Entanglement-enhanced transmission of classical information in Pauli channels with memory: Exact solution

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The amount of classical information that is reliably transmitted over two uses of general Pauli channels with memory, modeled as a correlated noise between a single pair of uses, is investigated. The maximum of the mutual information between the input and the output is proven to be achieved by a class of product states that is explicitly given in terms of the relevant channel parameters below some memory threshold, and by maximally entangled states above this threshold. In particular, this proves a conjecture on the depolarizing channel by Macchiavello and Palma [Phys. Rev. A **65**, 050301(R) (2002)]. Furthermore, it also shows that no other scenario can occur for Pauli channels as for example the existence of an intermediate optimal degree of entanglement reported for some Gaussian channels with memory.

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I. INTRODUCTION

The transmission of information over long distances in devices like optical fibers, or the storage of information in some type of memory, are tasks of quantum information processing that can be described by quantum channels. A major problem in quantum information theory is the evaluation of the classical capacity of quantum communication channels, which represents the amount of classical information that can be reliably transmitted by quantum states in the presence of a noisy environment. Early works in this direction were mainly devoted to memoryless channels for which consecutive signal transmissions through the channel are not correlated [1–4]. Recently, much attention was given to quantum channels with memory [5–14] in the hope that, by entangling multiple uses of the channel, a larger amount of classical information per use could be reliably transmitted. For bosonic continuous-variable memory channels, entangled states are shown to enhance the channel capacity [11–13] except in the absence of input energy constraints. Moreover, when the memory is modeled as a correlated noise, for each value of the noise correlation parameter, there exists an optimal degree of entanglement that maximizes the channel capacity [12]. For qubit channels with memory, it was shown that maximally entangled states enhance the two-use channel capacity with respect to product states if the correlation is stronger than some critical value. This was conjectured for the depolarizing channel with memory [5] and proven for a particular Pauli channel [6].

An open question is whether for some Pauli channels with memory the information transmission can be optimized by progressively entangling two uses of the channel, as occurs for some Gaussian channels where no critical threshold of correlations is present. We prove here that the states which optimize the transmission of classical information over two uses of any Pauli channel with memory modeled as a correlated noise, are a particular class of product states below some memory threshold, and maximally entangled states above that threshold. The optimal product states are explicitly given through the introduction of parameters that characterize the system in a global and more relevant way than the individual probabilities of each random process. The technique presented here is not specific to the problem of information transmission. In particular, it could be used for other optimization issues in quantum multipartite systems, which have attracted considerable attention in various areas of physics in recent years.

II. RELIABLY TRANSMITTED INFORMATION IN CHANNELS WITH CORRELATED NOISE

The action of n uses of a transmission channel on an initial state ρ is described by a completely positive map $\mathcal{E}$ which can be represented as an operator-sum

$$\rho \to \mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger, \quad \sum_k A_k^\dagger A_k = \mathrm{id}. \tag{1}$$

For memoryless channels, the amount of classical information that is reliably transmitted by quantum states through the channel is given by the Holevo-Schumacher-Westmoreland bound [1]

$$\chi(\mathcal{E}) = \max_{\{p_i,\rho_i\}} \left[S\left(\sum_i p_i \mathcal{E}(\rho_i) \right) - \sum_i p_i S(\mathcal{E}(\rho_i)) \right], \tag{2}$$

where $S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$ is the von Neumann entropy and the maximum is taken over all ensembles of input states ρ_i with *a priori* probabilities p_i. The n-shot classical capacity of the channel is this amount of reliably transmitted information per use,

$$\mathcal{C}_n(\mathcal{E}) = \frac{1}{n}\chi(\mathcal{E}), \tag{3}$$

whereas the classical capacity is defined as $\mathcal{C} = \sup_n \mathcal{C}_n$.

In the presence of memory, the χ quantity (2) is still an information characteristic of the channel. Here we focus on the case of two uses of a single qubit channel with memory as considered in Refs. [5,6],

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$$\mathcal{E}(\rho) = \sum_{i,j=0}^{3} p_{ij}\sigma_i \otimes \sigma_j \rho \sigma_i \otimes \sigma_j, \tag{4}$$

where σ_0 denotes the identity and $\{\sigma_1,\sigma_2,\sigma_3\}$ are the Pauli matrices. The memory is modeled as a correlated noise such that, with probability $\mu \in [0,1]$, the same random Pauli transformation is applied to both qubits while with probability $1-\mu$ the two rotations are uncorrelated,

$$p_{ij} = (1-\mu)q_i q_j + \mu q_i \delta_{ij}, \quad \sum_{j=0}^{3} q_j = 1. \tag{5}$$

As the maximally mixed state gives the largest possible entropy, $S(\frac{1}{4}\sigma_0 \otimes \sigma_0) = \log_2(4)$, the two-shot χ-quantity (2) is upper bounded by [6]

$$\chi(\mathcal{E}) \leqslant 2 - S(\mathcal{E}(\rho_\star)), \tag{6}$$

where $\rho_\star$ denotes an input state that minimizes the output entropy when transmitted through the channel $\mathcal{E}$. Equation (6) generalizes the relation between the one-shot classical capacity and the minimal output entropy which was proven to hold with an equality sign for covariant channels [7]. The upper bound (6) can be achieved in any channel whose action consists of random tensor products of Pauli transformations such as (4). In a nutshell, the argument amounts to constructing from $\rho_\star$ an ensemble with input states $\sigma_i \otimes \sigma_j \rho_\star \sigma_i \otimes \sigma_j$ which, as a result of the covariance, each have the same output entropy. On the other hand, for such an ensemble taken with equal *a priori* probabilities, one can show that the output state is maximally mixed. To optimize the transmission of information in Pauli channels with memory all that is required is thus to identify an *optimal* input state $\rho_\star$. Moreover, by the concavity of the von Neuman entropy, this search can be restricted to pure input states $\rho_\star = |\Psi_\star\rangle\langle\Psi_\star| \equiv \rho_{\Psi_\star}$ [6].

To date, the optimality of some input states has been conjectured [5] for the depolarizing channel $(q_0=1-p, q_1=q_2=q_3=p/3)$ and proven [6] only in one particular instance of a Pauli channel with memory $(q_0=q_3=p, q_1=q_2=\frac{1}{2}-p)$. To study the nature of the optimal states for arbitrary Pauli channels, we consider the two-qubit pure state obtained from the general superposition

$$|\Psi\rangle = c_{00}|00\rangle + c_{11}e^{i\varphi_{11}}|11\rangle + c_{10}e^{i\varphi_{10}}|10\rangle + c_{01}e^{i\varphi_{01}}|01\rangle. \tag{7}$$

The normalization implies the relation $c_{00}^2 + c_{11}^2 + c_{10}^2 + c_{01}^2 = 1$. This constraint is taken into account here by expressing the pertaining parameters in terms of three angles θ, ϕ, and ψ as follows:

$$c_{00} = \cos\frac{\phi+\psi}{2}\cos\frac{\theta}{2},$$

$$c_{11} = \sin\frac{\phi-\psi}{2}\sin\frac{\theta}{2},$$

$$c_{10} = \cos\frac{\phi-\psi}{2}\sin\frac{\theta}{2},$$

$$c_{01} = \sin\frac{\phi+\psi}{2}\cos\frac{\theta}{2}. \tag{8}$$

The density matrix ρ_Ψ can be expressed in terms of the tensor products of Pauli matrices as

$$\rho_\Psi = \frac{1}{4}\sum_{n,k=0}^{3} w_{nk}\sigma_n \otimes \sigma_k, \tag{9}$$

with the real coefficients $w_{nk} = \mathrm{Tr}(\rho_\Psi \sigma_n \otimes \sigma_k)$.

The interest of the decomposition (9) is that the action of the channel on $\sigma_n \otimes \sigma_k$ takes on the simple form

$$\mathcal{E}(\sigma_n \otimes \sigma_k) = \varepsilon_{nk}\sigma_n \otimes \sigma_k, \tag{10}$$

where ε_{nk} is a real number $\in [-1,1]$ which will be referred to as a *channel parameter*. It reads

$$\varepsilon_{kk'} = (1-\mu)\varepsilon_k \varepsilon_{k'} + \mu \varepsilon_{k''}, \tag{11}$$

where k'' is the index of the matrix $\sigma_{k''}$ to which $\sigma_k \sigma_{k'}$ is proportional: $\sigma_k\sigma_{k'} \propto \sigma_{k''}$. The channel parameter ε_k is defined as

$$\varepsilon_k = \sum_{j=0}^{3} q_j s_{jk}, \tag{12}$$

where $s_{jk}=+1$ if either $j=k$ or $j=0$ or $k=0$, and $s_{jk}=-1$ otherwise. Notice that $\varepsilon_0 = \varepsilon_{00} = 1$, $\varepsilon_{k0} = \varepsilon_k$, and $\varepsilon_{kn} = \varepsilon_{nk}$.

The ordering of the channel parameters ε_{nk} and ε_k will play a central role. To specify the ordering of ε_1, ε_2, and ε_3 we introduce the indices l, m, and s (which stand for large, medium, and small)

$$|\varepsilon_l| \geqslant |\varepsilon_m| \geqslant |\varepsilon_s|. \tag{13}$$

Since $\varepsilon_{kk} = (1-\mu)\varepsilon_k^2 + \mu$ it follows that $\varepsilon_{ll} \geqslant \varepsilon_{mm} \geqslant \varepsilon_{ss}$ for any μ. The following properties show that the channel parameters ε_l and ε_{ll} are larger in absolute value than the nondiagonal channel parameters ε_{nk} for any μ:

$$\varepsilon_l^2 \geqslant \varepsilon_{nk}^2, \quad n \neq k,$$

$$\varepsilon_{ll}^2 \geqslant \varepsilon_{nk}^2, \quad n,k \neq 0. \tag{14}$$

In contrast, the degree of correlation μ modifies the positions of the diagonal channel parameters ε_{kk} with respect to $|\varepsilon_l|$. When μ goes from 0 to 1, each ε_{kk} increases from ε_k^2 to 1. As $\varepsilon_k^2 \leqslant \varepsilon_l^2 \leqslant 1$, there is a value of μ where ε_{kk} crosses $|\varepsilon_l|$. This fact, together with the constraints imposed on the *weights* w_{nk}^2 of a pure state, is at the origin of the threshold phenomenon mentioned above.

In order to identify the states $\rho_{\Psi_\star}$ whose output entropy $S(\mathcal{E}(\rho_{\Psi_\star}))$ is minimal, the eigenvalues of $\mathcal{E}(\rho_\Psi)$ are to be considered. In terms of the decomposition (9) and of the mapping (10), the action of the channel (4) on a pure state reads explicitly

$$\mathcal{E}(\rho_\Psi) = \frac{1}{4}\sum_{n,k=0}^{3} \varepsilon_{nk} w_{nk} \sigma_n \otimes \sigma_k. \quad (15)$$

The roots of the pertaining characteristic equation $\lambda^4 - \lambda^3 + a_2\lambda^2 + a_1\lambda^1 + a_0 = 0$ are given by [15]

$$\lambda_{\eta,v} = \frac{1}{4}(1 + \eta R + v Q_\eta), \quad \eta, v = \pm 1, \quad (16)$$

where

$$R = \sqrt{1 - 4a_2 + \omega(\{a_i\})}$$

and

$$Q_\eta = \sqrt{2 - 4a_2 + \eta(4a_2 - 8a_1 - 1)/R - \omega(\{a_i\})}.$$

The function ω is the real root of a cubic equation which features solely the coefficients a_i. Owing to the symmetric structure of the roots in η, $v=\pm 1$, it can be shown that extrema in the eigenvalues can be achieved if and only if the coefficients a_i are each extremal. To minimize the output entropy, the quantities R and Q_η have to be maximized. As a_2 is positive and enters both R and Q_η with a negative sign, this coefficient has to be minimized. It reads $a_2 = \frac{1}{8}(3 - A - B - C)$ with

$$A = \sum_{k=1}^{3} \varepsilon_{kk}^2 w_{kk}^2,$$

$$B = \sum_{k=1}^{3} \varepsilon_k^2 (w_{0k}^2 + w_{k0}^2),$$

$$C = \sum_{n \neq k=1}^{3} \varepsilon_{nk}^2 (w_{nk}^2 + w_{kn}^2). \quad (17)$$

The optimal states are those that maximize $A+B+C$. Their identification rests on two elements: the ordering of the channel parameters ε_{kk} and $|\varepsilon_l|$ investigated above, and the allowed values of the weights w_{nk}^2 associated with the decomposition of a general pure state. First, note that $w_{00} = \mathrm{Tr}(\rho_\Psi) = 1$ and, by the Schwarz inequality, $|w_{nk}| \leqslant 1$. For a pure state the purity $\mathrm{Tr}(\rho_\Psi^2)$ is unity, which translates into the fact that the weights w_{nk}^2 involved in $A+B+C$ sum to 3,

$$\sum_{k=1}^{3} w_{kk}^2 + \sum_{n \neq k=0}^{3} w_{nk}^2 = 3. \quad (18)$$

In addition, the following inequality holds for any permutation of the indices 1, 2, 3:

$$w_{jj}^2 + w_{kk}^2 - w_{nn}^2 \leqslant 1. \quad (19)$$

Summing the three versions of this inequality implies that the sum of the weights featured in $A+B+C$ is at most 3,

$$\alpha \equiv w_{11}^2 + w_{22}^2 + w_{33}^2 \leqslant 3. \quad (20)$$

Similarly, it can be shown from the above parametrization that the sum of the weights entering B is at most 2,

$$\beta \equiv \sum_{n=1}^{3} (w_{n0}^2 + w_{0n}^2) \leqslant 2. \quad (21)$$

The optimization issue amounts to spreading three units over the 15 weights w_{nk}^2 featured in $A+B+C$ in such a way that the largest channel parameters contribute preferentially and the inequalities (19)–(21) are satisfied. The tight bounds $\alpha \leqslant 3$ and $\beta \leqslant 2$ imply that the three weights involved in A can be saturated for some states, whereas at most two of the six weights in B can be equal to unity. Recall that ε_l is the largest channel parameter of B and is also larger than any of the channel parameters of C for any degree of correlation μ. It follows that $B \leqslant 2\varepsilon_l^2$. The equality sign is attained if $w_{0l}^2 = w_{l0}^2 = 1$, which necessarily implies that $w_{ll}^2 = 1$ and $A+B+C = \varepsilon_{ll}^2 + 2\varepsilon_l^2$. This is the situation that prevails for $\mu = 0$ and certainly up to the value μ_0 such that $\varepsilon_{mm}^2 = \varepsilon_l^2$. On the other hand, when μ is larger than the value μ_1 for which $\varepsilon_{ss}^2 = \varepsilon_l^2$, then the three diagonal channel parameters ε_{kk} are larger than $|\varepsilon_l|$, so that the optimum is $w_{ll}^2 = w_{mm}^2 = w_{ss}^2 = 1$ and $A+B+C = \varepsilon_{ll}^2 + \varepsilon_{mm}^2 + \varepsilon_{ss}^2$. *A priori*, between μ_0 and μ_1, the optimal states could be different and feature, for instance, fractional weights w_{nk}^2. A detailed analysis reveals that the above optima extend, respectively, above μ_0 and below μ_1 until the value $\mu_\star$ such that $\varepsilon_{ss}^2 + \varepsilon_{mm}^2 = 2\varepsilon_l^2$.

Before illustrating this result, we proceed with its proof. For $0 \leqslant \mu \leqslant \mu_0$, one has $\varepsilon_l^2 \geqslant \varepsilon_{mm}^2$, the value μ_0 being determined by the equality sign. Applying the inequalities (14) and using (18) yields $A + C \leqslant \varepsilon_{ll}^2(3 - \beta)$ and $B \leqslant \varepsilon_l^2 \beta$ so that

$$A + B + C \leqslant 3\varepsilon_{ll}^2 + \beta(\varepsilon_l^2 - \varepsilon_{mm}^2) \leqslant \varepsilon_{ll}^2 + 2\varepsilon_l^2. \quad (22)$$

The second term on the right of the first inequality sign being positive in this interval of μ, it is majorized by taking the upper bound $\beta = 2$. The bound (22) is achieved if and only if $w_{ll}^2 = w_{0l}^2 = w_{l0}^2 = 1$. The optimal $|\Psi_\star\rangle$ is thus the tensor product of eigenstates of the Pauli matrix σ_l corresponding to the channel parameter ε_l of largest absolute value. The eigenvalues of $\mathcal{E}(\rho_{\Psi_\star})$, required to calculate $\chi(\mathcal{E})$ from (6), are of the form (16) with $R = \varepsilon_{ll}$ and $Q_\eta = (1+\eta)\varepsilon_l$.

Let $\mu_\star$ be the value of μ for which $\varepsilon_{ss}^2 + \varepsilon_{mm}^2 = 2\varepsilon_l^2$. For $\mu_0 \leqslant \mu \leqslant \mu_\star$, the ordering of the diagonal channel parameters with respect to $|\varepsilon_l|$ is therefore $\frac{1}{2}(\varepsilon_{ss}^2 + \varepsilon_{mm}^2) \leqslant \varepsilon_l^2 \leqslant \varepsilon_{mm}^2$. From (14) we obtain $B + C \leqslant \varepsilon_l^2(3 - \alpha)$, which entails that

$$A + B + C \leqslant \varepsilon_{ll}^2 + 2\varepsilon_l^2 + w_{ss}^2(\varepsilon_{ss}^2 + \varepsilon_{mm}^2 - 2\varepsilon_l^2) + (\varepsilon_l^2 - \varepsilon_{mm}^2) \times (1 - w_{ll}^2 - w_{mm}^2 + w_{ss}^2) \leqslant \varepsilon_{ll}^2 + 2\varepsilon_l^2. \quad (23)$$

In this interval of μ, the factors $\varepsilon_{ss}^2 + \varepsilon_{mm}^2 - 2\varepsilon_l^2$ and $\varepsilon_l^2 - \varepsilon_{mm}^2$ are negative while $1 - w_{ll}^2 - w_{mm}^2 + w_{ss}^2$ is positive or zero by (19). The bound (23) is thus realized if and only if $w_{0l}^2 = w_{l0}^2 = w_{ll}^2 = 1$. Notice that the optimal states coincides with those of (22) so that μ_0 turns out to be irrelevant.

For $\mu_\star \leqslant \mu \leqslant 1$, the ordering of the largest channel parameters is changed to $\varepsilon_l^2 \leqslant \frac{1}{2}(\varepsilon_{mm}^2 + \varepsilon_{ss}^2)$. This yields

$$A + B + C \leqslant 2\varepsilon_l^2 + \varepsilon_{ll}^2 + w_{ss}^2(\varepsilon_{mm}^2 + \varepsilon_{ss}^2 - 2\varepsilon_l^2) \leqslant \varepsilon_{ll}^2 + \varepsilon_{mm}^2 + \varepsilon_{ss}^2. \quad (24)$$

The first inequality comes from the first inequality of (23), where the last term, which is still negative or zero in the interval of μ considered here, has been upper bounded by taking $w_{ll}^2+w_{mm}^2-w_{ss}^2=1$. On the other hand, the term $w_{ss}^2(\varepsilon_{mm}^2+\varepsilon_{ss}^2-2\varepsilon_l^2)$ is now positive and upper bounded by setting $w_{ss}=1$, which gives the bound (24). It is achieved if and only if $w_{ss}^2=w_{mm}^2=w_{ll}^2=1$. The optimal input states $|\Psi_\star\rangle$ are thus the maximally entangled states $(|00\rangle \pm |11\rangle)/\sqrt{2}$ and $(|01\rangle \pm |10\rangle)/\sqrt{2}$. The eigenvalues of $\mathcal{E}(\rho_{\Psi_\star})$ are given by (16), where $R=\varepsilon_{33}$ and $Q_\eta=\varepsilon_{11}+\eta\varepsilon_{22}$. The threshold $\mu_\star$ separating the domain where the optimal states for the two-shot χ quantity are the tensor product of eigenstates of σ_l from the domain where the optimal states are the Bell states reads, with the notation $\delta_k \equiv 1-\varepsilon_k^2$,

$$\mu_\star = \frac{-\delta_m\varepsilon_m^2 - \delta_s\varepsilon_s^2 + \sqrt{2\varepsilon_l^2(\delta_m^2+\delta_s^2) - (\delta_m-\delta_s)^2}}{\delta_m^2+\delta_s^2}. \quad (25)$$

III. ILLUSTRATION

Consider two uses of a Pauli channel with correlated noise as given by (4) and (5). For the probabilities $q_0=0.2$, $q_1=0.1$, $q_2=0.3$, and $q_3=0.4$, the channel parameters defined in (12) are $\varepsilon_1=-0.4$, $\varepsilon_2=0$, and $\varepsilon_3=0.2$. The indices (large, medium, and small) specifying their ordering (13) are $l=1$, $m=3$, and $s=2$. From (25) the memory threshold is $\mu_\star \simeq 0.39$. Up to $\mu_\star$ the optimal states are the product states associated with the eigenstates of σ_1 since $l=1$. As a second example, take the channel for which the values of q_0 and q_2 are permuted. This yields $\varepsilon_1=-0.2$, $\varepsilon_2=0$, and $\varepsilon_3=0.4$ so that now $l=3$ and the optimal states below $\mu_\star$ are associated with the eigenstates of σ_3. Notice that σ_1 is the least likely transformation in the first example, whereas σ_3 is the most likely transformation in the second one. This shows that the individual probabilities q_i are not the relevant parameters, in contrast to the channel parameters ε_i introduced here. Above $\mu_\star$, the optimal input states for the information transmission are the Bell states in both cases.

IV. CONCLUSIONS

The amount of classical information reliably transmitted over two uses of arbitrary Pauli channels with memory modeled as a correlated noise between a single pair of uses is evaluated via the two-shot χ quantity, which is proven here to be $2-\sum_{\eta,\upsilon=\pm1}\lambda_{\eta,\upsilon}\log_2\lambda_{\eta,\upsilon}$ with $\lambda_{\eta,\upsilon}$ explicitly given by different functions of the degree of correlations μ for $0 \leqslant \mu \leqslant \mu_\star$ and for $\mu_\star \leqslant \mu \leqslant 1$. Below the memory threshold, the two-shot χ quantity is achieved by using the tensor product of the single-qubit density matrices pertaining to the eigenstates of the matrix σ_l whose associated channel parameter ε_l, defined above, has the largest absolute value. Above the threshold $\mu_\star$, the two-shot χ quantity is reached by maximally entangled states. Entanglement is therefore a useful resource to enhance the transmission of classical information in this class of quantum channels with memory.

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[1] A. S. Holevo, IEEE Trans. Inf. Theory **44**, 269 (1998); B. Schumacher and M. D. Westmoreland, Phys. Rev. A **56**, 131 (1997).
[2] V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, J. H. Shapiro, and H. P. Yuen, Phys. Rev. Lett. **92**, 027902 (2004).
[3] A. Sen(De), U. Sen, B. Gromek, D. Brußand, and M. Lewenstein, Phys. Rev. Lett. **95**, 260503 (2005).
[4] G. G. Amosov, Probl. Inf. Transm. **42**, 67 (2006).
[5] Ch. Macchiavello and G. M. Palma, Phys. Rev. A **65**, 050301(R) (2002).
[6] Ch. Macchiavello, G. M. Palma, and S. Virmani, Phys. Rev. A **69**, 010303(R) (2004).
[7] A. S. Holevo, e-print arXiv:quant-ph/0212025.
[8] G. Bowen and S. Mancini, Phys. Rev. A **69**, 012306 (2004).
[9] G. Bowen, I. Devetak, and S. Mancini, Phys. Rev. A **71**, 034310 (2005).
[10] D. Kretschmann and R. F. Werner, Phys. Rev. A **72**, 062323 (2005).
[11] V. Giovannetti and S. Mancini, Phys. Rev. A **71**, 062304 (2005).
[12] N. J. Cerf, J. Clavareau, Ch. Macchiavello, and J. Roland, Phys. Rev. A **72**, 042330 (2005).
[13] G. Ruggeri, G. Soliani, V. Giovannetti, and S. Mancini, Europhys. Lett. **70**, 719 (2005).
[14] E. Karpov, D. Daems, and N. J. Cerf, Phys. Rev. A **74**, 032320 (2006).
[15] *Handbook of Mathematical Functions*, edited by M. Abramowitz and A. Stegun (Dover, New York, 1972).