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Chapter 6

Optical quantum cloning

by

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§ 1. Introduction and history

1.1. The no-cloning theorem

The history of quantum cloning can be traced back to the controversial story of a paper by Herbert [1982] entitled "FLASH – A superluminal communicator based upon a new kind of measurement". In this paper, submitted in early 1981 to *Foundations of Physics*, Herbert was discussing an idealized laser gain tube which would produce, via *stimulated emission*, macroscopically distinguishable states of light from an incoming single photon in any polarization state. The claim was that the noise in this process would, in principle, not prevent *perfectly* identifying the polarization state of the photon. This process would supposedly open the way to faster-than-light communication, a possibility with which any physicist feels uncomfortable since it violates causality.

Today, more than twenty years later, it is publicly known that GianCarlo Ghirardi and Asher Peres were requested to review this paper. The first of them recommended its rejection, based on the argument that the linear nature of quantum mechanics must prevent such a process to exist, see van der Merwe [2002]. The second referee wrote, see Peres [2002], that he had realized the paper was wrong, but nevertheless recommended its publication because he expected that finding the error would raise a considerable interest! Herbert's paper was then published, and, funnily enough, the prediction of Peres happened to be true.

Soon afterwards, Wootters and Zurek [1982] published a paper in *Nature*, entitled "A single quantum cannot be cloned", which arrived at essentially the same conclusions as those drawn by Ghirardi in his anonymous referee report dated April 1981, which itself was turned into a paper two years later, see Ghirardi and Weber [1983]. Wootters and Zurek realized that, if one can build a "cloning machine" that produces several clones of the horizontal- and vertical-polarization states of an incoming photon, then circularly polarized states cannot yield circularly polarized clones. Instead, due to the linearity of quantum mechanics, one gets a linear superposition of vertically polarized clones and horizontally polarized clones. Indeed, if the cloning machine is such that

$$|H\rangle|C\rangle \to |H, H\rangle|C_H\rangle, \qquad |V\rangle|C\rangle \to |V, V\rangle|C_V\rangle, \tag{1.1}$$

where $|H\rangle$ and $|V\rangle$ are horizontal- and vertical-polarization states of the original photon, $|C\rangle$ is the initial state of the cloning machine, and $|C_H\rangle$ and $|C_V\rangle$ are the (arbitrary) final states of the cloning machine, then the left- and right-circularly polarized states, $|L\rangle = 2^{-1/2}(|H\rangle + \mathrm{i}|V\rangle)$ and $|R\rangle = 2^{-1/2}(|H\rangle - \mathrm{i}|V\rangle)$, are transformed as

$$
\begin{aligned}
&|L\rangle|C\rangle \to 2^{-1/2}\big(|H,H\rangle|C_H\rangle + \mathrm{i}|V,V\rangle|C_V\rangle\big) \neq |L,L\rangle|C_L\rangle,\\
&|R\rangle|C\rangle \to 2^{-1/2}\big(|H,H\rangle|C_H\rangle - \mathrm{i}|V,V\rangle|C_V\rangle\big) \neq |R,R\rangle|C_R\rangle,
\end{aligned}
\tag{1.2}
$$

$|C_L\rangle$ and $|C_R\rangle$ being (arbitrary) final states of the cloning machine. As a consequence, the cloning of circularly polarized states fails, even in the special case where $|C_H\rangle = |C_V\rangle$. This is the simplest explanation of what is known today as the *quantum no-cloning theorem*.

Independently of this story, a related paper by Dieks [1982] was published almost simultaneously in *Physics Letters*, also showing that the "FLASH" proposal by Herbert was flawed. Here, the proof relies on the existence of EPR states, see Einstein, Podolsky and Rosen [1935], which give rise to quantum correlations between spatially separated systems. If two photons are prepared in the EPR state

$$
|\mathrm{EPR}\rangle = 2^{-1/2}\big(|H,V\rangle - |V,H\rangle\big), \tag{1.3}
$$

it is well known that measuring the linear polarization of one of them in the horizontal–vertical basis allows one to immediately predict the outcome of a measurement of the linear polarization of the second one in the same basis, even if the measurement events are separated by a space-like interval. For example, if the first photon is found to be in the $|H\rangle$ state, then the second photon will necessarily be observed in the $|V\rangle$ state. This property holds for any measurement basis. It had been realized since the early times of quantum mechanics that this property, called *quantum entanglement*, does not permit superluminal communication. Indeed, the statistics of any measurement performed on one of the twin photons remains unchanged irrespectively of the measurement (or, more generally, the operation) applied on the second one. Dieks noticed, however, that if it was possible to perfectly clone one of the twin photons when the other had been measured, then superluminal communication would become possible; hence, cloning must be impossible.

Assume that Alice measures the first photon either in the horizontal–vertical linear polarization basis or in the left–right circular polarization basis depending on whether she wants to transmit a 0 or a 1 to Bob. In the former case, the second photon will be found by Bob to be in a balanced mixture of the $|H\rangle$ and $|V\rangle$ states, while in the second case it will be in a balanced mixture of the $|L\rangle$ and $|R\rangle$ states.

These two mixtures are indistinguishable (they are characterized by the same density operator, proportional to the identity I), which is why quantum mechanics is said to "coexist peacefully" with special relativity. However, if the second photon could be cloned perfectly, Bob would then get either a balanced mixture of $|H, H\rangle$ and $|V, V\rangle$, or a balanced mixture of $|L, L\rangle$ and $|R, R\rangle$. These mixtures being distinguishable, Bob would have a way to infer Alice's bit instantaneously (with some error, which however can be made arbitrarily small as the number of clones increases). Dieks concluded from this paradox that such a cloning transformation cannot be consistent with quantum mechanics.

It appears that the quantum no-cloning theorem is thus one of those scientific results that have been rediscovered several times, at least by Dieks, Ghirardi, Wootters and Zurek. Actually, it can be argued that it was already implicitly used by Stephen Wiesner in his famous paper entitled "Conjugate coding" written in the 1970s but published only in the 1980s (Wiesner [1983]) which is sometimes considered to be the founding paper of quantum information theory. In some sense, the no-cloning theorem was already intrinsically contained in the roots of quantum mechanics and is thus trivial; on the other hand, its discovery has contributed to revisiting quantum mechanics in an information-theoretic language, which has had a decisive influence on the dramatic development of quantum information science over the past decade.

1.2. Beyond the no-cloning theorem

Soon after the publication of the quantum no-cloning theorem, another paper appeared in *Nature*, written by Mandel [1983]. In this paper, entitled "Is a photon amplifier always polarization dependent?", Mandel drew attention to the physical origin of the impossibility of making a perfect amplifying apparatus for light, namely *spontaneous emission*. He showed that, if the amplifier is a single two-level atom with a dipole moment μ, then the amplification of an incoming photon with polarization vector ε depends on the scalar product between μ and ε. If the polarization vector ε of the incoming photon is parallel to the dipole moment μ, then the state $|1\rangle_\varepsilon$ will, after some interaction time, evolve into a state containing the desired two-photon state $|2\rangle_\varepsilon$ due to stimulated emission. On the contrary, if ε is orthogonal to μ, then the two-photon component of the resulting state corresponds to $|1\rangle_\varepsilon|1\rangle_{\bar{\varepsilon}}$, where $\bar{\varepsilon}$ is a polarization vector orthogonal to ε. This is due to spontaneous emission, which spoils the amplification since one of the two photons has the wrong polarization $\bar{\varepsilon}$. In other words, with such a simple one-atom amplifier, the final state depends on the polarization of the incoming photon.

Interestingly, Mandel noticed that if we consider a more elaborate amplifier made of two such atoms with orthogonal dipole moments (μ_1 and μ_2), it does become possible to amplify the photon independently of its polarization, although this process suffers from the unavoidable noise originating from spontaneous emission. Assuming that the two atoms interact similarly with the incoming photon, one understands intuitively that if one atom amplifies the photon "well" (when ε is close to μ_1), then the second atom amplifies it "poorly" (because ε is then approximately orthogonal to μ_2). The balance between these two effects results in an amplification that does not depend on ε. By filtering out the resulting two-photon component, one gets

$$|1\rangle_\varepsilon|0\rangle_{\bar{\varepsilon}} \to \frac{2}{3}|2\rangle_\varepsilon\langle 2| \otimes |0\rangle_{\bar{\varepsilon}}\langle 0| + \frac{1}{3}|1\rangle_\varepsilon\langle 1| \otimes |1\rangle_{\bar{\varepsilon}}\langle 1| \tag{1.4}$$

irrespective of ε. In some sense, the perfect cloning of polarization via stimulated emission works with probability $2/3$, while spontaneous emission blurs the polarization with probability $1/3$.

Mandel's paper remained mostly unnoticed and, remarkably, one had to wait more than ten years before the notion of *quantum cloning machine*, which was implicitly contained in this paper, became popular. In a seminal paper, Bužek and Hillery [1996] realized that, although *perfect* quantum cloning is ruled out by the no-cloning principle, some *imperfect* cloning may be possible. They found out that a qubit (two-level quantum system) that is in an unknown state can be approximately duplicated, resulting in two pretty good clones of the original state. This result holds in full generality, regardless of the physical variable carrying the qubit, so it goes much beyond the polarization-independent amplification of a single photon considered before. This paper had a considerable impact at the time because quantum information was born, and it had been realized how fruitful it is to investigate quantum mechanics using an information language.

Consider a qubit in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|0\rangle$ and $|1\rangle$ form an orthonormal basis of the Hilbert space, while α and β are arbitrary complex numbers satisfying $|\alpha|^2+|\beta|^2 = 1$. Bužek and Hillery [1996] addressed the following formal problem: find a transformation acting on an original qubit in state $|\psi\rangle$ together with an auxiliary system (commonly viewed as the cloning machine itself) that produces two clones with the same fidelity and is state-independent, or *universal*. If the cloning machine is initially put in state $|C\rangle$, then

$$|0\rangle|C\rangle \to |\Sigma_0\rangle, \qquad |1\rangle|C\rangle \to |\Sigma_1\rangle, \tag{1.5}$$

with the final states $|\Sigma_0\rangle$ and $|\Sigma_1\rangle$ belonging to the product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ denote the spaces of the two clones (called A and B)

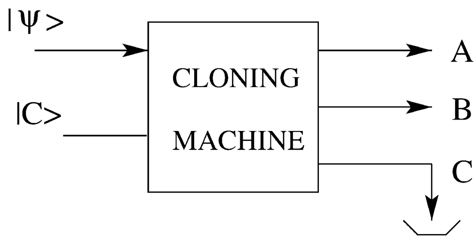


Fig. 1. Quantum cloning machine. The input of the transformation consists of the state $|\psi\rangle$ to be cloned and the initial state of the cloning machine $|C\rangle$. The two clones are contained in outputs A and B, while C refers to an ancilla, commonly viewed as the cloning machine itself.

and $\mathcal{H}_C$ denotes the space of the cloning machine C, see fig. 1. By linearity, an arbitrary qubit state $|\psi\rangle$ is cloned as

$$|\psi\rangle|C\rangle \to \alpha|\Sigma_0\rangle + \beta|\Sigma_1\rangle \equiv |\Sigma\rangle. \tag{1.6}$$

The fidelity of the clones, which measures the overlap between the input state and each clone, is given by

$$f_A(\psi) = \langle\psi|\,\mathrm{Tr}_{BC}(\Sigma)|\psi\rangle, \qquad f_B(\psi) = \langle\psi|\,\mathrm{Tr}_{AC}(\Sigma)|\psi\rangle, \tag{1.7}$$

where Tr denotes the trace and $\Sigma \equiv |\Sigma\rangle\langle\Sigma|$ is a short-hand notation for the density operator of a pure state. Bužek and Hillery [1996] showed that, under the constraint that $f_A(\psi) = f_B(\psi)$ is independent of ψ, quantum mechanics permits the existence of a cloning transformation which achieves a fidelity as high as

$$f^{\mathrm{univ}} = \frac{5}{6} \simeq 0.833. \tag{1.8}$$

This transformation, which is called a *quantum cloning machine*, is given by

$$\begin{aligned}
|0\rangle|C\rangle &\to |\Sigma_0\rangle \equiv \sqrt{\frac{2}{3}}|00\rangle_{AB}|0\rangle_C + \sqrt{\frac{1}{3}}\big|\Psi^+\big\rangle_{AB}|1\rangle_C,\\
|1\rangle|C\rangle &\to |\Sigma_1\rangle \equiv \sqrt{\frac{2}{3}}|11\rangle_{AB}|1\rangle_C + \sqrt{\frac{1}{3}}\big|\Psi^+\big\rangle_{AB}|0\rangle_C,
\end{aligned} \tag{1.9}$$

where $|\Psi^+\rangle = 2^{-1/2}(|01\rangle + |10\rangle)$ is one of the Bell states, while $|0\rangle_C$ and $|1\rangle_C$ denote two orthogonal states of the cloning machine. It is easy to check, by tracing over the cloning machine, that the two clones of an input state $|0\rangle$ are left in the joint state

$$\rho_{AB} = \mathrm{Tr}_C(\Sigma) = \frac{2}{3}|00\rangle\langle 00| + \frac{1}{3}\big|\Psi^+\big\rangle\big\langle\Psi^+\big| \tag{1.10}$$

which is equivalent to eq. (1.4) given the bosonic statistics of photons. More generally, if the input state is $|\psi\rangle$, the first term on the right-hand side of eq. (1.10) becomes a projector onto $|\psi\rangle^{\otimes 2}$, while the second term is some (ψ-depending) maximally-entangled state. Therefore, by tracing over one of the clones, the resulting state of the other clone is

$$\begin{aligned} \rho_A &= \mathrm{Tr}_{BC}(\Sigma) = \frac{2}{3}|\psi\rangle\langle\psi| + \frac{1}{6}I, \\ \rho_B &= \mathrm{Tr}_{AC}(\Sigma) = \frac{2}{3}|\psi\rangle\langle\psi| + \frac{1}{6}I, \end{aligned} \tag{1.11}$$

where I denotes the identity operator, confirming that the two clones are left in the same state. They can be viewed each as emerging from a quantum *depolarizing* channel: they are found in the right state $|\psi\rangle$ with probability $2/3$, while they are replaced by a random qubit $I/2$ with probability $1/3$.

Soon after the publication of this paper, it was proved that this machine is actually the *optimal* universal cloning machine, that is, the highest fidelity of cloning permitted by quantum mechanics is indeed $5/6$, see Bruss, DiVincenzo, Ekert, Fuchs, Macchiavello and Smolin [1998].

This discovery by Bužek and Hillery [1996] triggered an immense interest and initiated an entire subfield of quantum information science devoted to quantum cloning. In particular, further studies addressed cloning in dimensions larger than 2, state-dependent cloning (considering a restricted set of input states), the so-called N-to-M cloning (where one produces M identical clones out of N identical replicas of the original), asymmetric cloning (where the clones have unequal fidelities), the cloning of orthogonal qubit states, the cloning of continuous-variable states (such as coherent states), economical cloning (where no ancillary space is necessary), probabilistic cloning (which is not deterministic, i.e., it does not succeed with probability 100%), and even the cloning of quantum entanglement (instead of quantum states). These numerous results will be reviewed in Section 2.

Aside from its utmost importance for the foundations of quantum mechanics, the study of quantum cloning has drawn a lot of interest probably also because it is closely connected to quantum key distribution (QKD), see, e.g. the review by Dusek, Lutkenhaus and Hendrych [2006]. Indeed, in many cases, the cloning machine is known to be the most powerful eavesdropping strategy against QKD protocols: the eavesdropper duplicates the quantum state and sends one clone to the authorized party, while keeping the second clone for later measurement. The characterization of cloning machines is therefore crucial for assessing the security of these QKD protocols (this particular connection is outside the scope of the present review, and will not be discussed any further).

1.3. Quantum cloning without signaling

Before entering the detailed study of quantum cloning, it is interesting to backtrack for a moment and further discuss the proof of the quantum no-cloning theorem based on a pair of entangled photons due to Dieks [1982]. As explained earlier, if Alice measures her photon either in the horizontal–vertical linear polarization basis or in the left–right circular polarization basis, and if Bob is able to clone his photon perfectly, then he obtains two distinguishable two-photon mixtures, which apparently makes superluminal signaling possible.

A natural idea, due to Gisin [1998], is to assume that Bob's cloning machine must necessarily introduce some intrinsic noise, and determine the minimum amount of noise that must be added so that causality ceases to be violated. Remarkably, it so happens that the minimum noise needed to comply with causality exactly coincides with that of the optimal universal cloning machine. In other words, the upper bound on quantum cloning can be derived from simple principles.

As shown earlier, the two clones of the universal machine emerge each from a quantum depolarizing channel, see Bužek and Hillery [1996]. This channel can be interpreted as giving rise to a shrinking of the vector representing the qubit state in the Bloch sphere. Using the Bloch representation

$$\rho = \frac{I + \mathbf{m} \cdot \boldsymbol{\sigma}}{2}, \tag{1.12}$$

where $\mathbf{m}$ is a vector isomorphic to state ρ and $\boldsymbol{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices, we see from eq. (1.11) that a state associated with $\mathbf{m}$ yields two clones which are in a state associated to $2\mathbf{m}/3$, independently of the orientation of $\mathbf{m}$. Therefore, this universal cloning machine is sometimes also said to be *isotropic*.

Following Gisin [1998], consider that the (pure) state of the original qubit is associated with the (unit-norm) vector $\mathbf{m}$, and let us restrict our search to cloning machines that are symmetric and isotropic, that is, the clones are in the states

$$\rho_A(\mathbf{m}) = \rho_B(\mathbf{m}) = \frac{1 + \eta \mathbf{m} \cdot \boldsymbol{\sigma}}{2}, \tag{1.13}$$

where η is an unknown "shrinking factor" ($0 \leqslant \eta \leqslant 1$). It is easy to check that η is related to the fidelity by

$$f_A(\mathbf{m}) = f_B(\mathbf{m}) = \frac{1 + \eta}{2}. \tag{1.14}$$

Using eq. (1.13), the Hilbert–Schmidt decomposition of the joint state of the two clones can be written as

$$\rho_{AB}(\mathbf{m}) = \frac{I + \eta\mathbf{m}\cdot\boldsymbol{\sigma}\otimes I + I\otimes\eta\mathbf{m}\cdot\boldsymbol{\sigma} + \sum_{j,k} t_{j,k}\sigma_j\otimes\sigma_k}{4}, \tag{1.15}$$

where the matrix $t_{j,k}$ measures the quantum correlations between the clones. Gisin [1998] went on to derive constraints on $t_{j,k}$ that result from covariance and causality. The covariance property (which will be explained in detail later on) means, physically, that rotating the original qubit around, say, the z-axis before cloning must be equivalent to cloning the original qubit and then rotating each of the two clones by the same amount around the z-axis. Following Dieks' argument, the causality condition is taken into account by imposing that

$$\rho_{AB}(\mathbf{m}_1) + \rho_{AB}(-\mathbf{m}_1) = \rho_{AB}(\mathbf{m}_2) + \rho_{AB}(-\mathbf{m}_2) \tag{1.16}$$

which expresses the fact that the two-clone states corresponding to two indistinguishable mixtures of input states, $\{\mathbf{m}_1, -\mathbf{m}_1\}$ and $\{\mathbf{m}_2, -\mathbf{m}_2\}$, are themselves indistinguishable. Putting all these conditions on $t_{j,k}$ together, one can show that the maximum value of η that preserves the positivity of the two-clone state, $\rho_{AB}(\mathbf{m}) \geqslant 0$, is $\eta = 2/3$; hence $f_A = f_B = 5/6$. This provides an alternate proof of the optimality of the qubit universal cloner of Bužek and Hillery [1996].

For completeness, let us mention that such a use of the no-signaling condition has been criticized in Bruss, D'Ariano, Macchiavello and Sacchi [2000], the argument being that the linearity and trace-preservation properties of the cloning map (which, combined, imply the no-signaling condition) are not sufficient, strictly speaking, and need to be supplemented with the complete positivity condition in order to bound the cloning fidelity. This simple technique, however, has proved to be successful to recover conditions on probabilistic cloning, see Hardy and Song [1999], on asymmetric universal cloning, see Ghosh, Kar and Roy [1999], or even to find a new class of real cloning machines, see Navez and Cerf [2003].

1.4. Content of this review

The rest of this review will be devoted to the study of quantum cloning machines, as well as their optical realization. Let us sketch the content of the following sections. Section 2 provides an overview of the main papers that have been written in this context, focusing on the results but skipping the derivations. The numerous classes of quantum cloning machines will be presented (universal cloners, Pauli or Heisenberg cloners, phase-covariant cloners, Fourier-covariant cloners, group-covariant cloners, real cloners, entanglement cloners, continuous-variable cloners, probabilistic cloners, or economical cloners).

In Section 3 we will consider the issue of quantum cloning from a formal point of view, based on the description of the associated completely positive (CP) map and the notion of covariance. This study will be restricted to 1-to-2 cloning, and will focus on the isomorphism between CP maps and operators. It will be shown that finding the optimal cloning map reduces to a semidefinite programming problem, which can be solved efficiently by numerical methods. It will also be shown that the unitary realization of a cloning map based on the "double-Bell" ansatz provides a simple and efficient tool to investigate cloning analytically. Some examples of d-dimensional 1-to-2 cloners will be provided.

This formal study will be extended in Section 4 to N-to-M cloning machines in d dimensions, but will be restricted to the case of universal cloning. The derivation of the optimal cloning transformation as well as the optimality proof will be detailed. In addition, the extension to asymmetric cloning machines and the notion of universal-NOT gate will be discussed. The reader who is mainly interested in the optical realization of cloning machines and not so much in their theoretical derivation may skip Sections 3 and 4, and proceed immediately to the following sections.

In Section 5 the optical implementation of the universal quantum cloning machines will be analyzed in details. Cloning experiments relying on stimulated parametric down-conversion will be described first, followed by those relying on the symmetrization that can be obtained with a Hong–Ou–Mandel interferometer. Next, the optical realization of (universal) asymmetric cloning machines will be discussed, as well as the (universal) cloning of a pair of orthogonal qubits.

In Section 6 the phase-covariant cloning machines will be developed for qubits as well as d-dimensional systems, in a 1-to-2 or N-to-M configuration. The experimental realization of phase-covariant cloning for photonic qubits will be described.

In Section 7 the generalization of quantum cloning to states belonging to an infinite-dimensional Hilbert space will be considered. In particular, the cloning of coherent states of light by phase-insensitive amplification will be explained, as well as the experimental realization of continuous-variable cloning using linear optics, measurement, and feed-forward. The cloning of a finite-width distribution of coherent states will be analyzed, as well as the cloning of a pair of conjugate coherent states. Finally, Section 8 concludes.

§ 2. Overview of quantum cloning machines

2.1. Universal cloning machines

This section will be devoted to a summary of the various cloning machines that have been introduced in the literature, following the chronology as well as possible. Soon after the universal quantum cloning machine was discovered by Bužek and Hillery [1996], the question arose whether this machine was optimal. As already mentioned, this cloning machine is required to be *symmetric*, that is, the two clones must have equal fidelities $f_A(\psi) = f_B(\psi)\ \forall\psi$. In addition, it must be *universal* (or state-independent), which means that all states are cloned with the same fidelity, independent of ψ. It was proven by Bruss, DiVincenzo, Ekert, Fuchs, Macchiavello and Smolin [1998] that it is indeed the *optimal* symmetric universal duplicator for qubits, so that $f = \frac{5}{6}$ is indeed the highest fidelity allowed by quantum mechanics in this case. In the same paper, the concept of optimal *state-dependent* cloning machines was also introduced, that is, transformations that optimally duplicate only a particular subset of the input states. Almost simultaneously, Gisin and Massar [1997] introduced the concept of N-to-M quantum cloning machines, which transform N identical replicas of an arbitrary state, $|\psi\rangle^{\otimes N}$, into $M > N$ identical clones. They were able to prove for low N that the optimal universal N-to-M cloning of qubits is characterized by the fidelity

$$f^{\text{univ}}_{N\to M} = \frac{M(N+1)+N}{M(N+2)}. \tag{2.1}$$

Incidentally, this confirms the optimality of the 1-to-2 universal cloning machine with fidelity $f^{\text{univ}}_{1\to 2} = \frac{5}{6}$. The quantum network that realizes this 1-to-2 universal cloning of qubits was described by Bužek, Braunstein, Hillery and Bruss [1997], and was extended to 1-to-M universal cloning in Bužek and Hillery [1998b]. Note also that when the number of clones M increases for fixed N, the cloning fidelity decreases. This can simply be interpreted as a spreading of quantum information over more clones. In the limit $M \to \infty$, the cloning transformation tends to a measurement, which confirms that the optimal (state-independent) estimation of the state $|\psi\rangle^{\otimes N}$ of N identical qubits has a fidelity

$$f^{\text{univ}}_{N\to\infty} = \frac{N+1}{N+2} \tag{2.2}$$

as originally derived in Massar and Popescu [1995].

Then, in early 1998, the extension of quantum cloning machines to higher-dimensional spaces was considered independently by Bužek and Hillery [1998a], Cerf [1998] and Werner [1998]. The form of the optimal universal 1-to-2 cloner in dimension d was conjectured by Bužek and Hillery [1998a], Cerf [1998], while the derivation and full optimality proof of the universal d-dimensional N-to-M cloner was given by Werner [1998], Keyl and Werner [1999]. The optimal fidelity of the universal 1-to-2 cloner of d-dimensional states (or qudits) was shown to be

$$f^{\text{univ}}_{1\to 2}(d) = \frac{d+3}{2(d+1)} \tag{2.3}$$

while that for arbitrary N and $M > N$ is

$$f^{\text{univ}}_{N\to M}(d) = \frac{M(N+1)+(d-1)N}{M(N+d)}. \tag{2.4}$$

In Cerf [1998], the cloning of d-dimensional systems was actually investigated in a more general setting: a large class of symmetric or asymmetric, universal or state-dependent, 1-to-2 cloning machines was introduced in arbitrary dimension d. The optimality of this class of cloners was only conjectured, but, in the special case of a symmetric and universal cloner, eq. (2.3) was also derived. For the set of asymmetric universal 1-to-2 cloning machines, the balance between the fidelity of the two clones,

$$f^{\text{univ}}_{A}(d) = \eta_A + \frac{1-\eta_A}{d}, \qquad f^{\text{univ}}_{B}(d) = \eta_B + \frac{1-\eta_B}{d}, \tag{2.5}$$

was characterized by the simple relations

$$\eta_A = 1-\alpha^2, \qquad \eta_B = 1-\beta^2, \qquad \alpha^2 + \frac{2\alpha\beta}{d} + \beta^2 = 1, \tag{2.6}$$

where η_A and η_B are the "shrinking" factors associated with the clones (η is the probability that the input state emerges unchanged at the output of the quantum depolarizing channel). Here, α and β are positive real variables. It is instructive to notice that in the limit $d \to \infty$, the cloning of quantum information resembles the distribution of a resource that can strictly not be shared: the probability that $|\psi\rangle$ is found in one clone is complementary to the probability that it is found in the second clone, that is, $\eta_A + \eta_B = 1$.

Finally, even more general quantum cloning machines were obtained in the special case of qubits ($d = 2$) in an independent work by Niu and Griffiths [1998]. There, the 1-to-2 asymmetric and state-dependent cloning of a qubit was investigated in full generality, and, in particular, formulas (2.6) were recovered for $d = 2$ without any assumption. Note also that the universal cloning of mixed states in a symmetric subspace was studied by Fan [2003], while entanglement properties of cloning transformations were investigated by Bruss and Macchiavello [2003].

2.2. *Pauli and Heisenberg cloning machines*

The results of Cerf [1998] were later expanded for the case of qubits (Cerf [2000a]), and for the case of d-dimensional systems (Cerf [2000b]). The specificity of the approach to quantum cloning underlying these papers is that one considers the cloning of a system that is initially maximally entangled with another system instead of the cloning of a pure state. This second system acts as a "reference" by keeping a memory of the original state after the cloning has been achieved. The final state of the "reference", the two clones, and the cloning machine then fully characterizes the cloning transformation, as a consequence of the isomorphism between completely positive (CP) maps and operators (this will be explained in detail in Section 3.1). By choosing an appropriate form for this final state, one generates a large class of quantum cloning machines.

For qubits ($d = 2$), this class corresponds to the so-called *Pauli cloning machines*, whose clones emerge from two – possibly distinct – Pauli channels. In a Pauli channel, the input qubit undergoes one of the three Pauli rotations $\{\sigma_x, \sigma_y, \sigma_z\}$ or the identity I with respective probabilities $\{p_x, p_y, p_z, 1 - p_x - p_y - p_z\}$. For example, it was shown that the whole class of *symmetric* Pauli cloning machines corresponds to Pauli channels with probabilities $p_x = x^2$, $p_y = y^2$ and $p_z = z^2$, with x, y, z satisfying the condition

$$x^2 + y^2 + z^2 + xy + xz + yz = \frac{1}{2}. \tag{2.7}$$

The action of these Pauli cloners is easy to understand knowing that, if the original qubit is in an eigenstate of σ_x, namely $(|0\rangle \pm |1\rangle)/\sqrt{2}$, then it is rotated by an angle π around the y-axis (z-axis) under σ_y (σ_z) while it is left unchanged (up to a sign) by σ_x. Therefore, the cloning fidelity of the eigenstates of σ_x is $1 - p_y - p_z$. Similarly, the eigenstates of σ_y, namely $(|0\rangle \pm \mathrm{i}|1\rangle)/\sqrt{2}$, are cloned with fidelity $1 - p_x - p_z$, while the eigenstates of σ_z, namely $|0\rangle$ and $|1\rangle$, are cloned with fidelity $1 - p_x - p_y$. The universal 1-to-2 symmetric cloning machine simply corresponds to $p_x = p_y = p_z = \frac{1}{12}$. Note that these Pauli cloning machines appear to be a special case of the state-dependent cloning transformations considered in Niu and Griffiths [1998]. The quantum circuit for the asymmetric universal cloning of qubits was described in Bužek, Hillery and Knight [1998].

These considerations can be extended to d dimensions in order to obtain the set of so-called *Heisenberg cloning machines*, whose clones emerge from two – possibly distinct – Heisenberg channels. In a Heisenberg channel, the d-dimensional input state undergoes, according to some probability distribution, one of the d^2 error operators $E_{m,n}$ (with $0 \leqslant m, n \leqslant d - 1$) that form the discrete Weyl–Heisenberg group. It can be shown that the probability distribution of the $E_{m,n}$

errors for the first clone is dual, under a Fourier transform, to that of the second clone. This corroborates the fact that if one clone is close-to-perfect (its associated error distribution is peaked), then the second clone is very noisy (its associated error distribution is flat). More precisely, this fidelity balance between the two clones can be shown to result from a *no-cloning uncertainty principle*, akin to the Heisenberg principle, see also Cerf [1999]. The quantum circuit realizing these Heisenberg cloning machines was described by Braunstein, Bužek and Hillery [2001].

Recently, the optimality of this entire class of (Pauli or Heisenberg) quantum cloning machines has been proven rigorously by Chiribella, D'Ariano, Perinotti and Cerf [2005] in the following sense: under some general invariance conditions, the cloners of this class coincide with all the *extremal* cloners. Therefore, for a given (invariant) figure of merit, it is sufficient to search the optimal cloner within this class to be guaranteed that the solution thus found is the global optimal cloner.

2.3. Phase- and Fourier-covariant cloning machines

In 2000 an important class of state-dependent qubit cloning machines, named *phase-covariant cloning machines*, was introduced by Bruss, Cinchetti, D'Ariano and Macchiavello [2000]. It is defined as a transformation that clones all the *balanced* superpositions of basis states with the same (and highest) fidelity. These states

$$|\psi\rangle = \frac{|0\rangle + \mathrm{e}^{\mathrm{i}\phi}|1\rangle}{\sqrt{2}}, \tag{2.8}$$

with ϕ being an arbitrary phase, are located on the equator of the Bloch sphere. The optimal cloner also fulfills the covariance condition with respect to the rotation of ϕ, that is, cloning the rotated original qubit is equivalent to cloning the original qubit followed by a rotation of each of the clones. The optimal phase-covariant symmetric 1-to-2 cloner was found to have a fidelity

$$f^{\mathrm{pc}}_{1\to 2}(2) = \frac{1}{2} + \frac{1}{\sqrt{8}} \simeq 0.854, \tag{2.9}$$

which is higher than that of the corresponding universal cloner, $f^{\mathrm{univ}}_{1\to 2}(2) = 5/6$. In contrast, the resulting fidelity for the states $|0\rangle$ and $|1\rangle$, corresponding to the poles of the Bloch sphere, is equal to $3/4$, which is lower than $f^{\mathrm{univ}}_{1\to 2}(2)$. In some sense, it is possible to clone some restricted set of states (the equator) better at the expense of cloning some other states (near the poles) worse.

Interestingly, this phase-covariant cloner can be viewed simply as a special case of the Pauli cloners, see Cerf, Durt and Gisin [2002]. If we take $x = y = 1/\sqrt{8}$ and $z = \frac{1}{2} - 1/\sqrt{8}$, which satisfies eq. (2.7), we indeed recover the same cloner: the eigenstates of σ_x are cloned with fidelity $1 - p_y - p_z = \frac{1}{2} + 1/\sqrt{8}$, while the eigenstates of σ_y are cloned with fidelity $1 - p_x - p_z = \frac{1}{2} + 1/\sqrt{8}$. It was observed by Cerf, Durt and Gisin [2002] that imposing these 4 states lying symmetrically on the equator to be cloned with the same fidelity results in the phase-covariant cloner, which actually gives the same fidelity for all states on the equator (the deep reason for this equivalence was found by Chiribella, D'Ariano, Perinotti and Cerf [2005]). Finally, we verify that this Pauli cloner clones the eigenstates of σ_z with a lower fidelity $1 - p_x - p_y = \frac{3}{4}$.

One can summarize the results on qubit cloning machines by noting that the eigenstates of the three Pauli matrices play the role of three mutually unbiased (MU) bases for qubits (MU bases are such that the modulus of the scalar product of any two states taken from distinct bases is $1/\sqrt{d}$, with d the dimension). One can thus define three generic classes of qubit cloning machines, namely, the universal cloner (which can be obtained by imposing the states of 3 MU bases to be cloned with the same and highest fidelity), the phase-covariant cloner (if the states of only 2 MU bases are cloned equally), and some particular Pauli cloner (if the states of all 3 MU bases are cloned with unequal fidelities). The cloning of qubits having been essentially covered, it became natural to turn to the state-dependent cloning of qutrits ($d = 3$).

Cerf, Durt and Gisin [2002] defined four kinds of Heisenberg cloning machines for qutrits, depending on whether four, three, two or none of the MU bases are cloned with the same fidelity. If none of the MU bases are cloned with equal fidelities, one has a particular Heisenberg cloning machine. On the contrary, if all four MU bases are cloned with the same fidelity, one recovers eq. (2.3) for $d = 3$ in the case of symmetric cloning, that is, the qutrit universal cloner with fidelity

$$f_{1\to 2}^{\mathrm{univ}}(3) = \frac{3}{4}. \tag{2.10}$$

If three MU bases are requested to be cloned with the same fidelity, one gets the so-called *double-phase-covariant* qutrit cloner, with fidelity

$$f_{1\to 2}^{\mathrm{pc}}(3) = \frac{5 + \sqrt{17}}{12} \simeq 0.760, \tag{2.11}$$

slightly higher than $f_{1\to 2}^{\mathrm{univ}}(3)$. This cloner, which was independently derived by D'Ariano and Lo Presti [2001], has the property that it clones with the same and highest fidelity all the balanced superpositions

$$|\psi\rangle = \frac{1}{\sqrt{3}}\big(|0\rangle + \mathrm{e}^{\mathrm{i}\phi_1}|1\rangle + \mathrm{e}^{\mathrm{i}\phi_2}|2\rangle\big) \tag{2.12}$$

for arbitrary phases ϕ_1 and ϕ_2 (it is also covariant with respect to both ϕ_1- and ϕ_2-rotations). This can be understood by noting that if we complete the computational basis with any triplet of bases in order to make 4 MU bases, these 3 bases only consist of balanced superposition states. In analogy with the qubit case, it then appears that imposing these 3 bases to be cloned with the same (and highest) fidelity results in a cloning machine that clones all states (2.12) equally well, that is, the double-phase-covariant cloner.

Finally, we may impose that two MU bases that are dual under a Fourier transform are cloned with the same (and highest) fidelity, the other two being also cloned with an equal (albeit lower) fidelity. For example, the computational basis $\{|0\rangle, |1\rangle, |2\rangle\}$ and the dual basis $\| j\rangle\!\rangle = 3^{-1/2}\sum_{k=0}^{2}\gamma^{jk}|k\rangle$ with $j = 0, 1, 2$ and $\gamma = \mathrm{e}^{2\pi\mathrm{i}/3}$ form such a pair of MU bases. We then get the so-called *Fourier-covariant* cloner for qutrits, see Cerf, Durt and Gisin [2002], with fidelity

$$f_{1\to 2}^{\mathrm{Fourier}}(3) = \frac{1}{2} + \frac{1}{\sqrt{12}} \simeq 0.789 \tag{2.13}$$

which is even higher than $f_{1\to 2}^{\mathrm{pc}}(3)$ as expected since the considered set of input states is smaller than for the double-phase-covariant cloner. This cloner is covariant with respect to a Fourier transform, hence it clones two Fourier-conjugate bases with the same fidelity.

Note that, except in dimension 2, one cannot always map any two MU bases onto any other two MU bases, so that the Fourier-covariant cloner is not the unique transformation that clones equally well two MU bases. Indeed, Durt and Nagler [2003] showed that, in dimension 4, the cloner for two MU bases conjugate under a Fourier transform differs from the cloner for two MU bases conjugate under a double Hadamard transform. In the special case of qubits ($d = 2$), however, all pairs of MU bases are unitarily equivalent, so that the Fourier-covariant and phase-covariant cloners coincide, $f_{1\to 2}^{\mathrm{Fourier}}(2) = f_{1\to 2}^{\mathrm{pc}}(2)$.

2.4. Group-covariant cloning machines

In D'Ariano and Lo Presti [2001], a general method for optimizing the *group-covariant cloners* was derived. More specifically, they considered the optimal cloning transformations that are covariant under a proper subgroup Ω of the universal unitary group $U(d)$. For example, the universal qubit cloner is covariant with respect to $U(2)$, while the phase-covariant qubit cloner is covariant with respect to $U(1)$. They used this technique to derive the symmetric double-phase-covariant cloner for qutrits corresponding to eq. (2.11), as well as the 1-to-3 symmetric phase-covariant cloner for qubits, associated with the fidelity

$$f^{\mathrm{pc}}_{1\to 3}(2) = \frac{5}{6} \simeq 0.833. \tag{2.14}$$

Owing to the complexity of the group-theoretical parametrization of CP maps underlying this technique, its applicability seems rather limited. Nevertheless, using another method, Fan, Matsumoto, Wang and Wadati [2001] were able to derive the optimal 1-to-M symmetric phase-covariant cloning of qubits, yielding the fidelity

$$f^{\mathrm{pc}}_{1\to M}(2) = \begin{cases} \frac{1}{2} + \frac{\sqrt{M(M+2)}}{4M}, & M \text{ even}, \\ \frac{1}{2} + \frac{M+1}{4M}, & M \text{ odd}. \end{cases} \tag{2.15}$$

More recently, D'Ariano and Macchiavello [2003] succeeded in applying the theory of group-covariant cloning in order to confirm eq. (2.15), as well as to find a general expression for $f^{\mathrm{pc}}_{N\to M}(2)$ and the associated N-to-M cloner. This expression, which was partly conjectured in Fan, Matsumoto, Wang and Wadati [2001], is quite complex, and depends on whether N and M have the same parity. It was noticed that if the parities do not match then the cloner that optimizes the fidelity of each of the clones does not coincide with the optimal cloner with respect to the global fidelity (measuring how well the joint state of the clones approximates $|\psi\rangle^{\otimes M}$, if $|\psi\rangle$ is the state of the original). In the case of qutrits ($d = 3$), D'Ariano and Macchiavello [2003] also found the optimal 1-to-M symmetric double-phase-covariant cloner. The expression for its fidelity $f^{\mathrm{pc}}_{1\to M}(3)$ is rather complex, and depends on M modulo 3.

2.5. *High-d state-dependent cloning machines*

In parallel with this series of results on group-covariant cloning involving several originals and clones but in low dimensions, both the phase-covariant and Fourier-covariant 1-to-2 cloning machines were extended to arbitrary dimensions d. Cerf, Bourennane, Karlsson and Gisin [2002] derived the d-dimensional symmetric Fourier-covariant cloner, and showed it to be characterized by the fidelity

$$f^{\mathrm{Fourier}}_{1\to 2}(d) = \frac{1}{2} + \frac{1}{\sqrt{4d}}. \tag{2.16}$$

It clones equally well two MU bases that are conjugate under a Fourier transform, such as the computational basis $\{|0\rangle, \ldots, |d-1\rangle\}$ and the dual basis $\|j\rangle\rangle = d^{-1/2}\sum_{k=0}^{d-1}\gamma^{jk}|k\rangle$ with $j = 0, \ldots, d-1$ and $\gamma = \mathrm{e}^{2\pi\mathrm{i}/d}$. The asymmetric Fourier-covariant cloners were also characterized in the same paper.

Then, Fan, Imai, Matsumoto and Wang [2003] derived the d-dimensional symmetric multi-phase-covariant cloner, giving the fidelity

$$f^{\mathrm{pc}}_{1\to 2}(d) = \frac{1}{d} + \frac{d-2+\sqrt{d^2+4d-4}}{4d}. \tag{2.17}$$

It clones with the same (and highest) fidelity all balanced superpositions of the states of the computational basis, with arbitrary phases. This result was independently derived by Lamoureux and Cerf [2005], Rezakhani, Siadatnejad and Ghaderi [2005], who also extended it to asymmetric cloners. Note also that the role of multi-phase-covariant cloners in the context of entanglement-based QKD protocols was first studied by Durt, Cerf, Gisin and Zukowski [2003] for qutrits, then by Durt, Kaszlikowski, Chen and Kwek [2004] for d-dimensional systems.

2.6. *Cloning a pair of orthogonal qubits*

Another possible variant of the problem of cloning was studied by Fiurášek, Iblisdir, Massar and Cerf [2002], who introduced universal cloning machines that transform 2 qubits that are in an antiparallel joint state $|\psi\rangle|\psi^{\perp}\rangle$ into M clones of $|\psi\rangle$, with $\langle\psi|\psi^{\perp}\rangle = 0$. It was proven that for sufficiently large M such a cloner outperforms the standard 2-to-M cloner. One has the fidelity

$$f^{\mathrm{univ}}_{1,1\to M}(2) = \frac{1}{2} + \frac{\sqrt{(M+2)/(3M)}}{2} \tag{2.18}$$

which is greater than $f^{\mathrm{univ}}_{2\to M}(2)$ for $M > 6$. In some sense, it is better to replace one of the two original states $|\psi\rangle$ by its orthogonal state $|\psi^{\perp}\rangle$ if the goal is to produce $M > 6$ clones. This effect can be understood at the limit $M \to \infty$,

$$f^{\mathrm{univ}}_{1,1\to\infty}(2) = \frac{1}{2} + \frac{1}{2\sqrt{3}} \simeq 0.789, \tag{2.19}$$

that is, for the optimal measurement of a pair of antiparallel qubits. Indeed, it had been noticed earlier by Gisin and Popescu [1999] that measuring $|\psi\rangle|\psi^{\perp}\rangle$ yields more information than measuring $|\psi\rangle^{\otimes 2}$, with $f^{\mathrm{univ}}_{2\to\infty}(2) = \frac{3}{4}$. An interpretation of this property lies in the dimension of the Hilbert space spanned by $|\psi\rangle|\psi^{\perp}\rangle$, which is 4, while $|\psi\rangle^{\otimes 2}$ only spans the three-dimensional symmetric subspace of 2 qubits, $\mathcal{H}^{+}$.

2.7. Entanglement cloning machines

Another problem, related to quantum cloning, has been investigated by Lamoureux, Navez, Fiurášek and Cerf [2004]. They showed that the amount of entanglement contained in a two-qubit state cannot be cloned exactly, in analogy with the impossibility of cloning the state itself. If a cloning machine is devised that produces maximally entangled clones for maximally entangled qubit pairs at the input, then it cannot yield unentangled clones for all product states at the input. Nevertheless, the approximate cloning of entanglement is very well possible. Lamoureux, Navez, Fiurášek and Cerf [2004] defined a class of 1-to-2 *entanglement cloning machines* which are universal over the set of maximally entangled two-qubit states. The symmetric cloner of this class provides two clones of all maximally entangled two-qubit states with optimal fidelity

$$f_{1\to 2}^{\text{entang}}(2\times 2) = \frac{5+\sqrt{13}}{12} \simeq 0.717 \tag{2.20}$$

corresponding to an entanglement of formation 0.285 e-bits. In contrast, all product states are transformed into unentangled clones. This was recently extended to the cloning of entanglement for $(d\times d)$-dimensional systems by Karpov, Navez and Cerf [2005]. The fidelity of the optimal symmetric entanglement cloner that is universal over the set of maximally entangled $(d\times d)$-dimensional states is

$$f_{1\to 2}^{\text{entang}}(d\times d) = \frac{1}{4}\left[\frac{d^2+1}{d^2-1} + \sqrt{1+\frac{4}{d^2}\left(\frac{d^2-2}{d^2-1}\right)^2}\,\right]. \tag{2.21}$$

Note also that the broadcasting of entanglement via local cloning was investigated by Bužek, Vedral, Plenio, Knight and Hillery [1997].

2.8. Real cloning machines

Still another class of d-dimensional 1-to-2 cloners was introduced by Navez and Cerf [2003], and named *real cloning machines*. It is defined as a transformation that clones all *real* superpositions of the computational basis states with the same (and highest) fidelity. The optimal 1-to-2 symmetric real cloner in dimension d was shown to have fidelity

$$f_{1\to 2}^{\text{real}}(d) = \frac{1}{2} + \frac{2-d+\sqrt{d^2+4d+20}}{4(d+2)}. \tag{2.22}$$

Note that in dimension $d = 2$, the set of real states forms a circle in the Bloch sphere which is unitarily equivalent to the equator, so that we have $f_{1\to 2}^{\text{real}}(2) =$

$f^{\mathrm{pc}}_{1\to 2}(2) = f^{\mathrm{Fourier}}_{1\to 2}(2) = \frac{1}{2} + 1/\sqrt{8}$. For any dimension $d > 2$, one has

$$f^{\mathrm{univ}}_{1\to 2}(d) < f^{\mathrm{pc}}_{1\to 2}(d) < f^{\mathrm{real}}_{1\to 2}(d) < f^{\mathrm{Fourier}}_{1\to 2}(d). \tag{2.23}$$

Note that for $d = 4$ we have the identity $f^{\mathrm{real}}_{1\to 2}(4) = f^{\mathrm{entang}}_{1\to 2}(2 \times 2)$. This comes from the fact that the set of maximally-entangled two-qubit states is isomorphic to the set of four-dimensional real states.

2.9. *Highly-asymmetric cloning machines*

Iblisdir, Acín, Cerf, Filip, Fiurášek and Gisin [2005] introduced the concept of multipartite asymmetric N-to-M cloning machines (with $M > 2$). These machines are highly asymmetric in the sense that they produce M clones of unequal fidelities. A very general group-theoretical approach to the construction of the multipartite asymmetric cloning machines for qubits was then presented by Iblisdir, Acín and Gisin [2005]. It was applied to several particular examples such as the asymmetric $1 \to N+1$ cloning machine, which produces two kinds of clones, one clone with fidelity f^A and N clones with fidelity f^B. The optimal fidelities read

$$\begin{aligned} f^A_{1\to N+1} &= 1 - \frac{2}{3}x^2, \\ f^B_{1\to N+1} &= \frac{1}{2} + \frac{1}{3N}\left(x^2 + x\sqrt{(1-x^2)N(N+2)}\right), \end{aligned} \tag{2.24}$$

where $x \in (0, 1)$ parametrizes the class of optimal $1 \to N+1$ asymmetric cloners. Note that eq. (2.24) holds only for $N > 1$. It also was conjectured, based on exact analytical calculations for low N, that the optimal $N \to N+1$ asymmetric cloner, which produces, from N replicas of a qubit, N clones with fidelity f^A and a single clone with fidelity f^B, achieves

$$\begin{aligned} f^A_{N\to N+1} &= 1 - \frac{2}{N(N+2)}x^2, \\ f^B_{N\to N+1} &= 1 - \frac{1}{2}\left(\sqrt{\frac{N}{N+2}}x - \sqrt{1-x^2}\right)^2. \end{aligned} \tag{2.25}$$

The extension to d-dimensional systems was considered by Fiurášek, Filip and Cerf [2005] who investigated the universal asymmetric quantum triplicator, which produces, from a single replica of a qudit, three clones with three different fidelities f^A, f^B and f^C. A simple parametric description of the class of optimal universal highly-asymmetric triplicators was provided, extending eqs. (2.5) and (2.6).

It was proved that the optimal fidelities can be expressed as

$$
\begin{aligned}
f^{A}_{1\to1+1+1} &= 1 - \frac{d-1}{d}\left[\beta^2 + \gamma^2 + \frac{2\beta\gamma}{d+1}\right],\\
f^{B}_{1\to1+1+1} &= 1 - \frac{d-1}{d}\left[\alpha^2 + \gamma^2 + \frac{2\alpha\gamma}{d+1}\right],\\
f^{C}_{1\to1+1+1} &= 1 - \frac{d-1}{d}\left[\alpha^2 + \beta^2 + \frac{2\alpha\beta}{d+1}\right],
\end{aligned}
\tag{2.26}
$$

where the positive real parameters α, β, γ satisfy the normalization condition

$$
\alpha^2 + \beta^2 + \gamma^2 + \frac{2}{d}(\alpha\beta + \alpha\gamma + \beta\gamma) = 1. \tag{2.27}
$$

2.10. Continuous-variable cloning machines

Another interesting extension of quantum cloning, often referred to as *continuous-variable quantum cloning*, concerns the case of quantum systems lying in an infinite-dimensional Hilbert space. Cerf, Ipe and Rottenberg [2000] investigated the cloning of the set of coherent states $|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha a^\dagger}|0\rangle$, with $|0\rangle$ denoting the vacuum state, $a^\dagger$ being the bosonic creation operator, and $\alpha = (x + \mathrm{i}p)/\sqrt{2}$ being a c-number which defines the position (x, p) of $|\alpha\rangle$ in phase space. Here x and p are the so-called quadrature components. A set of 1-to-2 (symmetric or asymmetric) cloning machines that are covariant with respect to the Weyl group of displacements in phase space was derived. The symmetric 1-to-2 *Gaussian* cloner was found to have fidelity

$$
f^{\mathrm{CV}}_{1\to2} = \frac{2}{3} \simeq 0.667 \tag{2.28}
$$

and was conjectured to be optimal. It causes an independent Gaussian noise on x and p, with a variance equal to one shot-noise unit. Thus, the two clones are left in a thermal state (containing on average $\frac{1}{2}$ thermal photon) which is displaced by α. Let us also mention the independent derivation of this 1-to-2 Gaussian cloner as well as its extension to multiple clones ($M > 2$) by Lindblad [2000].

Cerf and Iblisdir [2000] later derived an upper bound on the fidelity of the symmetric N-to-M Gaussian cloners, based on a link with state estimation theory. Since it coincided with eq. (2.28) for $N = 1$ and $M = 2$, this proved that the above cloner is indeed the optimal cloner by means of a Gaussian operation. Cerf and Iblisdir [2001c] then showed that this 1-to-2 Gaussian cloner can be realized simply by use of an optical parametric amplifier of gain 2 followed by a balanced beamsplitter. The cloning noise then originates from the vacuum fluctuations of the ancillary modes that are coupled to the input mode.

Cochrane, Ralph and Dolinska [2004] showed that if the ensemble of input coherent states has a finite width, the 1-to-2 Gaussian cloning can be achieved with a higher fidelity. Clearly, if the task is to clone a coherent state drawn from a distribution that is peaked around the origin of phase space, the vacuum state is a very good approximation of the original state, so cloning with a fidelity close to one is possible. The above fidelity $f^{\mathrm{CV}}_{1\to 2}$ corresponds to the opposite situation of an infinitely wide input distribution, that is, an arbitrary input coherent state. For an input coherent state distributed according to a Gaussian distribution of zero mean and given variance, Cochrane, Ralph and Dolinska [2004] gave a closed formula for this fidelity as a function of the variance.

The optimal N-to-M Gaussian cloning transformation that achieves the above-mentioned upper bound was obtained by Braunstein, Cerf, Iblisdir, van Loock and Massar [2001] and Fiurášek [2001a], yielding

$$f^{\mathrm{CV}}_{N\to M} = \frac{MN}{MN + M - N}. \tag{2.29}$$

As for discrete-dimensional states, these cloners tend, at the limit $M \to \infty$, to the optimal measurement of $|\alpha\rangle^{\otimes N}$, with fidelity

$$f^{\mathrm{CV}}_{N\to\infty} = \frac{N}{N+1}. \tag{2.30}$$

The optical realization of these symmetric N-to-M cloners was also described there, while it was generalized to asymmetric 1-to-2 cloners in Fiurášek [2001a]. In the latter case, the balance between the fidelities of the two clones follows

$$f^{\mathrm{CV}}_{A} = \frac{1}{1+\sigma_A^2}, \qquad f^{\mathrm{CV}}_{B} = \frac{1}{1+\sigma_B^2}, \qquad \sigma_A\sigma_B = \frac{1}{2}, \tag{2.31}$$

which corresponds to the no-cloning uncertainty relation derived in Cerf, Ipe and Rottenberg [2000]. Here, σ_A^2 and σ_B^2 are the variances of the added noise on clone A and B, while one shot-noise unit is taken as $1/2$.

Finally, Cerf and Iblisdir [2001a] characterized a more general class of Gaussian cloners, which transform N replicas of an arbitrary coherent state $|\alpha\rangle$ and N' replicas of its phase-conjugate $|\alpha^*\rangle$ into M clones of $|\alpha\rangle$ and M' clones of $|\alpha^*\rangle$, with $N - N' = M - M'$. For well-chosen ratios N'/N, this cloner was shown to perform better than the $(N + N')$-to-M cloner. In addition, the special case of the *balanced* Gaussian cloner, with $N = N'$ and $M = M'$, was shown to be optimal among all cloners in this class in the sense that it yields the highest fidelity for fixed $N + N'$ and $M + M'$, namely

$$f^{\mathrm{CV}}_{N,N\to M,M} = \frac{4M^2N}{4M^2N + (M-N)^2}. \tag{2.32}$$

Interestingly, in the limit $M \to \infty$ we have

$$f^{\mathrm{CV}}_{N,N\to\infty,\infty} = \frac{4N}{4N+1}, \tag{2.33}$$

which means that the optimal measurement of $|\alpha\rangle^{\otimes N}|\alpha^*\rangle^{\otimes N}$ gives the same fidelity as the optimal measurement of $|\alpha\rangle^{\otimes 4N}$, instead of $|\alpha\rangle^{\otimes 2N}$ as a simple counting of states seems to imply. This advantage of phase conjugation was first noted by Cerf and Iblisdir [2001b].

Returning to the question of the symmetric N-to-M cloning of coherent states, Cerf, Krüeger, Navez, Werner and Wolf [2005] have recently investigated the question of whether the above Gaussian cloners really provide the absolute highest fidelity or, instead, transformations outside the realm of Gaussian operations need to be considered. Against all intuition it was shown that, provided M is finite, the cloning transformation that optimizes the single-clone fidelity is slightly non-Gaussian. For example, the optimal symmetric 1-to-2 non-Gaussian cloner of coherent states was shown to have fidelity

$$f^{\mathrm{CV,NG}}_{1\to 2} = 0.683 \tag{2.34}$$

strictly larger than $f^{\mathrm{CV}}_{1\to 2} = 2/3 \simeq 0.667$. In contrast, the optimal cloners of coherent states with respect to the global fidelity remain Gaussian. This discrepancy between optimal cloners with respect to single-clone or global fidelities is reminiscent of the situation for phase-covariant cloners in finite-dimensional spaces.

For a review on continuous-variable quantum cloning, see Cerf [2003] and Braunstein and van Loock [2005].

2.11. Probabilistic cloning machines

All cloning machines listed above are deterministic, i.e., they always produce (imperfect) clones. However, one can also consider probabilistic cloning machines, which sometimes fail to generate clones but, if they succeed, generate clones exhibiting higher fidelities than those achieved by the best deterministic cloners. The concept of probabilistic cloning was introduced by Duan and Guo [1998a, 1998b], Chefles and Barnett [1998, 1999] who investigated the cloning of a discrete finite set of pure states. They showed that a set of linearly independent states can be copied perfectly with some probability p. In particular, an exact 1-to-2 cloning of two generally nonorthogonal pure states $|\psi_1\rangle$ and $|\psi_2\rangle$ is possible with probability

$$p_{1\to 2} = \frac{1}{1+|\langle\psi_1|\psi_2\rangle|}. \tag{2.35}$$

The probabilistic cloning was then extended to infinite continuous sets of input states by Fiurášek [2004]. It was shown that the optimal universal cloning cannot be improved by using a probabilistic cloning strategy, due to a very high underlying symmetry of the problem. Nevertheless, if one considers cloning of some restricted set of states, then probabilistic cloning may become useful. A particular example is the optimal N-to-M phase-covariant cloning of qubits, where the optimal probabilistic cloner achieves the single-clone fidelity

$$f^{\mathrm{pc,prob}}_{N\to M}(2) = \frac{1}{2^M}\sum_{k=0}^{N}\binom{M}{k+\left[\frac{1}{2}(M-N)\right]},$$

where $[x]$ denotes the integer part of x. For $N>1$ the fidelity $f^{\mathrm{pc,prob}}_{N\to M}(2)$ is larger than the fidelity $f^{\mathrm{pc}}_{N\to M}(2)$ of the optimal deterministic phase-covariant cloning.

2.12. *Economical cloning machines*

The 1-to-2 cloning transformation for d-dimensional systems (qudits) can typically be expressed as a unitary operation on the Hilbert space of three qudits – the input, a blank copy, and an ancilla. The presence of an ancilla significantly affects the experimental implementation of the cloning operation, which becomes more complicated and sensitive to decoherence. These problems, which might drastically reduce the achieved cloning fidelity, may significantly be suppressed if an "economical" approach is followed, which avoids the ancilla. The 1-to-2 cloning is then realized as a unitary operation on two qudits only: the input and the blank copy. This is obviously much simpler to implement because it requires less qudits and two-qudit gates, and it requires to control the entanglement of a pair of qudits only. It is thus likely to be much less sensitive to noise and decoherence than its three-qudit counterpart.

To date, the only 1-to-2 cloning machine for which an economical realization is known is the phase-covariant qubit cloner due to Niu and Griffiths [1999], which optimally clones all states on the equator of the Bloch sphere, $|\psi\rangle_A = 2^{-1/2}(|0\rangle_A + \mathrm{e}^{\mathrm{i}\phi}|1\rangle_A)$. The qubit to be cloned is coupled to another qubit which becomes the second copy and is initially prepared in state $|0\rangle_B$. The unitary two-qubit transformation reads

$$|0\rangle_A|0\rangle_B \to |0\rangle_A|0\rangle_B, \qquad |1\rangle_A|0\rangle_B \to \frac{1}{\sqrt{2}}\big(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B\big),$$

for a symmetric phase-covariant cloner. It can easily be extended to an asymmetric setting as we will show in eq. (6.4).

The possibility of economically realizing various 1-to-2 cloning machines for qudits has been analyzed in detail by Durt, Fiurášek and Cerf [2005]. They showed that the economical universal cloning is not possible for any d. It was also argued that the optimal 1-to-2 phase-covariant cloning of qudits does not admit economical implementation for any $d > 2$, and this assertion was rigorously proved for $d \leqslant 7$. A suboptimal economical phase-covariant cloner was nevertheless constructed, which does not require an ancilla and achieves the fidelity

$$f_{1\to 2}^{\mathrm{pc,econ}}(d) = \frac{1}{2d^2}\big[d - 1 + \big(d - 1 + \sqrt{2}\big)^2\big], \tag{2.36}$$

which is only slightly below that of the optimal cloner. Similarly, it was argued that the 1-to-2 Fourier-covariant cloning cannot be realized economically, albeit in dimension $d = 2$ (in which case it is unitarily equivalent to the phase-covariant cloner).

The concept of economical cloning can be extended to N-to-M machines. As shown by Fan, Matsumoto, Wang and Wadati [2001], the optimal N-to-M phase-covariant cloning of qubits ($d = 2$), which maximizes the single-clone fidelity, admits an economical implementation for any N and $M > N$. Moreover, the economical phase-covariant cloning of d-dimensional systems (qudits) is also possible provided that $M = kd + N$, where k is an integer, see Buscemi, D'Ariano and Macchiavello [2005].

§ 3. One-to-two quantum cloning as a CP map

3.1. Isomorphism between CP maps and operators

A very useful characterization of cloning relies on the isomorphism between completely positive (CP) maps $\mathcal{S}: \mathcal{H}_{\mathrm{in}} \to \mathcal{H}_{\mathrm{out}}$ and positive semidefinite operators $S \geqslant 0$ acting on $\mathcal{H}_{\mathrm{in}} \otimes \mathcal{H}_{\mathrm{out}}$, where $\mathcal{H}_{\mathrm{in}}$ and $\mathcal{H}_{\mathrm{out}}$ denote, respectively, the input and output Hilbert spaces of $\mathcal{S}$, see Jamiolkowski [1972] and Choi [1975]. To construct this isomorphism, consider a maximally entangled state on $\mathcal{H}_{\mathrm{in}}^{\otimes 2}$,

$$\big|\Phi^+\big\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle|j\rangle, \tag{3.1}$$

where $d = \dim(\mathcal{H}_{\text{in}})$. If the map $\mathcal{S}$ is applied to the second subsystem of $|\Phi^+\rangle$ while the first one is left unchanged, then the resulting (generally mixed) quantum state is isomorphic to $\mathcal{S}$ and reads

$$S = [\mathcal{I} \otimes \mathcal{S}]\big(d\Phi^+\big), \tag{3.2}$$

where $\Phi^+ \equiv |\Phi^+\rangle\langle\Phi^+|$ and $\mathcal{I}$ stands for the identity map, while the prefactor d is introduced for normalization purposes. The map $\mathcal{S}$ can be characterized in terms of the state S as follows:

$$\rho_{\text{out}} = \mathcal{S}(\rho_{\text{in}}) = \text{Tr}_{\text{in}}\big[\big(\rho_{\text{in}}^{\text{T}} \otimes I_{\text{out}}\big)S\big], \tag{3.3}$$

where "in" labels the input space, I is the identity operator, and "T" denotes the transposition in the computational basis. If the map $\mathcal{S}$ is trace-preserving then S satisfies the condition

$$\text{Tr}_{\text{out}}[S] = I_{\text{in}}, \tag{3.4}$$

while the complete positivity condition on $\mathcal{S}$ translates into $S \geqslant 0$.

In the following, we shall make this description specific to the 1-to-2 quantum cloning machines, which produce two copies of a single d-dimensional system (qudit), see Fiurášek [2001b]. The output Hilbert space is endowed with a tensor product structure, $\mathcal{H}_{\text{out}} = \mathcal{H}_A \otimes \mathcal{H}_B$, where the subscripts A and B label the two clones. For each particular input state $|\psi\rangle$, the joint state of the clones is

$$\mathcal{S}(\psi) = \text{Tr}_{\text{in}}\big[\big(\psi_{\text{in}}^{\text{T}} \otimes I_{AB}\big)S\big], \tag{3.5}$$

where $\psi \equiv |\psi\rangle\langle\psi|$. It will be useful in the following to note that $\psi_{\text{in}}^{\text{T}}$ is a rank-one projector onto the state $|\psi_{\text{in}}^*\rangle$, where "*" denotes the complex conjugation in the computational basis.

Using eq. (3.5), the fidelity of the clones A and B is given by

$$\begin{aligned} F_A(\mathcal{S}, \psi) &= \text{Tr}\big[(\psi_A \otimes I_B)\mathcal{S}(\psi)\big] = \text{Tr}\big[\big(\psi_{\text{in}}^{\text{T}} \otimes \psi_A \otimes I_B\big)S\big], \\ F_B(\mathcal{S}, \psi) &= \text{Tr}\big[(I_A \otimes \psi_B)\mathcal{S}(\psi)\big] = \text{Tr}\big[\big(\psi_{\text{in}}^{\text{T}} \otimes I_A \otimes \psi_B\big)S\big]. \end{aligned} \tag{3.6}$$

The symmetric cloning machines are defined as the maps $\mathcal{S}$ verifying $F_A(\mathcal{S}, \psi) = F_B(\mathcal{S}, \psi)$ $\forall\psi$. Otherwise, the cloning machines are called asymmetric. When considering a universal cloning machine, we require that both $F_A(\mathcal{S}, \psi)$ and $F_B(\mathcal{S}, \psi)$ are independent of ψ, for all states ψ in $\mathcal{H}_{\text{in}}$. Other cloning machines, such as the phase-covariant, Fourier-covariant, or real cloning machines will correspond to a constant fidelity over a restricted set R of input states ψ.

3.2. Covariance condition

In what follows, we will always assume that the set R of input states ψ is invariant under the action of the group $G(\Omega)$ of unitaries $\{U_\omega \mid \omega \in \Omega\}$, that is,

$$U_\omega R U_\omega^\dagger = R \quad \forall \omega \in \Omega. \tag{3.7}$$

The universal cloning machine is the special case $U_\omega \in \mathrm{SU}(d)$. A useful figure of merit to measure the quality of cloning is the *global* fidelity,

$$F(\mathcal{S}, \psi) = \mathrm{Tr}\big[(\psi_A \otimes \psi_B)\mathcal{S}(\psi)\big] = \mathrm{Tr}\big[\big(\psi_{\mathrm{in}}^{\mathrm{T}} \otimes \psi_A \otimes \psi_B\big)S\big], \tag{3.8}$$

which measures how well the joint state of the two clones approximates $\psi^{\otimes 2}$. When looking for a cloning machine that optimally clones all the states of set R, one generally defines the cloning fidelity of map $\mathcal{S}$ as the infimum of the global fidelity over all input states ψ,

$$F(\mathcal{S}) = \inf_{\psi \in R} F(\mathcal{S}, \psi). \tag{3.9}$$

It has been shown by Werner [1998] that, by using the so-called *twirling* operation, there is no loss of generality in assuming the optimal cloning machine to be *covariant* with respect to the group $G(\Omega)$, hence the cloning fidelity to be state-independent within the set R. The twirling operation consists in randomly applying a unitary U_ω to the input state and then undoing this by applying the reverse unitary $U_\omega^\dagger$ to each of the two clones with the probability density $\mathrm{d}\omega$ equal to the Haar measure on the group $G(\Omega)$. This results in the twirled map

$$\mathcal{S}_{\mathrm{twirl}}(\psi) = \int_\Omega \mathcal{S}_\omega(\psi)\, \mathrm{d}\omega, \tag{3.10}$$

with the rotated map $\mathcal{S}_\omega$ being defined as

$$\mathcal{S}_\omega(\psi) = U_\omega^{\dagger\otimes 2}\mathcal{S}\big(U_\omega \psi U_\omega^\dagger\big)U_\omega^{\otimes 2}. \tag{3.11}$$

The core of the argument is that

$$\begin{aligned} F(\mathcal{S}) &= \inf_{\psi \in R} F(\mathcal{S}, \psi) \leqslant \inf_{\phi \in R} \int_\Omega F\big(\mathcal{S}, U_\omega \phi U_\omega^\dagger\big)\, \mathrm{d}\omega \\ &= \inf_{\phi \in R} \int_\Omega F(\mathcal{S}_\omega, \phi)\, \mathrm{d}\omega = \inf_{\phi \in R} F(\mathcal{S}_{\mathrm{twirl}}, \phi) = F(\mathcal{S}_{\mathrm{twirl}}), \end{aligned} \tag{3.12}$$

where we have used the invariance of R under the unitaries U_ω, the invariance of the trace function under U_ω, and the linearity of the fidelity in $\mathcal{S}$. As a result, the operation of twirling can only increase the cloning fidelity, so that the twirled map $\mathcal{S}_{\mathrm{twirl}}$ is at least as good as each of its constituent maps $\mathcal{S}_\omega$. Finally, as mentioned earlier, we note that $\mathcal{S}_{\text{twirl}}$ is *covariant* with respect to the group $G(\Omega)$, that is,

$$\mathcal{S}_{\text{twirl}}\big(U_\omega \psi U_\omega^\dagger\big) = U_\omega^{\otimes 2} \mathcal{S}_{\text{twirl}}(\psi) U_\omega^{\dagger\otimes 2} \qquad \forall \omega \in \Omega, \forall \psi \in R. \tag{3.13}$$

Physically, this covariance property means that rotating the original state is exactly equivalent to rotating the two clones by the same amount. This also implies that $F(\mathcal{S}_{\text{twirl}}, \psi)$ does not depend on ψ, within the set R.

In summary, we have shown that when looking for an optimal quantum cloning machine (i.e., a machine that maximizes the worst-case global fidelity), it is sufficient to consider cloning maps that are covariant with respect to the group under which the set of input states R is invariant. The cloning fidelity is therefore state-independent within the set R. Keyl and Werner [1999] proved that this reasoning also applies more generally to the quantum cloning machines that maximize the *single-clone* fidelities (F_A and F_B) instead of the global fidelity, provided that *universal* cloning machines are considered. This, however, does not hold for all quantum cloners (see, e.g., the case of phase-covariant or continuous-variable cloners we will be considering later).

3.3. Cloning as a semidefinite programming problem

Returning to the characterization of the map $\mathcal{S}$ via its associated operator S, we can now use the fact that the optimal cloning machine must have a state-independent fidelity over the set of input states considered. We can then turn to the average performance of the cloning machines, which is measured by the mean fidelities

$$F_A(\mathcal{S}) = \int_\psi F_A(\mathcal{S}, \psi)\, \mathrm{d}\psi, \qquad F_B(\mathcal{S}) = \int_\psi F_B(\mathcal{S}, \psi)\, \mathrm{d}\psi, \tag{3.14}$$

where the measure $\mathrm{d}\psi$ determines the kind of cloning machines we are dealing with. In particular, universal cloning machines correspond to choosing $\mathrm{d}\psi$ to be the invariant measure on the factor space $\mathrm{SU}(d)/\mathrm{SU}(d-1)$ induced by the Haar measure on the group $\mathrm{SU}(d)$. The fidelities (3.14) can be expressed as linear functions of the operator S,

$$F_A = \mathrm{Tr}[S R_A], \qquad F_B = \mathrm{Tr}[S R_B], \tag{3.15}$$

where we have defined the positive semidefinite operators

$$R_A = \int_\psi \psi_{\text{in}}^{\mathrm{T}} \otimes \psi_A \otimes I_B\, \mathrm{d}\psi, \qquad R_B = \int_\psi \psi_{\text{in}}^{\mathrm{T}} \otimes I_A \otimes \psi_B\, \mathrm{d}\psi. \tag{3.16}$$

We will see in the following sections how these operators can be calculated for different kinds of cloning machines. Note that for a *symmetric* cloning machine, one should simply maximize the average of the mean fidelities,

$$F(\mathcal{S}) = \frac{1}{2}\left[F_A(\mathcal{S}) + F_B(\mathcal{S})\right] = \mathrm{Tr}[SR], \tag{3.17}$$

with $R = (R_A + R_B)/2$. This can be justified by an argument similar to that used for the twirling operation. By averaging over the permutation between the two clones, one obtains a map whose mean fidelity can only be better than that of the original map. Therefore, we can restrict ourselves to cloning transformations that are covariant with respect to the interchange of the clones, hence satisfying $F_A(\mathcal{S}) = F_B(\mathcal{S})$.

From this argument based on twirling and permutation, we conclude that maximizing the mean fidelity, averaged over the two clones (with equal weights), should yield a cloning map which has a state-independent and clone-independent fidelity. The asymmetric cloners can also be obtained with the same maximization but by putting different weights in front of F_A and F_B.

An interesting point to note is that finding the optimal cloning map $\mathcal{S}$ reduces to a *semidefinite programming* problem, namely finding the operator S verifying $S \geqslant 0$ and $\mathrm{Tr}_{AB}[S] = I_{\mathrm{in}}$ that maximizes $\mathrm{Tr}[SR]$, with R depending on the considered cloning machine (Audenaert and De Moor [2002]). Very efficient numerical methods are available for solving semidefinite programs, see, e.g., Vandenberghe and Boyd [1996]. Even more importantly, it can be shown with the help of Lagrange duality lemma that the optimal cloning trace-preserving CP map, which maximizes $\mathrm{Tr}[SR]$, must satisfy

$$(R - \lambda_{\mathrm{in}} \otimes I_{AB})S = 0, \tag{3.18}$$

$$\lambda_{\mathrm{in}} \otimes I_{AB} - R \geqslant 0, \tag{3.19}$$

where $\lambda \geqslant 0$ is a positive semidefinite operator whose matrix elements represent the Lagrange multipliers accounting for the trace-preservation constraint $\mathrm{Tr}_{AB}[S] = I_{\mathrm{in}}$. Note that λ can be expressed in terms of the optimal CP map, $\lambda = \mathrm{Tr}_{AB}[SR]$. If both eqs. (3.18) and (3.19) are satisfied, then S is the optimal CP map maximizing $\mathrm{Tr}[SR]$, a property which is useful to prove and check the optimality of a given map S that is conjectured to be optimal.

The proof that eqs. (3.18) and (3.19) imply optimality is rather simple and we briefly sketch it here. Suppose that (3.19) is satisfied, then it holds for any trace-preserving CP map that $\mathrm{Tr}[S(\lambda \otimes I - R)] \geqslant 0$ and $\mathrm{Tr}[\lambda \otimes IS] = \mathrm{Tr}\,\lambda$, due to the trace-preservation condition. It follows that the fidelity is upper bounded by the trace of the Lagrange multiplier, $\mathrm{Tr}[RS] \leqslant \mathrm{Tr}[\lambda]$, and that the optimal map which satisfies (3.18) saturates this bound.

Very often a simpler method is sufficient to prove the optimality, namely, the fidelity can be bounded by the maximum eigenvalue $r_{\max}$ of R. Since $R \leqslant r_{\max} I$, we immediately have

$$F(S) \leqslant d r_{\max}, \tag{3.20}$$

where $d = \dim(\mathcal{H}_{\mathrm{in}})$. If there exists a CP map S which saturates (3.20), then this transformation is optimal. Note, however, that in certain cases such as the cloning of a pair of orthogonal qubits, the bound on fidelity (3.20) is not tight and cannot be saturated.

3.4. Double-Bell ansatz

Let us now consider the *unitary realization* of the cloning map $\mathcal{S}$. We know that any CP map can be realized physically by supplementing the input system with an ancilla (hence, extending the Hilbert space) and acting with a unitary operator in this extended space. Here, the ancilla can be viewed as the cloning machine itself, and it must be traced over after applying the unitary operator. The resulting map can be written as

$$\mathcal{S}(\psi_{\mathrm{in}}) = \mathrm{Tr}_{\mathrm{in},C}\big[\big(\psi_{\mathrm{in}}^{\mathrm{T}} \otimes I_{ABC}\big)\Sigma\big], \tag{3.21}$$

where C denotes the cloning machine and Σ is the operator that is isomorphic to this extended map $\mathcal{H}_{\mathrm{in}} \to \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Since this extended map is some unitary operation U_{ABC} in the extended space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, the operator Σ must be some (unnormalized) rank-one projector or pure state in the joint space $\mathcal{H}_{\mathrm{in}} \otimes \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. We thus have

$$\Sigma = |\sigma\rangle\langle\sigma|, \quad \text{with } |\sigma\rangle_{\mathrm{in},ABC} = \sqrt{d}(I_{\mathrm{in}} \otimes U_{ABC})\big|\Phi^+\big\rangle_{\mathrm{in},A}|0\rangle_{B,C}, \tag{3.22}$$

where $|0\rangle_{B,C}$ is the (arbitrary) initial state of the blank copy B and cloning machine C, and $|\Phi^+\rangle$ is defined as in eq. (3.1). The prefactor $\sqrt{d}$, with $d = \dim(\mathcal{H}_{\mathrm{in}})$, is introduced for normalization purposes, and the cloning map corresponds to eq. (3.5) with $S = \mathrm{Tr}_C\, \Sigma$.

Physically, the state $|\sigma\rangle$ has a very simple interpretation, see fig. 2. If we start with two qudits prepared in a maximally entangled state $|\Phi^+\rangle$ and process one of them in the quantum cloning machine while the other one is left unchanged (kept as a reference), then $|\sigma\rangle$ is the joint state of this “reference” qudit (denoted as “in” since it keeps a memory of the input state), the clones A and B, as well

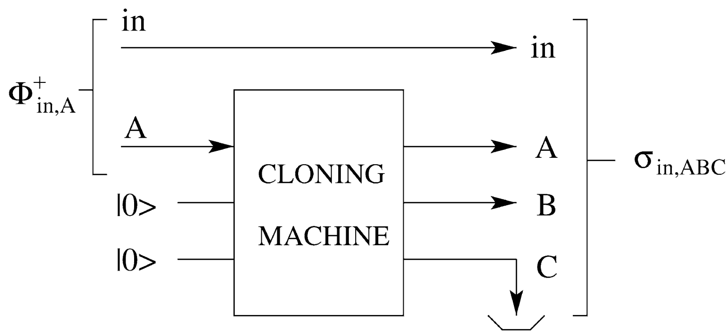


Fig. 2. Preparation of the state $|\sigma\rangle$ fully characterizing the cloning transformation. The input of the cloning machine is maximally entangled with a reference qudit labeled “in”. The two clones are contained in the outputs A and B, while C refers to an ancilla or the cloning machine itself.

as the cloning machine C. Remember that if we project the reference qudit onto the state $|\psi^*\rangle$, then, in the absence of cloning, qudit A is found in state $|\psi\rangle$. By causality, it is irrelevant whether this projection onto $|\psi^*\rangle$ is done before or after the cloning machine has been applied on qudit A. Therefore, projecting the reference qudit of state $|\sigma\rangle$ onto $|\psi^*\rangle$ yields the joint state of A, B and C that would have been obtained by cloning the state $|\psi\rangle$, namely

$$|\psi\rangle \to |\psi_{\text{out}}\rangle_{ABC} = {}_{\text{in}}\langle\psi^*|\sigma\rangle_{\text{in},ABC}. \tag{3.23}$$

We can say that $|\sigma\rangle$ fully encodes the information about the cloning of any state.

It was suggested by Cerf [1998, 2000a, 2000b] that a generic form for state $|\sigma\rangle$ involving a superposition of double-Bell states may encompass most of the interesting quantum cloning machines, including the universal or state-dependent – symmetric as well as asymmetric – cloners. This so-called *double-Bell ansatz* corresponds to taking $|\sigma\rangle = \sqrt{d}|\mathcal{A}\rangle$, with

$$|\mathcal{A}\rangle_{\text{in},A;B,C} = \sum_{m,n=0}^{d-1} a_{m,n}|\Phi_{m,n}\rangle_{\text{in},A}|\Phi^*_{m,n}\rangle_{B,C}, \tag{3.24}$$

where it is assumed that it is sufficient to use a Hilbert space for the cloning machine C which has the same dimension d as the input or the clones. Note that the Schmidt decomposition of $|\mathcal{A}\rangle$ for the partition “in” vs. ABC implies that $\dim(\mathcal{H}_C) \geqslant d$. In eq. (3.24), the $a_{m,n}$'s are complex amplitudes which satisfy the normalization condition $\sum_{m,n=0}^{d-1}|a_{m,n}|^2 = 1$, while $|\Phi_{m,n}\rangle$ denote Bell states in d dimensions. As the latter states play an important role in what follows, we will first discuss them in detail, as well as some useful related properties. Note also that $|\mathcal{A}\rangle$ is a quantum state of norm one, while $|\sigma\rangle$ has a norm equal to $\sqrt{d}$.

3.4.1. Useful properties of d-dimensional Bell states

A standard generalization of the Bell states in d dimensions is

$$|\Phi_{m,n}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \gamma^{nj} |j\rangle_1 |j+m\rangle_2, \tag{3.25}$$

where 1 and 2 denote two d-dimensional systems. These states form a set of d^2 maximally entangled states of systems 1 and 2, where $m, n \in \{0, 1, \ldots, d-1\}$ and $\gamma = \mathrm{e}^{2\pi \mathrm{i}/d}$ stands for the dth root of unity. Note that, in what follows, the "bra" and "ket" labels are always taken modulo d. In the case of qubits ($d = 2$), we recover the standard Bell states

$$\begin{aligned} |\Phi_{0,0}\rangle &= \left|\Phi^+\right\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \qquad |\Phi_{0,1}\rangle = \left|\Phi^-\right\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \\ |\Phi_{1,0}\rangle &= \left|\Psi^+\right\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \qquad |\Phi_{1,1}\rangle = \left|\Psi^-\right\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}. \end{aligned} \tag{3.26}$$

Taking the partial trace of any state $\Phi_{m,n} \equiv |\Phi_{m,n}\rangle\langle\Phi_{m,n}|$ over one of the two systems (1 or 2) results in the maximally mixed state,

$$\mathrm{Tr}_1(\Phi_{m,n}) = \mathrm{Tr}_2(\Phi_{m,n}) = \frac{I}{d} \qquad \forall m, n, \tag{3.27}$$

so that the states $|\Phi_{m,n}\rangle$ are indeed maximally entangled. It is easy to check that the states $|\Phi_{m,n}\rangle$ form a complete orthonormal basis in the d^2-dimensional Hilbert space considered here. The resolution of identity reads

$$\begin{aligned} \sum_{m,n=0}^{d-1} \Phi_{m,n} &= \frac{1}{d} \sum_{m,n} \sum_{j,j'} \gamma^{n(j-j')} |j\rangle\langle j'| \otimes |j+m\rangle\langle j'+m| \\ &= \sum_{m} \sum_{j,j'} \delta_{j,j'} |j\rangle\langle j'| \otimes |j+m\rangle\langle j'+m| = I_{12}, \end{aligned} \tag{3.28}$$

where we have used the identity $\frac{1}{d}\sum_{n=0}^{d-1} \gamma^{nj} = \delta_{j,0}$.

Let us focus on the Bell state with $m = n = 0$, that is,

$$|\Phi_{0,0}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle|j\rangle \tag{3.29}$$

which is another notation for the state $|\Phi^+\rangle$ as defined in eq. (3.1). This state is particularly useful because it satisfies the relation

$$(U^* \otimes U)|\Phi_{0,0}\rangle = |\Phi_{0,0}\rangle \tag{3.30}$$

for any unitary transformation U, as can readily be checked by using the unitarity condition $UU^\dagger = I$ and the completeness relation $\sum_j |j\rangle\langle j| = I$. Note that the symbol "*" denotes the complex conjugation operation in the computational basis $\{|j\rangle\}$; thus $|j^*\rangle = |j\rangle$. The identity (3.30), or equivalently

$$(I \otimes U)|\Phi_{0,0}\rangle = \big(U^{\mathrm{T}} \otimes I\big)|\Phi_{0,0}\rangle \tag{3.31}$$

corresponds to the following useful property: if the joint system 12 is prepared in the state $|\Phi_{0,0}\rangle$ and system 1 is projected onto $|\psi^*\rangle$, then the resulting state of system 2 is $|\psi\rangle$. Indeed, taking $|\psi\rangle = U|0\rangle$, we have $\langle\psi^*| = \langle 0|U^{\mathrm{T}}$, so that

$$\begin{aligned}
\big(|\psi^*\rangle\langle\psi^*| \otimes I\big)|\Phi_{0,0}\rangle &= \big(|\psi^*\rangle\langle 0| \otimes I\big)\big(U^{\mathrm{T}} \otimes I\big)|\Phi_{0,0}\rangle \\
&= \big(|\psi^*\rangle\langle 0| \otimes I\big)(I \otimes U)|\Phi_{0,0}\rangle \\
&= d^{-1/2}|\psi^*\rangle\big(U|0\rangle\big) = d^{-1/2}|\psi^*\rangle|\psi\rangle.
\end{aligned} \tag{3.32}$$

Interestingly, it makes no difference whether 1 prepares and sends the state $|\psi\rangle$ to 2, or 1 projects its part of a shared entangled state $|\Phi_{0,0}\rangle$ onto $|\psi^*\rangle$ so to create $|\psi\rangle$ at a distance on 2.

In what follows, we will also need the discrete group of Weyl–Heisenberg operators (also called *error operators*), namely

$$E_{m,n} = \sum_{j=0}^{d-1} \gamma^{jn}|j+m\rangle\langle j| \tag{3.33}$$

with $m, n \in \{0, 1, \ldots, d-1\}$, generalizing the Pauli matrices for more than two dimensions. For qubits ($d = 2$), we have

$$\begin{aligned}
&E_{0,0} = I, && E_{0,1} = \sigma_z, \\
&E_{1,0} = \sigma_x, && E_{1,1} = \sigma_x\sigma_z = -\mathrm{i}\sigma_y.
\end{aligned} \tag{3.34}$$

In arbitrary dimension, the error operator $E_{m,n}$ shifts the state by m units (modulo d) in the computational basis and multiplies it by a phase so as to shift its Fourier transform by n units (modulo d). Indeed, in the computational basis $\{|j\rangle\}$ we have $E_{m,0}|j\rangle = |j+m\rangle$, while in the dual basis $\{\||j\rangle\rangle = (1/\sqrt{d})\sum_{k=0}^{d-1}\gamma^{jk}|k\rangle\}$ we have $E_{0,n}\||j\rangle\rangle = \|j+n\rangle\rangle$. The error operators fulfill the following properties:

$$E^*_{m,n} = E_{m,-n}, \qquad E^{\mathrm{T}}_{m,n} = \gamma^{-mn}E_{-m,n}, \tag{3.35}$$

$$E^\dagger_{m,n} = \gamma^{mn}E_{-m,-n}, \qquad E_{m,n}E_{\mu,\nu} = \gamma^{n\mu}E_{m+\mu,n+\nu}. \tag{3.36}$$

Interestingly, the Bell states can be transformed into each other by applying an error operator locally (on one of the two systems, leaving the other one unchanged),

$$|\Phi_{m,n}\rangle = (I \otimes E_{m,n})|\Phi_{0,0}\rangle = \big(E^{\mathrm{T}}_{m,n} \otimes I\big)|\Phi_{0,0}\rangle. \tag{3.37}$$

This also implies that the Bell states are invariant (up to a phase) under correlated error operators ($E^*_{\mu,\nu} \otimes E_{\mu,\nu}$). We can check this by calculating

$$\begin{aligned}\big(E^*_{\mu,\nu} \otimes E_{\mu,\nu}\big)|\Phi_{m,n}\rangle &= (I \otimes E_{\mu,\nu}E_{m,n})\big(E^*_{\mu,\nu} \otimes I\big)|\Phi_{0,0}\rangle \\ &= (I \otimes E_{\mu,\nu}E_{m,n})\big(I \otimes E^\dagger_{\mu,\nu}\big)|\Phi_{0,0}\rangle \\ &= \gamma^{m\nu-n\mu}(I \otimes E_{m,n})|\Phi_{0,0}\rangle \\ &= \gamma^{m\nu-n\mu}|\Phi_{m,n}\rangle,\end{aligned} \tag{3.38}$$

where we have used property (3.37) as well as

$$\begin{aligned}E_{\mu,\nu}E_{m,n}E^\dagger_{\mu,\nu} &= \gamma^{\mu\nu}E_{\mu,\nu}E_{m,n}E_{-\mu,-\nu} = \gamma^{\mu\nu-n\mu}E_{\mu,\nu}E_{m-\mu,n-\nu} \\ &= \gamma^{\mu\nu-n\mu+\nu(m-\mu)}E_{m,n} = \gamma^{m\nu-n\mu}E_{m,n}.\end{aligned} \tag{3.39}$$

3.5. Heisenberg cloning machines

Returning to the double-Bell ansatz (3.24), the quantum cloning machine is thus completely characterized by the $d \times d$ matrix $\mathbf{a} = \{a_{m,n}\}$. The form (3.24) is particularly interesting because, when tracing over B and C, the systems "in" and A are left in a mixed state that is diagonal in the Bell basis,

$$\rho_{\text{in},A} = \sum_{m,n=0}^{d-1} |a_{m,n}|^2 \Phi_{m,n} \tag{3.40}$$

with $\Phi_{m,n} \equiv |\Phi_{m,n}\rangle\langle\Phi_{m,n}|$. Since the original system is maximally entangled with the reference system "in" (the initial state being $|\Phi_{0,0}\rangle$), this implies that clone A undergoes the error $E_{m,n}$ with probability $|a_{m,n}|^2$. It emerges from a Heisenberg channel characterized by the probability distribution $|a_{m,n}|^2$.

An important property of state $|\mathcal{A}\rangle$ is that, when interchanging clones A and B, it can be re-expressed as a superposition of double-Bell states albeit with different amplitudes,

$$|\mathcal{A}\rangle_{\text{in},B;A,C} = \sum_{m,n=0}^{d-1} b_{m,n}|\Phi_{m,n}\rangle_{\text{in},B}\big|\Phi^*_{m,n}\big\rangle_{A,C} \tag{3.41}$$

with

$$b_{m,n} = \frac{1}{d}\sum_{x,y=0}^{d-1} \gamma^{nx-my}a_{x,y}. \tag{3.42}$$

Again, when tracing over A and C, systems "in" and B are left in a Bell-diagonal mixed state,

$$\rho_{\text{in},B} = \sum_{m,n=0}^{d-1} |b_{m,n}|^2 \Phi_{m,n} \tag{3.43}$$

implying that clone B undergoes the error $E_{m,n}$ with probability $|b_{m,n}|^2$ (it emerges from another Heisenberg channel). Remarkably, eq. (3.42) implies that the matrix $\mathbf{b} = \{b_{m,n}\}$ is related to $\mathbf{a} = \{a_{m,n}\}$ by a (bivariate and d-dimensional) discrete Fourier transform, $\mathbf{b} = \mathcal{F}[\mathbf{a}]$. So, the cloning map can be characterized equivalently by the matrix $\mathbf{a}$ (characterizing the noise of clone A) or its Fourier transform $\mathbf{b}$ (characterizing the noise of clone B), and we see that the complementarity between these two clones simply originates from a Fourier transform: the more noisy clone A is, the less noisy is clone B. This leads to a no-cloning uncertainty relation, see Cerf [1999, 2000b].

Finally, we can use the ansatz (3.24) to express the map associated with an arbitrary Heisenberg cloner in the simple form

$$\begin{aligned} |\psi\rangle \to |\psi_{\text{out}}\rangle &= \sum_{m,n=0}^{d-1} a_{m,n} \operatorname{Tr}_{\text{in}}\bigl[\bigl(\psi_{\text{in}}^{\mathrm{T}} \otimes I_{ABC}\bigr)\bigl((\Phi_{m,n})_{\text{in},A} \otimes \bigl(\Phi_{m,n}^{\mathrm{T}}\bigr)_{BC}\bigr)\bigr] \\ &= \sum_{m,n=0}^{d-1} a_{m,n} E_{m,n} |\psi\rangle_A \otimes \bigl|\Phi_{m,n}^*\bigr\rangle_{BC}. \end{aligned} \tag{3.44}$$

Incidentally, we note here that by measuring the clone B together with the cloning machine C in the Bell basis, we get a pair of indices (m, n) which can be used to undo the noise on clone A simply by applying $E_{m,n}^{\dagger}$. This process, which bears some analogy with quantum teleportation, will be exploited in Section 5.3 in order to convert a symmetric cloner into an asymmetric cloner.

3.5.1. *Covariance with respect to the Weyl–Heisenberg group*

It can be proven that the Heisenberg cloning machines are covariant with respect to the discrete Weyl–Heisenberg group of error operators $\{E_{m,n}\}$. Recall that $E_{\mu,0}$ corresponds to a cyclic relabeling of the computational basis states, while $E_{0,\nu}$ corresponds to a cyclic relabeling of the dual basis states; $E_{\mu,\nu} = E_{\mu,0}E_{0,\nu}$ simply corresponds to a sequence of these cyclic permutations. Thus, Heisenberg cloners are covariant with respect to cyclic permutations of the basis states in the computational and dual basis.

Using eq. (3.5), it can easily be shown that the covariance condition of the cloning map $\mathcal{S}$ with respect to the unitary operator U, namely

$$\mathcal{S}\big(U\psi U^\dagger\big) = U^{\otimes 2}\mathcal{S}(\psi)U^{\dagger\otimes 2} \quad \forall\psi \in R, \tag{3.45}$$

translates into the condition

$$\big(U^* \otimes U^{\otimes 2}\big)S\big(U^{\mathrm{T}} \otimes U^{\dagger\otimes 2}\big) = S \quad \Longleftrightarrow \quad \big[S, U^* \otimes U^{\otimes 2}\big] = 0 \tag{3.46}$$

on the operator S that is isomorphic to $\mathcal{S}$. We may also impose that when the original is transformed according to the unitary U, the cloning machine is transformed according to the unitary U^*. This condition, named *strong covariance*, can be expressed as a constraint on state $|\sigma\rangle$ or $|\mathcal{A}\rangle$, namely

$$\big(U^* \otimes U^{\otimes 2} \otimes U^*\big)|\mathcal{A}\rangle_{\mathrm{in},A,B,C} = |\mathcal{A}\rangle_{\mathrm{in},A,B,C}. \tag{3.47}$$

It was shown recently that, provided that the set of input states is invariant with respect to the Weyl–Heisenberg group, the class of strongly covariant cloning maps is equivalent to the class of *extremal* covariant maps, see Chiribella, D'Ariano, Perinotti and Cerf [2005]. Thus, substituting covariance with strong covariance greatly simplifies the search for optimal cloners since, given that the covariant cloners form a convex set, it is sufficient to search among extremal cloners.

The strong covariance of the Heisenberg cloners can be checked by using condition (3.47) with $U = E_{\mu,\nu}$ for all μ and ν. This equation indeed holds for each component of $\mathcal{A}$, namely

$$\begin{aligned}
&\big(E^*_{\mu,\nu} \otimes E_{\mu,\nu} \otimes E_{\mu,\nu} \otimes E^*_{\mu,\nu}\big)|\Phi_{m,n}\rangle\big|\Phi^*_{m,n}\big\rangle \\
&\quad = \gamma^{m\nu-n\mu}|\Phi_{m,n}\rangle\gamma^{-(m\nu-n\mu)}\big|\Phi^*_{m,n}\big\rangle \\
&\quad = |\Phi_{m,n}\rangle\big|\Phi^*_{m,n}\big\rangle,
\end{aligned} \tag{3.48}$$

where we have used eq. (3.38). Thus, the Heisenberg cloning machines defined by the ansatz state $|\mathcal{A}\rangle$ for an arbitrary matrix $\mathbf{a}$ have the nice property that they keep the same form when making a cyclic permutation of the basis states (in both the computational and dual bases).

This covariance property also implies that the reduced cloning maps are *unital*. It is trivial to prove that applying an error operator $E_{m,n}$ chosen at random (uniformly among the d^2 possibilities) on an arbitrary state ρ always gives a maximally disordered state,

$$\frac{1}{d^2}\sum_{m,n=0}^{d-1} E_{m,n}\rho E^\dagger_{m,n} = \frac{I}{d}. \tag{3.49}$$

Consider an arbitrary input state of the cloner ψ. The covariance and the linearity of the reduced cloning map $\mathcal{S}_A$ or $\mathcal{S}_B$ imply that

$$\mathcal{S}_{A,B}\left(\frac{1}{d^2}\sum_{m,n=0}^{d-1} E_{m,n}\psi E_{m,n}^{\dagger}\right) = \frac{1}{d^2}\sum_{m,n=0}^{d-1} E_{m,n}\mathcal{S}_{A,B}(\psi)E_{m,n}^{\dagger} \tag{3.50}$$

so that, using eq. (3.49), we verify that the reduced cloning maps $\mathcal{S}_{A,B}$ are indeed unital:

$$\mathcal{S}_{A,B}\left[\frac{I}{d}\right] = \frac{I}{d}. \tag{3.51}$$

3.6. *Three special cases of Heisenberg cloners*

3.6.1. *Universal cloners*

Let us now discuss several interesting special cases of Heisenberg cloning machines. The first example is the *universal* (or isotropic) cloning machine, where the channel underlying each output is a quantum-depolarizing channel. This implies that all of the probabilities $p_{m,n} = |a_{m,n}|^2$ except $p_{0,0}$ must be equal. The same holds for the probabilities $q_{m,n} = |b_{m,n}|^2$ associated with the second clone. These conditions put very strong constraints on the matrix **a**, whose elements can be thus parametrized by two real coefficients v and x,

$$\begin{aligned} a_{m,n} &= (v-x)\delta_{n,0}\delta_{m,0} + x \\ &= \begin{pmatrix} v & x & \cdots & x \\ x & x & \cdots & x \\ \vdots & \vdots & \ddots & \vdots \\ x & x & \cdots & x \end{pmatrix}. \end{aligned} \tag{3.52}$$

The Fourier transform yields the matrix elements of **b**, namely

$$b_{m,n} = \left(v'-x'\right)\delta_{n,0}\delta_{m,0} + x', \tag{3.53}$$

with

$$x' = \frac{v-x}{d}, \qquad v' = \frac{v+(d^2-1)x}{d}. \tag{3.54}$$

The cloning is a trace-preserving operation so the condition $\mathrm{Tr}[\rho_{\mathrm{in},A}] = 1$ must be satisfied, which provides the normalization constraint

$$v^2 + \left(d^2-1\right)x^2 = 1. \tag{3.55}$$

If the input state is $|0\rangle$, the error operators $E_{0,n}$ $\forall n$, leave it unchanged up to a phase, while all the other $E_{m,n}$'s produce a state that is orthogonal to it. Therefore, the fidelities of the two clones for any input state can be expressed as

$$F_A = v^2 + (d-1)x^2, \qquad F_B = v'^2 + (d-1)x'^2. \tag{3.56}$$

Note that only a single free parameter x controls the asymmetry of the cloner. We can also characterize the cloner by the fidelity F_A of the first clone, namely,

$$x^2 = \frac{1-F_A}{d(d-1)}, \qquad v^2 = \frac{(d+1)F_A - 1}{d}. \tag{3.57}$$

The symmetric cloner is obtained by putting $x = x'$, which results in

$$x^2 = \frac{1}{2d(d+1)}, \qquad v^2 = \frac{d+1}{2d} \tag{3.58}$$

and is associated with the fidelity given in eq. (2.3).

Note that, as rigorously proved recently for any d by Fiurášek, Filip and Cerf [2005], this isotropic Heisenberg cloner represents the optimal asymmetric cloning machine which, for a fixed fidelity F_A of the first clone, maximizes the fidelity of the second clone F_B. Note also that the optimality of the Heisenberg cloners, based on the double-Bell ansatz, was explained by Chiribella, D'Ariano, Perinotti and Cerf [2005] as a consequence of the extremality of these cloners. It is worth stressing that by exploiting this double-Bell ansatz, these machines can be derived almost without any effort as they follow immediately from the general isotropy- and trace-preservation conditions.

3.6.2. Fourier-covariant cloners

As a second example, we shall consider the *Fourier-covariant* machine, which clones equally well two mutually unbiased bases, the computational basis $\{|k\rangle\}$ and the dual basis

$$\|l\rangle\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \mathrm{e}^{2\pi\mathrm{i}(kl/d)} |k\rangle. \tag{3.59}$$

The cloner copies equally well the states of both bases if the matrix **a** has the form

$$\begin{aligned} a_{m,n} &= (v - 2x + y)\delta_{m,0}\delta_{n,0} + (x-y)(\delta_{m,0} + \delta_{n,0}) + y \\ &= \begin{pmatrix} v & x & \cdots & x \\ x & y & \cdots & y \\ \vdots & \vdots & \ddots & \vdots \\ x & y & \cdots & y \end{pmatrix}, \end{aligned} \tag{3.60}$$

where the parameters satisfy the trace-preservation condition

$$v^2 + 2(d-1)x^2 + (d-1)^2 y^2 = 1. \tag{3.61}$$

The matrix $\mathbf{b}$ then has a similar form with v, x and y being replaced by

$$\begin{aligned} v' &= \frac{1}{d}\big[v + 2(d-1)x + (d-1)^2 y\big], \\ x' &= \frac{1}{d}\big[v + (d-2)x + (1-d)y\big], \\ y' &= \frac{1}{d}[v - 2x + y]. \end{aligned} \tag{3.62}$$

The fidelities of the two clones are again given by eq. (3.56) but now we have two free parameters, say x and y. To eliminate one of them, one has to maximize Bob's fidelity F_B for a given value of Alice's fidelity F_A (using the normalization relation), which is a simple constrained optimization problem. The resulting optimal asymmetric cloner is characterized by

$$v = F_A, \qquad x = \sqrt{\frac{F_A(1-F_A)}{d-1}}, \qquad y = \frac{1-F_A}{d-1}, \tag{3.63}$$

which depends on the single parameter F_A. The symmetric Fourier-covariant cloner can again be obtained by setting $x = x'$ and $y = y'$, which gives eq. (2.16) for the fidelity.

3.6.3. *Phase-covariant cloners*

As a third example, consider the *phase-covariant* machine, which optimally clones all balanced superpositions of the form

$$|\psi\rangle = d^{-1/2}\big[|0\rangle + \mathrm{e}^{\mathrm{i}\phi_1}|1\rangle + \cdots + \mathrm{e}^{\mathrm{i}\phi_{d-1}}|d-1\rangle\big], \tag{3.64}$$

where the ϕ_i's are arbitrary phases. Here, it can be easily shown that the matrix $\mathbf{a}$ must take the form

$$\begin{aligned} a_{m,n} &= (v-y)\delta_{m,0}\delta_{n,0} + (y-x)\delta_{m,0} + x. \\ &= \begin{pmatrix} v & y & \cdots & y \\ x & x & \cdots & x \\ \vdots & \vdots & \ddots & \vdots \\ x & x & \cdots & x \end{pmatrix} \end{aligned} \tag{3.65}$$

while the trace preservation condition is

$$v^2 + (d-1)y^2 + d(d-1)x^2 = 1. \tag{3.66}$$

The matrix **b** has the same form, albeit with v, x and y being replaced by

$$\begin{aligned}
v' &= \frac{1}{d}\big[v + d(d-1)x + (d-1)y\big],\\
x' &= \frac{1}{d}[v - y],\\
y' &= \frac{1}{d}\big[v - dx + (d-1)y\big].
\end{aligned} \tag{3.67}$$

If the input state is $\|0\rangle\rangle$ (all the states of the dual basis are balanced superpositions), the error operators $E_{m,0}$ $\forall m$, leave it unchanged up to a phase, while all the other $E_{m,n}$'s produce a state that is orthogonal to it. Therefore, the fidelities of the two clones are again given by eq. (3.56), and we have two free parameters, say x and y. We can eliminate one of them by maximizing F_B for a given F_A, which yields the optimal phase-covariant cloner. In the special case of a symmetric cloner, we have $x = x'$ and $y = y'$, resulting in the fidelity given by eq. (2.17).

§ 4. *N*-to-*M* universal quantum cloning

4.1. *Optimal cloning transformation*

In this section we will focus on universal (state-independent) cloning. An ideal universal $N \to M$ quantum cloning machine would be a device that prepares M exact clones of an arbitrary state $\psi \in \mathcal{H}$ from N copies of ψ. The input Hilbert space of the cloning transformation is the symmetric subspace $\mathcal{H}_+^{\otimes N}$ of N qudits, and $d = \dim \mathcal{H}$ denotes the dimension of the Hilbert space of the input states. As already explained above, exact deterministic quantum cloning is forbidden by the linearity of quantum mechanics, and only approximate copying with fidelity less than unity is possible.

As noted before, two different kinds of cloning fidelities are considered in the literature. The global fidelity compares the global state of M clones with the ideal output $\psi^{\otimes M}$. Let $\mathcal{S}$ denote the cloning CP map. Then the global fidelity of cloning the state ψ can be expressed as $F_{N\to M}^{\mathrm{univ,G}}(\mathcal{S}, \psi) = \mathrm{Tr}[\psi^{\otimes M}\mathcal{S}(\psi^{\otimes N})]$. Generally, the fidelity of the cloning can depend on ψ and one may define the cloning fidelity as the infimum of $F_{N\to M}^{\mathrm{univ,G}}(\mathcal{S}, \psi)$ over all input states ψ,

$$F_{N\to M}^{\mathrm{univ,G}}(\mathcal{S}) = \inf_{\psi} \mathrm{Tr}\big[\psi^{\otimes M}\mathcal{S}\big(\psi^{N}\big)\big]. \tag{4.1}$$

The single-clone fidelity quantifies how well each clone resembles the desired output ψ. For the kth clone we can write $F_{N\to M}^{\mathrm{univ,SC}}(S, \psi, k) = \mathrm{Tr}[\psi \, \mathrm{Tr}'_k \, \mathcal{S}(\psi^{\otimes N})]$, where Tr'_k denotes the trace over all M qudits except for the kth qudit. When judging the performance of the cloning machine, we should take the infimum of $F^{\text{univ,SC}}_{N\to M}(\mathcal{S}, \psi, k)$ over all input states and all M clones and define

$$F^{\text{univ,SC}}_{N\to M}(\mathcal{S}) = \inf_k \inf_\psi \mathrm{Tr}\big[\psi \, \mathrm{Tr}'_k \, \mathcal{S}\big(\psi^N\big)\big], \tag{4.2}$$

where $k \in \{1, \dots, M\}$.

The *universal* cloning machine should clone all quantum states equally well, so the fidelity should not depend on ψ. Any transformation $\mathcal{S}$ can be converted into a universal cloning transformation whose fidelity is state independent by a twirling operation that consists of applying randomly a unitary $U(\omega)$ to each input ψ and then undoing this by applying a unitary $U^\dagger(\omega)$ to each clone, with the probability density $\mathrm{d}\omega$ equal to the Haar measure on SU(2), see also Section 3.2. The effective map

$$\mathcal{S}_{\text{twirl}}(\psi) = \int_\Omega U^{\dagger\otimes M}(\omega)\mathcal{S}\big[\big(U(\omega)\psi U^\dagger(\omega)\big)^{\otimes N}\big]U^{\otimes M}(\omega)\,\mathrm{d}\omega \tag{4.3}$$

is covariant, i.e., $\mathcal{S}_{\text{twirl}}[(U\psi U^\dagger)^{\otimes N}] = U^{\otimes M}\mathcal{S}_{\text{twirl}}(\psi^{\otimes N})U^{\dagger\otimes M}$ and, consequently, $F^{\text{univ,G}}_{N\to M}$ does not depend on ψ. To guarantee the independence of the single-clone fidelity on the clone index k, it is also necessary to randomly permute the M clones after the twirling. The important feature of the twirling operation and the permutations is that they do not modify the *mean fidelity* calculated as the average of $F^{\text{univ,G}}_{N\to M}(\mathcal{S}, \psi)$ or $\frac{1}{M}\sum_{k=1}^{M} F^{\text{univ,SC}}_{N\to M}(\mathcal{S}, \psi, k)$ over all input states $|\psi\rangle = U(\omega)|\psi_0\rangle$ with the measure $\mathrm{d}\omega$.

Universal cloning has been studied extensively by many authors (Bužek and Hillery [1996, 1998a], Gisin and Massar [1997], Hillery and Bužek [1997], Bužek, Hillery and Knight [1998], Cerf [1998, 1999, 2000a, 2000b], Werner [1998], Niu and Griffiths [1998], Keyl and Werner [1999]). The task of cloning can be rephrased as diluting the quantum information carried by the N input qudits into M output qudits. Universal cloning should not prefer any direction in Hilbert space and should be isotropic. As shown by Werner [1998], the optimal universal cloning operation $\mathcal{S}_{\text{opt}}$ can be expressed as follows:

$$\mathcal{S}_{\text{opt}}\big(\psi^{\otimes N}\big) = \frac{D(N,d)}{D(M,d)}\Pi^+_{M,d}\big(\psi^{\otimes N} \otimes I^{\otimes(M-N)}\big)\Pi^+_{M,d}, \tag{4.4}$$

where $\Pi^+_{M,d}$ is the projector onto the fully symmetric (Bose) subspace of M qudits and

$$D(M,d) = \binom{d+M-1}{M} \tag{4.5}$$

is the dimension of this subspace. We can see from eq. (4.4) that the optimal cloning formally consists in attaching $M - N$ blank copies prepared in the maximally mixed state I/d to the input state $\psi^{\otimes N}$ and then projecting the whole state of M qudits onto the symmetric subspace of M qudits, $\mathcal{H}_+^{\otimes M}$. With proper normalization as given in eq. (4.4), $\mathcal{S}_{\rm opt}$ is a trace-preserving completely positive map and can be therefore realized deterministically.

The maximal global cloning fidelity achieved by the optimal cloner (4.4) reads

$$F_{N\to M}^{\rm univ,G} = \frac{D(N,d)}{D(M,d)} = \frac{(d+N-1)!M!}{(d+M-1)!N!}. \tag{4.6}$$

The density matrix of each output clone is a convex mixture of the input state ψ and the maximally mixed state I/d,

$$\rho = \eta\psi + \frac{1}{d}(1-\eta)I. \tag{4.7}$$

This expression reveals the high isotropy of universal symmetric quantum cloning which is fully characterized by a single parameter, namely the shrinking factor $\eta(N, M)$,

$$\eta(N,M) = \frac{N}{N+d}\frac{M+d}{M}. \tag{4.8}$$

The single-clone fidelity can be determined immediately from eq. (4.7) and we confirm eq. (2.4), that is,

$$F_{N\to M}^{\rm univ,SC} = \frac{MN+M+N(d-1)}{M(N+d)}. \tag{4.9}$$

4.1.1. Connection with quantum state estimation

There is a close relationship between optimal quantum cloning and optimal quantum state estimation. As shown by Bruss, Ekert and Macchiavello [1998] and Bruss [1999], in the limit of an infinite number of clones, $M \to \infty$, the single-clone fidelity $F_{N\to M}^{\rm univ,SC}$ becomes equal to the fidelity of the optimal estimation of the state ψ from N copies (Massar and Popescu [1995], Bruss and Macchiavello [1999], Hayashi, Hashimoto and Horibe [2004]):

$$F_{N\to\infty} = \frac{N+1}{N+d}. \tag{4.10}$$

Consequently, in the limit $M \to \infty$ the optimal cloning becomes equivalent to the optimal state estimation from N copies of ψ followed by the preparation of infinitely many copies of the estimated state. This relationship between optimal universal cloning and optimal state estimation can be explored to prove the optimality of the cloning transformation (4.4). It follows from the symmetry, isotropy and linearity of universal quantum cloning that the single-qudit outputs must have the form (4.7), and that for concatenated universal cloners the shrinking factors multiply. Since the concatenation of the optimal $N \to M$ and $M \to L$ cloners cannot be better than the optimal $N \to L$ cloner, we get

$$\eta(N, L) \geqslant \eta(N, M)\eta(M, L). \tag{4.11}$$

Taking the limit $L \to \infty$, and taking into account that the shrinking factor corresponding to the fidelity (4.10) reads $\eta(N, \infty) = N/(N + d)$, we get from the inequality (4.11) an upper bound on $\eta(N, M)$,

$$\eta(N, M) \leqslant \frac{\eta(N, \infty)}{\eta(M, \infty)} = \frac{N}{N + d}\frac{M + d}{M}, \tag{4.12}$$

which is saturated by the optimal universal cloning transformation (4.4).

4.1.2. *Unitary realization and quantum circuit*

So far the optimal cloning transformation was presented in the form of the rather abstract CP map (4.4). It holds that every trace-preserving CP map admits a unitary realization with the use of an ancilla system. The unitary realization of cloning requires $2(M - N)$ ancilla qudits: $M - N$ blank copies and $M - N$ additional ancillas. For the sake of presentation simplicity we will consider here the $N \to M$ cloning of qubits (Gisin and Massar [1997]). The unitary cloning transformation can be expressed in a covariant form:

$$\begin{aligned} &U|N\psi\rangle_{\mathrm{in}}|R\rangle_{\mathrm{anc}} \\ &\quad= \sum_{j=0}^{M} \alpha_j \big|(M - j)\psi, j\psi^{\perp}\big\rangle_{\mathrm{clones}} \big|(M - N - j)\psi^{\perp}, j\psi\big\rangle_{\mathrm{anc}}. \end{aligned} \tag{4.13}$$

Here $|k\psi, (N - k)\psi^{\perp}\rangle$ denotes a symmetric state of N qubits with k qubits in state $|\psi\rangle$ and $N - k$ qubits in an orthogonal state $|\psi^{\perp}\rangle$, $\langle\psi|\psi^{\perp}\rangle = 0$, $|R\rangle_{\mathrm{anc}}$ denotes the initial state of the ancilla qubits, and

$$\alpha_j = (-1)^j \binom{M - j}{N}^{1/2} \binom{M + 1}{M - N}^{-1/2}. \tag{4.14}$$

The generalization of the formula (4.13) to qudits with arbitrary d was obtained by Fan, Matsumoto and Wadati [2001]. In Section 5.1 we shall show that the transformation (4.13) arises naturally in stimulated amplification of light when the qubits are represented by the polarization states of single photons.

Quantum information theory teaches us that an arbitrary unitary operation U can be implemented as a sequence of single-qubit rotations and two-qubit

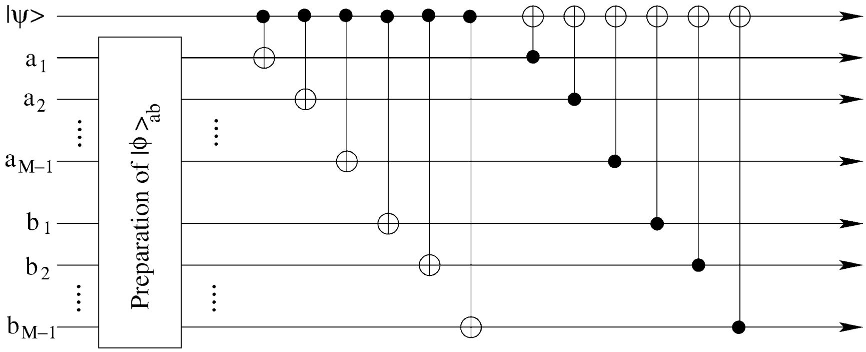


Fig. 3. Network for optimal universal $1 \to M$ cloning of qubits. The cloning requires $M-1$ blank copies $a_1, \ldots, a_{M-1}$ and $M-1$ ancilla qubits $b_1, \ldots, b_{M-1}$. The cloning consists of the preparation of an entangled state $|\Phi\rangle_{ab}$ of blank copies and ancillas followed by a sequence of $2(M-1)$ C-NOT gates between the input qubit a_0 and the blank copies and ancillas.

controlled-NOT gates, $U_{\mathrm{CNOT}} = |j\rangle_c|k\rangle_t = |j\rangle_c|k \oplus j\rangle_t$, where $\oplus$ denotes addition modulo 2, c is the control qubit and t is the target qubit. The quantum network for the optimal universal $1 \to M$ cloning of qubits (Bužek, Braunstein, Hillery and Bruss [1997], Bužek and Hillery [1998b]) is depicted in fig. 3. First, the $2(M-1)$ ancilla qubits $a_1, \ldots, a_{M-1}$ and $b_1, \ldots, b_{M-1}$ are prepared in an entangled state

$$|\Phi\rangle_{ab} = \frac{1}{M}\sqrt{\frac{2}{M+1}} \times \sum_{k=0}^{M-1} \big(e_k|M-1,k\rangle_a + f_k|M-1,k-1\rangle_a\big)|M-1,k\rangle_b, \quad (4.15)$$

with $e_k = M-k$ and $f_k = \sqrt{k(M-k)}$, where $|M-1,k\rangle$ denotes a symmetric state of $M-1$ qubits with k qubits in state $|1\rangle$ and $M-1-k$ qubits in state $|0\rangle$. The state (4.15) can be generated by a sequence of single-qubit rotations and C-NOT gates starting from any initial pure state of the ancilla. The cloning itself consists of a sequence of $2(M-1)$ C-NOT gates where the qubit a_0 that contains the state $|\psi\rangle$ to be copied serves as a control qubit and the ancillas are target qubits. This is followed by another sequence of $2(M-1)$ C-NOT gates where now the qubit a_0 is target and the ancilla qubits are controls. The M clones are stored in the qubits $a_0, \ldots, a_{M-1}$ while the qubits $b_1, \ldots, b_{M-1}$ represent the ancillas.

4.2. Optimality proof for $1 \to M$ cloning of qubits

The optimality of the $1 \to 2$ symmetric cloning machine for qubits was first proved by Bruss, DiVincenzo, Ekert, Fuchs, Macchiavello and Smolin [1998]. The optimality of the cloning transformation (4.4) for arbitrary number of inputs N, outputs M and dimension d was proved by Werner [1998] for the global fidelity and later by Keyl and Werner [1999] for the single-clone fidelity using powerful group-theoretical techniques. Here we shall present a simple optimality proof for the class of $1 \to M$ universal symmetric cloning machines for qubits. This proof follows the general concept outlined in Section 3.3 where it was shown that the fidelity is upper bounded by the maximum eigenvalue of a certain positive semidefinite operator. This optimality proof with single-clone fidelity being used as a figure of merit is similar to that of Gisin and Massar [1997]; it has been extended to global fidelity (Fiurášek [2001b]). The advantage of this approach is that it can easily be generalized to asymmetric cloning, as will be discussed in the next section.

Consider the maximization of the single-clone fidelity and let us assume that the output Hilbert space of the cloning map $\mathcal{S}$ is the symmetric subspace, since the desired outputs $|\psi\rangle^{\otimes M} \in \mathcal{H}_{+}^{\otimes M}$. Then all the clones have the same fidelity by construction and we can express the operator S that is isomorphic to the CP map $\mathcal{S}$ as follows:

$$S = \sum_{i,j=0}^{1} \sum_{k,l=0}^{M} S_{ik,jl} |i\rangle_{\text{in}}\langle j| \otimes |M,k\rangle_{\text{out}}\langle M,l|. \tag{4.16}$$

The mean single-clone fidelity can be calculated by averaging over the surface of the Poincaré sphere,

$$F_{N\to M}^{\text{univ,SC}} = \int_{\psi} \text{Tr}\big[\big(\psi^{\text{T}} \otimes \psi\big)\, \text{Tr}'_{\text{out}}(S)\big]\, \text{d}\psi, \tag{4.17}$$

where Tr'_{out} denotes tracing over all output qubits except for the first one, and

$$\int_{\psi} \text{d}\psi \equiv \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \sin\vartheta \, \text{d}\vartheta \, \text{d}\phi,$$
$$|\psi\rangle = \cos\frac{\vartheta}{2}|0\rangle + \text{e}^{\text{i}\phi} \sin\frac{\vartheta}{2}|1\rangle. \tag{4.18}$$

After the tracing and integration, we find that the mean single-clone fidelity is a linear function of S, $F_{N\to M}^{\text{univ,SC}} = \text{Tr}[S R_{\text{SC}}]$, where the positive semidefinite operator R_{SC} reads

$$R_{\mathrm{SC}} = \frac{1}{6M}\sum_{k=0}^{M}\big[(2M-k)|0\rangle\langle 0| + (M+k)|1\rangle\langle 1|\big]\otimes|M,k\rangle\langle M,k| + \frac{1}{6M}\sum_{k=0}^{M-1}\sqrt{(M-k)(k+1)}|1\rangle\langle 0|\otimes|M,k+1\rangle\langle M,k| + \mathrm{h.c.}$$

According to eq. (3.20) the fidelity $F_{N\to M}^{\mathrm{univ,SC}}$ is upper bounded by the maximum eigenvalue of R_{SC}, $F_{N\to M}^{\mathrm{univ,SC}} \leqslant 2r_{\mathrm{SC,max}}$. The matrix R_{SC} has a block diagonal structure and it is easy to show that all eigenstates of R_{SC} have the form $\alpha|0\rangle|M,k\rangle + \beta|1\rangle|M,k+1\rangle$. The calculation of the eigenvalues of R_{SC} thus reduces to finding roots of quadratic polynomials, and one finds that R_{SC} has only three different eigenvalues, $r_1 = (2M+1)/(6M)$, $r_2 = \frac{1}{3}$ and $r_3 = \frac{1}{6}$. This provides an upper bound $F_{N\to M}^{\mathrm{univ,SC}} \leqslant (2M+1)/(3M)$ which is saturated by the cloning machine (4.4). This proves that the machine (4.4) is optimal.

A similar chain of arguments can be used to demonstrate the optimality of the machine (4.4) when global fidelity is the figure of merit. The mean global fidelity can be written as $F_{N\to M}^{\mathrm{univ,G}} = \mathrm{Tr}[SR_{\mathrm{G}}]$, where

$$R_{\mathrm{G}} = \int_{\psi} \psi_{\mathrm{in}}^{\mathrm{T}} \otimes \psi_{\mathrm{out}}^{\otimes M}\,\mathrm{d}\psi. \tag{4.19}$$

With the help of Schur's lemma this integral can easily be evaluated, and one obtains $R_{\mathrm{G}} = \frac{1}{M+2}\Pi_{M+1}^{+,\mathrm{T}_1}$, where Π_{M+1}^{+} is a projector onto the symmetric subspace of $M+1$ qubits, and T_1 denotes partial transposition with respect to the first qubit. Again, the matrix R_G is block diagonal and its eigenvalues can easily be determined analytically. One finds that $r_{\mathrm{G,max}} = 1/(M+1)$ which implies $F_{1\to M}^{\mathrm{univ,G}} \leqslant 2/(M+1)$, and this bound is achieved by the cloner (4.4).

4.3. Universal asymmetric quantum cloning

Quantum cloning machines serve as universal distributors of quantum information among several parties. The symmetric cloner divides the information equally between all M copies but it is also possible to distribute the information unequally. A lot of attention has been devoted to universal asymmetric $1\to 2$ cloning machines for qubits (Cerf [1998], Bužek, Hillery and Bednik [1998], Niu and Griffiths [1998], Cerf [1999, 2000a]) and qudits (Cerf [2000b], Cerf, Bourennane, Karlsson and Gisin [2002]), which produce two clones A and B with different fidelities F_A and F_B. The optimal asymmetric cloner can be defined as a machine that for a given fixed fidelity F_A of the first clone maximizes the fidelity F_B of the second clone. Such machines can find application, e.g., in eavesdropping on quantum key distribution protocols, where they allow one to investigate the trade-off between the information gained by the eavesdropper and the disturbance observed at the receiver's station; see, e.g., Dusek, Lutkenhaus and Hendrych [2006].

In terms of the cloning CP map S, the mean fidelities of the two clones can be expressed as $F_A = \mathrm{Tr}[SR_A]$ and $F_B = \mathrm{Tr}[SR_B]$, where the positive semidefinite operators R_j are given by

$$R_A = \int_\psi \psi_{\mathrm{in}}^{\mathrm{T}} \otimes \psi_A \otimes I_B \,\mathrm{d}\psi, \qquad R_B = \int_\psi \psi_{\mathrm{in}}^{\mathrm{T}} \otimes I_A \otimes \psi_B \,\mathrm{d}\psi. \tag{4.20}$$

The optimal asymmetric cloning machine should maximize a convex mixture of the mean fidelities F_A and F_B (Fiurášek [2003], Lamoureux, Navez, Fiurášek and Cerf [2004], Fiurášek, Filip and Cerf [2005]),

$$F = pF_A + (1-p)F_B = \mathrm{Tr}[SR], \tag{4.21}$$

where $R = pR_A + (1-p)R_B$, and p is a parameter that controls the asymmetry of the cloner. The maximization of F for a given value of p can be equivalently rephrased as a maximization of F_B for a fixed value of F_A. After some algebra, we find

$$R = \frac{1}{d(d+1)}\big[I_{\mathrm{in},AB} + dp\Phi^+_{\mathrm{in},A} \otimes I_B + d(1-p)\Phi^+_{\mathrm{in},B} \otimes I_A\big]. \tag{4.22}$$

The maximum eigenvalue of R is d-fold degenerate, with corresponding eigenvector

$$|r_{\max}; k\rangle = \alpha\big|\Phi^+\big\rangle_{AR}|k\rangle_B + \beta\big|\Phi^+\big\rangle_{BR}|k\rangle_A, \tag{4.23}$$

where the coefficients $\alpha, \beta \geqslant 0$ are some functions of d and p. By properly normalizing the eigenstates (4.23) we get

$$\alpha^2 + \beta^2 + \frac{2\alpha\beta}{d} = 1. \tag{4.24}$$

The operator S isomorphic to the optimal cloning CP map $\mathcal{S}$ is proportional to the projector onto the subspace spanned by the eigenstates (4.23). The unitary realization of this map requires a single ancilla qudit C and can be written in a covariant way:

$$|\psi\rangle \to \alpha|\psi\rangle_A\big|\Phi^+\big\rangle_{BC} + \beta|\psi\rangle_B\big|\Phi^+\big\rangle_{AC}. \tag{4.25}$$

From this expression we can evaluate the fidelities of the two clones,

$$F_A = 1 - \frac{d-1}{d}\beta^2, \qquad F_B = 1 - \frac{d-1}{d}\alpha^2. \tag{4.26}$$

Note that the parameters α^2 and β^2 are the so-called *depolarizing fractions* as discussed by Cerf [1998, 2000a, 2000b]. The one-parametric class of optimal universal asymmetric $1 \to 2$ cloning machines is characterized by eqs. (4.25) and (4.26) together with the normalization condition (4.24).

4.4. Universal-NOT gate

The process of optimal quantum cloning is closely connected to another impossible operation in quantum mechanics, the so-called universal-NOT gate for qubits. The hypothetical universal-NOT gate would perfectly reverse any spin-$\frac{1}{2}$ state. This device should thus produce from the input qubit $|\psi\rangle$ an orthogonal state $|\psi_\perp\rangle$. However, this is impossible, because the transformation $|\psi\rangle \to |\psi_\perp\rangle$ is anti-unitary. More generally one can consider an extended scenario where N copies of the state $|\psi\rangle$ are available and the task is to prepare a single copy of the flipped spin $|\psi_\perp\rangle$. The best approximation to this forbidden operation was found by Gisin and Popescu [1999], Bužek, Hillery and Werner [1999, 2000].

The optimal universal NOT gate S_{UNOT} can be made covariant by twirling so that the fidelity $F = \langle\psi_\perp|\mathcal{S}_{\mathrm{UNOT}}(\psi^{\otimes N})|\psi_\perp\rangle$ does not depend on $|\psi\rangle$ and can be written as $F_{\mathrm{UNOT}} = \mathrm{Tr}[S_{\mathrm{UNOT}} R_{\mathrm{UNOT}}]$, where

$$\begin{aligned} R_{\mathrm{UNOT}} &= \int_\psi \left[\psi^{\otimes N}\right]^{\mathrm{T}} \otimes \psi_\perp \,\mathrm{d}\psi \\ &= \frac{1}{N+2}\big(U^{\otimes N} \otimes I\big)\Pi^+_{N+1}\big(U^{\dagger\otimes N} \otimes I\big), \end{aligned} \tag{4.27}$$

see Fiurášek [2001b]. The unitary operation $U = \mathrm{i}\sigma_y$ provides the link between the states $|\psi^*\rangle$ and $|\psi_\perp\rangle$, $|\psi^*\rangle = U|\psi_\perp\rangle$, $U|0\rangle = -|1\rangle$, $U|1\rangle = |0\rangle$. The fidelity of the U-NOT gate is bounded by the maximum eigenvalue of R_{UNOT}. Since this operator is proportional to a projector, we immediately find $r_{\mathrm{UNOT,max}} = 1/(N+2)$. The dimension of the input Hilbert space $\mathcal{H}_+^{\otimes N}$ is $d = N+1$, and we obtain

$$F_{\mathrm{UNOT}} = \frac{N+1}{N+2}. \tag{4.28}$$

Remarkably, this fidelity coincides with the optimal fidelity of the estimation of the state $|\psi\rangle$ from N copies. If we possess an estimate of $|\psi\rangle$ then we can also produce an estimate of $|\psi_\perp\rangle$ with the same fidelity, simply by flipping the estimated spin. This implies that the optimal U-NOT gate can be realized by performing the optimal estimation of the state $|\psi\rangle$ followed by the preparation of the flipped estimated state. In this way we can generate arbitrarily many approximate copies of $|\psi_\perp\rangle$, all with the same fidelity (4.28). Remarkably, the optimal cloning transformation (4.13) simultaneously also implements the optimal approximate U-NOT gate. This machine produces $M - N$ approximate anti-clones, which are stored in the ancillas, see Bužek, Hillery and Werner [1999], De Martini, Bužek, Sciarrino and Sias [2002].

It is possible to generalize the concept of U-NOT gate to qudits, by noting that the state $|\psi_\perp\rangle$ is unitarily equivalent to the state $|\psi^*\rangle$. The complex conjugation is well defined for any dimension d, and one can look for the transformation that optimally approximates the (generalized) transposition map $\psi^{\otimes N} \to \psi^* \equiv \psi^{\mathrm{T}}$. Using similar reasoning as before, one can prove that the maximal fidelity of the approximate transposition is equal to the fidelity (4.10) of the optimal state estimation from N copies, see Fiurášek [2004].

§ 5. Universal cloning of photons

5.1. Amplification of light

In quantum optics, single photons are very often used as carriers of quantum information. Photons represent ideal flying qubits; they can be transmitted over long distances via low-loss optical fibers and their interaction with the environment is very weak so they do not suffer from a significant decoherence. Quantum bits can be encoded into single photons in various ways. One natural option is to exploit the polarization degrees of freedom and to represent a qubit as a superposition of vertically ($|V\rangle$) and horizontally ($|H\rangle$) polarized photons, $|\psi\rangle = \alpha|H\rangle + \beta|V\rangle$. Another possibility is to use the so-called time-bin encoding where the photon can be located in one of d different time slots (Marcikic, de Riedmatten, Tittel, Scarani, Zbinden and Gisin [2002], de Riedmatten, Marcikic, Tittel, Zbinden, Collins and Gisin [2004]). Such encoding has been used advantageously for long-distance quantum key distribution. It is not restricted to qubits, and the photon can thus represent a d-dimensional system with arbitrary d (de Riedmatten, Marcikic, Scarani, Tittel, Zbinden and Gisin [2004]). Arbitrary time-bin qubits can be prepared using an unbalanced Mach–Zehnder interferometer.

The cloning of the quantum states of single photons requires that the number of output photons be higher than the number of input photons. This simple fact immediately leads to the insight that the optimal copying of photons can be performed by means of amplification of light (De Martini, Mussi and Bovino [2000], Simon, Weihs and Zeilinger [2000a, 2000b]). This is very natural because the goal of quantum cloning is to “amplify” the quantum information carried by the photons. Several physical mechanisms can be used for cloning, such as parametric down-conversion or amplification of light in atomic media. In all cases, the cloning is achieved due to the process of stimulated emission, which means that the medium emits preferably photons in the same quantum state as that of the input photons injected into the medium.

Most of the quantum cloning experiments based on stimulated amplification of light were carried out using the process of stimulated parametric down-conversion. Consider a nonlinear crystal with second-order nonlinearity $\chi^{(2)}$. In such a crystal, a single "blue" pump photon with frequency ω_{P} can be converted into two "red" photons with frequencies ω_{S} and ω_{I} such that $\omega_{\mathrm{S}} + \omega_{\mathrm{I}} = \omega_{\mathrm{P}}$, which expresses energy conservation. The two down-converted photons are referred to as signal (S) and idler (I), respectively, for historical reasons. An efficient down-conversion requires the conservation of momentum, which translates into the phase-matching condition $\mathbf{k}_{\mathrm{S}} + \mathbf{k}_{\mathrm{I}} = \mathbf{k}_{\mathrm{P}}$, where $\mathbf{k}_j$ stands for the wavevector of the jth photon and $|\mathbf{k}_j| = n_j\omega_j/c$, where n_j is the refraction index at frequency ω_j. Efficient phase matching in the nonlinear crystal can be achieved by exploiting the birefringence and using different polarizations for the pump, signal and idler beams. We can distinguish two different kinds of phase matching. In Type-I matching the pump beam is, say, vertically polarized, and both signal and idler are horizontally polarized. On the other hand, in Type-II matching, the signal and idler photons are orthogonally polarized. Besides their polarization states, the signal and idler beams can also be distinguished spatially. So, in nondegenerate Type-II down-conversion we deal with modes A_H and A_V for the signal beam and B_H and B_V for the idler beam. It is possible to arrange the configuration of the pump beam and nonlinear crystal and to select only certain directions in the output beams in such a way that the effective Hamiltonian describing this process reads

$$H = \mathrm{i}\kappa\left(a_V^\dagger b_H^\dagger - a_H^\dagger b_V^\dagger\right) + \mathrm{h.c.} \tag{5.1}$$

This Hamiltonian is obtained in the limit of strong coherent pumping, and the coupling constant κ is proportional to the pump-beam amplitude α_{P} and to the second-order nonlinearity $\chi^{(2)}$, while $a_j^\dagger$ is the creation operator for the jth mode. An essential feature of the Hamiltonian (5.1) is that it is invariant with respect to the simultaneous identical transformation of the polarization basis of signal and idler photons. Mathematically, we have $(U \otimes U)H(U^\dagger \otimes U^\dagger) = H$, where $Ua_V U^\dagger = u_{VV}a_V + u_{VH}a_H$, $Ua_H U^\dagger = u_{HV}a_V + u_{HH}a_H$, the matrix u_{ij} is unitary, and identical transformation rules hold for b_V and b_H. This covariance property guarantees that the cloning process is universal and the cloning fidelity

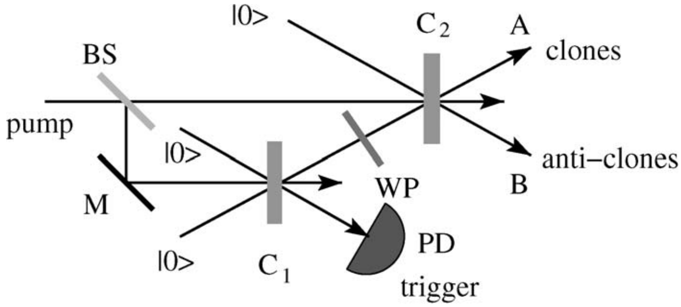


Fig. 4. Optimal universal cloning of photons by means of stimulated parametric down-conversion. BS: auxiliary beamsplitter; C_1, C_2 – nonlinear crystals; WP – wave plates for input-state preparation; M – mirror; PD – triggering single-photon detector for conditional preparation of N-photon Fock state.

is the same for all input states. It therefore suffices to consider only one particular input state.

The cloning of polarization states of photons via stimulated down-conversion is sketched in fig. 4. The signal mode is initially prepared in the N-photon state $|\psi\rangle_a^{\otimes N}$. This can be achieved in practice, e.g., by means of spontaneous parametric down-conversion in crystal C_1 and conditioning on observing N photons in the output idler mode with photodetector PD. After the passage through crystal C_2, $M - N$ photon pairs can be generated with a certain probability. If this happens, then M clones are present in mode A while mode B contains $M - N$ anti-clones. Note that the cloning is only probabilistic and we cannot predict a priori the number of clones that will be generated. The particular $N \to M$ cloning events can be selected only a posteriori by accepting events with M photons detected in mode A or $M - N$ photons in mode B.

Let us start with a simple example of $1 \to 2$ cloning to illustrate all the main features. In this case, the input state is given by a single photon in mode A and a vacuum in mode B. As already explained, without loss of generality we can assume that the photon is vertically polarized and we have $|\psi_{\rm in}\rangle = |1\rangle_{aV}|0\rangle_{aH}|0\rangle_{bV}|0\rangle_{bH}$. The generation of the second clone requires that a single photon pair is emitted in the nonlinear crystal. In the first-order perturbation theory, the output state is given by

$$\begin{aligned} &H|1\rangle_{aV}|0\rangle_{aH}|0\rangle_{bV}|0\rangle_{bH} \\ &\quad\propto \sqrt{2}|2\rangle_{aV}|0\rangle_{aH}|0\rangle_{bV}|1\rangle_{bH} - |1\rangle_{aV}|1\rangle_{aH}|1\rangle_{bV}|0\rangle_{bH}. \end{aligned} \tag{5.2}$$

Notice the prefactor $\sqrt{2}$ which arises because the emission of the second vertically polarized photon in mode A is stimulated by the presence of a vertically polarized photon in this spatial mode. This cloning is optimal since it yields the maximum fidelity. We can immediately see that the global fidelity is $\frac{2}{3}$. To determine the single-clone fidelity we note that with probability $\frac{2}{3}$ both photons in mode A are vertically polarized and with probability $\frac{1}{3}$ only one photon is vertically polarized. So, the probability that one randomly chosen photon in spatial mode A is vertically polarized is $\frac{2}{3} \times 1 + \frac{1}{3} \times \frac{1}{2} = \frac{5}{6}$, which is the maximal single-clone fidelity for $1 \to 2$ cloning of qubits.

Several experiments on cloning via parametric down-conversion have been reported (De Martini, Mussi and Bovino [2000], Lamas-Linares, Simon, Howell and Bouwmeester [2002], Pelliccia, Schettini, Sciarrino, Sias and De Martini [2003], Sias, Sciarrino and De Martini [2003], De Martini, Pelliccia and Sciarrino [2004]). The experimental set-up used by Lamas-Linares, Simon, Howell and Bouwmeester [2002] is shown in fig. 5. A nonlinear BBO crystal is pumped by a second harmonic of a Ti:sapphire laser which emits 120-fs-long pulses. A tiny part of the coherent master laser beam is split on the first beamsplitter BS and used as a seed for the down-conversion. With probability $p \ll 1$, the beam contains exactly one photon. This beam is fed to the BBO crystal and the output is analyzed us-

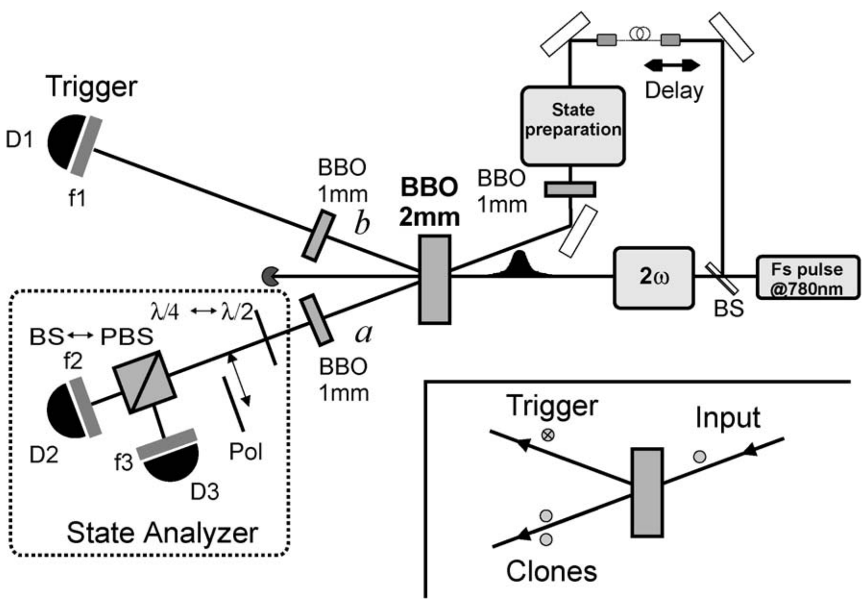


Fig. 5. Experimental set-up for optimal universal cloning by means of stimulated parametric down-conversion. The input single-photon state to be cloned is obtained from a weak coherent laser beam. (After Lamas-Linares, Simon, Howell and Bouwmeester [2002].)

ing a sequence of wave plates, polarizing beamsplitters (PBS) and single-photon photodetectors. The probability of pair generation in the crystal $p_2 \ll p$ which guarantees that the dominant event leading to two photons in mode a and one photon in mode b is when a single photon was in the weak coherent beam a and a single pair was emitted in the crystal. The conditioning on observing a click of the trigger detector D1 is important since it eliminates events with two photons in mode a and no pair generated in the crystal.

In the experiment, one measures the number of coincidence clicks of the photodetectors D2 and D3 as a function of the time delay between the input photon beam in mode a and the pump beam. If those two beams do not overlap in the BBO crystal, then there is no stimulated down-conversion and the polarization of the second photon emitted in mode a is fully random. If the two beams overlap, then stimulated amplification sets on and the second photon is emitted preferably with the same polarization as the input photon. Optimal cloning is achieved when the overlap is perfect. A detector setting with a PBS was used to measure the number of orthogonally polarized photon pairs $N(1,1)$. To detect the number of pairs with the same polarization $N(2,0)$, the PBS was replaced by a polarizer followed by an ordinary beamsplitter. The observed coincidence rates as a function of the time delay are shown in fig. 6 for three different polarizations. We see that $N(1,1)$ does not depend on the delay as expected, while $N(2,0)$ decreases with increasing delay. The average experimental cloning fidelity determined from these data reads $F \approx 0.81$ which is very close to the theoretical maximum $\frac{5}{6} \approx 0.833$.

An improved experimental set-up involving double passage of the pump beam through the nonlinear crystal was developed by Pelliccia, Schettini, Sciarrino, Sias and De Martini [2003] and De Martini, Pelliccia and Sciarrino [2004]), see fig. 7. In this set-up, the photon to be cloned is generated during the first passage of the pump pulse through the crystal. Since the signal and idler beams are entangled, projecting the idler beam onto state $|\psi\rangle$ prepares the signal in state $|\psi_\perp\rangle$. The click of the trigger detector D_T heralds the preparation of a single photon in the mode labeled $-k_1$ in fig. 7. This photon is then cloned by sending it again through the nonlinear BBO crystal. The delay between the pump and signal is controlled by moving the mirror M_P. In this experiment, the states of both clones and the anti-clone were analyzed simultaneously and it was demonstrated that this device accomplishes jointly the optimal $1 \to 2$ cloning and also the optimal universal NOT gate for qubits. The attained fidelities were $F_{\mathrm{CLON}} = 0.81$ and $F_{\mathrm{UNOT}} = 0.62$.

The stimulated down-conversion can be used to probabilistically implement any $N \to M$ cloning of qubits (Simon, Weihs and Zeilinger [2000a]) and even qudits (Kempe, Simon and Weihs [2000], Fan, Weihs, Matsumoto and Imai [2002]).

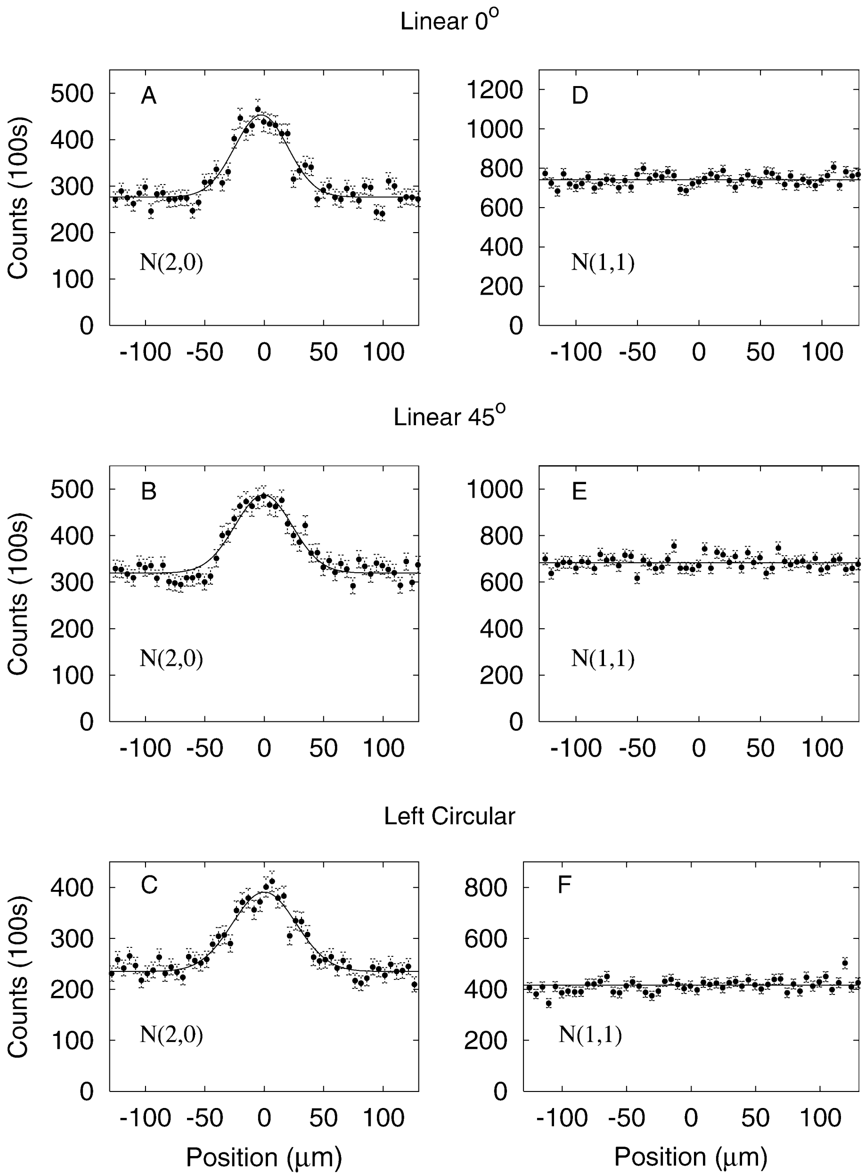


Fig. 6. Observed coincidence rates as functions of the position of the movable mirror of two clones in identical polarization states (panels A, B, C) and in orthogonal polarization (panels D, E, F) for three different input polarizations. (After Lamas-Linares, Simon, Howell and Bouwmeester [2002].)

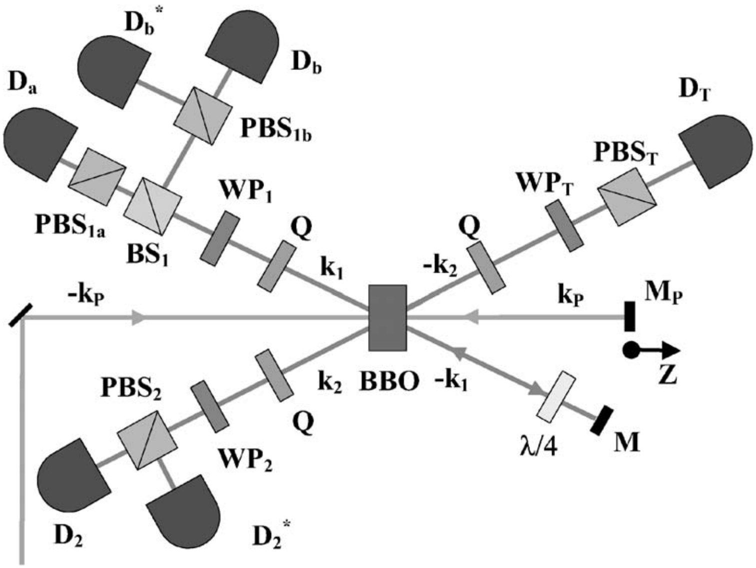


Fig. 7. Simultaneous contextual realization of the optimal cloning machine and the universal NOT gate. Q – quarter-wave plates; WP – half-wave plates; PBS – polarizing beamsplitters; BS – beamsplitters, BBO – nonlinear crystal; D – single-photon detectors; M – movable mirror. (After De Martini, Pelliccia and Sciarrino [2004].)

The unitary transformation induced by the Hamiltonian H can be written in a factorized form as follows:

$$\mathrm{e}^{-\mathrm{i}Ht} = \mathrm{e}^{\lambda(a_V^\dagger b_H^\dagger - a_H^\dagger b_V^\dagger)} \left(1-\lambda^2\right)^{n_{\mathrm{tot}}/2+1} \mathrm{e}^{-\lambda(a_V b_H - a_H b_V)}, \tag{5.3}$$

where $\lambda = \tanh(\kappa t)$, t is an effective interaction time and $n_{\mathrm{tot}} = a_V^\dagger a_V + a_H^\dagger a_H + b_V^\dagger b_V + b_H^\dagger b_H$ is the total number of photons in spatial modes a and b. Since the Hamiltonian H is covariant it is enough to consider the input state $|\psi_{\mathrm{in}}\rangle = |N\rangle_{a_V}|0\rangle_{a_H}|0\rangle_{b_V}|0\rangle_{b_H}$. With the help of the factorization (5.3) we find that the corresponding output state reads

$$\mathrm{e}^{-\mathrm{i}Ht}|\psi_{\mathrm{in}}\rangle = \left(1-\lambda^2\right)^{N/2+1} \sum_{M=N}^{\infty} \lambda^{M-N} |\Psi_M\rangle, \tag{5.4}$$

where

$$|\Psi_M\rangle = \sum_{k=0}^{M-N} (-1)^k \sqrt{\binom{M-k}{N}} \times |M-k\rangle_{a_V}|k\rangle_{a_H}|k\rangle_{b_V}|M-N-k\rangle_{b_H}. \tag{5.5}$$

We can see that the output state (5.4) is a weighted superposition of states $|\Psi_M\rangle$ with different numbers of clones M. The state $|\Psi_M\rangle$ and hence the fidelity of $N \to M$ cloning is independent of the coupling strength λ, and only the probability of generating exactly M clones depends on λ. One can also immediately see that the state (5.5) coincides (up to an irrelevant overall normalization factor) with the outcome of the optimal cloning transformation (4.13), hence the universal cloning via parametric down-conversion is optimal.

We now extend the concept of cloning via amplification to qudits represented by a single photon in d different spatial modes or time bins. The use of time-bin encoding seems to be particularly advantageous since only a single nonlinear Type-I-matched crystal is required, and the pump beam should consist of a sequence of d pulses. We associate creation operators $a_j^\dagger$ and $b_j^\dagger$ with the jth time bins of signal and idler beams, respectively. The Hamiltonian governing the evolution of this system can be expressed as

$$H_d = \mathrm{i}\kappa \sum_{j=1}^{d} \left(a_j^\dagger b_j^\dagger - a_j b_j\right). \tag{5.6}$$

This Hamiltonian is invariant with respect to simultaneous unitary transformations of the signal and idler modes, $(U \otimes U^*) H (U^\dagger \otimes U^{\mathrm{T}}) = H$, where $U \in \mathrm{SU}(d)$. This covariance property guarantees that the cloning is universal and the cloning fidelity does not depend on the input state, so it suffices to consider the input state $|\psi_{\mathrm{in,d}}\rangle = |N\rangle_{a_1}|0\rangle_{a_2}\cdots|0\rangle_{a_d}|0\rangle_{b_1}|0\rangle_{b_2}\cdots|0\rangle_{b_d}$. The unitary operation $\exp(-\mathrm{i}Ht)$ can again be factorized, similarly as in eq. (5.3), and we get $\mathrm{e}^{-\mathrm{i}H_d t}|\psi_{\mathrm{in,d}}\rangle = (1-\lambda^2)^{N/2+d}\sum_{M=N}^{\infty}\lambda^{M-N}|\Psi_{M,d}\rangle$, where the state containing M clones reads

$$\begin{aligned}|\Psi_{M,d}\rangle = \sum_{\mathbf{m}} \sqrt{\binom{N+m_1}{N}}\, |N+m_1\rangle_{a_1}|m_2\rangle_{a_2}\cdots|m_d\rangle_{a_d} \\ \times |m_1\rangle_{b_1}|m_2\rangle_{b_2}\cdots|m_d\rangle_{b_d}.\end{aligned} \tag{5.7}$$

In this formula, $\sum_{\mathbf{m}}$ indicates summation over all vectors $\mathbf{m} = (m_1, \ldots, m_d)$ satisfying $\sum_{j=1}^{d} m_j = M - N$. The optimality of this cloning transformation can be proved by explicit evaluation of the fidelity. It can be shown that there are $\binom{M-N+d-2-m}{d-2}$ different terms in eq. (5.7) with $N+m$ photons in mode a_1, each with weight $\binom{N+m}{m}$. The average single-clone fidelity can be thus expressed as

$$F = \frac{1}{\mathcal{N}} \sum_{m=0}^{M-N} \binom{N+m}{N}\binom{M-N+d-2-m}{d-2}\frac{N+m}{M}, \tag{5.8}$$

where the normalization factor is given by

$$\mathcal{N} \equiv \sum_{m=0}^{M-N} \binom{N+m}{N} \binom{M-N+d-2-m}{d-2} = \binom{M+d-1}{N+d-1}. \tag{5.9}$$

The summation in eq. (5.8) can be performed with the help of the identity given in eq. (5.9), and one recovers the optimal fidelity (4.9).

Instead of parametric down-conversion it is also possible to amplify the light by sending it through an inverted atomic medium (Simon, Weihs and Zeilinger [2000a], Kempe, Simon and Weihs [2000], Fan, Weihs, Matsumoto and Imai [2002]). The atoms should possess d different ground states $|g_j\rangle$ and an excited state $|e\rangle$. We assume that each atomic transition $|e\rangle \to |g_j\rangle$ is strongly coupled to a single optical mode a_j and the qudits are represented by single photons in those d modes. The universality of the cloning requires that the coupling strength κ must be the same for all d transitions $|e\rangle \to |g_j\rangle$. In the interaction picture and in the rotating-wave approximation, the interaction of light with atoms is governed by the Jaynes–Cummings Hamiltonian,

$$H_{\mathrm{JC}} = \kappa \sum_{k=1}^{L} \sum_{j=1}^{d} a_j^\dagger |g_{jk}\rangle\langle e_k| + \text{h.c.}, \tag{5.10}$$

where L is the number of atoms and $|g_{jk}\rangle$ stands for the ground state $|g_j\rangle$ of the kth atom. This Hamiltonian satisfies the covariance property $U \otimes U^* H U^\dagger \otimes U^{\mathrm{T}} = H$, where $U a_j^\dagger U^\dagger = u_{jk} a_k^\dagger$, $U|e_k\rangle = |e_k\rangle$ and $U|g_{jk}\rangle = \sum_{l=1}^{d} u_{jl} |g_{lk}\rangle$.

Suppose that all $L = M - N$ atoms are initially prepared in the excited state and that all N input photons are in mode a_1. The joint atoms–photons input state reads $|\psi_{\mathrm{in},LA}\rangle = |N\rangle_{a_1} |0\rangle_{a_2} \cdots |0\rangle_{a_d} |e_1\rangle \cdots |e_L\rangle$. If each atom emits a photon during the passage of the light through the atoms then M clones are generated and all atoms end up in ground states.

In the weak-coupling regime we can express the output state conditional on all atoms being in some ground state using the Lth-order perturbation theory:

$$|\psi_{\mathrm{out}}\rangle \propto \left(\sum_{j=1}^{d} a_j^\dagger b_j^\dagger c \right)^L |\psi_{\mathrm{in},LA}\rangle, \tag{5.11}$$

where the operator $b_j^\dagger c$ is defined as $b_j^\dagger c = \sum_{k=1}^{L} |g_{jk}\rangle\langle e_k|$. Note that with this notation, the Hamiltonian (5.10) becomes similar to the down-conversion Hamiltonian (5.6). Since the atoms are supposed to be identical, the photons emitted by them do not carry any information about which atom emitted which photon. Consequently, if all atoms emit photons, then the atoms relax to symmetric ground state. Suppose that m_j photons were emitted to mode a_j, with $j = 1, \dots, d$. The corresponding symmetrized ground atomic state reads

$$|g_{\mathbf{m}}\rangle = C_{L\mathbf{m}}^{-1} \sum_{\pi(\mathbf{k})} |g_{1k_1}\rangle \cdots |g_{1k_{m_1}}\rangle |g_{2k_{m_1+1}}\rangle \cdots |g_{2k_{m_1+m_2}}\rangle \cdots |g_{dk_{L-m_d+1}}\rangle \cdots |g_{dk_L}\rangle, \tag{5.12}$$

where $\sum_{\pi(\mathbf{k})}$ denotes summation over all $L!$ values of the subscripts k_l, $l = 1, \dots, L$, which can be obtained as permutations of $\{1, \dots, L\}$. The normalization coefficient

$$C_{L\mathbf{m}}^2 = m_1! m_2! \cdots m_d! L! \tag{5.13}$$

is chosen such that $\langle g_{\mathbf{m}} | g_{\mathbf{m}} \rangle = 1$. After some algebra, one finds that the output state (5.11) can be expressed as

$$\begin{aligned} |\psi_{\text{out}}\rangle &\propto \sum_{\mathbf{m}} \frac{C_{L\mathbf{m}} L!}{m_1! m_2! \cdots m_d!} a_1^{\dagger m_1} a_2^{\dagger m_2} \cdots a_d^{\dagger m_d} |N\rangle_{a_1} |0\rangle_{a_2} \cdots |0\rangle_{a_d} |g_{\mathbf{m}}\rangle \\ &\propto \sum_{\mathbf{m}} \sqrt{\binom{N+m_1}{N}} |N+m_1\rangle_{a_1} |m_2\rangle_{a_2} \cdots |m_d\rangle_{a_d} |g_{\mathbf{m}}\rangle. \end{aligned}$$

Since this state is fully equivalent to the state (5.7), the cloning is optimal. Although this result was obtained within the framework of perturbation theory, a detailed analysis reveals that it holds for any interaction strength. It can also be shown that if only $M' - N < L$ atoms emit photons and the rest of the atoms remain in the excited state, then M' optimal photonic clones are generated (Fan, Weihs, Matsumoto and Imai [2002]).

A proof-of principle experiment on cloning via stimulated emission was reported by Fasel, Gisin, Ribordy, Scarani and Zbinden [2002] utilizing a commercially available polarization-insensitive erbium-doped fiber amplifier. The amplifier was injected with a weak vertically polarized coherent signal with mean photon number $\bar{n}_{\text{in}}$. After the amplification, the output mean numbers $\bar{n}_V$ and $\bar{n}_H$ of vertically and horizontally polarized photons were measured. The fidelity of the amplification process can be simply defined as $F = \bar{n}_V / (\bar{n}_V + \bar{n}_H)$. The output mean intensities depend linearly on the input intensity (Shimoda, Takahasi and Townes [1957]),

$$\bar{n}_V = G\bar{n}_{\text{in}} + \frac{1}{Q}(G-1), \qquad \bar{n}_H = \frac{1}{Q}(G-1). \tag{5.14}$$

Here G is the gain of the amplifier and Q is a factor depending on the properties of the amplification process. The term $G\bar{n}_{\text{in}}$ represents the amplified injected input signal while $(G-1)/Q$ represents the noise arising due to spontaneous emission. For quantum-noise-limited amplification, $Q = 1$. From eqs. (5.14) we can express G in terms of $\bar{n}_{\text{in}}$, $\bar{n}_{\text{out}} = \bar{n}_V + \bar{n}_H$ and Q, and we find $G = (Q\bar{n}_{\text{out}} + 2)/(Q\bar{n}_{\text{in}} + 2)$. On inserting this expression into the formula for the fidelity, we obtain

$$F = \frac{Q\bar{n}_{\text{out}}\bar{n}_{\text{in}} + \bar{n}_{\text{out}} + \bar{n}_{\text{in}}}{Q\bar{n}_{\text{out}}\bar{n}_{\text{in}} + 2\bar{n}_{\text{out}}}. \tag{5.15}$$

If we formally replace $\bar{n}_{\text{in}}$ with N (the number of input copies), and $\bar{n}_{\text{out}}$ with M (the number of output clones), then for $Q = 1$ the formula (5.15) becomes the optimal fidelity of $N \to M$ cloning of qubits. Experimentally, $G = 1.3$ and $Q = 0.8$ was observed, quite close to the optimal value $Q = 1$. For instance, the fidelity of $1 \to 2$ cloning for $Q = 0.8$ inferred from eq. (5.15) reads $F = 0.821$ which is only slightly lower than the optimal fidelity $F = \frac{5}{6} \approx 0.833$.

5.2. *Symmetrization*

We have seen in Section 4.1 that the optimal universal $N \to M$ quantum cloning can be accomplished by symmetrizing the state of N input copies and $M-N$ maximally mixed states. Since photons are bosons, the projection onto the symmetric subspace can be easily carried out with the use of linear optics, namely by mixing the M photons on an array of $M - 1$ beamsplitters and selecting only the events when all photons are collected in a single spatial mode.

Let us first illustrate this method on the example of $1 \to 2$ cloning of polarization states of photons (Ricci, Sciarrino, Sias and De Martini [2004], Irvine, Lamas-Linares, de Dood and Bouwmeester [2004], Sciarrino, Sias, Ricci and De Martini [2004b]). The set-up is schematically illustrated in fig. 8a. The photon in mode A whose state is to be cloned is combined on a balanced beamsplitter BS_1 with a blank copy photon prepared in a maximally mixed state. Only the cases when both photons leave the beamsplitter in the left output mode are post-selected, and the two clones are spatially separated by an auxiliary balanced beamsplitter BS_2. At the heart of cloning via symmetrization is the Hong–Ou–Mandel effect (Hong, Ou and Mandel [1987]). If two photons with identical polarization state interfere on a balanced beamsplitter, then they both end up in the same spatial mode and one does not observe any coincidences of one photon in mode A and one in mode B. So, for the input $|\psi\rangle_A|\psi\rangle_B$, there is probability $\frac{1}{2}$ of having two photons in the left output port and probability $\frac{1}{2}$ of splitting them in the two output modes A' and B'. Altogether, the conditional transformation reads

$$|\psi\rangle_A|\psi\rangle_B \to \frac{1}{2}|\psi\rangle_{A'}|\psi\rangle_{B'}. \tag{5.16}$$

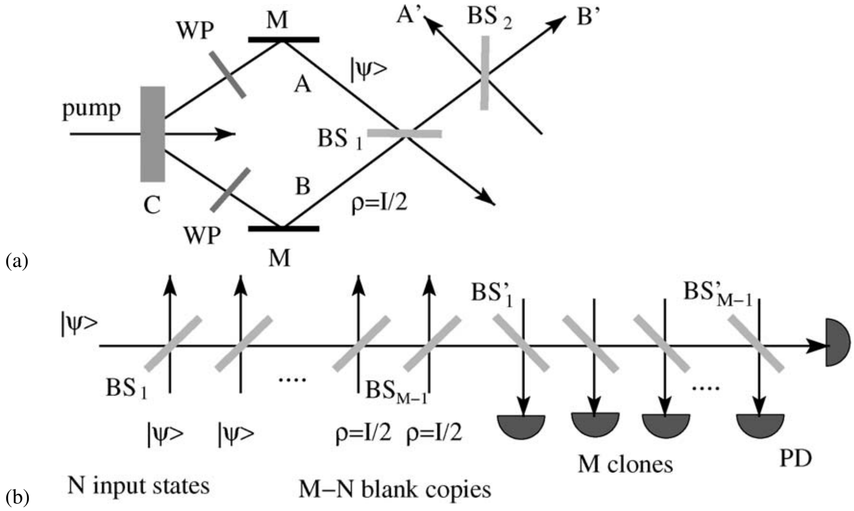


Fig. 8. Optimal cloning of polarization states of photons via projection onto the symmetric subspace. (a) Optimal $1 \to 2$ universal cloning based on the interference of two photons on a balanced beamsplitter BS_1. (b) Extension to optimal $N \to M$ cloning. The N input states and $M-N$ blank copies in maximally mixed states are combined on an array of $M-1$ beamsplitters BS_j, and the clones are then separated on another array of $M-1$ beamsplitters BS'_j.

On the other hand, if the two photons are initially in orthogonal polarization states, $|\psi\rangle_A|\psi_\perp\rangle_B$, then they are distinguishable and do not interfere on BS_1. With probability $\frac{1}{4}$, the photon in mode A is reflected and the photon in mode B is transmitted and they are both in the left output. Again, there is probability $\frac{1}{2}$ that the two photons will be divided on a balanced beamsplitter BS_2. Since the photon in state $|\psi\rangle$ can be either reflected or transmitted on BS_2, the final state of photons in modes A' and B' is a balanced superposition of these two possibilities, namely a symmetric state,

$$|\psi\rangle_A|\psi_\perp\rangle_B \to \frac{1}{4}\big(|\psi\rangle_{A'}|\psi_\perp\rangle_{B'} + |\psi_\perp\rangle_{A'}|\psi\rangle_{B'}\big). \tag{5.17}$$

Since the projector onto the symmetric subspace acts as $\Pi_+|\psi\rangle|\psi\rangle = |\psi\rangle|\psi\rangle$ and $\Pi_+|\psi\rangle|\psi_\perp\rangle = \frac{1}{2}(|\psi\rangle|\psi_\perp\rangle + |\psi_\perp\rangle|\psi\rangle)$, it immediately follows from eqs. (5.16) and (5.17) that the set-up shown in fig. 8(a) implements with probability $\frac{1}{4}$ the projection onto the symmetric subspace followed by a spatial separation of the two photons.

The maximally mixed polarization state in mode B can be obtained for instance by preparing the blank copy photon in state $|V\rangle$ or $|H\rangle$ with probability $\frac{1}{2}$ each. Another, more intriguing option is to send into port B one part of the maximally entangled two-photon singlet state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle_B|\psi_\perp\rangle_C - |\psi_\perp\rangle_B\psi\rangle_C)$. In this case, if the symmetrization succeeds, then we obtain in the spatial mode C the optimal anti-clone of $|\psi\rangle$, i.e. a state that has a fidelity $\frac{2}{3}$ with $|\psi_\perp\rangle$.

The optimal $1 \to 2$ cloning based on symmetrization has been experimentally demonstrated by two groups (Ricci, Sciarrino, Sias and De Martini [2004], Irvine, Lamas-Linares, de Dood and Bouwmeester [2004]). In both experiments, the input photon whose state was cloned was obtained from a weak coherent beam, and it was combined on a balanced beamsplitter with one photon from a maximally entangled singlet state generated in a nonlinear crystal by means of spontaneous parametric down-conversion. The triple-coincidence events were selected where there were two clones and one anticlone present, and the intensity of the weak coherent beam was adjusted such that the dominant contribution to the triple-coincidence events originated from the cases when there was a single photon in the coherent beam and a single entangled photon pair was generated in the nonlinear crystal. The observed mean cloning fidelities in these two experiments were $F = 0.82$ (Ricci, Sciarrino, Sias and De Martini [2004]) and $F = 0.81$ (Irvine, Lamas-Linares, de Dood and Bouwmeester [2004]), respectively. The simpler set-up depicted in fig. 8(a), involving only two photons, was also implemented experimentally (Sciarrino, Sias, Ricci and De Martini [2004a]). A single photon pair was generated in a nonlinear crystal. One photon representing the input was prepared in the state $|\psi\rangle$ using wave plates while the other photon was randomly prepared in the state $|V\rangle$ or $|H\rangle$. This experiment is much simpler than the previous one, because only two-photon coincidence events were observed instead of tree-photon coincidences. This resulted in a much higher rate of cloning, and also in better visibility and mean cloning fidelity $F = 0.826$ very close to the theoretical maximum $F = 0.833$.

An extension of the symmetrization procedure to M photons is illustrated in fig. 8(b). The photons are combined on an array of $M - 1$ beamsplitters BS_j, and the symmetrization succeeds if all M photons are bunched in the same spatial mode (Sciarrino, Sias, Ricci and De Martini [2004b]). To confirm this we can split the output signal into M different spatial modes using another array of $M - 1$ beamsplitters BS'_j and post-select only events with each of M photodetectors PD registering one photon.

We now demonstrate that the array of beamsplitters accomplishes the desired projection onto the symmetric subspace. The symmetric two-mode L-photon states $|L, k\rangle$ with $L - k$ photons polarized vertically and k photons polarized horizontally form a basis in the symmetric space $\mathcal{H}_+^{\otimes L}$. We prove our claim by induction. Consider the Lth beamsplitter BS_L in the scheme of fig. 8(b). The state impinging from the left is a symmetric L-photon state while a single photon impinges on BS_L from the bottom. The beamsplitter BS_L does not need to be balanced but its transmittance t and reflectance r should be independent of the polarization. In the Heisenberg picture, the mixing of the modes on the beamsplitter is described by linear input output canonical transformations of the creation operators,

$$a^\dagger_{V,\mathrm{out}} = r a^\dagger_{V,\mathrm{in}} + t b^\dagger_{V,\mathrm{in}}, \qquad b^\dagger_{V,\mathrm{out}} = r b^\dagger_{V,\mathrm{in}} - t a^\dagger_{V,\mathrm{in}}, \tag{5.18}$$

and similar formulas hold for horizontal polarization. The state transformation on a beamsplitter can be most easily determined by expressing all states in terms of the creation operators acting on the vacuum,

$$\begin{aligned} |L,k\rangle &= \frac{1}{\sqrt{k!(L-k)!}} a^{\dagger k}_{H,\mathrm{in}} a^{\dagger L-k}_{V,\mathrm{in}} |\mathrm{vac}\rangle, \\ |V\rangle &= b^\dagger_{V,\mathrm{in}}|\mathrm{vac}\rangle, \qquad |H\rangle = b^\dagger_{H,\mathrm{in}}|\mathrm{vac}\rangle. \end{aligned} \tag{5.19}$$

From eqs. (5.18) we express the "in" operators as linear combinations of the "out" operators and substitute into the formulas (5.19). Using this technique it is easy to show that if all $L+1$ photons bunch in the right output mode then the following conditional transformation takes place:

$$\begin{aligned} |L,k\rangle|V\rangle &\to t_L^L r_L \sqrt{L+1-k}\,|L+1,k\rangle, \\ |L,k\rangle|H\rangle &\to t_L^L r_L \sqrt{k+1}\,|L+1,k+1\rangle. \end{aligned} \tag{5.20}$$

Consider now the projection of the states $|L,k\rangle|V\rangle$ and $|L,k\rangle|H\rangle$ onto the symmetric subspace of $L+1$ photonic qubits. One finds that

$$\begin{aligned} \Pi_{+,L+1}|L,k\rangle|V\rangle &= \sqrt{\frac{L+1-k}{L+1}}\,|L+1,k\rangle, \\ \Pi_{+,L+1}|L,k\rangle|H\rangle &= \sqrt{\frac{k+1}{L+1}}\,|L+1,k+1\rangle. \end{aligned} \tag{5.21}$$

The transformations (5.20) and (5.21) are equivalent up to a state-independent prefactor $\sqrt{L+1}\,t_L^L r_L$, which proves that the array of $M-1$ beamsplitters BS in fig. 8(b) projects the input states onto the symmetric subspace of M qubits. The probability of success of the projection can be determined by comparing the coefficients in eqs. (5.20) and (5.21), and we find

$$P = P_S M! \prod_{j=1}^{M-1} T_j^j (1-T_j), \tag{5.22}$$

where $T_j = t_j^2$ and $P_S = \mathrm{Tr}[\Pi_M^+ \rho_{\mathrm{in}}]$ is the overlap of the input M-photon state ρ_{in} with projector onto the symmetric subspace. The optimal transmittance T_j of the

jth beamsplitter leading to maximal P can be obtained by maximizing $T_j^j(1-T_j)$, which yields $T_{j,\mathrm{opt}} = j/(j+1)$. Note that $T_{j,\mathrm{opt}}$ does not depend on the input N-photon state. On inserting the optimal T_j into eq. (5.22) we get

$$P_{\mathrm{opt}} = P_S \frac{M!}{M^M}. \tag{5.23}$$

Recently, the optimal universal $1 \to 3$ and $2 \to 3$ cloning of polarization states of photons via symmetrization was demonstrated experimentally by Masullo, Ricci and De Martini [2004]. The three photons used in the experiment consisted of a pair of photons generated in the process of spontaneous parametric down-conversion and a single photon in a very weak coherent beam. These three photons were combined on two beamsplitters, and only the events where all photons bunched in a single spatial mode were chosen by post-selection. Wave plates, a polarizing beamsplitter, an array of beamsplitters and photodetectors were employed to analyze the clones. The experimentally observed fidelity of the $1 \to 3$ cloning was $F_{1\to 3}^{\exp} = 0.758$, very close to the theoretical maximum $F_{1\to 3}^{\mathrm{th}} = \frac{7}{9} \approx 0.778$. The observed fidelity of $2 \to 3$ cloning, $F_{2\to 3}^{\exp} = 0.894$, was also close to the optimum value $F_{2\to 3}^{\mathrm{th}} = \frac{11}{12} \approx 0.917$.

The symmetrization on a beamsplitter can be naturally extended to qudits. Symmetrization of two photonic qudits represented by a state of a photon in d different spatial modes would require an array of d balanced beamsplitters, each mixing the jth mode of the first and second qudits. It may be more advantageous to work with time-bin qudits, where the symmetrization would require only one balanced beamsplitter where the two photons would interfere. Similarly as before, only the events when the two photons bunch and leave the beamsplitter in the same spatial port have to be post-selected.

Note finally that the cloning of the quantum state of a single photon using linear optics was also demonstrated by Huang, Li, Li, Zhang, Jiang and Guo [2001] using a different approach. In their scheme, both clones were represented by the quantum state of just a single photon in several modes, so that the two clones could not be physically separated.

5.3. *Universal asymmetric cloning of photons*

So far, we have presented various optical implementations of symmetric cloning machines. In this section we will consider the optimal $1 \to 2$ asymmetric cloning of qubits. We will describe two methods, both based on the interference of photons on unbalanced beamsplitters. The first approach, introduced in Section 3.5, is to start from the output of the optimal symmetric cloner and convert it into an output of the optimal asymmetric cloner, which is given by

$$|\Psi\rangle = \frac{1}{\sqrt{2-2p+2p^2}} \times \big[|\psi\rangle_A|\psi\rangle_B|\psi_\perp\rangle_C - p|\psi\rangle_A|\psi_\perp\rangle_B|\psi\rangle_C - (1-p)|\psi_\perp\rangle_A|\psi\rangle_B|\psi\rangle_C\big]. \tag{5.24}$$

Here $p \in [0, 1]$ is an asymmetry parameter and the fidelities of the clones in qubits A and B read

$$F_A = 1 - \frac{(1-p)^2}{2(1-p+p^2)}, \qquad F_B = 1 - \frac{p^2}{2(1-p+p^2)}. \tag{5.25}$$

The symmetric cloner is recovered when $p = \frac{1}{2}$ and we have

$$|\Psi\rangle_{\text{sym}} = \frac{1}{\sqrt{6}}\big[2|\psi\rangle_A|\psi\rangle_B|\psi_\perp\rangle_C - |\psi\rangle_A|\psi_\perp\rangle_B|\psi\rangle_C - |\psi_\perp\rangle_A|\psi\rangle_B|\psi\rangle_C\big]. \tag{5.26}$$

Suppose first that the second clone (qubit B) and the anti-clone (qubit C) are projected on the singlet state $|\Psi^-\rangle$. We obtain

$$I_A \otimes \Pi^-_{BC}|\Psi\rangle_{\text{sym}} = \frac{1}{2}\sqrt{3}|\psi\rangle_A\big|\Psi^-\big\rangle_{BC}, \tag{5.27}$$

where $\Pi^-_{BC} = |\Psi^-\rangle\langle\Psi^-|$. The original input state $|\psi\rangle$ is perfectly recovered in qubit A. The projection on a singlet forms a part of the Bell measurement, i.e. a measurement in the basis of four maximally entangled Bell states. There is an interesting analogy between eq. (5.27) and the process of quantum teleportation (Bennett, Brassard, Crepeau, Jozsa, Peres and Wootters [1993], Bouwmeester, Pan, Mattle, Eibl, Weinfurter and Zeilinger [1997], Boschi, Branca, De Martini, Hardy and Popescu [1998], Marcikic, de Riedmatten, Tittel, Zbinden and Gisin [2003]). Indeed, as implied by eq. (3.44), the cloning can be deterministically reversed by performing a Bell measurement on one of the clones and the anti-clone and applying an appropriate correcting unitary to the first clone (Bruss, Calsamiglia and Lütkenhaus [2001]). In the Bell measurement, the singlet is detected with probability $\frac{3}{4}$ while each of the triplet Bell states is detected with probability $\frac{1}{12}$, independently of the input state.

This full reversal of cloning can be generalized to a partial reversal which converts the symmetric cloner to asymmetric one (Filip [2004a]). The idea is to apply to qubits B and C a filter $\Pi^-_{BC} + a\Pi^+_{BC}$, where $a \in [0, 1]$ controls the asymmetry.

If $a = 0$ we get projection on singlet and full reversal, while for $a = 1$ the two qubits are multiplied by the identity and nothing happens. Let us now consider arbitrary a. The state after filtering,

$$|\Psi_{\text{proj}}\rangle = I_A \otimes \big(\Pi^-_{BC} + a\Pi^+_{BC}\big)|\Psi_{\text{sym}}\rangle, \tag{5.28}$$

can be expressed, after normalization, as follows,

$$\begin{aligned}|\Psi_{\text{proj}}\rangle = {} & \frac{1}{\sqrt{6(a^2+3)}} \\ & \times \big[(3+a)|\psi\rangle_A|\psi\rangle_B|\psi_\perp\rangle_C \\ & \quad - (3-a)|\psi\rangle_A|\psi_\perp\rangle_B|\psi\rangle_C - 2a|\psi_\perp\rangle_A|\psi\rangle_B|\psi\rangle_C\big].\end{aligned} \tag{5.29}$$

We can immediately see that this state coincides with the outcome of the optimal asymmetric cloner (5.24) and $p = (3-a)/(3+a)$.

For optical polarization qubits, the filtration (5.28) can be implemented by letting the two photons interfere on an unbalanced beamsplitter and post-selecting only the events when a single photon is detected in each output port. There are two ways for the photons to exit the beamsplitter in different spatial modes: either both photons are reflected or both are transmitted. Unitarity dictates that these two alternatives acquire a mutual phase shift π. If the two photons are in the same state $|\psi\rangle$, then these two alternatives interfere destructively, while if the photons are in orthogonal polarization states there is no interference. The resulting conditional transformation reads

$$\begin{aligned}&|\psi\psi\rangle_{BC} \to (R-T)|\psi\psi\rangle_{BC}, \\ &|\psi\psi_\perp\rangle \to R|\psi\psi_\perp\rangle - T|\psi_\perp\psi\rangle.\end{aligned} \tag{5.30}$$

It follows that the unbalanced beamsplitter applies the filter $\Pi^- + a\Pi^+$ with $a = R - T$. A schematic set-up of the proposed asymmetric cloning experiment is shown in fig. 9. Optimal symmetric cloning is accomplished by stimulated parametric down-conversion as discussed in detail in Section 5.1. At the output, the two clones are separated on an auxiliary balanced beamsplitter, and one of the clones is combined with the anti-clone on an unbalanced beamsplitter. Successful asymmetric cloning is heralded by a coincident observation of a single photon in each of the modes A, B and C.

The second scheme for optimal asymmetric cloning (Filip [2004b]) very closely resembles the scheme for teleportation of polarization states of photons, see fig. 10. The only difference is that the balanced beamsplitter used in teleportation to perform a Bell analysis is replaced by an unbalanced beamsplitter that conditionally applies the filter $\Pi_- + a\Pi_+$. The cloning succeeds if a single photon is detected in each of the modes A, B and C. The initial state in the scheme

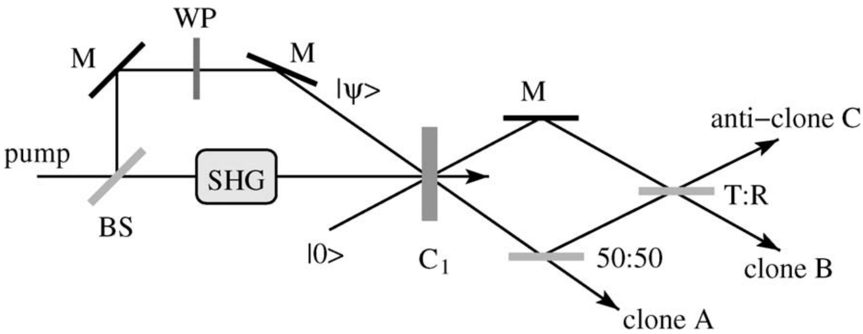


Fig. 9. Optimal universal asymmetric cloning from symmetric cloning. The input photon to be cloned is obtained from a weak coherent beam. BS – auxiliary beamsplitter; SHG – second-harmonic generation; C_1 – nonlinear crystal; 50:50 – balanced beamsplitter; T:R – unbalanced beamsplitter; WP – wave plates for input-state preparation; M – mirrors.

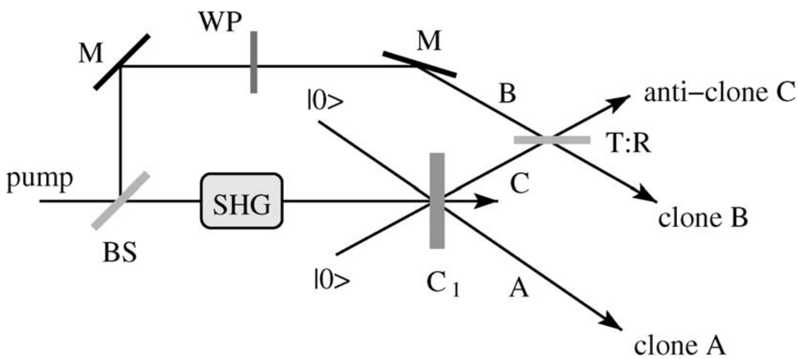


Fig. 10. Partial quantum state teleportation as an optimal asymmetric cloning. The input photon to be cloned is obtained from a weak coherent beam. BS – auxiliary beamsplitter; SHG – second-harmonic generation; C_1 – nonlinear crystal; T:R – unbalanced beamsplitter; WP – wave plates for input-state preparation; M – mirrors.

shown in fig. 10 is $|\psi\rangle_B|\Psi^-\rangle_{AC}$, and after the interference on a beamsplitter and post-selection we get

$$\left|\widetilde{\Psi}_{\mathrm{proj}}\right\rangle = I_A \otimes \left(a\Pi^+_{BC} + \Pi^-_{BC}\right)|\psi\rangle_B\left|\Psi^-\right\rangle_{AC}. \tag{5.31}$$

After some algebra we arrive at

$$\begin{aligned}\left|\widetilde{\Psi}_{\mathrm{proj}}\right\rangle \propto\ & \frac{1}{\sqrt{2(1+3a^2)}} \\ & \times \left[(1+a)|\psi\rangle_A|\psi\rangle_B|\psi_\perp\rangle_C \right. \\ & \quad \left. -(1-a)|\psi\rangle_A|\psi_\perp\rangle_B|\psi\rangle_C - 2a|\psi_\perp\rangle_A|\psi\rangle_B|\psi\rangle_C\right].\end{aligned} \tag{5.32}$$

This is again the output state of the optimal asymmetric cloning machine with $p = (1-a)/(1+a)$, so the asymmetric cloning can be implemented by means of a partial teleportation. An interesting feature of this scheme is that one of the clones is teleported from Alice to Bob so we can speak about cloning at a distance.

The universal asymmetric cloning of polarization states of single photons has been demonstrated experimentally by Zhao, Zhang, Zhou, Chen, Lu, Karlsson and Pan [2005] following the scheme illustrated in fig. 10. In that experiment, a Mach–Zehnder interferometer acted as an effective unbalanced beamsplitter whose transmittance could be controlled by changing the relative path difference between the two arms of the interferometer. In this way it was possible to demonstrate the whole class of asymmetric $1 \to 2$ cloning machines.

5.4. Cloning of orthogonally polarized photons

It was shown in Section 4.1 that the optimal universal quantum cloning and optimal quantum state estimation are closely related, and that in the limit of an infinite number of clones the fidelity of cloning is equal to the fidelity of optimal state estimation. In this context, a very interesting and surprising observation was made by Gisin and Popescu [1999], who found that the state of a single qubit can be estimated better from the state $|\psi\rangle|\psi_\perp\rangle$ than from the state $|\psi\rangle|\psi\rangle$. Picturing the qubits as spin-$\frac{1}{2}$ particles, we can say that the information about the direction is encoded better in two anti-parallel spins than in two parallel ones. The fidelity of the estimation of $|\psi\rangle$ from a single copy of the two-qubit state $|\psi\rangle|\psi_\perp\rangle$ reads (Gisin and Popescu [1999], Massar [2000]),

$$F_\perp = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right) \approx 0.789, \tag{5.33}$$

which is slightly higher than the fidelity of optimal estimation from $|\psi\rangle|\psi\rangle$, $F = \frac{3}{4}$. Motivated by this observation we may expect that this advantage of two anti-parallel spins over two parallel ones extends also to cloning. This is indeed the case, provided that the number of clones is large enough. We shall now describe the optimal universal cloning transformation which produces M approximate clones of the state $|\psi\rangle$ from a single replica of $|\psi\rangle|\psi_\perp\rangle$ and maximizes the single-clone fidelity.

Making the natural assumption that the output Hilbert space is the symmetric subspace of M qubits, the optimal cloning CP map $\mathcal{S}$ can be determined analytically for any M (Fiurášek, Iblisdir, Massar and Cerf [2002]). The mean single-clone fidelity can be expressed as $F = \mathrm{Tr}[SR]$, where R has a rather complicated form and can be found in (Fiurášek, Iblisdir, Massar and Cerf [2003]). In contrast to the universal cloning with input state $|\psi\rangle^{\otimes N}$, the maximum fidelity cannot be determined from the maximum eigenvalue of R, and one has to solve the extremal equations (3.18) and prove the optimality by checking that the inequality (3.19) is satisfied.

Since the input state of the cloner can be obtained as an orbit of the group SU(2), $|\psi\rangle|\psi_\perp\rangle = U \otimes U|0\rangle|1\rangle$, the optimal cloner is covariant and can be expressed as follows,

$$|\psi, \psi_\perp\rangle \to \sum_{j=0}^{M} \alpha_{j,M} \big|(M-j)\psi, j\psi_\perp\big\rangle \otimes \big|(M-j)\psi_\perp, j\psi\big\rangle, \tag{5.34}$$

where the coefficients $\alpha_{j,M}$ are given by

$$\alpha_{j,M} = (-1)^j \left[\frac{1}{\sqrt{2(M+1)}} + \frac{\sqrt{3}(M-2j)}{\sqrt{2M(M+1)(M+2)}} \right]. \tag{5.35}$$

The cloning machine (5.34) is symmetric with respect to the interchange of $|\psi\rangle$ and $|\psi_\perp\rangle$. The cloner requires an ancilla whose size is the same as the size of the output Hilbert space, i.e., the ancilla Hilbert space is also a symmetric subspace of M qubits. The ancilla contains M approximate copies of the state $|\psi_\perp\rangle$, and the fidelity of these anti-clones is the same as the fidelity of the clones. The single-clone fidelity can be calculated as weighted average of the coefficients $\alpha_{j,M}^2$,

$$F_\perp(M) = \sum_{j=0}^{M} \frac{M-j}{M} \alpha_{j,M}^2, \tag{5.36}$$

and after a simple algebra we arrive at

$$F_\perp(M) = \frac{1}{2}\left(1 + \sqrt{\frac{M+2}{3M}}\right). \tag{5.37}$$

The fidelity monotonically decreases with increasing number of clones M, and in the limit $M \to \infty$ we recover the fidelity (5.33) of the optimal state estimation from $|\psi\rangle|\psi_\perp\rangle$. Upon comparing the fidelity $F_\perp(M)$ with the fidelity of the optimal cloner for a pair of identical qubits, $F_\parallel(M) = (3M+2)/(4M)$, we see that $F_\parallel(M) \geqslant F_\perp(M)$ for $M \leqslant 6$, while $F_\perp(M) > F_\parallel(M)$ for $M > 6$ and the cloner (5.34) outperforms the standard universal cloner.

We have seen in Section 5.1 that the optimal universal cloning of polarization states of photons can be realized by means of stimulated parametric down-conversion. It turns out that the optimal cloning with a pair of orthogonal qubits as the input can be performed in the same way, if the photons in states $|\psi\rangle$ and $|\psi_\perp\rangle$ are fed to the input signal and idler ports of the amplifier, respectively, as

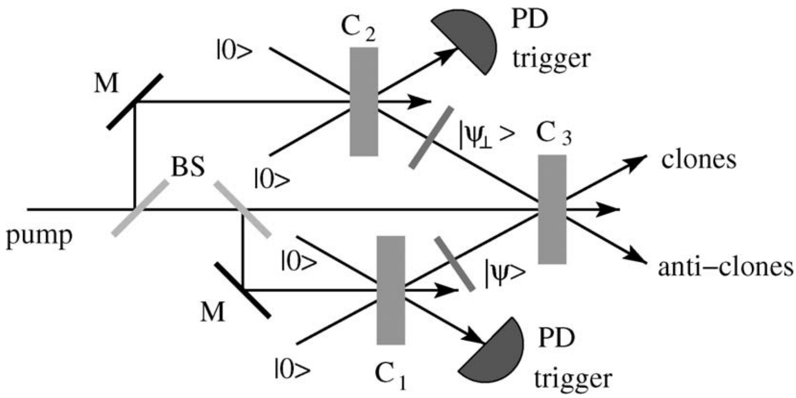


Fig. 11. Cloning of a pair of orthogonal qubits by means of stimulated parametric down-conversion. BS – auxiliary beamsplitters; C_j – nonlinear crystals; WP – wave plates for input-state preparation; M – mirrors; PD – triggering single-photon detectors for conditional preparation of single-photon states.

schematically illustrated in fig. 11. We assume that the parametric amplification in the nonlinear crystal C_3 is governed by the singlet-type Hamiltonian (5.1), which is invariant under the simultaneous rotation of the signal and idler qubits, $(U \otimes U)H(U^\dagger \otimes U^\dagger) = H$. Assuming the input state $|1\rangle_{a_V}|0\rangle_{a_H}|0\rangle_{b_V}|1\rangle_{b_H}$, the output state after the amplification in the crystal C_3 reads

$$|\Psi_{\text{out}}\rangle = \sum_{M=0}^{\infty} \lambda^{M-1}\big(1-\lambda^2\big)|\Psi_{\perp,M}\rangle, \tag{5.38}$$

where the state with M clones and M anti-clones is given by

$$|\Psi_{\perp,M}\rangle = \sum_{j=0}^{M}(-1)^j\big[(M-j)\big(1-\lambda^2\big)-\lambda^2\big] \times |M-j\rangle_{a_V}|j\rangle_{a_H}|j\rangle_{b_V}|M-j\rangle_{b_H}, \tag{5.39}$$

with $\lambda = \tanh(\kappa t)$ and t the effective interaction time.

In contrast to universal cloning with N identical photons at the input, the state $|\Psi_{\perp,M}\rangle$ depends on the strength of the parametric amplification λ. The cloner that produces M copies is obtained by post-selecting only the events with exactly M photons detected in signal and idler spatial modes, which corresponds to the selection of the state $|\Psi_{\perp,M}\rangle$ from the superposition (5.38). The fidelity of the cloner depends on λ,

$$F_\perp(M,y) = \frac{3y^2 - 2y(2M+1) + \frac{3}{2}M(M+1)}{6y^2 - 6My + M(2M+1)}, \tag{5.40}$$

where $y = \lambda^2/(1-\lambda^2) = \sinh^2(\kappa t)$. The optimal parametric gain which maximizes the fidelity (5.40) can be found by solving the equation $\frac{\partial F_{\perp}(M,y)}{\partial y} = 0$, which yields

$$y_{\mathrm{opt}} = \frac{M}{2} - \frac{1}{2}\sqrt{\frac{M(M+2)}{3}}. \tag{5.41}$$

On inserting the optimal y into eq. (5.40) we recover the fidelity (5.37), hence the optimal cloning of a pair of orthogonal qubits can be achieved by means of stimulated parametric down-conversion with properly chosen gain.

§ 6. Phase-covariant cloning of photons

In Sections 4 and 5 we have focused on the implementation of universal cloning machines that clone all states equally well. In many situations, however, one deals only with a subset of states. An archetypal example is the Bennett–Brassard 1984 (BB84) protocol for quantum key distribution (Bennett and Brassard [1984]), which utilizes four non-orthogonal states $|0\rangle$, $|1\rangle$, $|0\rangle + |1\rangle$ and $|0\rangle - |1\rangle$. If we restrict the range of admissible input states of the cloning machine, then we can expect that the machine will exhibit better performance than the universal cloner and will reach higher fidelity. In this section we shall study phase-covariant cloning machines which optimally clone all states that are balanced superpositions of the computational basis states,

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \mathrm{e}^{\mathrm{i}\phi_j} |j\rangle, \tag{6.1}$$

where the phases ϕ_j can be arbitrary, see Section 3.6.3.

6.1. Phase-covariant cloning of qubits

The simplest and perhaps most important example is the $1 \to 2$ phase-covariant cloning machine which can be used as the optimal individual eavesdropping attack on the BB84 protocol (Fuchs, Gisin, Griffiths, Niu and Peres [1997]). In contrast to universal cloners, the optimal cloning transformation here depends on whether single-clone fidelity or global fidelity is taken as the figure of merit to be maximized. In the context of eavesdropping on the quantum key distribution protocol, it is natural to consider the single-clone fidelities, since they quantify the amount of information transmitted to the receiver and gained by the eavesdropper.

The optimal symmetric $1 \to 2$ cloning transformation for qubits that maximizes the single-clone fidelity has the following form (Bruss, Cinchetti, D'Ariano and Macchiavello [2000], Bruss and Macchiavello [2001], Fan, Weihs, Matsumoto and Imai [2002], Cerf, Durt and Gisin [2002], Karimipour and Rezakhani [2002]):

$$\begin{aligned}|0\rangle|A_{\text{in}}\rangle &\to \frac{1}{\sqrt{2}}|0\rangle_A|0\rangle_B|0\rangle_C + \frac{1}{2}\big(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B\big)|1\rangle_C,\\ |1\rangle|A_{\text{in}}\rangle &\to \frac{1}{\sqrt{2}}|1\rangle_A|1\rangle_B|1\rangle_C + \frac{1}{2}\big(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B\big)|0\rangle_C.\end{aligned} \tag{6.2}$$

It is a special case of the Pauli cloner, see Section 2.2. The two clones are stored in qubits A and B, and the fidelity of each clone reads $F_{1\to 2}^{\text{pc}} = (1+1/\sqrt{2})/2 \approx 0.855$, which is indeed slightly higher than the fidelity of the optimal universal $1 \to 2$ cloner for qubits, $F_{1\to 2}^{\text{univ}} = \frac{5}{6} \approx 0.833$. Note that besides a blank copy qubit, the transformation also requires another ancilla qubit C. However, in contrast to universal cloning, this ancilla is not necessary and one can design a simplified cloning transformation which achieves the same fidelity and requires only two qubits: the input and a blank copy (Niu and Griffiths [1999], Durt and Du [2004]). This is very important from the experimental point of view since it is much easier to realize a two-qubit transformation than a three-qubit transformation. The economic phase-covariant cloner can be obtained by projecting the ancilla C on the basis state $|0\rangle$ (or $|1\rangle$). If we project on $|0\rangle$, then we get

$$\begin{aligned}|0\rangle_A|0\rangle_B &\to |0\rangle_A|0\rangle_B,\\ |1\rangle_A|0\rangle_B &\to \frac{1}{\sqrt{2}}\big(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B\big).\end{aligned} \tag{6.3}$$

An alternative economic cloning transformation can be obtained from eq. (6.3) by exchanging 0 and 1. Interestingly, the cloning machine (6.3) is optimal not only for the states on the equator of the Bloch sphere but also for all the states on the northern hemisphere, i.e., all states $\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$ with $\theta \leqslant \pi/2$ (Fiurášek [2003]). The optimal asymmetric cloning machine which produces two clones with different fidelities F_A and F_B is obtained by breaking the symmetry in the output superposition of $|10\rangle$ and $|01\rangle$,

$$\begin{aligned}|0\rangle_A|0\rangle_B &\to |0\rangle_A|0\rangle_B,\\ |1\rangle_A|0\rangle_B &\to \cos\vartheta|0\rangle_A|1\rangle_B + \sin\vartheta|1\rangle_A|0\rangle_B,\end{aligned} \tag{6.4}$$

and the two fidelities can be expressed as follows:

$$F_A = \frac{1}{2}(1+\sin\vartheta), \qquad F_B = \frac{1}{2}(1+\cos\vartheta), \quad \vartheta \in \left[0, \frac{1}{2}\pi\right]. \tag{6.5}$$

The phase-covariant cloning machine that maximizes the global two-qubit fidelity has a structure that is qualitatively similar to the cloner (6.2),

$$
\begin{aligned}
|0\rangle|A_{\text{in}}\rangle &\to \frac{1}{\sqrt{3}}\big[|0\rangle_A|0\rangle_B|0\rangle_C + \big(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B\big)|1\rangle_C\big],\\
|1\rangle|A_{\text{in}}\rangle &\to \frac{1}{\sqrt{3}}\big[|1\rangle_A|1\rangle_B|1\rangle_C + \big(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B\big)|0\rangle_C\big],
\end{aligned}
\tag{6.6}
$$

and it reaches fidelity $F_{1\to 2}^{\mathrm{pc,G}} = \frac{3}{4}$ which is again higher than the global fidelity of the universal cloner, $F_{1\to 2}^{\mathrm{univ,G}} = \frac{2}{3}$.

The phase-covariant cloning can be extended to the case when we possess N copies of the state and would like to prepare M clones, $M > N$. The optimal $1 \to M$ phase-covariant cloning machine was determined by Fan, Matsumoto, Wang and Wadati [2001], who considered the single-clone fidelity as the figure of merit. The structure of the cloning transformation depends of the parity of M. If M is even, there exist two independent cloning transformations,

$$
|0\rangle \to |M, M/2 - 1\rangle, \qquad |1\rangle \to |M, M/2\rangle \tag{6.7}
$$

and

$$
|0\rangle \to |M, M/2\rangle, \qquad |1\rangle \to |M, M/2 + 1\rangle, \tag{6.8}
$$

where $|M, k\rangle$ is a symmetric state of M qubits with k qubits in state $|1\rangle$ and $M-k$ qubits in state $|0\rangle$. Note that there are in fact infinitely many cloning transformations since any convex mixture of the operations (6.7) and (6.8) is also optimal. On the other hand, if M is odd then we get only one optimal transformation:

$$
|0\rangle \to \big|M, (M-1)/2\big\rangle, \qquad |1\rangle \to \big|M, (M+1)/2\big\rangle. \tag{6.9}
$$

The resulting fidelity is

$$
F = \begin{cases} \frac{1}{2} + \frac{\sqrt{M(2+M)}}{4M}, & M \text{ even},\\ \frac{1}{2} + \frac{M+1}{4M}, & M \text{ odd}. \end{cases} \tag{6.10}
$$

The optimality of the cloning transformations (6.7)–(6.9) can be proved using the method that was employed in Section 4.2 to prove the optimality of the $1 \to M$ universal cloning machine for qubits. In particular, the single-clone fidelity can be expressed as $F = \mathrm{Tr}[SR]$, where S is the operator isomorphic to the cloning CP map and

$$
\begin{aligned}
R = \frac{1}{4} I \otimes \Pi_{+,M} &+ \frac{1}{4}\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)\\
&\otimes \sum_{k=0}^{M-1} D_{M,k}\big(|M, k+1\rangle\langle M, k| + |M, k\rangle\langle M, k+1|\big),
\end{aligned}
\tag{6.11}
$$

where $D_{M,k} = \sqrt{(M-k)(k+1)}/M$. The fidelity is upper bounded by the maximum eigenvalue $r_{\max}$ of R, $F \leqslant 2r_{\max}$, and this bound is saturated by the above phase-covariant cloners.

Fan, Matsumoto, Wang and Wadati [2001] also conjectured the structure of the general optimal $N \to M$ phase-covariant cloning transformation for qubits. The proposed generalization is straightforward, namely, every input symmetric N-qubit state $|N,k\rangle$ is transformed to an M-qubit symmetric state $|M,k+j\rangle$ with the constant j adjusted such that the fidelity is maximized. If N and M have the same parity, $M = N + 2L$, then the suggested cloning map is $|N,j\rangle \to |M,j+L\rangle$, and the corresponding fidelity is

$$F^{\mathrm{pc}}_{N\to M} = \frac{1}{2} + \frac{1}{M2^N}\sum_{j=0}^{N-1}\sqrt{\binom{N}{j}\binom{N}{j+1}} \times \sqrt{(N+L-j)(L+j+1)}. \tag{6.12}$$

When M and N have different parities, $M = N + 2L + 1$, then the two possible cloning transformations are either $|N,j\rangle \to |M,j+L\rangle$ or $|N,j\rangle \to |M,j+L+1\rangle$, and the corresponding fidelity is

$$F^{\mathrm{pc}}_{N\to M} = \frac{1}{2} + \frac{1}{M2^{N+1}}\sum_{j=0}^{N-1}\sqrt{\binom{N}{j}\binom{N}{j+1}} \times \Big[\sqrt{(N+L-j+1)(L+j+1)} + \sqrt{(L+j+2)(N+L-j)}\Big]. \tag{6.13}$$

The optimality of the fidelity (6.12) was proved by D'Ariano and Macchiavello [2003] exploiting the generic theory of covariant cloning machines, see D'Ariano and Lo Presti [2001]. In contrast, if N and M have different parities, the optimal phase-covariant cloning transformation found by D'Ariano and Macchiavello [2003] differs from eq. (6.13).

6.2. *Phase-covariant cloning of qudits*

Going beyond the cloning of qubits, the $1 \to 2$ phase-covariant cloning of qudits (6.1) was investigated by Fan, Imai, Matsumoto and Wang [2003], Lamoureux and Cerf [2005] and Rezakhani, Siadatnejad and Ghaderi [2005]. It can be shown that the optimal cloning transformation for qudits (6.1) has the structure

$$|j\rangle \to \alpha|jj\rangle_{AB}|j\rangle_C + \frac{\beta}{\sqrt{2(d-1)}}\sum_{l\neq j}^{d-1}\big(|jl\rangle_{AB} + |lj\rangle_{AB}\big)|l\rangle_C, \tag{6.14}$$

where $\alpha^2 + \beta^2 = 1$. The two clones are contained in qudits A and B while the qudit C serves as an ancilla. Note that eq. (6.14) is a direct extension of the cloning transformation for qubits (6.2). The coefficients α and β have to be optimized such that the cloning fidelity is maximized. After some algebra one arrives at

$$\alpha = \left(\frac{1}{2} - \frac{d-2}{2\sqrt{d^2+4d-4}}\right)^{1/2},$$
$$\beta = \left(\frac{1}{2} + \frac{d-2}{2\sqrt{d^2+4d-4}}\right)^{1/2}, \tag{6.15}$$

and the fidelity reads

$$F = \frac{1}{4} + \frac{1}{2d} + \frac{\sqrt{d^2+4d-4}}{4d}. \tag{6.16}$$

In contrast to the phase-covariant cloning of qubits, we cannot get rid of the ancilla C because if we project the ancilla on the computational basis state $|k\rangle$ then the conditional map is not unitary. So, for $d > 2$ it seems impossible to implement the optimal phase-covariant $1 \to 2$ cloning in an economic way, without ancilla, see Durt, Fiurášek and Cerf [2005].

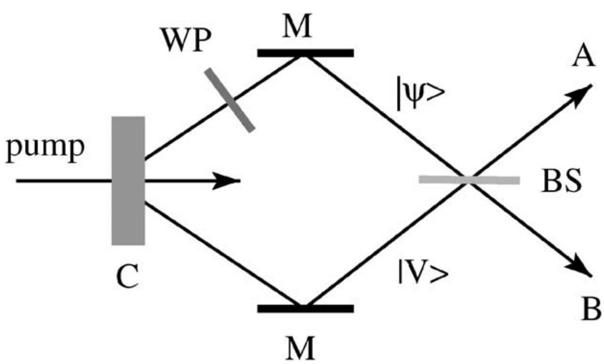
6.3. *Optical phase-covariant cloning*

In contrast to universal cloning, the optical experimental implementation of phase-covariant cloning machines has received much less attention. This may come as a surprise in view of the apparent simplicity of the optimal cloning transformation (6.3). However, the phase-covariant cloning exhibits much less symmetry than the universal copying, and methods such as stimulated amplification or symmetrization cannot readily be extended to implement the $1 \to 2$ phase-covariant cloning machine.

It is nevertheless possible to conditionally realize the $1 \to 2$ phase-covariant cloning of photonic qubits with linear optics (Fiurášek [2003]). As usual, the qubits are encoded into polarization states of single photons, and the state to be cloned is a balanced superposition of vertical and horizontal polarization, $|\psi\rangle = (|V\rangle + e^{i\phi}|H\rangle)/\sqrt{2}$. Besides the input state, the cloning requires also a second photon, the blank copy which we assume to be initially prepared in the state $|V\rangle$. Written in the basis of polarization states, the cloning transformation (6.3) becomes

$$|VV\rangle \to |VV\rangle, \qquad |HV\rangle \to \frac{1}{\sqrt{2}}\big(|HV\rangle + |VH\rangle\big). \tag{6.17}$$

The cloning machine is shown schematically in fig. 12(a). The input photon and the blank copy are combined on an unbalanced beamsplitter whose transmittance t_j and reflectance r_j for the vertical $(j = V)$ and horizontal $(j = H)$ polarizations are different. Only the events when the two photons leave the beamsplitter in different output ports are post-selected. The principle of operation of the cloner is easy to grasp. If the input $|\psi\rangle$ is in state $|V\rangle$, the two photons at the output must be in state $|VV\rangle$ since the blank copy is initially in the state $|V\rangle$. On the other hand, if the input to be cloned would be in the state $|H\rangle$ then the beamsplitter would produce a superposition of $|HV\rangle$ and $|VH\rangle$. By properly choosing r_j this superposition can be made balanced and the conditional map becomes exactly the unitary (6.17).



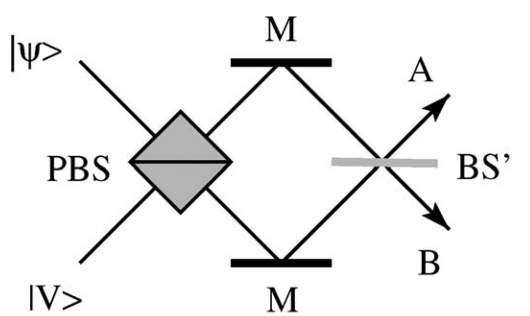


Fig. 12. Phase-covariant cloning using interference of two photons on an unbalanced beamsplitter. (a) Scheme with a single beamsplitter BS that differently reflects vertical and horizontal polarizations. (b) Alternative set-up involving a polarizing beamsplitter PBS and an unbalanced beamsplitter BS$'$ whose reflectance does not depend on the polarization.

The mixing of the modes on a beamsplitter is governed by the linear canonical transformations

$$a_{j,\mathrm{out}}^{\dagger} = r_j a_j^{\dagger} + t_j b_j^{\dagger}, \qquad b_{j,\mathrm{out}}^{\dagger} = r_j b_j^{\dagger} - t_j a_j^{\dagger}, \tag{6.18}$$

with $j = V, H$ and $r_j^2 + t_j^2 = 1$. The conditional transformation corresponding to selecting only the events with one photon in the left output arm (mode A) and one photon in the right output arm (mode B) reads

$$\begin{aligned} &|VV\rangle \to \left(r_V^2 - t_V^2\right)|VV\rangle, \\ &|HV\rangle \to r_H r_V |HV\rangle - t_H t_V |VH\rangle. \end{aligned} \tag{6.19}$$

This transformation becomes fully equivalent to eq. (6.17) if the following conditions are satisfied:

$$r_V^2 - t_V^2 = \sqrt{2} r_H r_V = -\sqrt{2} t_H t_V. \tag{6.20}$$

These constraints imply that $r_H = t_V$, $t_H = -r_V$ and $(r_V^2 - t_V^2) = \sqrt{2} r_V t_V$. On combining this equation with the normalization $r_V^2 + t_V^2 = 1$ we can determine r_V. After simple algebra we obtain

$$r_V^2 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right). \tag{6.21}$$

The probability of successfully realizing the phase-covariant cloner is given by

$$P = \left(r_V^2 - t_V^2\right)^2 = \frac{1}{3}. \tag{6.22}$$

The required beamsplitter with different transmittances for vertical and horizontal polarizations can be simulated by a Mach–Zehnder interferometer with polarization-dependent phase shifters in its arms, such as Soleil–Babinet compensators, so that the phase shift and, consequently, the splitting ratio could be controlled independently for vertical and horizontal polarizations. The set-up could also be modified to work with a beamsplitter whose reflectance is the same for both vertical and horizontal polarizations. This alternative configuration is depicted in fig. 12(b). The signal and blank copy photons are first combined on a polarizing beamsplitter PBS that reflects vertically polarized photons and transmits horizontally polarized photons. The two beams are then recombined on a beamsplitter with reflectance r. If the signal photon is initially vertically polarized, then a vertically polarized photon enters each input port of BS. If the signal photon is polarized horizontally, then it is switched to the right arm and two photons in orthogonal polarization states impinge on the right input port of BS. The polarizing beamsplitter ensures that the role of the transmittance and reflectance for the horizontally polarized photon is interchanged with respect to the scheme shown in fig. 12(a). It can easily be shown that this set-up leads to the cloning transformation (6.17) provided that the reflectance of the beamsplitter is equal to $r^2 = (1 + 1/\sqrt{3})/2$.

In the experiment, it may not be easy to precisely control the transmittance. It is therefore important to investigate how the performance of the set-up shown in fig. 12(b) depends on the reflectance r of the beamsplitter. The cloning transformation remains phase covariant and the cloning fidelity F is the same for both clones and does not depend on ϕ. However, F becomes a function of r. After some algebra one arrives at the formula for the fidelity of cloning of equatorial qubits,

$$F = \frac{1}{2}\left[1 + \frac{2r(2r^2 - 1)\sqrt{(1 - r^2)}}{2r^4 - 2r^2 + 1}\right]. \tag{6.23}$$

It turns out that the cloning is rather robust with respect to the variations of the reflectance of the beamsplitter, and a cloning fidelity $F > 0.8$ can be achieved for a broad range of beamsplitter reflectances $0.7 \leqslant r^2 \leqslant 0.9$.

In the experiment, the required pair of photons can be produced in spontaneous Type-I parametric down-conversion and the desired initial states of the photons can be prepared with the use of wave plates. After cloning, the states of the two clones can be analyzed by a sequence of wave plates, polarizing beamsplitters, and single-photon detectors, similarly as in the experiments on universal cloning.

6.4. Experimental 1-to-3 phase-covariant cloning

Remarkably, while the optimal $1 \to 2$ phase-covariant cloning transformation (6.2) or (6.3) has not yet been implemented for optical qubits, the optimal $1 \to 3$ phase-covariant cloning of the polarization state of a single photon has been demonstrated experimentally by Sciarrino and De Martini [2004]. The set of cloned states included all linear polarization states $\cos\theta|V\rangle + \sin\theta|H\rangle$. The first step in the copying process consisted of the optimal $1 \to 2$ universal cloner described in Section 5.1 which produced two clones and one anti-clone. In the next step, the anti-clone was converted into a clone by applying a unitary transformation σ_y with the help of a half-wave plate. The final step was to symmetrize the state of the three clones by combining the two clones and the anti-clone on a balanced beamsplitter and selecting only the events where all three photons ended up in the same output spatial mode. In this way three copies of equal fidelity were produced.

The experimentally observed fidelities were $F^{\mathrm{pc}}_{1\to 3}(|+\rangle) = 0.76$ for the state $|+\rangle = 2^{-1/2}(|V\rangle + |H\rangle)$ and $F^{\mathrm{pc}}_{1\to 3}(|H\rangle) = 0.80$. This should be compared with the theoretical maximum $F^{\mathrm{pc}}_{1\to 3} = \frac{5}{6} \approx 0.833$. It is also instructive to make a comparison with the fidelity of the optimal universal $1 \to 3$ cloner, $F^{\mathrm{univ}}_{1\to 3} = \frac{7}{9} \approx 0.778$. One can conclude that the experimental phase-covariant cloning machine operates very close to its theoretical limit, and for certain inputs it achieves better fidelity than what would be possible with universal cloning machine.

§ 7. Cloning of optical continuous variables

In Sections 4–6 we have considered the cloning of quantum states in finite-dimensional Hilbert spaces. During recent years, however, quantum information processing in systems with infinite-dimensional Hilbert space, such as modes of the electromagnetic field, has attracted a great deal of attention (see, e.g., Braunstein and van Loock [2005]). In this approach, the quantum information is usually encoded into two noncommuting quadrature operators x and p which satisfy canonical commutation relations $[x, p] = \mathrm{i}$. Since these operators have continuous spectra, one speaks of quantum information processing with continuous variables.

The universal cloning machine for states belonging to infinite-dimensional Hilbert space can be formally obtained as a limit of the universal cloning machine for qudits when $d \to \infty$. One finds that the single-clone fidelity of the universal $1 \to 2$ cloner is $\frac{1}{2}$, which means that the optimal cloning can be achieved by a very simple strategy where the input state is sent with probability $\frac{1}{2}$ to the first or second output, while the other output is prepared in maximally mixed state. Besides being rather trivial, this universal cloner is not of great practical interest because most of the quantum information protocols with continuous variables involve only the so-called Gaussian states. These states have a Gaussian Wigner function and their great advantage is that they can be generated and manipulated relatively easily in the laboratory with the help of linear optical interferometers and optical parametric amplifiers which produce squeezed and entangled Gaussian states.

7.1. Cloning of coherent states

Among the Gaussian states, the coherent state is perhaps the best known example. The coherent state $|\alpha\rangle$ can be defined as a displaced vacuum state $D(\alpha)|0\rangle$, where $D(\alpha) = \exp(\alpha a^{\dagger} - \alpha^{*} a)$ is the displacement operator. The coherent state is the eigenstate of the annihilation operator, $a|\alpha\rangle = \alpha|\alpha\rangle$ and it is also a minimum uncertainty state. The variance of all rotated quadratures $x_\theta = x\cos\theta + p\sin\theta$ is the same and equal to $\frac{1}{2}$. The Glauber P-distribution of a coherent state is a Dirac delta function, so that coherent states are not usually considered as non-classical states in the quantum-optical sense. Still, they are pure quantum states and they carry quantum noise. This makes these states suitable for applications such as quantum key distribution. It has been shown theoretically (Grosshans and Grangier [2002], Grosshans, Cerf, Wenger, Tualle-Brouri and Grangier [2003], Grosshans and Cerf [2004], Iblisdir, Van Assche and Cerf [2004]) and demonstrated experimentally (Grosshans, Van Assche, Wenger, Tualle-Brouri, Cerf and Grangier [2003]) that secure key distribution can be achieved with coherent states and balanced homodyne detection.

Let us first consider the optimal Gaussian cloning of coherent states introduced by Cerf, Ipe and Rottenberg [2000] and Lindblad [2000]. In this scenario, the class of admissible cloning transformations is restricted to Gaussian operations, which preserve the Gaussian shape of the Wigner function. Intuitively, one could expect that the Gaussian cloning should be optimal. This is indeed true if the figure of merit is the global M-clone fidelity or if the quality of the clones is quantified in terms of the noise added to the two quadratures x and p (Cerf and Iblisdir [2000]). However, it has been realized recently that, remarkably, the single-clone fidelity of the $1 \to 2$ cloning of coherent states is maximized by a non-Gaussian cloner (Cerf, Krüeger, Navez, Werner and Wolf [2005]).

We begin with the Gaussian $1 \to 2$ cloning. We require that the mean values of the quadratures of the two clones A and B are equal to the mean values of the quadratures x_{in} and p_{in} of the input coherent state α. This guarantees that the cloning transformation is invariant with respect to the displacements and the cloning fidelity does not depend on the amplitude α. In the Heisenberg picture, the most general Gaussian cloning transformation can be written in the form

$$x_A = x_{\text{in}} + \tilde{x}_A, \qquad x_B = x_{\text{in}} + \tilde{x}_B, \tag{7.1}$$

$$p_A = p_{\text{in}} + \tilde{p}_A, \qquad p_B = p_{\text{in}} + \tilde{p}_B. \tag{7.2}$$

The operators $\tilde{x}_A$, $\tilde{p}_A$, $\tilde{x}_B$ and $\tilde{p}_B$ represent the noise that is added to the two copies during the cloning process, and they all commute with x_{in} and p_{in}. The quadrature operators x_A, p_A, x_B and p_B must satisfy the canonical commutation relations, which implies

$$[\tilde{x}_A, \tilde{p}_B] = -\mathrm{i}, \qquad [\tilde{x}_B, \tilde{p}_A] = -\mathrm{i}. \tag{7.3}$$

The Heisenberg uncertainty relation gives a lower bound on the products of the variances of the noise operators,

$$\big\langle(\Delta\tilde{x}_A)^2\big\rangle\big\langle(\Delta\tilde{p}_B)^2\big\rangle \geqslant \frac{1}{4}, \qquad \big\langle(\Delta\tilde{x}_B)^2\big\rangle\big\langle(\Delta\tilde{p}_A)^2\big\rangle \geqslant \frac{1}{4}. \tag{7.4}$$

as shown in Cerf, Ipe and Rottenberg [2000] and Grosshans and Grangier [2001]. The cloning should add noise isotropically, that is, the variance of the x and p quadratures of each clone should be the same. Since the noise operators are not correlated with x_{in} and p_{in}, the variance of the quadratures (7.2) of the two clones is the sum of two variances, and the isotropy condition is satisfied if

$$\big\langle(\Delta\tilde{x}_A)^2\big\rangle = \big\langle(\Delta\tilde{p}_A)^2\big\rangle = \bar{n}_A, \qquad \big\langle(\Delta\tilde{x}_B)^2\big\rangle = \big\langle(\Delta\tilde{p}_B)^2\big\rangle = \bar{n}_B. \tag{7.5}$$

The two uncertainty relations (7.4) boil down to a single constraint

$$\bar{n}_A\bar{n}_B \geqslant \frac{1}{4}. \tag{7.6}$$

The state of each clone is a mixed Gaussian state, namely a coherent state with added thermal noise with mean number of thermal photons equal to $\bar{n}_j$, $j = A, B$. The fidelity of cloning can be most easily calculated from the Husimi Q-function, which is defined as the overlap of the density matrix with the coherent state, $Q(\beta) = \pi^{-1}\langle\beta|\rho|\beta\rangle$. The Q-function of the jth clone reads (Fiurášek [2001a]),

$$Q_j(\beta) = \frac{1}{\pi(1+\bar{n}_j)}\exp\left[-\frac{|\beta-\alpha|^2}{1+\bar{n}_j}\right]. \tag{7.7}$$

The fidelity can be calculated as $F_j(\alpha) = \pi Q_j(\alpha) = 1/(1+\bar{n}_j)$. The best trade-off between the fidelities of the two clones is obtained when the equality holds in eq. (7.6), and we get

$$F_A = \frac{2}{2+\mathrm{e}^{2\gamma}}, \qquad F_B = \frac{2}{2+\mathrm{e}^{-2\gamma}}, \tag{7.8}$$

where γ is a parameter which controls the asymmetry of the cloning. The fidelity of the optimal $1 \to 2$ symmetric ($\gamma = 0$) Gaussian cloner is $F = 2/3$.

7.2. *Cloning by phase-insensitive amplification*

If the coherent states are carried by optical modes, then the cloning can be realized with the use of a phase-insensitive amplification of light (Cerf and Iblisdir [2001c], Braunstein, Cerf, Iblisdir, van Loock and Massar [2001], Fiurášek [2001a], Cerf, Iblisdir and Van Assche [2002]). This is a natural and intuitive result, because the idealized perfect cloning amounts to noiseless amplification of

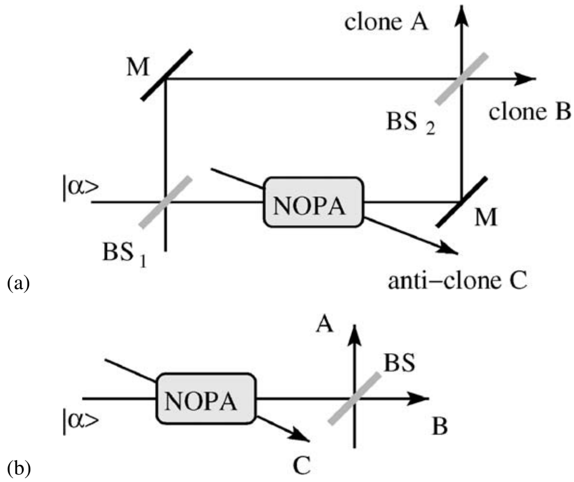


Fig. 13. Cloning of coherent states in a nondegenerate optical parametric amplifier. (a) Asymmetric $1 \to 2$ cloner consisting of a Mach–Zehnder interferometer with a non-degenerate parametric amplifier (NOPA) in one of its arms. The amplification gain and the splitting ratios determine the asymmetry of the cloner. (b) Simplified scheme of a symmetric cloner.

the coherent state, $|\alpha\rangle \to |\sqrt{2}\alpha\rangle$. The optimal amplification that adds the minimum amount of noise can be performed, e.g., in a nondegenerate optical parametric amplifier (NOPA), which transforms the input annihilation operator a as $a_{\mathrm{out}} = \sqrt{G}a_{\mathrm{in}} + \sqrt{G-1}c_{\mathrm{in}}^{\dagger}$, with c the annihilation operator of the idler mode in the NOPA. The set-up for asymmetric cloning of coherent states is shown in fig. 13(a). It consists of a Mach–Zehnder interferometer with an amplifier in one of its arms. The signal is initially divided into two beams and one beam is amplified such that the total mean intensity is twice the input intensity. The two clones are obtained by recombining the two beams on the second beamsplitter. The splitting ratios of the unbalanced beamsplitters BS_1 and BS_2 and the intensity gain G of the amplifier can be expressed in terms of the asymmetry parameter γ as follows:

$$
\begin{aligned}
r_1 &= -\frac{\sqrt{2}\sinh\gamma}{\sqrt{1+2\sinh^2\gamma}}, \\
G &= 1+\cosh(2\gamma), \\
r_2 &= \frac{\mathrm{e}^{2\gamma}}{\sqrt{1+\mathrm{e}^{4\gamma}}}.
\end{aligned}
\tag{7.9}
$$

The set-up becomes particularly simple for a symmetric cloner. In this case the first beamsplitter disappears and the whole input signal is amplified with gain $G = 2$ and then divided into two modes on a balanced beamsplitter, see fig. 13(b).

The procedure for symmetric cloning can be readily extended to the optimal symmetric $N \to M$ Gaussian cloning of coherent states. The cloning consists of three steps. First, the whole signal is collected in a single mode using an array of $N-1$ beamsplitters with properly chosen transmittances, $|\alpha\rangle^{\otimes N} \to |\sqrt{N}\alpha\rangle$. Next, the collected signal is amplified with a gain $G = M/N$. Finally, the amplified signal is distributed among the M modes with the help of another array of $M - 1$ unbalanced beamsplitters such that the mean complex amplitude in all modes is the same and equal to α. The fidelity of this cloner does not depend on α, and each clone is in a coherent state with thermal noise described by the Husimi function (7.7). The total mean number of thermal photons in all modes is $G-1 = M/N-1$, and the noise is equally divided into M modes, hence the thermal noise in each clone is $\bar{n} = 1/N - 1/M$. On inserting this into the expression for the fidelity, $F = 1/(1+\bar{n})$, we obtain

$$F = \frac{MN}{MN + M - N}. \tag{7.10}$$

In the limit of an infinite number of copies, $M \to \infty$, we get $F = N/(N+1)$ which is the fidelity of optimal estimation of a coherent state from N copies. Similarly as in the case of universal cloning of qubits (Bruss, Ekert and Macchiavello [1998]), the connection between optimal cloning and optimal state estimation can be exploited to prove that eq. (7.10) is the maximal fidelity of the Gaussian $N \to M$ cloning of coherent states (Cerf and Iblisdir [2000]). As shown by van Loock and Braunstein [2001], it is also possible to clone coherent states via an extended continuous-variable teleportation. The telecloning requires a specific multimode entangled Gaussian state that can be generated by mixing single-mode squeezed vacuum states on an array of unbalanced beamsplitters (van Loock and Braunstein [2000]).

As noted before, the Gaussian machine depicted in fig. 13 is not the optimal one if the single-clone fidelity is taken as the figure of merit (Cerf, Krüeger, Navez, Werner and Wolf [2005]). The optimal non-Gaussian cloner can achieve a fidelity $F_{\max} = 0.6826$, which is slightly higher than the maximum fidelity achievable by Gaussian transformations, $F = \frac{2}{3} \approx 0.6667$. Interestingly, the optimal non-Gaussian cloner can be obtained from the set-up shown in fig. 13(b) if the input ports of the idler mode of the amplifier and the auxiliary mode of the beamsplitter are fed with a specific non-Gaussian entangled state $|\psi\rangle_{BC} = \sum_{n=0}^{\infty} c_n |2n\rangle_B |2n\rangle_C$. The coefficients c_n can be optimized in order to maximize the cloning fidelity which results in the above value $F_{\max} = 0.6826$. It should be stressed that while the non-Gaussian cloner maximizes the fidelity, the variance of the quadratures of the clones is higher than for the optimal Gaussian cloner. In applications such as quantum key distribution with coherent states and balanced homodyning, where the quantum channel between Alice and Bob is characterized in terms of the first and second moments of the transmitted quadratures, the aim of the eavesdropper is to minimize the quadrature variance instead of the fidelity, and the Gaussian cloning (or its variant entangling Gaussian cloner in case of reverse reconciliation protocol) can be the most dangerous individual eavesdropping attack.

7.3. Experimental cloning of coherent states

The first proposal for an experimental continuous-variable cloning machine, due to D'Ariano, De Martini and Sacchi [2001], seemed quite involved as it required a network of parametric amplifiers. Recently, the optimal Gaussian $1 \to 2$ cloning of coherent states was experimentally demonstrated by Andersen, Josse and Leuchs [2005]. The distinct feature of this experiment is that it does not require an amplifier, the latter being replaced by a clever combination of measurement and feedback. A simplified scheme of the experimental set-up is shown in fig. 14. Mode $a_{\rm in}$ contains the coherent state to be cloned. The beam is split into two parts on a balanced beamsplitter whose auxiliary input port ν_1 is in vacuum state. The output annihilation operators thus read

$$a' = \frac{1}{\sqrt{2}}(a_{\rm in} + \nu_{1,\rm in}), \qquad \nu_1' = \frac{1}{\sqrt{2}}(a_{\rm in} - \nu_{1,\rm in}).$$

The output beam ν_1' is sent to an eight-port homodyne detector, which consists of a balanced beamsplitter followed by two balanced homodyne detectors. This detector effectively measures the operator $\lambda = \nu_1' + \nu_2^\dagger$, where $\nu_2^\dagger$ is the creation operator of an auxiliary vacuum mode. After the measurement, the mode a' is displaced by the amount λ which is in practice achieved by mixing this beam with a strong coherent beam with amplitude $\lambda/\sqrt{1-T}$ on a highly unbalanced beamsplitter with transmittance $T \approx 99\%$. The resulting displaced beam is effectively the amplified input,

$$a_{\rm disp} = \sqrt{2}a_{\rm in} + \nu_2^\dagger.$$

The cloning is finished by dividing the amplified beam into two parts with the help of another balanced beamsplitter, thereby preparing the two clones of the input coherent state. The fidelity observed in the experiment was about 65%, very close to the optimal value $2/3 \approx 0.667$.

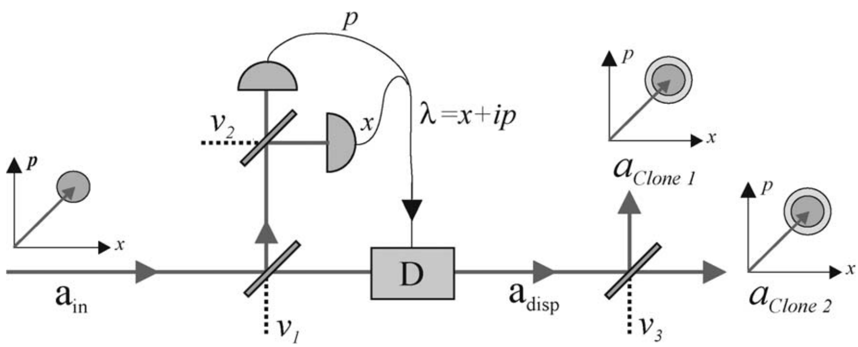


Fig. 14. Cloning of coherent states using only linear optics and homodyne detection. ν_1, ν_2 and ν_3 denote vacuum auxiliary input modes, D is a displacement proportional to the measurement λ. (After Andersen, Josse and Leuchs [2005].)

7.4. *Gaussian distribution with finite width*

Up to now, we have assumed that the distribution of the coherent states that should be cloned is uniform over the entire phase space. However, this is clearly an idealization, since the mean energy of the input state would be infinite. A more realistic scenario, considered by Cochrane, Ralph and Dolinska [2004], is that the coherent states are drawn from a Gaussian distribution with width σ and centered on vacuum, so that the a priori probability that the cloned state is $|\alpha\rangle$ is given by

$$P(\alpha) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{|\alpha|^2}{2\sigma^2}\right). \tag{7.11}$$

This occurs for instance in quantum key distribution with coherent states (Grosshans, Van Assche, Wenger, Tualle-Brouri, Cerf and Grangier [2003]).

If the width σ of the distribution (7.11) is finite, then we possess some information that can be explored in order to increase the average cloning fidelity. Also, the probability (7.11) is not invariant with respect to the displacements, so there is no reason to search for a covariant cloner. The fidelity of the cloner may depend on the input state, and the figure of merit that should be maximized is the average fidelity,

$$\mathcal{F} = \int P(\alpha) F(\alpha)\, \mathrm{d}^2\alpha. \tag{7.12}$$

It turns out that the optimal finite-width symmetric $1 \to 2$ Gaussian cloning transformation is still amplification followed by beamsplitting on a balanced beamsplitter. However, the gain G depends on σ. After the amplification and beamsplitting, the coherent amplitude in each mode is $\alpha\sqrt{G/2}$ and the mean number of chaotic photons in each mode is $\bar{n} = (G-1)/2$. The fidelity of cloning a particular coherent state $|\alpha\rangle$ reads

$$F(\alpha) = \frac{2}{G+1}\exp\left[-\frac{2(1-\sqrt{G/2}\,)^2}{G+1}|\alpha|^2\right]. \tag{7.13}$$

After averaging over the Gaussian distribution (7.11) we arrive at the expression for the mean fidelity,

$$\mathcal{F} = \frac{2}{G+1+2\sigma^2(2+G-2\sqrt{2G}\,)}. \tag{7.14}$$

We have to find the maximum of $\mathcal{F}$ under the constraint $G \geqslant 1$. It turns out that there are two different solutions depending on the value of σ. If $\sigma^2 > \sigma_{\text{th}}^2 = (1+\sqrt{2}\,)/2$, then it is optimal to amplify the signal and the optimal gain is

$$G = \frac{8\sigma^4}{(1+2\sigma^2)^2}. \tag{7.15}$$

On the other hand, if $\sigma^2 < \sigma_{\text{th}}^2$ then it is optimal to simply divide the input signal into two beams without any amplification, and $G = 1$. The resulting cloning fidelity is

$$\mathcal{F} = \begin{cases} \frac{4\sigma^2+2}{6\sigma^2+1}, & \sigma^2 \geqslant \sigma_{\text{th}}^2, \\ \frac{1}{1+(3-2\sqrt{2})\sigma^2}, & \sigma^2 < \sigma_{\text{th}}^2. \end{cases} \tag{7.16}$$

The average fidelity increases monotonically with decreasing width of the distribution (7.11), and in the limit $\sigma \to 0$ we get $\mathcal{F} = 1$, as expected.

7.5. Cloning of conjugate coherent states

In Section 5.4 we discussed a cloning machine for a pair of orthogonal qubits. This device possesses a natural and very interesting continuous-variable analogue, namely, one can consider a cloning machine for coherent states $|\alpha\rangle$ whose input consists of N copies of the state $|\alpha\rangle$ and N' copies of the complex conjugate coherent state $|\alpha^*\rangle$. This problem was analyzed in detail in an even more general setting by Cerf and Iblisdir [2001a].

Without any loss of generality, we can assume that a pair of arrays of beamsplitters is used to collect all signal into two modes, and the input state of the cloning machine thus reads $|\sqrt{N}\alpha\rangle_A|\sqrt{N'}\alpha^*\rangle_B$. The goal of cloning is to produce M copies of $|\alpha\rangle$ with minimum added noise. This could again be accomplished with the help of a non-degenerate parametric amplifier. While mode A represents the signal input similarly as before, mode B is sent to the idler input port of the amplifier. Assuming amplification with intensity gain G, the output annihilation operator of the signal mode is given by $a_{\text{out}} = \sqrt{G}a_{\text{in}} + \sqrt{G-1}b_{\text{in}}^{\dagger}$. Note that both terms a_{in} and $b^{\dagger}$ in the above formula contribute to the total coherent signal in a_{out}. If the cloning should be performed with unity gain, then G must satisfy

$$\sqrt{M} = \sqrt{G}\sqrt{N} + \sqrt{G-1}\sqrt{N'}, \tag{7.17}$$

and we can easily determine G by solving the above quadratic equation.

A careful analysis reveals that for certain values of N, N' and M the cloning with conjugate inputs could be more efficient than the standard cloning of coherent states. To be fair, we should compare the cloning fidelities for inputs consisting either of $N+N'$ copies of $|\alpha\rangle$ or of N copies of $|\alpha\rangle$ and N' copies of $|\alpha^*\rangle$. The advantage of dealing with complex conjugate inputs could be most easily illustrated in the limit of an infinite number of clones, $M \to \infty$, where the optimal cloning becomes equivalent with optimal state estimation. It has been shown by Cerf and Iblisdir [2001b] that when possessing a single copy of $|\alpha\rangle|\alpha^*\rangle$ we can estimate $|\alpha\rangle$ with fidelity $F_{\text{c.c.}} = \frac{4}{5}$, which is strictly higher than the estimation fidelity $F = \frac{2}{3}$ corresponding to the input state $|\alpha\rangle^{\otimes 2}$. In the former case the optimal detection strategy is the nonlocal continuous-variable Bell measurement where the quadratures $x_A + x_B$ and $p_A - p_B$ are measured simultaneously.

§ 8. Conclusions

The quantum no-cloning theorem is a crucial aspect of modern quantum mechanics and one of the cornerstones of quantum information theory. Besides its fundamental interest for the foundations of quantum physics, the impossibility of exactly copying an unknown quantum state is crucial for the security of quantum key distribution protocols. Going beyond the no-cloning theorem, it is possible to design approximate quantum cloning machines, which enable the copying of quantum information in an optimal – albeit imperfect – way, an issue which has attracted considerable attention over the last decade.

This review aims at providing an exhaustive overview of the various quantum cloning machines that have been introduced since the concept was put forward by Bužek and Hillery [1996]. The mathematical description of quantum cloning machines based on the isomorphism between maps and operators is developed in detail. Special attention is also devoted to the experimental optical implementations of these machines. The cloning of single photons has now been accomplished by several groups, and these experiments represent a very valuable contribution to the toolbox of available optical methods for quantum information processing.

In the course of years, quantum cloning has grown into a genuine subfield of quantum information sciences, which is still currently very active on both the theoretical and experimental side. The advanced methods of preparation, manipulation, and measurement of quantum states of light, whose development has been stimulated to a large extent by the perspectives of quantum information processing, have recently enabled the demonstration of even more complex cloning machines. In the years to come, we anticipate many new achievements and breakthroughs in quantum information sciences, and there is no doubt that quantum cloning will play an important role in these future developments.

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