Optimal multicopy asymmetric Gaussian cloning of coherent states

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(Received 25 October 2006; published 29 May 2007)

We investigate the asymmetric Gaussian cloning of coherent states which produces M copies from N input replicas in such a way that the fidelity of each copy may be different. We show that the optimal asymmetric Gaussian cloning can be performed with a single phase-insensitive amplifier and an array of beam splitters. We obtain a simple analytical expression characterizing the set of optimal asymmetric Gaussian cloning machines and prove the optimality of these cloners using the formalism of Gaussian completely positive maps and semidefinite programming techniques. We also present an alternative implementation of the asymmetric cloning machine where the phase-insensitive amplifier is replaced with a beam splitter, heterodyne detector, and feedforward.

DOI: 10.1103/PhysRevA.75.052335 PACS number(s): 03.67.Hk, 42.50.−p

I. INTRODUCTION

The perfect copying of unknown quantum states is forbidden by the linearity of quantum mechanics [1]. This observation lies at the heart of quantum communication protocols, such as quantum key distribution (QKD) which allows the provably secure sharing of a secret key between two distant partners (see, e.g., [2]). Any eavesdropping on a QKD system introduces noise into the transmission, which can be detected by the legitimate users. The optimal individual eavesdropping attacks on many QKD protocols consist of the optimal (approximate) copying of the quantum states transmitted in the channel, where one copy is sent to the legitimate receiver while the other copy is kept by the eavesdropper and measured upon at a later stage. After the seminal paper by Bužek and Hillery [3], where the concept of universal quantum cloning machine was introduced, the issue of quantum copying has attracted considerable attention (see, e.g., [4,5]). This effort culminated in the recent years with the experimental demonstration of optimal $1\rightarrow2$ cloning machines for polarization states of photons based either on parametric amplification [6–8] or on the symmetrization of the multiphoton state on an array of beam splitters [9,10]. The latter technique was also exploited lately to realize the universal symmetric cloning machine for qubits that produces three clones [11].

In applications such as QKD, one is often interested in asymmetric cloning where the fidelities of the clones are different. This is indeed necessary to study the trade-off between the information gained by the eavesdropper and the noise detected by the legitimate users. The optimal $1\rightarrow2$ asymmetric cloning of qubits and qudits has been studied in detail [12–14] and very recently an experimental demonstration of $1\rightarrow2$ asymmetric cloning of polarization states of photons [15] based on partial teleportation [16] was reported. Going beyond two copies, multipartite asymmetric cloning machines have been introduced in [17], which produce M copies with different fidelities F_j ($j=1,\ldots,M$). Several examples of such multipartite asymmetric cloners for qubits and qudits were presented in [18,19].

In the context of the rapid development of quantum information processing with continuous variables [20], the cloning of coherent states has been extensively studied over the last years [21,22]. It was shown that the optimal $N\rightarrow M$ symmetric cloning of optical coherent states that preserves the Gaussian shape of the Wigner function can be accomplished with the help of a phase-insensitive amplifier followed by an array of beam splitters that distributes the amplified signal into M modes [23,24]. It is also possible to exploit the off-resonant interaction of light beams with atomic ensembles and perform the cloning of coherent states into an atomic memory [25]. The cloning of a finite distribution of coherent states was studied [26], and a reversal of cloning by means of local operations and classical communication was suggested in [27].

On the experimental side, the $1\rightarrow2$ optimal Gaussian cloning of coherent states of light was recently demonstrated in [28]. There, the phase-insensitive optical amplifier was replaced with a clever combination of beam splitters, homodyne detection, and feedforward, which effectively simulated the amplification process [29–32]. Using homodyne detectors with very low electronic noise, it was possible to achieve a cloning fidelity of about 65%, very close to the theoretical maximum of $2/3$. In another experiment, the $1\rightarrow2$ telecloning of coherent states of light was also realized [33].

In this paper, we extend the concept of multipartite asymmetric cloning to continuous variables and present the optimal multipartite asymmetric Gaussian cloning machines for coherent states. These devices produce M approximate replicas of the coherent state $|\alpha\rangle$ from N input replicas, such that the fidelity F_j of each clone is generally different, and, for a given set of $F_1,\ldots,F_{M-1}$, the fidelity of the Mth clone F_M is the maximum possible. The multicopy asymmetric $1\rightarrow M$ cloning of a coherent state was previously studied by Ferraro and Paris in the context of telecloning [34]. Here, we rigorously prove that their scheme is optimal, and present a generic optimal asymmetric cloning machine for any number N of input replicas, as well as its optical implementation. We also consider the related problem of the optimal partial state estimation of coherent states.

The rest of the paper is structured as follows. In Sec. II, we present an optical cloning scheme based on phase-insensitive amplification and passive linear optics. We also derive the trade-off between the fidelities (or, equivalently,

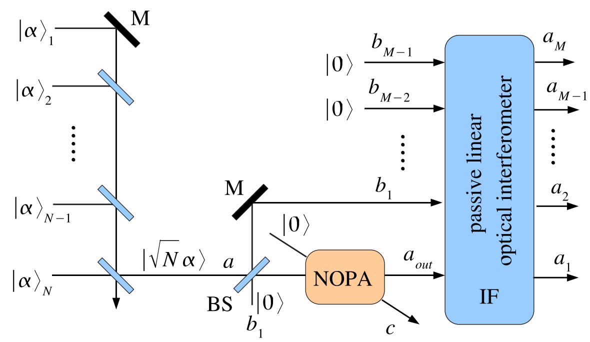

FIG. 1. (Color online) Optimal Gaussian $N \to M$ fully asymmetric cloning of coherent states. See text for details.

added thermal noises), which fully characterizes the class of the optimal multipartite asymmetric Gaussian cloners. In Sec. III, we describe an alternative cloning scheme where the amplification is replaced by measurement and feedforward. In Sec. IV, we briefly discuss the relationship between the optimal asymmetric cloning and optimal partial measurement of coherent states. Then, the proof of the optimality of the asymmetric cloning machine is given in Sec. V. Finally, Sec. VI contains a brief summary and conclusions.

II. ASYMMETRIC GAUSSIAN CLONING OF COHERENT STATES

In what follows, we restrict ourselves to Gaussian cloning transformations. Note that it has been found in [35] that the optimal $1 \to 2$ cloning transformation for coherent states (i.e., the transformation that maximizes the single-clone fidelity) is in fact non-Gaussian, though the gain in fidelity is tiny. Nevertheless, it should be stressed that, while this non-Gaussian cloner is optimal in terms of fidelity, it adds more noise to the clones (as measured by the quadrature variances) than the optimal Gaussian cloner. In potential applications of cloning such as eavesdropping on QKD protocols with coherent states and homodyne detection [36,37], one is often interested in minimizing the quadrature variance. In such a case, the Gaussian cloning turns out to be the most dangerous attack [38]. Finding the multipartite generalization of asymmetric Gaussian cloning is therefore a very interesting question.

As we will prove in Sec. V, the optimal Gaussian cloning machine has the simple structure depicted in Fig. 1, which is a direct generalization of the $1 \to 2$ asymmetric cloner [24]. The signal contained in N input replicas of the coherent state $|\alpha\rangle$ is first collected into a single mode by an array of $N-1$ unbalanced beam splitters [23,24]. After this, a single mode a carries all the signal, and is in a coherent state $|\sqrt{N}\alpha\rangle$. This mode is sent on an unbalanced beam splitter BS with amplitude transmittance t and reflectance r, which divides the signal into two modes a and b_1. Mode a is then amplified in a phase-insensitive amplifier (NOPA) with amplitude gain g. The modes a and b_1 together with $M-2$ auxiliary modes b_j, with $j=2,\ldots,M-1$, are combined in a passive linear M-port interferometer (IF) whose output modes contain the M clones. The interferometer is designed in such a way that the coherent component in each output mode is equal to α. In the Heisenberg picture, the overall input-output transformation describing the cloner depicted in Fig. 1 reads

$$a_j = \frac{1}{\sqrt{N}} a + \sum_{k=1}^{M-1} \kappa_{jk} b_k + \sqrt{n_j} c^\dagger, \tag{1}$$

where $c^\dagger$ is the creation operator of the idler port of the amplifier and b_k are the annihilation operators of the $M-1$ auxiliary modes, initially in the vacuum state. Here, n_j represents the amount of noise added to the jth clone, and the κ_{jk} coefficients are chosen in such a way that the canonical commutation relations are preserved.

It follows from the canonical transformations (1) that each clone is in a mixed Gaussian state with coherent amplitude α and added thermal noise characterized by a mean number of thermal photons n_j. The Husimi Q-function of the jth clone reads

$$Q_j(\beta) = \frac{1}{\pi(n_j+1)} \exp\left(-\frac{|\alpha-\beta|^2}{n_j+1}\right). \tag{2}$$

The fidelity of the jth clone is proportional to the value of the Husimi Q-function at $\beta=\alpha$, and is therefore a monotonic function of the added thermal noise,

$$F_j = \frac{1}{1+n_j}. \tag{3}$$

The higher the thermal noise n_j, the lower is the fidelity F_j, and *vice versa*. The cloning is covariant and isotropic, i.e., the fidelity does not depend on the input state $|\alpha\rangle$ and the added noise is the same for each quadrature. These are natural conditions that the optimal cloning machine should satisfy.

The shot-noise limited amplification is governed by the transformation

$$a_{\text{out}} = g(ta - rb_1) + \sqrt{g^2-1}\, c^\dagger \tag{4}$$

and the total mean number of thermal photons produced during the amplification is $n_{\text{tot}}=g^2-1$. Since the linear interferometer does not add any noise, we have

$$g = \sqrt{1+n_{\text{tot}}}, \tag{5}$$

where $n_{\text{tot}}=\Sigma_{j=1}^{M} n_j$. The total intensity of the coherent signal after amplification is $N(r^2+g^2t^2)|\alpha|^2$, which should be equal to $M|\alpha|^2$ if we require the coherent component of each clone to be equal to α. From this, we can determine the transmittance of BS, namely

$$t = \sqrt{\frac{M-N}{n_{\text{tot}} N}}. \tag{6}$$

The multiplets of n_j cannot be arbitrary. Indeed, the M-port interferometer (IF) in Fig. 1 is described by a unitary matrix V, such that $a_j=\Sigma_{j=1}^{M-1} v_{jk} b_k + v_{jM} a_{\text{out}}$. The unitarity of V imposes a constraint on n_j which can be expressed as

$$\left(\sum_{k=1}^{M}\sqrt{n_k}\right)^2 = (M-N)\left(\sum_{j=1}^{M} n_j + 1\right). \tag{7}$$

This formula provides a simple analytical parametrization of the set of optimal $N \to M$ asymmetric Gaussian cloning machines for coherent states.

In the special case of a $1 \to 2$ asymmetric Gaussian cloner, Eq. (7) reduces to

$$n_1 n_2 = (1/2)^2, \tag{8}$$

which coincides with the no-cloning uncertainty relation that was displayed in [22,24]. (Note that 1/2 corresponds here to one shot-noise unit.) Interestingly, if we consider a $1 \to 3$ cloner and assign to the first two clones the fidelity of the optimal $1 \to 2$ symmetric cloner, that is, $n_1 = n_2 = 1/2$, we obtain by solving Eq. (7) that the noise of the third clone is not infinite, $n_3 = 2$. As noticed in [18], this means that some quantum information remains available beyond the one contained in the two clones (it actually corresponds to the information hidden in the anticlone). In the case where $N=1$ but M is arbitrary, we recover the expression that was derived by Ferraro and Paris [34]. Finally, note that if one clone is perfect, e.g., $n_M = 0$, then Eq. (7) is transformed into the same equation but for a $(N-1) \to (M-1)$ cloner, which means that one of the input replicas is simply redirected to the perfect clone while the cloning of the $N-1$ remaining input replicas into the $M-1$ other clones is simply governed by the same relation.

III. OPTIMAL ASYMMETRIC CLONING VIA MEASUREMENT AND FEEDFORWARD

In the experimental demonstration of the optimal $1 \to 2$ cloning of coherent states of light carried out in [28], the amplification in a phase-insensitive amplifier was replaced by a clever combination of a partial measurement and feedforward that effectively simulates the amplification process [29]. Later on it was shown experimentally that an amplifier of arbitrary gain g can be simulated in this way [30] and it was noted that any $N \to M$ symmetric Gaussian cloning of coherent states can be implemented by measurement and feedforward [31,32]. Here we show that this holds true also for the optimal asymmetric cloning. The scheme shown in Fig. 1 can be straightforwardly transformed into a setup which involves only passive linear optics, balanced homodyne detection, and coherent displacement of the beams proportional to the measurement outcomes. The resulting configuration is illustrated in Fig. 2.

We assume that all of the available signal has been collected into a single mode a which is thus in the coherent state $|\sqrt{N}\alpha\rangle$. The beam is divided into two parts on a beam splitter BS_g with amplitude transmittance $\tilde{t}$ and reflectance $\tilde{r}$. The reflected part is fed into a heterodyne detector consisting of a balanced beam splitter whose auxiliary input port c is in the vacuum state and two balanced homodyne detectors BHD measuring the x and p quadratures, respectively. This detector effectively measures the operator $o = \tilde{r}a + \tilde{t}b + c^\dagger$. The portion of the beam transmitted through BS_g, characterized by

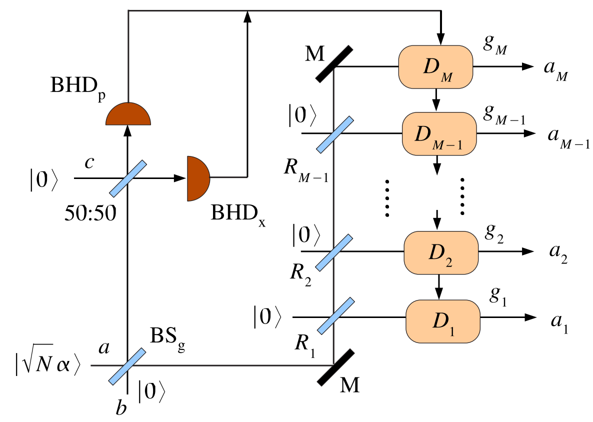

FIG. 2. (Color online) Setup for the multipartite asymmetric Gaussian cloning of coherent states using homodyne detection and feedforward.

$\tilde{t}a - \tilde{r}b$, is divided into M modes a_j by an array of $M-1$ beam splitters with reflectances r_k. Each mode a_k is then coherently displaced by amount $g_j o$, where g_j is the electronic gain of the corresponding feed forward. The added thermal noise in the output mode a_j reads $n_j = g_j^2$, which immediately fixes all electronic gains,

$$g_j = \sqrt{n_j}. \tag{9}$$

As shown in [30], the optical amplification gain g is obtained in the feedforward scheme if the beam is split on a beam splitter with reflectance $\sqrt{1-1/g^2}$, and the reflected part is heterodyne measured. Since in the scheme of Fig. 1 only an (amplitude) fraction t of the input beam a is actually amplified, we see that by replacing the amplifier with a beam splitter of reflectance $\sqrt{1-1/g^2}$, the fraction of the beam that is sent to the heterodyne detector is $t\sqrt{1-1/g^2}$. Thus, in Fig. 2, the reflectance of the beam splitter BS_g must be $\tilde{r} = t\sqrt{1-1/g^2}$, which yields

$$\tilde{r} = \sqrt{\frac{M-N}{(1+n_{\mathrm{tot}})N}}. \tag{10}$$

It remains to determine the reflectances r_k of the final array of beam splitters. They are fixed by the condition that the coherent amplitude of each clone is α. After some algebra we find that

$$r_j = \frac{\sqrt{1+n_{\mathrm{tot}}} - \sqrt{(M-N)n_j}}{\sqrt{(2+n_{\mathrm{tot}})N - M}} \prod_{k=1}^{j-1} (1-r_k^2)^{-1/2}. \tag{11}$$

From this formula, all r_k's can be calculated in an iterative way, starting from r_1 [which is given by Eq. (11) for $j=1$ with the product over k replaced by 1].

IV. OPTIMAL CLONING AND OPTIMAL PARTIAL ESTIMATION OF COHERENT STATES

There is a close relationship between optimal cloning and optimal state estimation. An interesting scenario that recently attracted a lot of attention consists in the partial estimation of a state, which yields the classical estimate of the state as well as the perturbed quantum state [39–44]. According to the fact that in the limit of an infinite number of copies, the optimal cloning becomes equivalent to optimal state estimation [45], this optimal partial estimation can be viewed as a limiting case of an asymmetric cloning producing one (quantum) copy with fidelity F and infinitely many (classical) copies with fidelity G [17]. From the analytical formula (7), we can thus rigorously derive the optimal trade-off between the fidelities F and G in the partial Gaussian estimation of coherent states. We set $N=1$, $n_1=n_F$, $n_2=n_3=\cdots=n_M=n_G$ and take the limit $M\to\infty$, which results in

$$n_G=\frac{(n_F+1)^2}{4n_F}. \tag{12}$$

In the limit of an undisturbed quantum copy ($n_F=0$), we have an infinitely noisy state estimation, as expected. We also note that $n_F=n_G=1$ is a solution of Eq. (12), which corresponds to the optimal (full) estimation of coherent states. Equation (12) also translates in the following relation between the fidelities:

$$G=\frac{4F(1-F)}{4F(1-F)+1}. \tag{13}$$

This agrees with the trade-off derived in [39] which confirms that the experimentally demonstrated partial measurement of coherent states in that work was indeed optimal among all Gaussian strategies. Note that using a non-Gaussian protocol a slightly better trade-off between F and G could be achieved [46].

V. PROOF OF OPTIMALITY

In what follows, we will prove the optimality of the asymmetric cloner defined in Secs. II and III. Let us first note that the fidelities are monotonic functions of n_j, so that instead of maximizing the fidelities F_j we can equivalently minimize the added thermal noise n_j. The design of the optimal cloner can be thus rephrased as the minimization of a cost function [17],

$$C(n_j)=\sum_{j=1}^{M}x_jn_j, \tag{14}$$

which is a linear convex mixture of n_j, $x_j>0$. The ratios of the coefficients x_j control the asymmetry of the cloning machine.

The most general Gaussian operation is a trace-preserving Gaussian completely positive (CP) map [21], and we must minimize (14) over all such maps. At the level of covariance matrices γ, the Gaussian CP map acts as

$$\gamma_{\text{out}}=S\gamma_{\text{in}}S^T+G. \tag{15}$$

The covariance matrix of N modes is defined as $\gamma_{jk}=\langle\Delta r_j\Delta r_k+\Delta r_k\Delta r_j\rangle$, where $r=(x_1,\ldots,x_N,p_1\ldots,p_N)$ is the vector of quadrature operators, $[x_j,p_k]=i\delta_{jk}$. The first moments transform under the Gaussian CP map according to $\langle r_{\text{out}}\rangle=S\langle r_{\text{in}}\rangle$.

The matrices S and G must satisfy the complete positivity constraint

$$A\equiv G+iK\geqslant 0,\quad K=J_{M_{\text{out}}}-SJ_{M_{\text{in}}}S^T, \tag{16}$$

where the matrix

$$iJ_M=i\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix} \tag{17}$$

comprises the commutators of the quadrature operators, while I denotes the identity matrix of dimension M, and M_{in} and M_{out} are the number of input and output modes, respectively. The cloning machine of interest has effectively only a single input mode a (as we collect the N input signals into a single mode) and M output modes, so that $M_{\text{in}}=1$ and $M_{\text{out}}=M$. The $M\times 2$ matrix S is fixed by the condition that the first moments should be preserved by cloning. We obtain

$$S^T=\frac{1}{\sqrt{N}}\begin{pmatrix}1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1\end{pmatrix}. \tag{18}$$

We must minimize $C(n_j)$ over the set of all Gaussian completely positive maps (15) with the matrix S given by (18), that is, we must optimize over all G's satisfying (16). Clearly, if G_1 and G_2 satisfy (16), then any convex combination $pG_1+(1-p)G_2$ with $p\in[0,1]$ also does. Thus, we have a convex optimization problem. Moreover, since $n_j=(G_{jj}+G_{M+j,M+j}+2/N-2)/4$, the cost function (14) is linear in the matrix elements of G. Hence, the problem amounts to minimizing,

$$\tilde{C}(G)=\sum_{j=1}^{M}x_j(G_{jj}+G_{M+j,M+j}) \tag{19}$$

under the constraints (16), which is an instance of a semidefinite program [47]. We shall now prove the optimality of (1) by deriving a lower bound on $\tilde{C}(G)$ which is saturated by (1).

The specific feature of the transformation (1) is that only a single creation operator $c^\dagger$ is admixed to the annihilation operators. This operator is responsible for the added noise in cloning. Since all modes are initially in coherent states, the normally ordered moments of the operators a_j for the clones can be easily calculated,

$$\langle\Delta a_j\Delta a_k\rangle=\langle\Delta a_j^\dagger\Delta a_k^\dagger\rangle=0,$$

$$\langle\Delta a_j^\dagger\Delta a_k\rangle=\sqrt{n_jn_k}.$$

The covariance matrix of the M clones is then fully determined by the added noises n_j,

$$\gamma_{\text{out}}=\begin{pmatrix}I+2F & 0\\ 0 & I+2F\end{pmatrix}, \tag{20}$$

where F is a symmetric $M\times M$ matrix with elements $F_{jk}=\sqrt{n_jn_k}$. Since the matrix S is fixed and the input state of the cloner is a coherent state with covariance matrix $\gamma_{\text{in}}=I$, the matrix G_{opt} corresponding to transformation (1) can be determined from Eq. (15). This yields

$$G_{\text{opt}} = \begin{pmatrix} I + 2F - \frac{1}{N}H & 0 \\ 0 & I + 2F - \frac{1}{N}H \end{pmatrix}, \tag{21}$$

where H is a matrix whose elements are all equal to 1, $H_{jk} = 1$.

Since the transformation (1) can be associated with a CP map, the matrix G_{opt} must satisfy the inequality (16). For the particular S matrix (18), this gives

$$A_{\text{opt}} \equiv \begin{pmatrix} I + 2F - \frac{1}{N}H & i\left(I - \frac{1}{N}H\right) \\ -i\left(I - \frac{1}{N}H\right) & I + 2F - \frac{1}{N}H \end{pmatrix} \geqslant 0. \tag{22}$$

We can transform the matrix A_{opt} to a block-diagonal form with the help of the unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & iI \\ iI & I \end{pmatrix}, \tag{23}$$

which gives $UA_{\text{opt}}U^{\dagger} = \text{diag}(2F + 2I - 2H/N, 2F)$, so that there remains to prove that $F + I - H/N \geqslant 0$ and $F \geqslant 0$. The matrices F and H both have rank one, and can be written in Dirac notation as $F = |f\rangle\langle f|$, where $|f\rangle = \Sigma_{j=1}^{M} \sqrt{n_j}|j\rangle$, and $H = |h\rangle\langle h|$, where $|h\rangle = \Sigma_{j=1}^{M}|j\rangle$. The condition $F \geqslant 0$ is satisfied by definition, while solving $F + I - H/N \geqslant 0$ yields the nontrivial constraint on n_j's. Actually, it is sufficient to check the latter positivity condition in the two-dimensional subspace spanned by the (unnormalized) vectors $|f\rangle$ and $|h\rangle$, and one can prove that it is indeed satisfied provided that Eq. (7) holds.

We now derive a tight lower bound on $\tilde{C}(G)$, which is saturated by (1). Suppose that we find a positive semidefinite matrix $Z \geqslant 0$ such that it satisfies the conditions

$$\text{Tr}[ZG] = \tilde{C}(G) \tag{24}$$

and

$$ZA_{\text{opt}} \equiv Z(G_{\text{opt}} + iK) = 0, \tag{25}$$

where $iK = iJ_M - iSJ_1S^T$. Then, the Gaussian CP map with matrix G_{opt} is the optimal one that minimizes $\tilde{C}(G)$. Since for every admissible G we have $G + iK \geqslant 0$, it follows from $Z \geqslant 0$ that $\text{Tr}[Z(G + iK)] \geqslant 0$, which implies that $\tilde{C}(G) \geqslant -i\,\text{Tr}[ZK]$, $\forall\, G$. Equation (25) implies that this lower bound is saturated by G_{opt}, which is therefore optimal.

The matrix Z can be determined from Eqs. (24) and (25). We find that

$$Z = \begin{pmatrix} X & iY \\ -iY & X \end{pmatrix}, \tag{26}$$

where $X = \text{diag}(x_1, \ldots, x_M)$ is fixed by Eq. (24), while Y is a real symmetric matrix that satisfies

$$Y(I - N^{-1}H) + X(I + 2F - N^{-1}H) = 0,$$

$$XF = YF, \tag{27}$$

as a consequence of Eq. (25). Note that $X > 0$ by definition because $x_j > 0$. Since the matrix $I - N^{-1}H$ is invertible, we can express the matrix Y in terms of X using the first condition of (27),

$$Y = -X[I + 2F(I - (M - N)^{-1}H)]. \tag{28}$$

The second condition of (27) is then satisfied for any X provided that (7) holds.

In order to further simplify the matrix Y, we need to establish the relationship between x_j and n_j. Without loss of generality, we can restrict ourselves to the cloning machines that satisfy (7), and minimize the cost $C(n_j)$ under the constraint (7). Using the standard method of Lagrange multipliers, we obtain the extremal equations for the optimal n_j's for a given set of x_j's,

$$x_j\sqrt{n_j} - \lambda(M - N)\sqrt{n_j} + \lambda\sum_{k=1}^{M}\sqrt{n_k} = 0, \tag{29}$$

with λ being the Lagrange multiplier. With the help of these formulas, we can show that

$$XF[I - (M - N)^{-1}H] = \frac{1}{\lambda(M - N)}XFX, \tag{30}$$

so that

$$Y = -X - \frac{2}{\lambda(M - N)}XFX. \tag{31}$$

The matrix XFX is symmetric, hence $Y = Y^T$ as required.

The last step of the proof is to show that the matrix Z is positive semidefinite. We first apply a transformation that preserves the positive semidefiniteness, $\tilde{Z} = VZV^{\dagger}$, where $V = \text{diag}(X^{-1/2}, X^{-1/2})$,

$$\tilde{Z} = \begin{pmatrix} I & -iI - i2\eta X^{1/2}FX^{1/2} \\ iI + i2\eta X^{1/2}FX^{1/2} & I \end{pmatrix},$$

where $\eta = 1/[\lambda(M - N)]$. We multiply Eq. (30) with X^{-1} and take the trace, so we find that

$$\eta\,\text{Tr}(X^{1/2}FX^{1/2}) = -1, \tag{32}$$

where we made use of Eq. (7). Since F is proportional to rank-one projector the normalization (32) implies that $\eta X^{1/2}FX^{1/2} = -|\phi\rangle\langle\phi| \equiv -\Phi$, where $|\phi\rangle$ is a normalized real vector, $\langle\phi|\phi\rangle = 1$. We can convert $\tilde{Z}$ to block diagonal form with the unitary (23), $U\tilde{Z}U^{\dagger} = \text{diag}(2\Phi, 2I - 2\Phi)$, which is obviously positive semidefinite. This concludes the proof of optimality of the cloning machine (1).

VI. CONCLUSIONS

In summary, we have proposed a multipartite asymmetric Gaussian cloning machine for coherent states. The machine produces M approximate copies of a coherent state from N replicas of this state in such a way that each copy can have a different fidelity. A simple analytical formula characterizing the set of optimal Gaussian asymmetric cloning machines has been derived, and it was shown that the asymmetric cloning can be realized by amplifying of a part of the input signal followed by mixing the amplified signal and the bypass signal together with auxiliary vacuum modes on an array of beam splitters with carefully chosen transmittances. An alternate implementation is also described, where the amplifier is replaced by a passive optical circuit supplemented with feedforward. We hope that our study of multipartite asymmetric cloning will trigger further investigations of optimal quantum information distribution in continuous-variable quantum communication networks.

ACKNOWLEDGMENTS

The authors acknowledge financial support from the EU under projects COVAQIAL and SECOQC. One of the authors (J.F.) acknowledges support from the Ministry of Education of the Czech Republic (Contract Nos. LC06007 and MSM6198959213) and from GACR (202/05/0498). One of the authors (N.J.C.) acknowledges support from the IUAP programme of the Belgian government under Grant No. V-18.

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