(t) = \langle \varphi_k|h(t)|\varphi_l\rangle \in \mathcal{N}(0,\sigma_{kl}^2). \tag{4}$$

More specifically, we assume that the matrix $h(t)$ is drawn from a Gaussian orthogonal ensemble (GOE), so that the variance σ_{kl}^2 of its elements is defined by $\sigma_{kl}^2=(1+\delta_{kl})\sigma^2$, where σ^2 is an overall variance (see [11] for more details on random matrix ensembles). Moreover, any two distinct elements of a GOE matrix are taken as independent random variables,

$$\langle h_{kl}(t)h_{k'l'}(t)\rangle = 0$$

$$\Leftrightarrow$$

$$(k,l) \neq (k',l') \text{ and } (k,l) \neq (l',k'). \tag{5}$$

Even though the above assumptions are not based on a specific physical source of noise, they may be justified by considering that the noise is generally caused by many independent sources of error which, combined together, finally result in a random Hamiltonian drawn from a GOE as a consequence of the central-limit theorem.

Furthermore, we assume that the random matrix elements $h_{kl}(t)$ evolve in time as some stationary random process with an autocorrelation function [12]

$$R(\tau) = \langle h_{kl}(t+\tau)h_{kl}(t)\rangle. \tag{6}$$

For instance, a very typical noise model we can use is a white noise with a high-frequency cutoff ω_0 (see, for instance, [13]), which yields

$$R(\tau) = \sigma_{kl}^2 \frac{\sin \omega_0 \tau}{\omega_0 \tau}. \tag{7}$$

However, to be slightly more general, we will only assume later on that the autocorrelation function is of the type

$$R(\tau) = \sigma_{kl}^2 f(\omega_0 \tau), \tag{8}$$

where $f(x)$ verifies $f(-x)=f(x)$ and $f(x) \leqslant f(0)=1$, as well as some other regularity conditions (see next sections). Thus, we only need to assume that $R(\tau)$ is a function of $\omega_0 \tau$.

Finally, as we will be interested in the scaling of the perturbed Hamiltonian-based algorithm as a function of the size of the problem, N, we need to properly define the dependence of the noise term in N. For the scaling analysis to be sensible, we must keep a constant signal-to-noise ratio as N increases; that is, the eigenvalues of $h(t)$ should scale similarly to those of $\bar{H}(t)$. As a result of Wigner's semicircular law, we know that the density of eigenvalues of GOE matrices for $N \gg 1$ is given by

$$\rho(E) \xrightarrow[N\to\infty]{} \begin{cases} \dfrac{1}{4\sigma^2\pi}\sqrt{4\sigma^2 N - E^2} & \text{if } |E| \leqslant \sqrt{4\sigma^2 N}, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Therefore, to keep a constant signal-to-noise ratio, we have to impose that $\sigma^2 = \bar{E}^2/4N$, where $\bar{E}$ is of the order of the eigenvalues of $\bar{H}(t)$.

III. TIME-INDEPENDENT HAMILTONIAN EVOLUTION WITH NOISE

A. Perturbation theory

Let us study first the simplest case of a time-independent Hamiltonian evolution. The solution of the unperturbed Schrödinger equation is

$$|\bar{\psi}(t)\rangle = \sum_k \bar{b}_k e^{-iE_k t/\hbar}|\varphi_k\rangle, \tag{10}$$

where $|\varphi_k\rangle$ and E_k are the eigenstates and eigenvalues of the unperturbed Hamiltonian and the amplitudes $\bar{b}_k$ follow from the initial conditions. By use of perturbation theory, we can study the effect of a small time-dependent perturbation $\varepsilon h(t)$ on the Hamiltonian $\bar{H}$. Expanding the solution of the perturbed equation in the basis formed by the solutions of the unperturbed equation, that is,

$$|\psi(t)\rangle = \sum_k b_k(t) e^{-iE_k t/\hbar}|\varphi_k\rangle, \tag{11}$$

and introducing this expression into the Schrödinger equation, we get

$$\dot{b}_k = -i\frac{\varepsilon}{\hbar}\sum_l b_l e^{i\omega_{kl}t}h_{kl}(t), \tag{12}$$

where $\omega_{kl}=(E_k-E_l)/\hbar$. Using the same initial state as for the noiseless evolution—i.e., $b_k(0)=\bar{b}_k$—we obtain the system of equations

$$b_k(t) = \bar{b}_k - i\frac{\varepsilon}{\hbar}\sum_l \int_0^t b_l(t_1)e^{i\omega_{kl}t_1}h_{kl}(t_1)dt_1. \tag{13}$$

Using standard perturbation theory (see, e.g., [14]), this may be solved iteratively, building step by step the expansion of $b_k(t)$ in increasing orders in ε. From this solution, one can derive (an expansion of) the error probability p_{err} introduced by the perturbation $\varepsilon h(t)$. As the matrix elements of $h(t)$ are random variables, so will p_{err} be, and we will only have access to its statistics. In particular, we will focus on its mean $\langle p_{\text{err}}\rangle$. Using our assumption that $h(t)$ is a random matrix drawn from a GOE, we can show that

$$\langle p_{\text{err}}\rangle = \varepsilon^2\left\{\sum_{k,l}|\bar{b}_k|^2(1-|\bar{b}_l|^2)I_{kl}^- - \sum_{k\neq l}(\bar{b}_k^*\bar{b}_l)^2 I_{kl}^+\right\} + O(\varepsilon^3), \tag{14}$$

where we have introduced the integrals

$$I_{kl}^{\pm} = \frac{\sigma_{kl}^2}{\hbar^2}\int\int_0^T dt_1 dt_2 e^{i\omega_{kl}(t_1\pm t_2)}f(\omega_0(t_1-t_2)), \tag{15}$$

which correspond to the coupling between the states $|\varphi_k\rangle$ and $|\varphi_l\rangle$ that is effected by the perturbation. Our problem now is to evaluate these integrals. We see that they only depend on the noise model via the autocorrelation function $f(x)$ and the high-frequency cutoff ω_0, while they depend on each particular instance of the problem via the spectrum of the Hamiltonian or the frequencies ω_{kl} (as well as the computation time T). Therefore, $I_{kl}^{\pm}$ vary for different instances of a problem. However, we can derive some general expressions, which remain valid for a fairly large class of problems.

First of all, since $I_{kl}^{\pm}$ are integrals over a domain of size T^2 and as the amplitude of their integrand is bounded by 1, we immediately see that, whatever the values of ω_{kl} and ω_0, we have the upper bound

$$|I_{kl}^{\pm}| \leqslant \frac{\sigma_{kl}^2 T^2}{\hbar^2}. \qquad (16)$$

Furthermore, we note that the I_{kl}^+ couplings only appear between the eigenstates that are initially populated and may therefore be viewed as the interferences caused by the noise between these states. As there is in general a small and fixed number of eigenstates $|\varphi_k\rangle$ that are populated ($\bar{b}_k \neq 0$) in the algorithm,[1] Eq. (14) implies that there will be a fixed number of I_{kl}^+ terms contributing to the expression of $\langle p_{\mathrm{err}}\rangle$. In contrast, the number of I_{kl}^- terms, corresponding the the coupling of the initially populated states to all others, will in general grow with the dimension N of the Hilbert space. Therefore, the scaling of the average error probability $\langle p_{\mathrm{err}}\rangle$ will mostly depend on the integrals I_{kl}^-, which is why we now focus on these in what follows. By changing the integration variables to $u=t_1-t_2$ and $v=t_1+t_2$, we get

$$I_{kl}^- = 2\frac{\sigma_{kl}^2}{\hbar^2} \int_0^T dv \int_0^v du \, \cos(\omega_{kl} u) f(\omega_0 u), \qquad (17)$$

which is the integral of a modulated oscillation.

For a white noise (7), we get by direct integration

$$I_{kl}^- = \frac{\sigma_{kl}^2}{\hbar^2 \omega_0} \left[\frac{1-\cos(\omega_{kl}-\omega_0)T}{\omega_{kl}-\omega_0} + T\mathrm{Si}(\omega_{kl}-\omega_0)T \right.$$
$$\left. - \frac{1-\cos(\omega_{kl}+\omega_0)T}{\omega_{kl}+\omega_0} + T\mathrm{Si}(\omega_{kl}+\omega_0)T \right], \qquad (18)$$

where $\mathrm{Si}(x)$ is the sine integral function. Depending on the value of ω_0, we may consider two limiting regimes: for a high cutoff frequency $\omega_0 \gg \omega_{kl}$, we get

$$I_{kl}^- = \frac{\sigma_{kl}^2}{\hbar^2 \omega_0^2} O\left(\left(1+\frac{\omega_{kl}}{\omega_0}\right)\omega_0 T \right), \qquad (19)$$

while for a low cutoff frequency $\omega_0 \ll \omega_{kl}$, we have

$$I_{kl}^- = \frac{\sigma_{kl}^2}{\hbar^2 \omega_{kl}^2} O\left(1 + \frac{\omega_0}{\omega_{kl}} \right). \qquad (20)$$

Although Eqs. (19) and (20) are only valid, strictly speaking, for a white noise, we obtain similar results for a general function $f(x)$. In the high-ω_0 regime, since the autocorrelation function $R(\tau)$ usually tends to zero as τ increases [i.e., $h_{kl}(t+\tau)$ becomes less and less correlated with $h_{kl}(t)$ for increasing τ], Eq. (19) follows from the approximation $\cos(\omega_{kl}u)f(\omega_0 u)=1+O(\omega_{kl}/\omega_0)$. In the low-$\omega_0$ regime, we must integrate a rapidly oscillating function over many periods, which is treated in the Appendix. Under very general

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[1]In particular we will see that for the analog quantum search, there are only two populated levels along the evolution: namely, the ground and first excited states

regularity conditions on $f(x)$,[2] we may use the lemma 2 twice and finally recover Eq. (20). It is interesting to note that in the low-frequency regime, the coupling integral does not depend on the computation time T. We will see that, at least for the algorithms considered here, this causes a very different behavior of the scaling for this regime as compared to the high-ω_0 regime.

B. Analog quantum search

Let us recall the principle of Farhi and Gutmann's analog quantum search [10]. Suppose that we may apply an oracle Hamiltonian

$$H_f = \bar{E}(I - |m\rangle\langle m|) \qquad (21)$$

to the system, with $\bar{E}$ denoting an energy scale of the system. The problem is to prepare the system in the (unknown) solution state $|m\rangle$. In [10], it is shown that this may be achieved by preparing the system in the uniform superposition of all states,

$$|\psi_0\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}|k\rangle, \qquad (22)$$

and applying the constant Hamiltonian $\bar{H}=H_0+H_f$, where

$$H_0 = \bar{E}(I - |\psi_0\rangle\langle\psi_0|), \qquad (23)$$

during a time

$$T = \frac{\pi\hbar}{2\bar{E}}\sqrt{N}. \qquad (24)$$

This results in a quadratic speed-up with respect to a classical search in an unstructured database of size N.

In order to study the robustness of this quantum algorithm against a stationary Gaussian noise as defined in Sec. II, let us first consider the spectrum of the Hamiltonian $\bar{H}=H_0+H_f$ (see Fig. 1). We assume, for simplicity and without loss of generality, that the problem admits the solution $m=0$. The two lowest eigenvalues of $\bar{H}$—that is, $E_0=(1-x)\bar{E}$ and $E_1=(1+x)\bar{E}$ with $x=1/\sqrt{N}$—are nondegenerate and correspond to the ground and first-excited states,

$$|\varphi_0\rangle = \sqrt{\frac{1+x}{2}}|0\rangle + \frac{x}{\sqrt{2(1+x)}}\sum_{k=1}^{N-1}|k\rangle, \qquad (25)$$

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[2]Note that in order to satisfy the hypotheses of lemma 2 in the Appendix, $f(x)$ has to be infinitely differentiable. While this is generally the case for the widely used noise models such as a white noise, $f(x)$ may be discontinuous in $x=0$ for some specific models, such as $f(x)=\exp(-|x|)$, which could follow from a Poissonian process. This discontinuity in the first derivative of $f(x)$ is actually linked to the fact that the spectral density of the noise does not rapidly converge to zero for very high frequencies and therefore that the noise has some probability to vary arbitrarily fast. We will not consider this case here, but only mention that this would yield a drastically different behavior in the low-ω_0 regime.
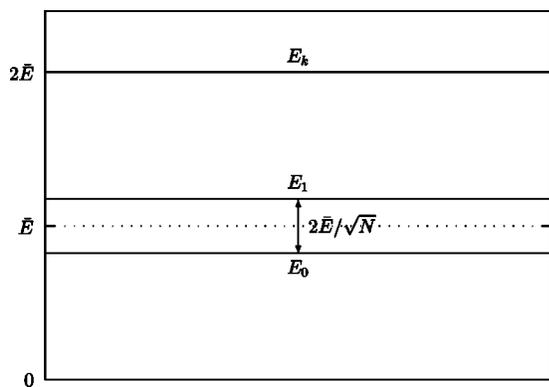
FIG. 1. Spectrum of the unperturbed Hamiltonian $\bar{H}=H_0+H_f$.

$$|\varphi_1\rangle = \sqrt{\frac{1-x}{2}}|0\rangle - \frac{x}{\sqrt{2(1-x)}}\sum_{k=1}^{N-1}|k\rangle, \qquad (26)$$

whereas the $N-2$ times degenerate eigenvalue $E_2=2\bar{E}$ corresponds to the eigenstates

$$|\varphi_k\rangle = \frac{1}{\sqrt{2}}(|k\rangle - |1\rangle), \quad k = 2, \ldots, N. \qquad (27)$$

Expressing $|\psi(t=0)\rangle=|\psi_0\rangle$ in terms of the eigenstates $|\varphi_k\rangle$ of $\bar{H}$, we get

$$|\psi(0)\rangle = \sqrt{\frac{1+x}{2}}|\varphi_0\rangle - \sqrt{\frac{1-x}{2}}|\varphi_1\rangle. \qquad (28)$$

As a consequence, the instantaneous state of the noiseless algorithm $|\bar{\psi}(t)\rangle$ is given by Eq. (10) with

$$\bar{b}_0 = \sqrt{\frac{1+x}{2}}, \quad \bar{b}_1 = -\sqrt{\frac{1-x}{2}}, \qquad (29)$$

and $\bar{b}_k=0$ for $k \geq 2$. Only two states are populated during the algorithm, and the average error probability (14) becomes

$$\langle p_{\mathrm{err}}\rangle = \varepsilon^2\{(N-2)[|\bar{b}_0|^2 I_{02}^- + |\bar{b}_1|^2 I_{12}^-] + |\bar{b}_0|^2|\bar{b}_1|^2(I_{00}^- + I_{11}^-)$$
$$+ (|\bar{b}_0|^4 + |\bar{b}_1|^4)I_{01}^- - 2\,\mathrm{Re}[(\bar{b}_0^*\bar{b}_1)^2 I_{01}^+]\}, \qquad (30)$$

where we have used the normalization condition $|\bar{b}_0|^2+|\bar{b}_1|^2=1$ and the fact that $I_{kl}^-=I_{k2}^-$ for $l \geq 2$. For this algorithm, the bound (16) gives $|I_{kl}^\pm(t)| \leq \pi^2/8$, which is independent of N. Therefore, only the first term of Eq. (30), which represents the coupling of the ground and first-excited states (the only initially populated states) to the $N-2$ others, can grow with N and must be taken into account in the scaling analysis. Let us focus on this term in the two limiting regimes considered above.

For a noise with a high cutoff frequency, Eq. (19) yields
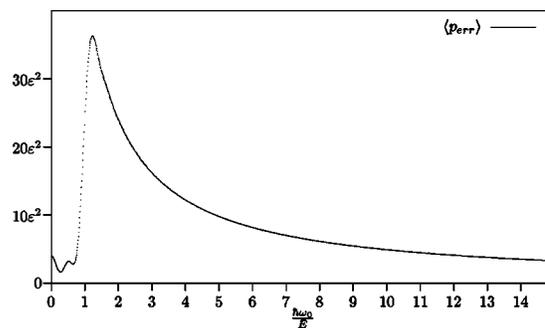
FIG. 2. Average error probability $\langle p_{\mathrm{err}}\rangle$ (to the second order) due to a noise modeled as in Sec. II with an autocorrelation function $f(x)=\sin x/x$ for the analog search among $N=100$ elements. Note that $\langle p_{\mathrm{err}}\rangle$ stays very small as long as $\hbar\omega_0/\bar{E}<1$, at which point it shows a sudden increase. For larger values of $\hbar\omega_0$, it tends progressively back to a low value. The peak at $\hbar\omega_0 \sim \bar{E}$ scales as $\sqrt{N}$.

$$I_{k2}^- = \frac{\bar{E}}{\hbar\omega_0}O\left(\frac{1}{\sqrt{N}}\left(1+\frac{\bar{E}}{\hbar\omega_0}\right)\right) \quad (k=0,1), \qquad (31)$$

which is valid if $\hbar\omega_0 \gg \bar{E}$. Clearly, I_{k2}^- should be of order $1/N$ for $\langle p_{\mathrm{err}}\rangle$ not to grow with N, which imposes the condition

$$\hbar\omega_0 \gg \bar{E}\sqrt{N}. \qquad (32)$$

Thus, in this regime, the cutoff frequency of the noise must increase as the square root of the size of the problem in order to keep a probability of error of constant order. In the case of a noise with a low cutoff frequency, Eq. (20) yields

$$I_{k2}^- = \frac{1}{N}O\left(1+\frac{\hbar\omega_0}{\bar{E}}\right) \quad (k=0,1), \qquad (33)$$

so we see that $\langle p_{\mathrm{err}}\rangle$ will not grow with N as long as $\hbar\omega_0 \ll \bar{E}$. Interestingly, this upper bound on the cutoff frequency does not depend on the size of the problem.

We conclude that the influence of noise on the analog quantum search algorithm is negligible if the noise varies either very slowly or very rapidly with respect to the natural time scale of the problem $\hbar/\bar{E}$. For the typical case of a white noise with a high-frequency cutoff, the exact integration (18) shows that for intermediate values of the cutoff frequency $\omega_0 \sim \bar{E}/\hbar$, the average error probability scales as $\varepsilon^2\sqrt{N}$ for a given signal-to-noise ratio ε (see Fig. 2). This means that there exists a forbidden band for the cutoff frequency ω_0 if we want to keep our algorithm robust with respect to a constant noise (i.e., a noise that scales with N so to keep a constant signal-to-noise ratio). Alternatively, we see that the signal-to-noise ratio ε should scale as $N^{-1/4}$ in order to keep the error probability $\langle p_{\mathrm{err}}\rangle$ constant for a cutoff frequency ω_0 of order $\bar{E}/\hbar$. In other words, for an increasing problem size N, the noise variance σ^2 must decrease faster than $\bar{E}^2/4N$ for the algorithm to remain immune to noise. Finally, let us mention that $\varepsilon \sim N^{-1/4}$ coincides with the result of Shenvi et al. [9], even though their noise model was less

general than ours since they only considered an error in the magnitude of the oracle Hamiltonian, modeled as a Markovian stochastic variable with Gaussian distribution.

IV. ADIABATIC EVOLUTION WITH NOISE

A. Adiabatic approximation

Let us recall the adiabatic approximation, which is at the basis of the quantum algorithms by adiabatic evolution. Qualitatively speaking, the idea is as follows: if a quantum system is prepared in its ground state and its Hamiltonian varies "slowly enough," it remains in a state close to the instantaneous ground state of the Hamiltonian at any time. To be more precise, let us consider the Schrödinger equation for a time-dependent Hamiltonian (see [14] for details),

$$i\hbar \frac{d}{dt}|\bar{\psi}(t)\rangle = \bar{H}(t)|\bar{\psi}(t)\rangle. \tag{34}$$

To solve this equation, we express its solution $|\bar{\psi}(t)\rangle$ in the basis formed by the instantaneous eigenstates $|\varphi_k(t)\rangle$ of the Hamiltonian $\bar{H}(t)$,

$$|\bar{\psi}(t)\rangle = \sum_k \bar{b}_k(t)\exp\left(-i\int_0^t \frac{E_k(t_1)}{\hbar}dt_1\right)|\varphi_k(t)\rangle, \tag{35}$$

where $E_k(t)$ are the corresponding instantaneous eigenvalues of $\bar{H}(t)$. By inserting this expression into Eq. (34), we find the system of differential equations

$$\dot{\bar{b}}_k(t) = \sum_{l\neq k} \bar{b}_l(t)\exp\left(i\int_0^t \omega_{kl}(t_1)dt_1\right)\frac{\left\langle\varphi_k(t)\left|\frac{d\bar{H}}{dt}\right|\varphi_l(t)\right\rangle}{E_k(t)-E_l(t)}. \tag{36}$$

If the quantum system is initially in its ground state $|\bar{\psi}(0)\rangle = |\varphi_0(0)\rangle$, these equations can be integrated, giving

$$\bar{b}_k(t) = \delta_{0k} + \sum_{l\neq k}\int_0^t \bar{b}_l(t_1)\exp\left(i\int_0^{t_1}\omega_{kl}(t_1')dt_1'\right)$$
$$\times\frac{\left\langle\varphi_k(t_1)\left|\frac{d\bar{H}}{dt_1}\right|\varphi_l(t_1)\right\rangle}{E_k(t_1)-E_l(t_1)}dt_1. \tag{37}$$

As in perturbation theory, these equations may be solved iteratively, which gives after one iteration

$$\bar{b}_k^{(1)}(t) = \int_0^t \exp\left(i\int_0^{t_1}\omega_{k0}(t_1')dt_1'\right)\frac{\left\langle\varphi_k(t_1)\left|\frac{d\bar{H}}{dt_1}\right|\varphi_0(t_1)\right\rangle}{E_k(t_1)-E_0(t_1)}dt_1 \tag{38}$$

for $k\neq 0$. Now if the variation $d\bar{H}/dt$ of the Hamiltonian is slow enough or, more specifically, if

$$|A_k(t)| \equiv \hbar\frac{\left|\left\langle\varphi_k(t)\left|\frac{d\bar{H}}{dt}\right|\varphi_0(t)\right\rangle\right|}{[E_k(t)-E_0(t)]^2} \leq \delta_k \ll 1 \tag{39}$$

and under suitable regularity conditions, we may integrate Eq. (38) by parts as shown in the Appendix, which yields

$$\bar{b}_k^{(1)}(t) = -i\left[A_k(t_1)\exp\left(i\int_0^{t_1}\omega(t')dt'\right)\right]_0^t + O(\delta_k^2). \tag{40}$$

We see that, at the first order, the amplitudes $|\bar{b}_k(t)|$ are bounded by $2\delta_k$. This first-order (so-called *adiabatic*) approximation is acceptable if $|\bar{b}_0(t)|^2$ stays close to 1—that is, if the probability $\bar{p}(t)=\Sigma_{k\neq 0}|\bar{b}_k(t)|^2$ of hopping to any excited state remains small. In other words, if the adiabatic condition

$$4\sum_{k\neq 0}\sup_{[0,t]}|A_k(t')|^2 \leq \delta^2 \tag{41}$$

is satisfied, then $\bar{p}(t)\leq \delta^2\ll 1$, with $\delta\ll 1$ being a "slowness" parameter.

Now, suppose that a time-dependent perturbation $\varepsilon h(t)$ adds to the Hamiltonian $\bar{H}(t)$. We again express the solution of the perturbed Schrödinger equation in the basis formed by the instantaneous eigenstates of $\bar{H}(t)$,

$$|\psi(t)\rangle = \sum_k b_k(t)\exp\left(-i\int_0^t \frac{E_k(t_1)}{\hbar}dt_1\right)|\varphi_k(t)\rangle, \tag{42}$$

which transforms Eq. (37) into

$$b_k(t) = \delta_{0k} + \sum_{l\neq k}\int_0^t b_l(t_1)\exp\left(i\int_0^{t_1}\omega_{kl}(t_1')dt_1'\right)$$
$$\times\frac{\left\langle\varphi_k(t_1)\left|\frac{d\bar{H}}{dt_1}\right|\varphi_l(t_1)\right\rangle}{E_k(t_1)-E_l(t_1)}dt_1$$
$$-i\frac{\varepsilon}{\hbar}\sum_l\int_0^t b_l(t_1)\exp\left(i\int_0^{t_1}\omega_{kl}(t_1')dt_1'\right)\langle\varphi_k(t_1)|h(t_1)$$
$$\times|\varphi_l(t_1)\rangle dt_1. \tag{43}$$

We may again solve this system of equations iteratively, which gives after one iteration

$$b_k^{(1)}(t) = \bar{b}_k^{(1)}(t) - i\frac{\varepsilon}{\hbar}\int_0^t \exp\left(i\int_0^{t_1}\omega_{k0}(t_1')dt_1'\right)\langle\varphi_k(t_1)|h(t_1)$$
$$\times|\varphi_0(t_1)\rangle dt_1 \tag{44}$$

for $k\neq 0$. As before, this first-order approximation remains valid provided that the probability $p(t)=\Sigma_{k\neq 0}|b_k(t)|^2$ of hopping to any excited state remains small.

Let us now evaluate the average error probability at the end of the evolution $t=T$ using the same model as before for the perturbation $h(t)$. Defining the error probability as $p_{\text{err}}=p(T)$, we have

$$\langle p_{\mathrm{err}}\rangle = \bar{p}_{\mathrm{err}} + \frac{\varepsilon^2}{\hbar^2}\sum_{k\neq 0}\int\int_0^T dt_1 dt_2 \exp\left(i\int_{t_2}^{t_1}\omega_{k0}(t')dt'\right)$$

$$\times \langle\langle\varphi_k(t_1)|h(t_1)|\varphi_0(t_1)\rangle\langle\varphi_k(t_2)|h(t_2)|\varphi_0(t_2)\rangle\rangle + O(\varepsilon^3), \tag{45}$$

where $\bar{p}_{\mathrm{err}}=\bar{p}(T)$ is the error probability of the adiabatic evolution. Let us note that $|\varphi_k(t_1)\rangle \neq |\varphi_k(t_2)\rangle$ in general, so that we do not immediately recover the autocorrelation function of one particular matrix element of $h(t)$. However, as the different matrix elements of $h(t)$ are independent in a particular basis, we have

$$\langle\langle\varphi_k(t_1)|h(t_1)|\varphi_0(t_1)\rangle\langle\varphi_k(t_2)|h(t_2)|\varphi_0(t_2)\rangle\rangle$$
$$= [\langle\varphi_k(t_2)|\varphi_k(t_1)\rangle\langle\varphi_0(t_1)|\varphi_0(t_2)\rangle + \langle\varphi_k(t_2)|\varphi_0(t_1)\rangle$$
$$\times\langle\varphi_k(t_1)|\varphi_0(t_2)\rangle] \times \sigma_{k0}^2 f(\omega_0(t_1-t_2)). \tag{46}$$

If $|\varphi_k(t)\rangle$ varies sufficiently smoothly for $0 \leq t \leq T$, then the first factor is of order $1 - O((t_1-t_2)^2/T^2)$. Thus, we may approximate it by 1 as long as $\omega_0 T \gg 1$—that is, if the noise varies quickly compared to the adiabatic evolution. In that case, we get for the average error probability

$$\langle p_{\mathrm{err}}\rangle = \bar{p}_{\mathrm{err}} + \varepsilon^2\sum_{k\neq 0} I_{k0}^- + O((\delta+\varepsilon)^3), \tag{47}$$

where the integrals

$$I_{k0}^- = \frac{\sigma_{k0}^2}{\hbar^2}\int\int_0^T dt_1 dt_2 \exp\left(i\int_{t_2}^{t_1}\omega_{k0}(t')dt'\right)f(\omega_0(t_1-t_2)), \tag{48}$$

represent the coupling of the ground state to the excited states induced by the perturbation. The adiabatic condition generalizes in the case of such a perturbation to

$$\sum_{k\neq 0}(4\sup_{[0,T]}|A_k(t)|^2 + \varepsilon^2 I_{k0}^-) \leq \delta^2. \tag{49}$$

Similarly to the case of the perturbed time-independent Hamiltonian evolution, the effect of the perturbation on the adiabatic evolution mainly depends on the coupling integrals I_{k0}^-, which are bounded by

$$|I_{k0}^-| \leq \frac{\sigma_{k0}^2 T^2}{\hbar^2} \tag{50}$$

and can be approximated as

$$I_{k0}^- = \frac{\sigma_{k0}^2}{\hbar^2\omega_0^2}O\left(\left(1 + \frac{\omega_{k0}^{\max}}{\omega_0}\right)\omega_0 T\right), \quad \omega_0 \gg \omega_{k0}^{\max}, \tag{51}$$

or

$$I_{k0}^- = \frac{\sigma_{k0}^2}{\hbar^2\omega_{k0}^{\min 2}}O\left(1 + \frac{\omega_0}{\omega_{k0}^{\min}}\right), \quad \omega_0 \ll \omega_{k0}^{\min}, \tag{52}$$

in the limiting regimes of high or low cutoff frequency ω_0, respectively, where $\omega_{k0}^{\min} \leq \omega_{k0}(t) \leq \omega_{k0}^{\max}$ for $t \in [0,T]$.

B. Adiabatic quantum search

The principle of the adiabatic quantum search [1] is to apply the Hamiltonian $H_0=\bar{E}(I-|\psi_0\rangle\langle\psi_0|)$ to a system prepared in its ground state $|\psi_0\rangle$ and then to progressively switch the Hamiltonian H_0 to the Hamiltonian $H_f = \bar{E}(I-|m\rangle\langle m|)$, where m is the solution of the search problem. If this switch is done slowly enough, the system will stay in the instantaneous ground state of the Hamiltonian and thus end up in the ground state of H_f—i.e., the solution state $|m\rangle$. The instantaneous Hamiltonian is chosen as

$$\tilde{H}(s) = (1-s)H_0 + sH_f, \tag{53}$$

where $s=s(t)$ is an evolution function which must be optimized so as to reduce the computation time while respecting the adiabatic condition (41) (see [15,16] for details). In this case, this condition may be rewritten as

$$\frac{ds}{dt} \leq \frac{\delta}{2\hbar}\frac{[E_1(t)-E_0(t)]^2}{|\langle\varphi_1(t)|H_f-H_0|\varphi_0(t)\rangle|}. \tag{54}$$

Without loss of generality, we may once again suppose that $m=0$. The instantaneous eigenstates of $\tilde{H}(s)$ are

$$|\varphi_0(s)\rangle = \frac{\sqrt{N}[E_1(s)-s]|\psi_0\rangle + s|0\rangle}{\sqrt{E_1(s)^2 + (N-1)[E_1(s)-s]^2}}, \tag{55}$$

$$|\varphi_1(s)\rangle = \frac{\sqrt{N}[E_0(s)-s]|\psi_0\rangle + s|0\rangle}{\sqrt{E_0(s)^2 + (N-1)[E_0(s)-s]^2}}, \tag{56}$$

$$|\varphi_k(s)\rangle = \frac{1}{\sqrt{2}}(|k\rangle - |1\rangle), \quad k \geq 2, \tag{57}$$

where

$$E_0(s) = \frac{\bar{E}}{2}\left[1 - \sqrt{1 - 4\frac{N-1}{N}s(1-s)}\right], \tag{58}$$

$$E_1(s) = \frac{\bar{E}}{2}\left[1 + \sqrt{1 - 4\frac{N-1}{N}s(1-s)}\right], \tag{59}$$

$$E_k(s) = \bar{E}, \quad k \geq 2, \tag{60}$$

are the instantaneous eigenvalues of $\tilde{H}(s)$ (see Fig. 3). Since $\|H_f-H_0\| \leq \bar{E}$, taking an evolution function $s(t)$ that satisfies

$$\frac{ds}{dt} = \frac{\delta}{2\hbar\bar{E}}[E_1(s)-E_0(s)]^2 \tag{61}$$

complies with the adiabatic condition (54) and then leads to a computation time

$$T = \frac{\pi}{\delta}\frac{\hbar}{\bar{E}}\sqrt{N}. \tag{62}$$

Consider that some noise, modeled again as a stationary Gaussian random process, perturbs the evolution. Equation (47) then reads
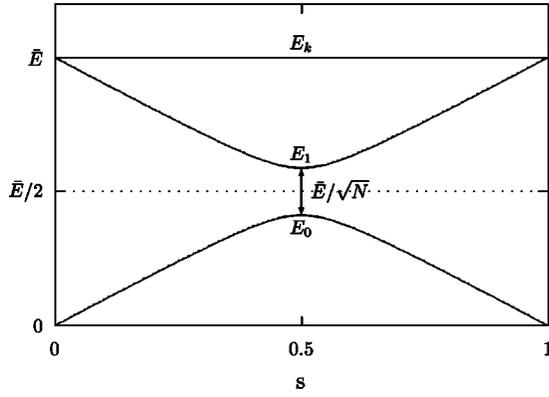
FIG. 3. Instantaneous eigenvalues of $\tilde{H}(s)$ for $N=32$.

$$\langle p_{\text{err}} \rangle \leq \bar{p}_{\text{err}} + \varepsilon^2 [I_{10}^- + (N-2)I_{k0}^-] + O((\delta + \varepsilon)^3), \quad (63)$$

where $I_{k0}^- \leq \pi^2/(64\delta^2)$ as a result of Eq. (50). Let us emphasize that, while it was only the excitation to the first-excited state that was critical for the noiseless adiabatic algorithm, in this case it is the coupling of the ground state to all excited states that could make the algorithm fail, since their number grows as the size N of the problem. Moreover, this bound already suggests that the coupling integrals I_{k0}^-—and therefore the error probability could increase when the evolution slows down (δ decreases) which means there must be a compromise between a slow evolution, very close to perfect adiabaticity, and a fast evolution, more robust to noise.

As before, let us consider the two limiting regimes of a high or a low cutoff frequency ω_0. In the case of a high cutoff frequency ($\hbar\omega_0 \gg \bar{E}$), Eq. (51) yields

$$I_{k0}^- = \frac{\bar{E}}{\hbar\omega_0} O\left(\frac{1}{\delta\sqrt{N}} \left(1 + \frac{\bar{E}}{\hbar\omega_0} \right) \right) \quad \forall k \neq 0, \quad (64)$$

exactly as for the analog quantum search except for the factor $1/\delta$. The latter factor shows that in order to keep the algorithm robust to noise, the cutoff frequency has to increase not only as the size of the database N grows (just as for the analog quantum search), but also as the evolution slows down (δ decreases). More precisely, we see that the perturbed adiabatic condition (49) is satisfied only if

$$\hbar\omega_0 \gg \frac{\varepsilon^2}{\delta^3} \bar{E} \sqrt{N} \quad (65)$$

[compare with Eq. (32)]. When the cutoff frequency becomes very low ($\hbar\omega_0 \ll \bar{E}$), Eq. (52) implies that the coupling integrals behave as

$$I_{k0}^- = \frac{1}{N} O\left(1 + \frac{\hbar\omega_0}{\bar{E}} \right) \quad (66)$$

for all excited states except the first one (i.e., for $k \geq 2$). For the first-excited state ($k=1$), we have $\hbar\omega_{10}^{\min} \sim \bar{E}/\sqrt{N}$, so Eq. (52) does not yield a useful result. Instead, we simply use the general bound $I_{k0}^- \leq \pi^2/(64\delta^2)$. Therefore, the adiabatic con-

dition is satisfied here as long as $\hbar\omega_0 \ll \bar{E}$ (just as for the analog quantum search), but also if $\varepsilon \ll \delta$.

In summary, we recover essentially the same effects for the adiabatic quantum search as for the analog quantum search; that is, the influence of noise becomes negligible only in the case of a very high or a very low cutoff frequency ω_0, apart from the influence of the slowness parameter δ. Regarding this latter parameter, we see that while decreasing δ gets the noiseless evolution closer to adiabaticity and therefore reduces the error probability without noise, in the presence of noise it imposes that ε decreases—i.e., that the signal-to-noise ratio increases—in both regimes of a high or low cutoff frequency (or that the high cutoff frequency increases as $1/\delta^3$).

V. CONCLUSION

We have studied the resistance of Hamiltonian quantum algorithms (including adiabatic algorithms) to a noise that is modeled as a random matrix whose elements are stationary Gaussian random processes within a fixed bandwidth. This statistical noise model is generic and should therefore make our analysis valid over a large class of physical systems, regardless of the exact origin of the added noise. Another main advantage of this noise model is that it makes it possible to perform a fully analytical scaling analysis. Our general result is that the Hamiltonian algorithms are resistant to noise (i.e., the error probability $\langle p_{\text{err}} \rangle$ does not increase with increasing problem sizes N) as long as the cutoff frequency of the noise is either very high or very low with respect to the inverse of the characteristic time scale of the system $\bar{E}/\hbar$. Asides from the influence of the slowness parameter δ in the case of the adiabatic algorithms, this resistance is essentially similar for adiabatic and time-independent Hamiltonian algorithms. Our results are in good agreement with the numerical study of Childs *et al.* [8]. They even corroborate the results of Shenvi *et al.* [9], although their noise model was rather different, which supports the idea that using random matrix theory provides a rather general description of noise.

Roughly speaking, the two limiting regimes of high or low cutoff frequencies can be understood in the following way. If the frequency components of the noise are much below the inverse of the characteristic time scale of the system, it is intuitively clear that the noise cannot effect transitions to undesired states. On the contrary, if the noise spectrum spreads over a band which is much broader than the inverse of the characteristic time scale of the system, then the noise spectral density is low around the frequencies that effect undesired transitions. In the intermediate region, we found that the error probability unfortunately scales as $\sqrt{N}$,[3] which implies that some error correction is needed to make Hamiltonian algorithms scalable. This last point is particularly important as it is plausible that the source of noise occurring in a physical system typically varies on a time

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[3]This scaling has been analytically obtained for a time-independent Hamiltonian and is strongly believed to generalize to the case of an adiabatic evolution.

scale comparable to the natural time scale of the system, so that the less favorable regime ($\hbar\omega_0 \sim \bar{E}$) may be the most common situation.

Coming back to the two limiting regimes, let us notice that the high cutoff frequency increases towards higher frequencies when N raises [see Eqs. (32) and (65)]. Consequently, the Hamiltonian algorithms will in practice not be scalable in the high cutoff frequency regime too, since the noise should spread over an increasingly large spectrum to keep the error probability low. In this case, some kind of error correction should also be implemented if the size of the problem becomes too large. The case of a low cutoff frequency, however, is more favorable. Indeed, the situation is quite different here as the error probability stays small as soon as the cutoff frequency is lower than some fixed value, even when the size of the problem increases. This fault tolerance may be explained by the fact that the spectral density of noise does not contain frequencies close to resonances and thus will not efficiently couple different eigenstates of the Hamiltonian.

It should be emphasized that it is not the possible excitation to one particular state that makes the algorithm fail, but the fact the dimension of the Hilbert space increases with the problem size and hence the number of states that could be accidentally populated as well. This means, in the case of adiabatic computation, that even if the gap between the ground and first-excited states decreases, the algorithm may remain robust to a noise with a low cutoff frequency (even if this frequency remains constant) as long as the gap between the ground states and the other excited states remains lower bounded. Therefore, the algorithm would remain scalable in the case of a low cutoff frequency as long as the natural frequencies of the Hamiltonian are much larger than the frequencies contained in the noise. Of course, throughout this analysis, we always made the assumption that the signal-to-noise ratio remains essentially constant when the size of the Hilbert space where the computation takes place becomes large, which may practically not be the case. Thus, even in this low cutoff frequency regime, it may be necessary to devise error correction techniques for Hamiltonian quantum algorithms. Nonetheless, in the case of adiabatic algorithms, this conclusion may be overpessimistic since, as long as the noise is sufficiently slowly varying (that is, the cutoff frequency is low), the perturbed evolution may remain adiabatic so that the system would stay in the ground state of the perturbed Hamiltonian. Our perturbative method did not reproduce this effect because we developed the solution of the *perturbed* Schrödinger equation in the basis formed by the eigenstates of the *unperturbed* Hamiltonian. Actually, it was shown, at least numerically, that some slowly varying noise could actually help the computation, even for a high noise intensity [8].

Note added. Our result is related to a recent paper [17], which also considers the use of random matrix theory in adiabatic quantum computing but with a distinct goal: namely, to analyze the spectral statistics of the Hamiltonian over a large class of problems along an adiabatic (but noiseless) evolution.

APPENDIX

In this appendix, we give a useful tool to evaluate integrals of an oscillating function such as

$$\int_a^b dx F(x) e^{i\omega x}. \tag{A1}$$

The basic idea relies on Riemann-Lebesgue's lemma.

Lemma 1 (Riemann-Lebesgue). If $F(x)$ is an integrable function on $[a,b]$, then

$$\lim_{\omega \to \infty} \int_a^b dx F(x) e^{i\omega x} = 0.$$

This lemma suggests that the integral (A1) will be *relatively small* if ω is *sufficiently large*. The purpose of this appendix is to quantify this idea.

First of all, as long as $F(x)$ is differentiable on $[a,b]$, we may integrate Eq. (A1) by parts:

$$\int_a^b dx F(x) e^{i\omega x} = -\frac{i}{\omega}\left[F(x)e^{i\omega x}\right]_a^b + \frac{i}{\omega}\int_a^b dx \frac{dF}{dx}(x) e^{i\omega x},$$

where $[f(x)]_a^b = f(b) - f(a)$, and, using this last equation iteratively, we show that for an N-times differentiable function $F(x)$ on $[a,b]$,

$$\int_a^b dx F(x) e^{i\omega x} = -\sum_{n=0}^{N-1}\left(\frac{i}{\omega}\right)^{n+1}\left[\frac{d^n F}{dx^n}(x) e^{i\omega x}\right]_a^b$$
$$+ \left(\frac{i}{\omega}\right)^N \int_a^b dx \frac{d^N F}{dx^N}(x) e^{i\omega x}. \tag{A2}$$

The order of the error introduced by neglecting the last term may be evaluated as follows:

$$\left|\left(\frac{i}{\omega}\right)^N \int_a^b dx \frac{d^N F}{dx^N}(x) e^{i\omega x}\right| \leq \frac{1}{\omega^N}\int_a^b dx \left|\frac{d^N F}{dx^N}(x)\right|. \tag{A3}$$

We see that the accuracy of this approximation increases with the oscillation frequency ω. Moreover, if $(1/\omega^N)d^N F/dx^N \to 0$ for $N \to \infty$, this error approaches zero as N increases and we prove the following lemma:

Lemma 2. Let the function $F(x)$ be infinitely differentiable on $[a,b]$. If

$$\frac{1}{\omega^N}\frac{d^N F}{dx^N}(x) \xrightarrow[N \to \infty]{} 0 \quad \forall \, x$$

for *some real* ω, *then*

$$\int_a^b dx F(x) e^{i\omega x} = -\sum_{n=0}^{\infty} \left(\frac{i}{\omega}\right)^{n+1} \left[\frac{d^n F}{dx^n}(x) e^{i\omega x}\right]_a^b.$$

While this result is helpful to study a time-independent Hamiltonian evolution, in the case of an adiabatic evolution the typical frequencies become time dependent. However, using the same method, we easily generalize this lemma to the case of a varying frequency $\omega(x)$.

Lemma 3. Let the function $F(x)$ be infinitely differentiable on $[a,b]$. If

$$\frac{1}{\omega(x)^N} \frac{d^N F}{dx^N}(x) \xrightarrow[N\to\infty]{} 0 \quad \forall\, x$$

for some real differentiable function $\omega(x)$ *on* $[a,b]$, *then*

$$\int_a^b dx F(x) \exp\left(i \int_0^x \omega(x') dx'\right)$$

$$= \sum_{n=0}^{\infty} \left\{ -\left[\left(\frac{i}{\omega(x)}\right)^{n+1} \frac{d^n F}{dx^n}(x) \exp\left(i\int_0^x \omega(x') dx'\right)\right]_a^b \right.$$

$$\left. + \int_a^b dx \frac{d}{dx}\left(\frac{i}{\omega(x)}\right)^{n+1} \frac{d^n F}{dx^n}(x) \exp\left(i\int_0^x \omega(x') dx'\right) \right\}.$$

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