Tripartite entanglement in parametric down-conversion with spatially structured pump

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Most investigations of multipartite entanglement have been concerned with temporal modes of the electromagnetic field and have neglected its spatial structure. We present a simple model which allows us to generate tripartite entanglement between spatial modes by parametric down-conversion with two symmetrically tilted plane waves serving as a pump. The characteristics of this entanglement are investigated. We also discuss the generalization of our scheme to $2N+1$ partite entanglement using $2N$ symmetrically tilted plane pump waves. Another interesting feature is the possibility of *entanglement localization* in just two spatial modes. © 2010 Optical Society of America

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1. INTRODUCTION

Multipartite entanglement, as the name suggests, is the entanglement between more than two parties. The existence of this type of entanglement for both discrete and continuous quantum variables is one of the most striking predictions of quantum mechanics and is very unusual from the point of view of classical physics. It is, therefore, not surprising that nowadays multipartite entanglement is a very active research area of quantum optics and quantum information. The interest in multipartite entanglement is motivated not only by its fundamental nature but also by its potential for applications in quantum communication technologies.

Numerous theoretical publications have addressed the characterization of multipartite entanglement. Coffman, Kundu, and Wootters [1] have established for a three-qubit system and conjectured for N-qubit systems the so-called monogamy of quantum entanglement, constraining the maximum entanglement between partitions of a multiparty system. Since then, Adesso and Illuminati [2–5] have introduced the continuous-variable tangle as a measure of multipartite entanglement for continuous variable (CV) multimode Gaussian states and have demonstrated that it satisfies the Coffman–Kundu–Wootters monogamy inequality. The conjecture of Ref. [1] has been proven by Osborne and Verstraete [6]. The corresponding proof for Gaussian states is in Hiroshima, Adesso, and Illuminati [7].

Among the potential applications of CV multipartite entanglement, let us mention quantum teleportation networks, quantum telecloning, controlled quantum dense coding, and quantum secret sharing [8].

Several generation schemes for CV multipartite entanglement have been proposed in the literature theoretically and realized experimentally in recent years. Among the first ones was the "passive" optical scheme using squeezed states mixed with beam splitters [9]. Then came the "active" schemes in which multipartite entanglement is created as a result of parametric interaction of several optical waves such as cascaded/concurrent [10–12], interlinked [13,14], or consecutive parametric interactions [15]. All of these schemes for generation of CV multipartite entanglement are considering the temporal modes of the electromagnetic field, and they neglect its spatial structure. The question arises on whether the spatial modes can also serve for the creation of CV multipartite entanglement [16–19]. In Ref. [3], a physical implementation is given in terms of $2N-1$ beam splitters and $2N$ single-mode squeezed input states, itself based on a previous work [20].

In this paper we propose a simple scheme that corresponds to the "active" creation of tripartite entanglement between spatial modes of the electromagnetic field in the process of parametric interaction. We create spatial tripartite entanglement by pumping a nonlinear parametric medium by a coherent combination of several tilted-plane monochromatic waves, which we call a spatially structured pump. Since the pump photon can be extracted from any of these waves, and the pair of down-converted photons emitted in different directions according to the phase-matching condition, our scheme allows for the creation of tripartite entanglement between spatial modes of the down-converted field. An interesting feature of our scheme is the possibility of localizing the created spatial tripartite entanglement in just two well-defined spatial modes formed as a linear combination of all the modes participating in the down-conversion process.

In Section 2, we shall consider explicitly the case of tripartite entanglement, which is produced by parametric down-conversion with two symmetrically tilted plane waves serving as a pump. The genuine tripartite entanglement is analyzed in Subsection 2.A, and its local-

ization on two modes is studied in Subsection 2.B. We then discuss in Section 3 a generalization of our scheme to $2N+1$ partite entanglement using $2N$ symmetrically tilted pump waves and draw some conclusions.

2. SPATIAL TRIPARTITE ENTANGLEMENT

A. Parametric Down-conversion with Two Symmetrically Tilted Plane Waves

The scheme of parametric down-conversion used here is shown in Fig. 1.

We consider two plane pump waves with wave vectors $\mathbf{k}_p(\pm q_0)$ where $\pm q_0$ are the pertaining transverse components (which are vertical on Fig. 1) at the input to the crystal. The pump field then reads

$$E_p(x) = \alpha(e^{iq_0 x} + e^{-iq_0 x}), \tag{1}$$

with α the amplitude of the pump, chosen as a real number in our case. We shall denote the corresponding spatial Fourier components of the corresponding field operators as follows:

$$\hat{a}(z,q) \equiv \hat{a}_0(z),$$

$$\hat{a}(z, \pm q_0 + q) \equiv \hat{a}_\pm(z). \tag{2}$$

These modes are depicted in Fig. 1: the long (blue) arrows pertains to $\hat{a}_\pm(0)$, the short one to $\hat{a}_0^\dagger(0)$.

Due to the parametric interaction in the nonlinear medium, these operators are coupled and obey the following equations in the rotating wave approximation [21]:

$$\frac{d}{dz}\hat{a}_0 = \alpha\lambda(\hat{a}_+^\dagger + \hat{a}_-^\dagger)e^{i\Delta z},$$

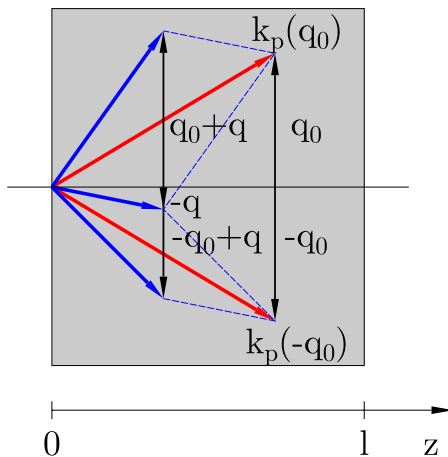$$\frac{d}{dz}\hat{a}_+ = \alpha\lambda\hat{a}_0^\dagger e^{i\Delta z},$$

Fig. 1.   (Color online) Scheme for generating tripartite entanglement between spatial modes. The tilted pumps have wave vectors $\mathbf{k}_p(\pm q_0)$. The transverse (vertical) components are $\pm q_0$.

$$\frac{d}{dz}\hat{a}_- = \alpha\lambda\hat{a}_0^\dagger e^{i\Delta z}, \tag{3}$$

where λ is the coupling constant and Δ is the phase mismatch.

The solution to this set of equations for the fields at the output of the crystal, $z=l$, can be written analytically as follows:

$$\hat{a}_+(l) = \frac{1}{2}[U(l) + 1]\hat{a}_+(0) + \frac{1}{2}[U(l) - 1]\hat{a}_-(0) + \frac{1}{\sqrt{2}}V(l)\hat{a}_0^\dagger(0),$$

$$\hat{a}_-(l) = \frac{1}{2}[U(l) - 1]\hat{a}_+(0) + \frac{1}{2}[U(l) + 1]\hat{a}_-(0) + \frac{1}{\sqrt{2}}V(l)\hat{a}_0^\dagger(0),$$

$$\hat{a}_0(l) = U(l)\hat{a}_0(0) + \frac{1}{\sqrt{2}}V(l)[\hat{a}_+^\dagger(0) + \hat{a}_-^\dagger(0)], \tag{4}$$

with the functions

$$U(z) = e^{i\Delta z/2}\left(\cosh(\Gamma z) - i\frac{\Delta}{2\Gamma}\sinh(\Gamma z)\right),$$

$$V(z) = e^{i\Delta z/2}\frac{\sqrt{2}\alpha\lambda}{\Gamma}\sinh(\Gamma z), \tag{5}$$

where $\Gamma = \sqrt{2\alpha^2\lambda^2 - \Delta^2/4}$. We shall be concerned, for simplicity, with the case of zero phase mismatch, $\Delta=0$, and will be considering only the zeroth spatial Fourier components of the field, $q=0$. Physically, this corresponds to photodetection of the light field by a single large photodetector without spatial resolution. In this case, we have $U(l)=\cosh r$ and $V(l)=\sinh r$, with $r=\sqrt{2}\alpha\lambda l$ being the squeezing parameter.

When the input states of the three interacting modes are vacuum states, the output states remain Gaussian. Hence, they are completely described by the output covariance matrix σ associated with the following quadrature components:

$$\hat{x}_0(\ell) \equiv 2\,\mathrm{Re}\,\hat{a}_0(\ell),$$

$$\hat{p}_0(\ell) \equiv 2\,\mathrm{Im}\,\hat{a}_0(\ell),$$

$$\hat{x}_\pm(\ell) \equiv 2\,\mathrm{Re}\,\hat{a}_\pm(\ell),$$

$$\hat{p}_\pm(\ell) \equiv 2\,\mathrm{Im}\,\hat{a}_\pm(\ell). \tag{6}$$

Its matrix elements read

$$\sigma_{ij} = \mathrm{Tr}[\rho(\Delta\hat{\xi}_i\Delta\hat{\xi}_j + \Delta\hat{\xi}_j\Delta\hat{\xi}_i)/2], \tag{7}$$

where ρ is the output density matrix and $\Delta\hat{\xi}_i = \hat{\xi}_i - \langle\hat{\xi}_i\rangle_\rho$ with $\hat{\xi}_i$ the ith component of the vector $\hat{\xi} = (\hat{x}_0(\ell), \hat{p}_0(\ell), \hat{x}_+(\ell), \hat{p}_+(\ell), \hat{x}_-(\ell), \hat{p}_-(\ell))$. Taking the solution (4) into account, the covariance matrix at the output of the crystal reads explicitly

$$\sigma = \begin{pmatrix} \cosh(2r) & 0 & \dfrac{\sinh(2r)}{\sqrt{2}} & 0 & \dfrac{\sinh(2r)}{\sqrt{2}} & 0 \\ 0 & \cosh(2r) & 0 & -\dfrac{\sinh(2r)}{\sqrt{2}} & 0 & \dfrac{-\sinh(2r)}{\sqrt{2}} \\ \dfrac{\sinh(2r)}{\sqrt{2}} & 0 & \cosh^2 r & 0 & \sinh^2 r & 0 \\ 0 & -\dfrac{\sinh(2r)}{\sqrt{2}} & 0 & \cosh^2 r & 0 & \sinh^2 r \\ \dfrac{\sinh(2r)}{\sqrt{2}} & 0 & \sinh^2 r & 0 & \cosh^2 r & 0 \\ 0 & -\dfrac{\sinh(2r)}{\sqrt{2}} & 0 & \sinh^2 r & 0 & \cosh^2 r \end{pmatrix}. \tag{8}$$

Notice that this matrix is *bisymmetric* as it is invariant under the permutation of the quadratures pertaining to the modes $+$ and $-$: $\hat{x}_+(\ell), \hat{p}_+(\ell) \leftrightarrow \hat{x}_-(\ell), \hat{p}_-(\ell)$. Hence, it has the tripartite entanglement structure of covariance matrices associated to bisymmetric $(1+2)$ mode Gaussian states considered in Ref. [3]. Actually, we can show that the output state obtained here exhibits genuine tripartite entanglement in the sense of Ref. [20]. For that purpose, we have to verify that the following condition on covariance matrix elements is violated:

$$C \equiv \left\langle \left\{ \hat{x}_0(\ell) - \frac{\hat{x}_+(\ell) + \hat{x}_-(\ell)}{\sqrt{2}} \right\}^2 \right\rangle_\rho + \left\langle \left\{ \hat{p}_0(\ell) \right. \right.$$
$$\left. \left. + \frac{\hat{p}_-(\ell) + \hat{p}_-(\ell)}{\sqrt{2}} \right\}^2 \right\rangle_\rho \geqslant \frac{1}{2}. \tag{9}$$

From (8) one deduces that

$$C = 4\{\cosh(2r) - \sinh(2r)\} = 4e^{-2r}, \tag{10}$$

which is smaller than $1/2$ if the squeezing parameter $r > 3/2 \ln 2$. Recalling that $r = \sqrt{2}\alpha\lambda\ell$ therefore entails that the output state ρ produced by the above parametric down-conversion process with two symmetrically tilted plane waves exhibits genuine tripartite entanglement when the pump amplitude α, the coupling parameter λ, and the crystal length ℓ are such that

$$\alpha\lambda\ell > \frac{3\ln 2}{2\sqrt{2}} \approx 0.735. \tag{11}$$

B. Two-Mode Entanglement Localization
We shall show that the tripartite entanglement generated by the above process can be concentrated on two modes by local unitary transformations as was discovered by Serafini, Adesso, and Illuminati [3]. Let us consider the following quadratures:

$$\hat{x}'_0(\ell) \equiv \hat{x}_0(\ell), \quad \hat{p}'_0(\ell) \equiv \hat{p}_0(\ell),$$

$$\hat{x}'_1(\ell) \equiv \frac{\hat{x}_+(\ell) + \hat{x}_-(\ell)}{\sqrt{2}}, \quad \hat{p}'_1(\ell) \equiv \frac{\hat{p}_+(\ell) + \hat{p}_-(\ell)}{\sqrt{2}},$$

$$\hat{x}'_2(\ell) \equiv \frac{\hat{x}_+(\ell) - \hat{x}_-(\ell)}{\sqrt{2}}, \quad \hat{p}'_2(\ell) \equiv \frac{\hat{p}_+(\ell) - \hat{p}_-(\ell)}{\sqrt{2}}, \tag{12}$$

which are obtained from the previous ones by a unitary transformation corresponding to the action of a beam-splitter. Accordingly, the output covariance matrix σ is thus transformed by congruence,

$$\sigma' = S^{\mathrm{T}}\sigma S, \tag{13}$$

where S is the following symplectic transformation,

$$S = I_1 \oplus \begin{pmatrix} \dfrac{1}{\sqrt{2}} & 0 & \dfrac{1}{\sqrt{2}} & 0 \\ 0 & \dfrac{1}{\sqrt{2}} & 0 & \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & 0 & -\dfrac{1}{\sqrt{2}} & 0 \\ 0 & \dfrac{1}{\sqrt{2}} & 0 & -\dfrac{1}{\sqrt{2}} \end{pmatrix}, \tag{14}$$

with I_1 the 2×2 identity matrix. The new covariance matrix reads then

$$\sigma' = \begin{pmatrix} \cosh(2r) & 0 & \sinh(2r) & 0 \\ 0 & \cosh(2r) & 0 & -\sinh(2r) \\ \sinh(2r) & 0 & \cosh(2r) & 0 \\ 0 & -\sinh(2r) & 0 & \cosh(2r) \end{pmatrix} \oplus I_1. \tag{15}$$

The separability of the pertaining state ρ' can be determined by considering a partition of the system in two subsystems and investigating the positivity of the partially transposed matrix $\tilde{\rho}'$, obtained upon transposing the vari-

ables of only one of the two subsystems. The positivity of partial transposition (PPT) is a necessary condition for the separability of any bipartite quantum state [22,23]. It is also sufficient for the separability of $(1+n)$ mode Gaussian states [24,25].

The covariance matrix $\tilde{\sigma}'$ of the partially transposed state $\tilde{\rho}'$ with respect to one subsystem is obtained [24] by changing the signs of the quadratures $\hat{p}_i$ belonging to that subsystem. Here, because of the block-diagonal structure of σ', we can restrict the analysis to the first block and consider the transposition of the mode 1, i.e., change the sign of $\hat{p}'_1$,

$$\tilde{\sigma}' = \begin{pmatrix} \cosh(2r) & 0 & \sinh(2r) & 0 \\ 0 & \cosh(2r) & 0 & \sinh(2r) \\ \sinh(2r) & 0 & \cosh(2r) & 0 \\ 0 & \sinh(2r) & 0 & \cosh(2r) \end{pmatrix} \oplus I_1.$$

(16)

The symplectic eigenvalues $\{\tilde{\nu}_i\}$ of the covariance matrix $\tilde{\sigma}'$ are $\tilde{\nu}_\pm = e^{\pm 2r}$ and 1. The necessary and sufficient PPT condition for the separability of the state ρ' amounts to having $\tilde{\nu}_i \geqslant 1$ for all i. We can therefore focus on the smallest eigenvalue $\tilde{\nu}_-$. The extent to which the separability of the state is violated, is measured by $\mathcal{E}_N(\rho)$, the logarithmic negativity of ρ', defined as the logarithm of the trace norm of $\tilde{\rho}'$,

$$\mathcal{E}_{\mathcal{N}}(\rho') = \ln \|\tilde{\rho}'\|_1 = \max(0, -\ln \tilde{\nu}_-).$$

(17)

Using the calculated symplectic eigenvalues $\{\tilde{\nu}_i\}$ we obtain a very simple expression for the logarithmic negativity,

$$\mathcal{E}_{\mathcal{N}}(\rho') = 2r.$$

(18)

This implies that the central mode 0 is entangled with the mode 1, which is precisely the combination of modes $+$ and $-$ that was used in the condition (9) in order to show the existence of genuine tripartite entanglement.

3. DISCUSSION AND CONCLUSION

Our scheme can be generalized to multipartite entanglement localization using $2N$ symmetrically tilted plane waves as pump modes in the parametric down-conversion process. An illustration is provided in Fig. 2 for the cases $N=2$ and 4, which correspond to 4 and 8 pump waves.

4 pump waves
5 coupled modes

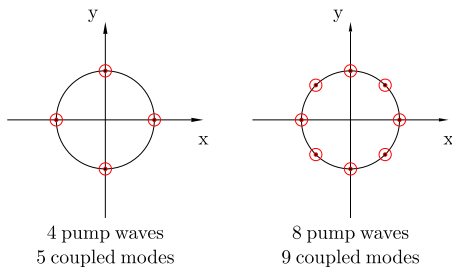8 pump waves
9 coupled modes

Fig. 2.   (Color online) Generalization of the scheme for $2N$ symmetrically tilted pump waves with $N=2$ and 4. The little circles represent the projections of the pump wave vectors in the xy plane of the crystal entering face.

The projections of the pump wave vectors in the xy plane of the crystal entering face are depicted as little circles. In this case, the system contains $2N+1$ coupled modes which are described by equations similar to (3). We can solve them in the rotating wave approximation and investigate the properties of the multipartite entanglement in this case (the details will be provided in a forthcoming publication). The covariance matrix σ at the output of the crystal is again bisymmetric as those considered in Ref. [3]. By an orthogonal transformation of the field operators, which generalizes the one achieved in the tripartite case, the covariance matrix σ can be brought to a form similar to Eq. (15), namely, the tensor product of a 4×4 covariance matrix and a $2(2N-1) \times 2(2N-1)$ identity matrix. The 4×4 covariance matrix has exactly the same form as the one in Eq. (8) with r replaced by $\sqrt{N}r$. Hence, it corresponds to that of two entangled modes with the squeezing parameter enhanced by $\sqrt{N}$ and $2N-1$ modes in the vacuum state. For the logarithmic negativity, we indeed obtain $\mathcal{E}_N(\rho') = 2\sqrt{N}r$. Physically, this means that by mixing different spatial modes with beam splitters, we can "localize" the entanglement distributed initially among all the $2N+1$ spatial modes in only two well-defined modes, formed by the linear combinations of the initial ones. Interestingly, this entanglement localization results in an enhancement of the entanglement by a factor $\sqrt{N}$.

To conclude, our "active" optical scheme relies on the use of a spatially structured pumping of the parametric medium. It provides a realistic proposal for the experimental realization of multipartite entanglement as well as its localization. The possibility of entanglement localization was predicted mathematically in Ref. [3], and a physical implementation was also given in terms of $2N-1$ beam splitters and $2N$ single-mode squeezed inputs [20]. We have also established the scaling behavior of the logarithmic negativity with the number of modes. Note that, due to the asymmetric coupling between the modes in our scheme, this scaling is a relatively slowly growing function of the number of modes. An interesting matter is to determine the ultimate rate at which the logarithmic negativity may grow with the number of modes. This subject will be addressed elsewhere.

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