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Entanglement May Enhance Channel Capacity in Arbitrary Dimensions

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Abstract. We consider explicitly two examples of d-dimensional quantum channels with correlated noise and show that, in agreement with previous results on Pauli qubit channels, there are situations where maximally entangled input states achieve higher values of the output mutual information than product states. We obtain a strong dependence of this effect on the nature of the noise correlations as well as on the parity of the space dimension, and conjecture that when entanglement gives an advantage in terms of mutual information, maximally entangled states achieve the channel capacity.

1. Introduction

The evaluation of the amount of *classical information* which can be reliably transmitted by *quantum states* is a major problem of quantum information. Early works in this direction, devoted mainly to memoryless channels, for which consecutive signal transmissions through the channel are not correlated, allowed the determination of capacities in many instances [1 − 7] and proved their additivity. Recently much attention was given to quantum channels with memory [10 − 14] in the hope that the capacity of such channels be superadditive and could be enhanced by using entangled input states. Contrary to this expectation, for bosonic memory channels without input energy constraints, the additivity conjecture was proven leaving no hope to enhance the channel capacity using entangled inputs [15]. However, for more realistic situation where the energy of input Gaussian states is finite, it was shown that entangling two consecutive uses of the channel with memory introduced by a correlated noise enhances the overall channel capacity [16, 17]. For each value of the noise correlation parameter, there exists an optimal degree of entanglement (not necessarily maximal entanglement) that maximizes the channel capacity. Other examples of quantum channels with memory introduced by a correlated noise include qubit Pauli channels [18, 19]. For these channels it was shown that if the noise correlations are stronger than some critical value, maximally entangled input states enhance the channel capacity compared to product input states. Quantum channels with correlated noise in dimensions $d > 2$ were not considered in the literature in this context, except for [20], which appeared after our work was completed and presented, where a class of d-dimensional quantum channels with memory is considered for which maximally entangled states maximize the channel capacity beyond some memory threshold. The d-dimensional channel corresponds to a kind of intermediate system between the qubit and the Gaussian channel. Therefore, we expect to find new features that this intermediate dimensionality can add to the known facts. We shall consider d-dimensional quantum channels which are generalizations of the Pauli qubit channels studied in [18, 19]. We start with the introduction of the classical capacity of quantum channels, consider explicitly two examples of d-dimensional quantum channels with memory and present results on their capacity.

2. Capacity of Quantum Channels with Correlated Noise

The action of a transmission channel on an initial quantum state described by a density operator ρ is given by a linear completely positive (CP) map $\mathcal{E} : \rho \to \mathcal{E}(\rho)$. The amount of classical information which can be reliably transmitted through a quantum channel is given by the Holevo-Schumacher-Westmoreland bound [1, 2] as the maximum of mutual information

$$\chi(\mathcal{E}) \;=\; \max_{\{P_i,\rho_i\}} I(\mathcal{E}) \tag{1}$$

taken over all possible ensembles $\{P_i, \rho_i\}$ of input states ρ_i with *a priori* probabilities $P_i > 0$, $\sum_i P_i = 1$. The mutual information of an ensemble $\{P_i, \rho_i\}$ is defined as

$$I(\mathcal{E}(\{P_i,\rho_i\})) \;=\; \Big[S\Big(\sum_i P_i \mathcal{E}(\rho_i)\Big) - \sum_i P_i S(\mathcal{E}(\rho_i)) \Big] \,, \tag{2}$$

where $S(\rho) = -\operatorname{Tr}[\rho \log_2 \rho]$ is the von Neumann entropy. If we find a state ρ_* which minimizes the output entropy $S(\mathcal{E}(\rho_*))$ and replace the first term in (2) by the largest possible entropy given by the entropy of the maximally mixed state, we obtain the following bound

$$\chi(\mathcal{E}) \;\leq\; \log_2(d) - S(\mathcal{E}(\rho_*)) \,, \tag{3}$$

where d^2 is the dimension of ρ. With the help of the arguments of [19], it can be shown [23] that this bound is tight for the channels which we consider.

In general, being a linear CP map, any quantum channel can be represented by an operator-sum: $\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger$, $\sum_k A_k^\dagger A_k = \mathbb{1}$. In order to describe n uses of the same quantum channel, we have to consider the Hilbert space of the initial states, which is a tensor product such that $\rho \in \mathcal{H}^{\otimes n}$. The repeated use ($n$ times) of the channel is a CP map $\mathcal{E}_n$

$$\mathcal{E}_n(\rho) \;=\; \sum_{k_1,\dots k_n} A_{k_1 \dots k_n} \, \rho \, A^\dagger_{k_1 \dots k_n} \tag{4}$$

represented by the Kraus operators acting in d^n-dimensional space. The channel is memoryless if the Kraus operators can be factorized according to

$$\mathcal{E}_n(\rho) = \sum_{k_1,\ldots,k_n} p_{k_1\ldots k_n} (A_{k_1} \otimes \ldots \otimes A_{k_n})\, \rho \left(A_{k_1}^\dagger \otimes \ldots \otimes A_{k_n}^\dagger\right), \quad (5)$$

where each operator A_k represents one single use of the channel and the normalized probability distribution $\sum_{k_1,\ldots,k_n} p_{k_1\ldots k_n} = 1$ is factorized as well into probabilities which are independent for each use of the channel. On the other hand, a memory effect is introduced when correlations are present between consecutive uses of the channel, e.g., when each use of the channel depends on the preceding one in such a way that $p_{k_1\ldots k_n}$ is given by a product of conditional probabilities $p_{k_1\ldots k_n} = p_{k_1} p_{k_2|k_1} \cdots p_{k_n|k_{n-1}}$. Indeed, the correlations between the consecutive uses of the channel act as if the channel "remembers" the first signal and acts on the second one using this "knowledge". This type of channels is called a Markov channel as the probability $p_{k_1\ldots k_n}$ corresponds to a Markov chain of order 2.

Following [18, 19] one can introduce a Markov type of memory effect by choosing probabilities p_{ij} which include a correlated noise: $p_{ij} = (1-\mu)\, p_i p_j + \mu\, p_i \delta_{ij}$. The memory parameter $\mu \in [0,1]$ characterizes the correlation "strength". Indeed, for $\mu = 0$, the probabilities of two subsequent uses of the channel are independent, whereas for $\mu = 1$, the correlations are the strongest ones. We shall consider channels given by a product of pairwise correlated channels $\mathcal{E}_{2n} = \mathcal{E}_2^{\otimes n}$ where $\mathcal{E}_2$ is determined by (5). These channels are not strictly Markov: due to the pairwise correlations, $2n$ quantum states sent are split into n consecutive pairs. The actions of the channel on the states belonging to the same pair are correlated according to p_{ij}, whereas, the actions of the channel on the states from different pairs are uncorrelated. However, even these "limited" (within each pair) correlations result in the advantages of using entangled input states as it was shown in [18, 19].

3. Model and Results

We shall study d-dimensional Heisenberg channels [21] which may be considered as a generalization Pauli qubit channels. The Kraus operators are therefore given by the "error" or "displacement" operators acting on d-dimensional states,

$$U_{m,n} = \sum_{k=0}^{d-1} e^{\frac{2\pi i}{d} kn} |k+m\rangle\langle k|, \quad (6)$$

where the index m characterizes the cyclic shift of the computational basis vectors by analogy with the bit-flip, and the index n characterizes the phase shift. The displacement operators form a Heisenberg group [22] with commutation relation

$$U_{m,n} U_{m',n'} = e^{2\pi i (m'n - mn')/d} U_{m',n'} U_{m,n}. \quad (7)$$

For two uses of a d-dimensional Heisenberg channel the CP map (5) becomes

$$\mathcal{E}_2(\rho) \;=\; \sum_{m,n,m',n'=0}^{d-1} p_{m,n,m',n'} \times (U_{m,n} \otimes U_{m',n'})\, \rho\, (U_{m,n}^{\dagger} \otimes U_{m',n'}^{\dagger}), \tag{8}$$

where ρ is a $d^2 \times d^2$ density matrix. The Markov-type joint probability reads

$$p_{m,n,m',n'} \;=\; (1-\mu) q_{m,n}\, q_{m',n'} + \mu\, q_{m,n}\, \delta_{m,m'} \left((1-\nu)\, \delta_{n,n'} + \nu\, \delta_{n,-n'}\right). \tag{9}$$

Notice the presence of a product of two Kronecker deltas representing the noise correlations separately for displacements (index m) and for phase shifts (index n). In addition, we introduce both, phase correlations ($\delta_{n,n'}$) as well as phase anticorrelations ($\delta_{n,-n'}$) with a new parameter ν characterizing the type of the phase correlations in the channel. For $d = 2$ such a distinction disappears as phase correlations $\delta_{n,n'}$ and phase anticorrelations $\delta_{n,-n'}$ coincide. Note that for infinite dimensional bosonic Gaussian channel the phase anticorrelations provided an enhancement of the channel capacity by entangled states [16]. We consider two types of d-dimensional channels, characterized by the following sets of $q_{m,n}$:

A. Quantum depolarizing (QD) channel:

$q_{m,n} = p$ if $m = n = 0$ and $q_{m,n} = q$ otherwise. As $q = (1-p)/(d^2-1)$ the channel is characterized by a single parameter $\eta = p - q \in [-1/(d^2-1), 1]$ reminiscent of the "shrinking factor" of the two-qubit QD channel.

B. Quasi-classical depolarizing (QCD) channel:

$q_{m,n} = q_m$ with $q_m = p$ if $m = 0$ and $q_m = q$ otherwise. The probabilities of the displacements of the same mode m are equal regardless of the phase shift (determined by n) and the probability of "zero" displacement ($m = 0$) differs from the others that are equal. We call this channel quasi-classical as a classical depolarizing channel changes the amplitude of the modes of a classical signal with some probability, but there is no quantum phase in classical signals. Since $q = (1-dp)/(d(d-1))$ the channel is characterized by the parameter $\eta = d(p-q) \in [-1/(d-1), 1]$.

For these d-dimensional channels, following [19] and using covariance, we have proven [23] that the bound (3) is tight, i.e., in order to determine the capacity of the channels we have to find an *optimal* state ρ_* that minimizes the output entropy $S(\mathcal{E}_2(\rho_*))$.

By analogy with the two-dimensional case [18, 19] we look for an optimal ρ_* as a pure input state $\rho_{\text{in}} = |\psi_0\rangle\langle\psi_0|$, where

$$|\psi_0\rangle \;=\; \sum_{j=0}^{d-1} \alpha_j e^{i\phi_j} |j\rangle|j\rangle\,, \quad \alpha_j \geq 0\,, \quad \sum_{j=0}^{d-1} \alpha_j^2 = 1\,. \tag{10}$$

This ansatz allows us to go from a product state to a maximally entangled state by changing the parameters α_j and ϕ_j. Indeed, the choice $\alpha_j = \delta_{j,0}$ and $\phi_j = 0$ results

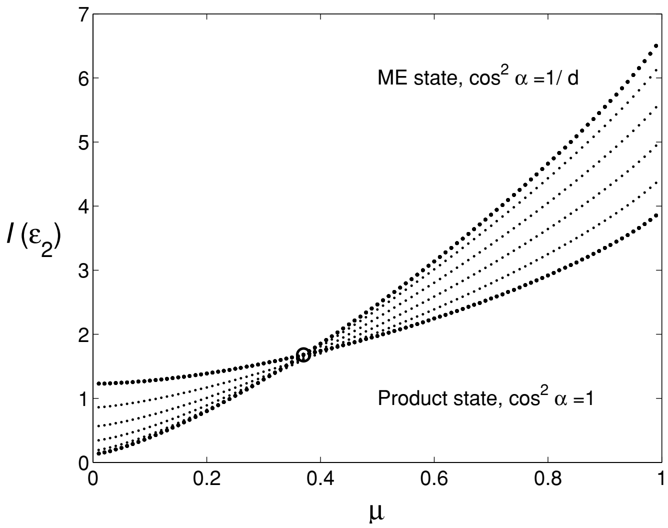

Fig. 1: Mutual information $I(\mathcal{E}_2(\rho_\alpha))$ as a function of the memory parameter μ for a QCD channel with $\eta = 0.4$ for different values of the optimization parameter α.

in a product state whereas the choice $\alpha_j = 1/\sqrt{d}$ and $\phi_j = 0$ results in a maximally entangled state. Taking into account the form (6) of the displacement operators $U_{m,n}$, the probability distribution $p_{m,n,m',n'}$ (9) and the probability parameters $q_{m,n}$ for both channels, we evaluate the action of the channel given by (8) on the initial state $|\psi_0\rangle\langle\psi_0|$ in the form (10). Then we diagonalize the output states and find their von Neumann entropy that allows us to obtain the mutual information according to (2). We evaluate the action of the QD channel and of the QCD channel given by (8) − (9) on a pure initial state given by (10). The analytic results of these evaluations are presented elsewhere [23]. Here we shall present and discuss some figures displaying these analytic results, but before we discuss whether these results provide an *optimal* ρ_*.

The task of finding an *optimal* ρ_* becomes easier for the quasi-classical depolarizing channel because we can restrict our search from the whole space to a certain subclass. In order to show this we note, following [19], that the averaging operation $\mathcal{F}$

$$\mathcal{F}(\rho) \;=\; \frac{1}{d}\sum_{n=0}^{d-1}(U_{0,n}\otimes U_{0,n})\,\rho\,(U_{0,n}^{\dagger}\otimes U_{0,n}^{\dagger}) \tag{11}$$

does not affect the QCD-channel in the sense that $\mathcal{E}_2 \circ \mathcal{F} = \mathcal{E}_2$. Then, if ρ_* is an *optimal* state then $\mathcal{F}(\rho_*)$ is also an optimal state. Therefore, we can restrict our search from the whole space $\mathcal{H}^{\otimes 2}$ to $\mathcal{F}(\mathcal{H}^{\otimes 2})$.

Finally, using (6), it is straightforward to show that any state from $\mathcal{F}(\mathcal{H}^{\otimes 2})$ is a convex combination of pure states $|\psi_m\rangle\langle\psi_m|$, where

$$|\psi_m\rangle = \sum_{j=0}^{d-1} \alpha_j e^{i\phi_j}|j\rangle|j+m\rangle\,, \quad \alpha_j \in \mathbb{R}\,, \quad \sum_{j=0}^{d-1}\alpha_j^2 = 1\,. \tag{12}$$

Restricting our search to the states of the form (12) we reduce the number of real optimization parameters from $(2d)^2$ to $2d$, which can still be a large number. In order to reduce this number to 1, we consider the following ansatz

$$|\psi(\alpha)\rangle = \cos\alpha|00\rangle + \frac{\sin\alpha}{\sqrt{d-1}}\sum_{j=1}^{d-1}|j\,j\rangle\,, \tag{13}$$

interpolating between the product state ($\cos\alpha = 0$) and the maximally entangled state ($\cos^2\alpha = 1/d$). Using the one-parameter family of input states $\rho_\alpha = |\psi(\alpha)\rangle\langle\psi(\alpha)|$, in Fig. 1 we present the mutual information $I(\mathcal{E}_2(\rho_\alpha))$ for different values of α. The mutual information is monotonously modified when α goes from a product state to a maximally entangled state, whereas the crossover point μ_c stays intact. However, we cannot guarantee that no other entangled state minimizes the entropy $S(\mathcal{E}_2(\rho))$ and provides therefore the maximum of the mutual information.

In the sequel, we present numerical graphs based on our analytic results obtained for product states and maximally entangled states as candidates for the *optimal* ρ_*. The mutual information $I(\mathcal{E}_2)$ is depicted in Fig. 2 as a function of the memory parameter μ for both these states for QD channel for various dimensions. The curves in Fig. 2 (a) correspond to the strongest phase anti-correlations expressed by $\delta_{n,-n'}$ in (9) and correlation parameter $\nu = 1$. The curves in Fig. 2 (b) correspond to the strongest phase correlations expressed by $\delta_{n,n'}$ in (9) and the correlation parameter $\nu = 0$.

In both figures for all drawn dimensions we see a crossover point, the "μ" coordinate of which we denote μ_c. For $\mu \in [0, \mu_c[$ the product states provide higher value of mutual entropy and for $\mu \in]\mu_c, 1]$ the maximally entangled states do. In addition, in Fig. 2 (a) we observe that with an increasing dimension the crossover points move toward smaller μ thus widening the interval where maximally entangled states provide higher values of the mutual information than product states do. For this reason we call the $\nu = 1$ version of the channel "entanglement-friendly". An opposite effect can be seen in Fig. 2 (b) where the crossover point moves toward higher values of μ with the increasing dimension of the space of states thus shrinking the interval where entangled states perform better. The $\nu = 0$ version of the channel is thus called "entanglement-non-friendly". We note that for $d = 2$ this difference between the two types of channels disappears and we recover the result obtained in [18].

In order to see the effect of the phase correlations for higher dimensions we draw in Fig. 3 (a) the μ coordinate of the crossover point, μ_c, as a function of d. We observe that only strongly anticorrelated phases ($\nu \approx 1$) provide "entanglement-friendly" channel so that with increasing dimension the interval of μ's that are favorable for entangled states increases. In addition, even for $\nu = 1$ this increase

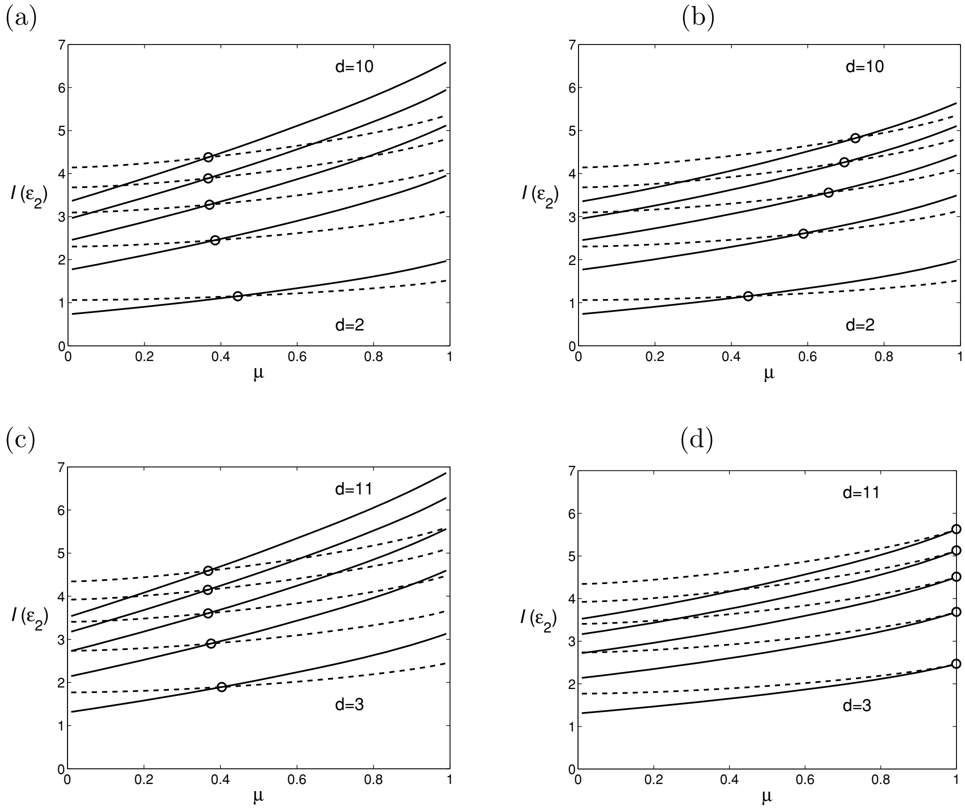

Fig. 2: Mutual information $I(\mathcal{E}_2(\rho))$ as a function of the memory parameter μ for a QD ($\eta = 0.8$) entanglement-friendly (a, c) with $\nu = 1$ and entanglement-non-friendly (b, d) with $\nu = 0$ channels for different *even* (a, b) and *odd* (c, d) dimensions d.

continues only up to certain d, after which the interval of μ begins to shrink with increasing d.

We note that μ_c depends also on the "shrinking factor" η. With increasing η the slope of the curves, which are drawn in Fig. 3 (a) for $\eta = 0.8$ would become steeper and the upper curves, corresponding to the small values of the phase correlations parameter ν, would cross the level $\mu_c = 1$ at some $d < 100$. Hence for higher dimensions there is no values of μ for which entangled input states may have any advantage at all.

For the QCD channel the result is similar, hence we present it only on Fig. 3 (c) which shows that for the "entanglement-non-friendly" version ($\nu = 0$) the advantages of entangled states completely disappear in higher dimensions.

As the result, we conclude that for even dimensions, the advantages of entangled states are more essential for low (but not always lowest) dimensions, anticor-

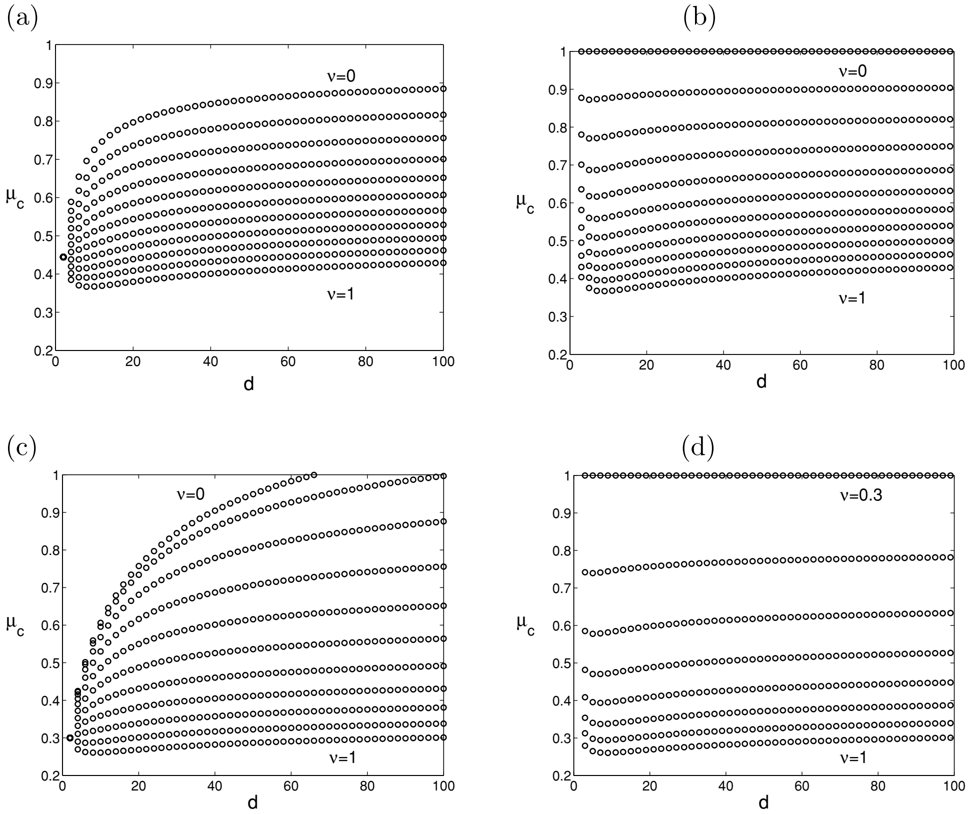

Fig. 3: Crossover point μ_c vs. *even* (a,c) and *odd* (b,d) space dimension for a QD channel (a,b) with $\eta = 0.8$ and a QCD channel (c,d) with $\eta = 0.3$ for different phase correlations ν.

related phases, smaller values of p, and a QD channel.

Our analytic formulas for *even* and *odd* dimensions are different. However, due to the factor $(1-\nu)$ the case of entanglement-friendly ($\nu = 1$) version of both the QD and QCD channels for *odd* dimensions differs from Fig. 2 (a) only in the position of the curves whereas qualitatively the pictures are the same as displayed in Fig. 2 (c). In the entanglement-non-friendly version ($\nu = 0$) of the QD channel in *odd* dimensions shown in Figs. 2 (d) and 3 (b) the crossover points lay on the line $\mu = 1$. Therefore, effectively there is no crossover as μ cannot be larger than 1. In this case for all μ the maximally entangled input states do not provide higher values of the mutual information than the product states do. For the QCD *odd*-dimensional channel the picture is similar however, the upper horizontal line $\mu = 1$ is achieved even for a non vanishing value of the "friendness" parameter, $\nu = 0.3$.

4. Conclusion

We have considered two examples of d-dimensional quantum channels with a memory effect modeled by a correlated noise. We have shown the existence of the crossover points separating the intervals of the memory parameter μ where ensembles of maximally entangled input states or product input states provide higher values of the mutual information. This result is the same as in the 2-dimensional case. However, it always holds only for channels (which we call "entanglement-friendly") with a particular kind of phase correlations, namely anticorrelations. For these channels the crossover point moves with increasing d towards lower values of the memory parameter thus widening the range of correlations where maximally entangled input states enhance the mutual information. For usual phase correlations the situation is opposite, namely, for higher dimensions of the space the crossover point is shifted towards $\mu_c = 1$ so that only for higher degrees of correlations maximally entangled input states have advantages. In addition, for these "entanglement-non-friendly" channels the crossover point completely disappears for higher dimensions so that product input states always provide higher values of the mutual information than maximally entangled input states. Therefore, we conclude that the type of phase correlations strongly affects this entanglement-assisted enhancement of the channel capacity.

We have observed that the parity of the dimension of the space of initial states makes an important difference in the "entanglement-non-friendly" channels. Not only the curves of the mutual information vs. the memory parameter for odd dimensions are shifted with respect to the curves for even dimensions, but also for $\nu = 0$ in all odd dimensions maximally entangled input states are always worse than product states. Strikingly, the channels with anticorrelated noise do not feel the parity of the space at all (in Fig. 3, compare the curves $\nu = 1$ from (a) with (b) and from (c) with (d)). However, any non vanishing degree of the "entanglement-non-friendly" correlations reveals the parity effect.

The anticorrelated phases remind us the bosonic Gaussian channels considered in [16] where the p quadratures are correlated while the q quadratures are anticorrelated. However, the existence of the crossover point is a significant difference with the case of the Gaussian channels for which each value of the noise correlation parameter determines an optimal degree of entanglement (different from maximal entanglement) maximizing the mutual information. A challenging open problem is to find a link between these results for d-dimensional channels and the results obtained in [15, 16, 17] for Gaussian channels with finite energy input signals.

Although we have shown that for certain cases of d-dimensional channels maximally entangled states provide higher values of mutual information than product states, a full proof of the optimality of maximally entangled input states is still missing. However if it is true, the presented parametrization illustrating a "monotonous" deformation of the curves of mutual information vs. the memory parameter during the transition from product states to maximally entangled states shows that at μ_c the optimal state "jumps" from the product to the maximally entangled state. The "sharp" character of this transition is due to the fact that the crossover points stay intact during the deformation of the curves.

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