Uncertainty, Entropy and non-Gaussianity for mixed states

Aikaterini Mandilara, Evgueni Karpov and Nicolas J. Cerf

Quantum Information and Communication, École Polytechnique, CP 165/59, Université Libre de Bruxelles, 1050 Brussels, Belgium

ABSTRACT

In the space of mixed states the Schrödinger-Robertson uncertainty relation holds though it can never be saturated. Two tight extensions of this relation in the space of mixed states exist; one proposed by *Dodonov* and *Man'ko,* where the lower limit on the uncertainty depends on the purity of the state, and another where the uncertainty is bounded by the von Neumann entropy of the state proposed by *Bastiaans.* Driven by the needs that have emerged in the field of quantum information, in a recent work we have extended the purity-bounded uncertainty relation by adding an additional parameter characterizing the state, namely its degree of non-Gaussianity. In this work we alternatively present a extension of the entropy-bounded uncertainty relation. The common points and differences between the two extensions of the uncertainty relation help us to draw more general conclusions concerning the bounds on the non-Gaussianity of mixed states.

Keywords: Uncertainty principle, mixed states, non-Gaussian states

1. INTRODUCTION

More than 80 years have been passed from the first inception of the Heisenberg's principle, one of the fundamental ideas in quantum mechanics. Using his famous microscope,[1] Heisenberg argued that the uncertainty in localizing a particle is related to the uncertainty of its momentum in the following way:

$$\Delta x \Delta p \approx \hbar \tag{1}$$

where $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$.

Since its first formulation and up to our days, a lot of work have been dedicated to the extension and alternative formulation,[2] conceptual[3] and practical interpretation[4] of the uncertainty principle. The first refinement on Heisenberg's uncertainty relation (see[2] for a more detailed review) came shortly after by Kennard[5] and Weyl,[6] who formally derived the exact inequality

$$\Delta x \Delta p \geq \hbar/2. \tag{2}$$

employing the canonical commutation relation of the conjugate variables. Contrary to the initial formulation due to Heisenberg, Eq.(2) does not carry any information about simultaneous measurement on the position and momentum of a particle and only tells us that it is impossible to prepare states in which position and momentum are simultaneously arbitrarily well localized.[4]

A few years later the relation Eq.(2) was generalized by Scrödinger and Robertson[7] independently, for arbitrary observables (Hermitian operators) A and B to the following one

$$\Delta A \Delta B \geq \hbar \left| \langle [A, B] \rangle \right| /2. \tag{3}$$

We should note here that the uncertainty relation Eq.(2) for the canonically conjugated variables of position and momentum can be easily extended to a more general form that takes into account the existence of correlations between two variables,

$$\sqrt{(\Delta x)^2 (\Delta p)^2 - (\Delta xp)^2} \geq \hbar/2 \tag{4}$$

Further author information: (Send correspondence to A.M)
A.M.: E-mail: admandil@ulb.ac.be

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where $\Delta xp = \frac{1}{2}\langle xp+px\rangle - \langle x\rangle\langle p\rangle$. The inequality Eq.(4) is largely known nowadays as the *Scrödinger-Robertson uncertainty relation* and its lower bound is reached exclusively by the so-called *Gaussian* pure states, i.e., the states with Wigner function[8] of Gaussian form. The Gaussian set is composed by all coherent and squeezed-displaced states which are often encountered in the field of quantum optics.

Apart from the uncertainty relation where the uncertainty, "spread", is quantified by the standard deviation there is another family of uncertainty relations where the spread is characterized by different kind of entropies. Such inequalities are generally named as *entropic uncertainty relations* and have regained interest in the context of quantum information.[9] However in this work we are going to confine ourselves to the "standard" uncertainty relations.

One important question that arises is how the Schrödinger-Robertson uncertainty relation is modified for mixed states. This question has been answered by Bastiaans[10] and Dodonov-Man'ko[11] by identifying a relation between the lower limit of the uncertainty, Eq.(4), and the degree of mixedeness of a state. Bastiaans has employed as a measure of mixedeness the von Neumann entropy, while Dodonov-Man'ko the purity of the state. Both extensions exhibit similar behavior; the lower limit in the uncertainty is increasing as the degree of mixedeness is increasing. However, concerning the states which saturate the inequalities, so-called minimizing states[2](MS) there is a striking difference. The extended relation built on the measure of the von Neumann entropy, the so-called *entropy-bounded uncertainty relation*,[10] is minimized by Gaussian mixed states (thermal states) and while the *purity-bounded uncertainty relation*[11] is minimized by non-Gaussian states.

In a recent work[12] we have further extended the purity-bounded uncertainty relation by adding one extra parameter characterizing the mixed state namely its degree of non-Gaussianity. The idea of constructing uncertainty relations depending on several parameters characterizing the state is not new, however with our choice we arrive to some interesting results. Among these is an uncertainty relation for pure states that depends on the degree of non-Gaussianity and which is saturated, among others, by all number states. In addition this, as we call it, *non-Gaussianity bounded uncertainty relation*, can be employed to derive bounds on the degree of non-Gaussianity for mixed states with Wigner functions without negative domain.[13]

In this work after we introduce in Sec. 2 the needed quantities, we present in Sec. 3, in parallel, the extensions of the purity and entropy bounded uncertainty relations and we express them in a compact way using parametric relations. In Sec. 4 following similar steps as in,[12] we further extend the entropy-bounded uncertainty relation adding one more degree of freedom which characterizes the deviation of the state from a reference Gaussian state. In the last section, Sec. 5, we compare the new obtained results with those in[12] and we conclude.

2. QUANTITIES AND MEASURES

Let us start with a general density matrix $\hat{\rho}$ defining a quantum mixed state. To quantify the degree of mixedeness of the state $\hat{\rho}$ one may employ its *purity* $\mu = \mathrm{Tr}\left(\hat{\rho}^2\right)$. The range of values for μ is $[0,1]$, where the value 1 corresponds to pure states the value 0 to totally mixed ones. In a equivalent way, one may use the von Neumann entropy of the state, $S = -Tr\left(\hat{\rho}\ln\hat{\rho}\right)$ which can be seen as an extension of Shannon's entropy. The entropy S takes the value 0 for pure states and it tends to ∞ for totally mixed states. Throughout this work we consider quantum systems with only one pair of conjugate variables, like one-particle systems or one-mode states in quantum optics. In this particular case there is a relation connecting the von Neumann entropy with the purity of the state,

$$S = \frac{1-\mu}{2\mu}\ln\left(\frac{1+\mu}{1-\mu}\right) - \ln\left(\frac{2\mu}{1+\mu}\right). \tag{5}$$

In the figures presented in this work we employ the quantity $\exp\left[-S\right]$ instead of S, since this is bounded between 0 and 1.

A second basic characteristic of the state $\hat{\rho}$ that we would like to introduce is the *covariance matrix* γ of the state $\hat{\rho}$ defined as

$$\gamma_{ij} = \mathrm{Tr}(\{(\hat{r}_i - d_i), (\hat{r}_j - d_j)\}\hat{\rho}) \tag{6}$$

where $\hat{\mathbf{r}}$ is the vector of the canonically conjugated observables position and momentum (or quadrature operators in quantum optics) $\hat{\mathbf{r}} = (\hat{x}, \hat{p})^T$, $\mathbf{d} = Tr(\hat{\mathbf{r}}\hat{\rho})$ is the displacement vector, and $\{\cdot, \cdot\}$ is the anticommutator. We can put the displacement vector to zero with no loss of generality since the purity (as well as the quantities we will be interested in) is invariant with respect to $\mathbf{d}$.

The importance of the covariance matrix stems from the fact that it is measurable experimentally and that it is tightly connected, as we show below, with the uncertainty relation. Since we consider here one-dimensional quantum systems the covariance matrices we are dealing with are 2×2. The covariance matrix of $\hat{\rho}$ uniquely determines a Gaussian state $\hat{\rho}_G$ which henceforth will refer as the *reference Gaussian state* for the state $\hat{\rho}$. The purity μ_G of $\hat{\rho}_G$, exactly defines the inverse of the "*uncertainty*" of the state (the left-hand side of Eq.(4)) as,

$$\mu_G = 1/\sqrt{\gamma_{11}\gamma_{22} - |\gamma_{12}|^2}. \tag{7}$$

More precisely $\sqrt{(\Delta x)^2 (\Delta p)^2 - (\Delta xp)^2} = 1/2\mu_G$ and in this work, we often abbreviate the uncertainty by the quantity μ_G which is bounded between the values 0 and 1.

A third quantity that is needed for our purposes is a measure of *non-Gaussianity.* The problem of characterizing the non-Gaussianity of a state strongly resembles an older one, namely that of quantifying the non-classical character of a state.[14] In analogy with the common definitions of non-classicality,[15] the non-Gaussianity of a pure state is to be defined as the minimum of the distance from the given state to all Gaussian states. The employed measure of distance can be any construction that satisfies the usual requirements of a distance[16] and, in addition, that is invariant under symplectic (Gaussian) transformations. However, when looking at a computable measure of non-Gaussianity, this definition, even if correct, is not easily applicable.

In this work we choose to use as a measure of non-Gaussianity simply the deviation of the state from its reference Gaussian state. This can be quantified in different ways, and the simplest one is by the trace overlap of the states $\hat{\rho}$ and $\hat{\rho}_G$, $\mathrm{Tr}(\hat{\rho}\hat{\rho}_G)$. If a state is Gaussian then $\mathrm{Tr}(\hat{\rho}\hat{\rho}_G) = \mu = \mu_G$, while for non-Gaussian states it can be larger or smaller than this value. Using the Cauchy-Schwarz inequality and the expression of the trace overlap in Wigner representation

$$\mathrm{Tr}(\hat{\rho}\hat{\rho}_G) = 2\pi \int\int W(x,p)\, W_G(x,p)\, dxdp, \tag{8}$$

one may show[17] that the upper, not-necessarily reachable, limit is $\sqrt{\mu\mu_G}$. The lower limit of the trace overlap has been shown to be 0 and it is reached by states with "spread" tending to infinity. However the exact reachable limits of $\mathrm{Tr}(\hat{\rho}\hat{\rho}_G)$ for different values of μ and μ_G is within the objectives of this work.

Furthermore, the trace overlap can be employed together with μ_G, μ for evaluating a recently suggested measure of non-Gaussianity, the normalized Hilbert-Schmidt distance,[2,18]

$$\delta = \frac{1}{2\mu}\mathrm{Tr}\left((\hat{\rho} - \hat{\rho}_G)^2\right) = \frac{\mu_G + \mu - 2\mathrm{Tr}(\hat{\rho}\hat{\rho}_G)}{2\mu}. \tag{9}$$

This measure has the main properties of the distance, it takes the value 0 for the thermal states and while its maximum value is 1/2.[12]

Finally we note that for the rest of this work we set $\hbar = 1$.

3. PURITY- AND ENTROPY- BOUNDED UNCERTAINTY RELATIONS FOR MIXED STATES

In this section we derive the purity- and entropy-bounded uncertainty relations in a unified framework and we express them using compact parametric relations. The steps of derivation can be easily generalized to the case where a higher number of parameters is employed to characterize the mixed state.

The objective is to identify the lower bounds on the uncertainty $\sqrt{(\Delta x)^2 (\Delta p)^2 - (\Delta xp)^2}$ (upper limits for μ_G, Eq.(7)) given the degree of mixedeness of the state $\hat{\rho}$, as well as, the mixed states which reach the limits (MS).

Without loss of generality we assume that the covariance matrix γ of $\hat{\rho}$ is diagonal ($\Delta xp = 0$) and symmetric in the variables x and p ($\Delta x = \Delta p$). This can be easily achieved by applying symplectic transformations on a state $\hat{\rho}$. Such transformations do not alter the degree of mixedeness (eigenvalues) of the state.

We have an extremization problem to solve under specific constraints, and the natural method to apply is that of Lagrange multipliers. Within this method the solution is identical if the quantity to be extremized is exchanged with one of the constraints. In view of this statement, the constraints that we impose are summarized below,

1. The state is normalized
$$\mathrm{Tr}\left(\hat{\rho}\right) = 1. \tag{10}$$
2. The reference Gaussian state is a thermal non-displaced one,
$$\mathrm{Tr}\left(\hat{\rho}\hat{x}\right) = \mathrm{Tr}\left(\hat{\rho}\hat{p}\right) = \mathrm{Tr}\left(\hat{\rho}\hat{x}\hat{p}\right) = 0 \tag{11}$$
and its purity (or uncertainty) is fixed
$$\mathrm{Tr}\left(\hat{\rho}\left(2\hat{n}+1\right)\right) = \frac{1}{\mu_G}. \tag{12}$$

The quantity to be extremized is the degree of mixedeness quantified by the purity $\mu = \mathrm{Tr}\left(\hat{\rho}^2\right)$ or the von Neumann entropy $S = -Tr\left(\hat{\rho}\ln\hat{\rho}\right)$. In the first case, the extremum solution we obtain after differentiation is of the form,
$$\hat{\rho}_{exP} = \alpha_0 + \alpha_1\hat{x} + \alpha_2\hat{p} + \alpha_3\hat{x}\hat{p} + \alpha_4\hat{n} \tag{13}$$
and in the case of the entropy
$$\hat{\rho}_{exS} = \exp[b_0 + b_1\hat{x} + b_2\hat{p} + b_3\hat{x}\hat{p} + b_4\hat{n}]. \tag{14}$$

For these solutions, the constraints Eqs.(11) can be only satisfied if $\alpha_{1,2,3} = 0$ and $b_{1,2,3} = 0$, implying that the extremum solution for both problems is diagonal in the eigenbasis of the harmonic oscillator. We proceed by working in the number state basis to identify the exact coefficients of the solutions Eqs.(13) and (14).

3.1 Purity-bounded uncertainty relation

According our argument above, we may confine ourselves to states which are finite convex combinations of the number states
$$\hat{\rho} = \sum_{n=0}^{N} P_n|n\rangle\langle n|. \tag{15}$$
The quantity which we would like to extremize is the purity,
$$\sum_{n=0}^{N} P_n^2 = \mu\,, \tag{16}$$
under the constraints of normalization
$$\sum_{n=0}^{N} P_n = 1 \tag{17}$$
where $0 \leq P_n \leq 1$, and uncertainty,
$$\sum_{n=0}^{N} P_n(n+1/2) = 1/2\mu_G\,. \tag{18}$$
The Lagrange multipliers method provide the following solution, (see Eq.(13)),
$$P_n^{ex} = A_1 + A_2 n \tag{19}$$

where $A_1$and A_2 are to be determined by the conditions Eqs.(17) - (18). However the parameter N, the upper limit in the summation, remains undetermined and to resolve this ambiguity we introduce a continuous parameter y that satisfies

$$A_1 + A_2 y = 0. \tag{20}$$

This additional equation permits us to define N as the integer part of y, $N = \lfloor y \rfloor$.

We employ the conditions, Eqs. (17) and (20) to express $A_1$and A_2 in terms of N and y. We substitute the resulting expressions into Eqs.(16),(18), thus obtaining a parametric relation for the extremal solution,

$$\mu_G^{ex} = \tfrac{3(N-2y)}{N(5+4N)-6(1+N)y}, \tag{21}$$

$$\mu^{ex} = \tfrac{2\left(N+2N^2-6Ny+6y^2\right)}{3(1+N)(N-2y)^2}. \tag{22}$$

where $N = \lfloor y \rfloor$ and $y \in [1, \infty)$. Moreover, by setting $N = y$ one can arrive to an approximate formula

$$\mu^{ex} = \frac{8\mu_G^{ex}}{9 - (\mu_G^{ex})^2}$$

that reproduces the curve of Eqs.(21)-(22) in very good approximation when $0 \leq \mu_G \leq 3/5$. For the rest of the region, $3/5 \leq \mu_G \leq 1$, one should better employ the exact solution

$$\mu_{N=1} = \frac{1 - 4\mu_G + 5\mu_G^2}{2\mu_G^2}. \tag{23}$$

The obtained results Eqs.(21)-(22) give the lower extreme bound (see Fig.1(a)) on the plane μ_G and μ and are equivalent to the expressions derived in.[11] In their derivation, the aim is to obtain a function of the uncertainty μ_G in terms of the purity. This requires an inversion step that is valid under specific constraints and what the authors obtain is sequence of functions each one valid, for a different N (segment of μ). This inversion step in our case would mean to solve Eq.(22) for y and substitute the result into Eq.(21).

The states which minimize (MS) this extended uncertainty relation can be easily derived,

$$\hat{\rho}_{ex}(y) = \sum_{n=0}^{N} \left(\frac{2(n-y)}{(1+N)(N-2y)} \right) |n\rangle\langle n| \tag{24}$$

where $N = \lfloor y \rfloor$ and $y \in [1, \infty)$.

3.2 Entropy-bounded uncertainty relation

As in the previous case, we express the trial solution in the number states basis

$$\hat{\rho} = \sum_{n=0}^{N} P_n |n\rangle\langle n| \tag{25}$$

and we impose the constraints

$$\sum_{n=0}^{N} P_n = 1 \tag{26}$$

and

$$\sum_{n=0}^{N} P_n (n + 1/2) = 1/2\mu_G\,. \tag{27}$$

The quantity to extremize now is the von Neumann entropy

$$-\sum_{n=0}^{N} P_n \ln P_n = S\,. \tag{28}$$

After differentiation, the solution that we obtain is

$$P_n^{ex} = \exp\left(B_1 + B_2 n\right). \tag{29}$$

Contrary to the case of the purity, the coefficients P_n^{ex} we obtain are positive for every n and therefore the upper limit of summation, N, should be taken to ∞. In other words, in this case, there is no need for an ansatz i.e. introduce the variable y, since the obtained solution corresponds always to a positive density matrix. Here, though, one should take care so that the matrix obtained is normalizable and for this to be true, the values of $B's$ in Eq.(29) should both be negative. After substituting Eq.(P_n^{ex}) into the Eqs.(26)-(27) we obtain the following indirect relations

$$\frac{e^{B_1}}{1-e^{B_2}} = 1 \tag{30}$$

$$\frac{e^{B_1}\left(1+e^{B_2}\right)}{\left(1-e^{B_2}\right)^2} = 1/\mu_G \tag{31}$$

$$-\frac{e^{B_1}\left(B_1+e^{B_2}\left(B_2-B_1\right)\right)}{\left(1-e^{B_2}\right)^2} = S. \tag{32}$$

We use the Eq.(30) to express B_1 in terms of B_2, thus obtaining the following indirect extreme relation between the uncertainty and the von Neumann entropy

$$\mu_G^{ex} = \frac{\left(1-e^{-B}\right)^2}{\left(1-e^{-2B}\right)}, \tag{33}$$

$$S^{ex} = \frac{Be^{B}-\left(1-e^{-B}\right)\ln\left(1-e^{-B}\right)}{1-e^{-B}}. \tag{34}$$

where $B \in (0,\infty)$. The states which extremize the entropy bounded uncertainty relation are thermal states,

$$\hat{\rho}_{ex} = \sum_{n=0}^{\infty}\left(1-e^{-B}\right)\exp\left(-Bn\right)|n\rangle\langle n| \tag{35}$$

where $B \in (0,\infty)$ has the role of the inverse temperature β.

In Fig.1 we present the dependence of the uncertainty of a mixed state on the degree of mixedeness, quantified by the measures of purity and von Neumann entropy. The two curves exhibit the same behavior, that is the uncertainty is getting larger as the degree of mixedeness is increasing. This similar behavior is justified by the monotonic relation between the entropy and the purity given by the Eq.(5). On the other hand the MS for the two relations are not the same even though they have some similar features. Being mixtures of the number states, both classes of MS are phase-independent in the phase-space representation. In addition, both have Wigner representation strictly positive.[2] However for the purity-bounded uncertainty relation the MS are not Gaussian states, as it is the case for pure states and the entropy-bounded uncertainty relation. Gaussian states tend to be considered extremal within all continuous variable states if one imposes constraints on the covariance matrix.[19] The purity-bounded uncertainty relation can be considered as a counter-example to this statement.

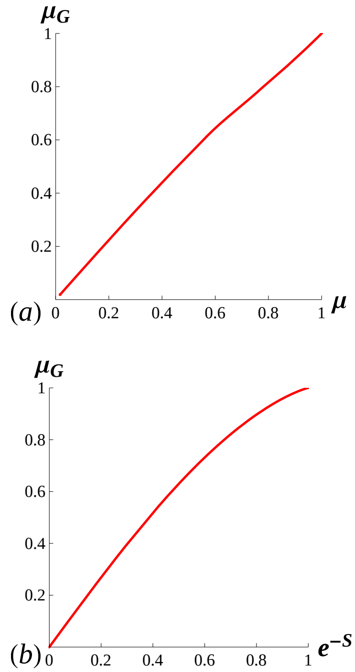


Figure 1. a) The purity-bounded uncertainty relation provides a lower bound on the determinant of the covariance matrix of a state (upper bound on μ_G Eq.(7)) as a function of its purity μ. (b) The entropy-bounded uncertainty relation, where the upper bound on μ_G depends on the von Neumann entropy S of the state.

4. EXTENDING FURTHER THE ENTROPY-BOUNDED UNCERTAINTY RELATION

In a recent work[12] we have extended the purity-bounded uncertainty relation by adding one more parameter characterizing the state namely the degree of non-Gaussianity quantified by δ Eq.(9). The motivation behind that work was to extend in a parallel way, to the space of mixed states, two basic theorems of quantum mechanics which for pure states bare a striking similarity, the Hudson's theorem and the uncertainty principle. According to Hudson's theorem the only pure states with everywhere positive Wigner function are the Gaussian ones. These are also the states which minimize the Schrödinger-Robertson uncertainty relation Eq.(4).

More explicitly, in[17] we have formulated an extension of Hudson's theorem, by identifying the maximum degree of deviation from a Gaussian state which can be reached by a mixed state with positive Wigner function, given its purity and its uncertainty. This problem does not have a simple solution and in[17] we have only identified the bounds for any continuous classical distribution which may not necessarily correspond to a valid Wigner function, and therefore the bounds are not tight for the set of quantum states. In [12] we have extended the purity-bounded uncertainty so that we are able to draw more complete conclusions on the overlap between states minimizing that uncertainty relation and the set of states with positive Wigner function.[13] According to our results the overlap of the two sets is maximal for the Gaussian states and as the degree of non-Gaussianity is increasing the overlap is gradually decreasing and after some certain degree, the two sets have no overlap at all.

In this work, we want to investigate how a similar extension for the entropy-bounded uncertainty relation stands and we follow similar steps, as in,[12] to derive it. However employing the entropy, we are confined to work with the quantity of trace overlap $\mathrm{Tr}\,(\hat{\rho}\hat{\rho}_G)$ that is not a measure of non-Gaussianity in a formal sense. Furthermore, as we are going to show that contrary to the extension in,[12] the solution to this problem, in order to be visualized, requires numerical methods.

To derive the extended relation we simply add the constraint of fixed trace overlap $\mathrm{Tr}\,(\hat{\rho}\hat{\rho}_G)$ to the extremization problem analysed in Sec. 3. Given that, up to a normalization factor, $\hat{\rho}_G = \exp(F\hat{n})$, the additional constraint can be simply expressed as

$$\mathrm{Tr}\left(\hat{\rho}\exp\left(F\hat{n}\right)\right) = Ov. \tag{36}$$

The inverse temperature F of the reference Gaussian state $\hat{\rho}_G$ is related to the purity μ_G Eq.(12) with a relation that we derive below. For simplicity reasons and without any effect on the solution, we also omit the constraint of normalization Eq.(10) on the state $\hat{\rho}$.

Under the two constraints, Eqs.(12), (36) the extremization procedure on the quantity of von Neumann entropy Eq.(28) provides a solution of the form

$$\hat{\rho}_{ex} = \exp[A\hat{n} + B\exp\left(F\hat{n}\right)] \tag{37}$$

Obviously the solution is diagonal in the number state representation. We express Eq.(37) in this basis and we include the normalization to arrive to,

$$\hat{\rho}_{ex} = \frac{\sum_{n=0}^{\infty}\exp[An + B\exp\left(Fn\right)]|n\rangle\langle n|}{\sum_{n=0}^{\infty}\exp[An + B\exp\left(Fn\right)]}. \tag{38}$$

Let us now, impose the constraints Eqs.(12), (36) to identify the coefficients A, B and F.

In view of Eq.(38), the constraint $\mathrm{Tr}\left(\hat{\rho}\left(2\hat{n}+1\right)\right) = \frac{1}{\mu_G}$ becomes

$$\frac{\sum_{n=0}^{\infty}2n\exp[An + B\exp\left(Fn\right)]}{\sum_{n=0}^{\infty}\exp[An + B\exp\left(Fn\right)]} + 1 = \frac{1}{\mu_G}. \tag{39}$$

The corresponding, normalized now, Gaussian state is

$$\hat{\rho}_G = \frac{\sum_{n=0}^{\infty}\exp[Fn]|n\rangle\langle n|}{\sum_{n=0}^{\infty}\exp[Fn]}, \tag{40}$$

It should have the same covariance as $\hat{\rho}$,

$$\begin{aligned}\mathrm{Tr}\left(\hat{\rho}_G\left(2\hat{n}+1\right)\right) &= \\ \frac{\sum_{n=0}^{\infty}2n\exp[Fn]}{\sum_{n=0}^{\infty}\exp[Fn]} + 1 &= \\ 2\frac{1}{e^{-F}-1} + 1 &= \tfrac{1}{\mu_G}.\end{aligned} \tag{41}$$

From Eqs.(39) and (41), we conclude that

$$\frac{\sum_{n=0}^{\infty}n\exp[An + B\exp\left(Fn\right)]}{\sum_{n=0}^{\infty}\exp[An + B\exp\left(Fn\right)]} = \frac{1}{e^{-F}-1}. \tag{42}$$

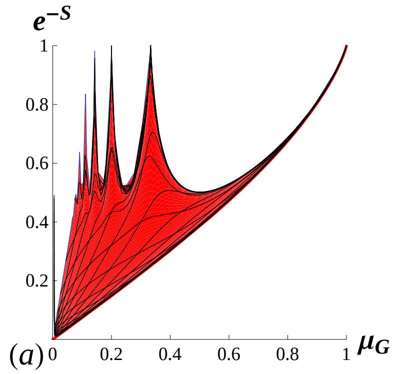
Similarly, the overlap constraint $\mathrm{Tr}\left(\hat{\rho}\exp\left(F\hat{n}\right)\right) = Ov$, with the help of Eq.(38), is re-expressed as

$$\left(1-e^F\right)\frac{\sum_{n=0}^{\infty}\exp[Fn]\exp[An + B\exp\left(Fn\right)]}{\sum_{n=0}^{\infty}\exp[An + B\exp\left(Fn\right)]} = Ov \tag{43}$$

and the von Neumann entropy $S = -Tr\left(\hat{\rho}\ln\hat{\rho}\right)$ as

$$-\sum_{m=0}^{\infty}\frac{\exp[Am + B\exp\left(Fm\right)]}{\sum_{n=0}^{\infty}\exp[An + B\exp\left(Fn\right)]}\ln\left(\frac{\exp[Am + B\exp\left(Fm\right)]}{\sum_{n=0}^{\infty}\exp[An + B\exp\left(Fn\right)]}\right) = S. \tag{44}$$

Then Eqs.(42),(43) and (44), permit us to connect the quantities of interest : uncertainty, overlap and entropy via the three parameters A, B and F. The range of the parameters though, is not arbitrary. The solution, Eq.(37), always corresponds to a positive matrix but to be a normalizable one, a constraint $A < 0$ should be



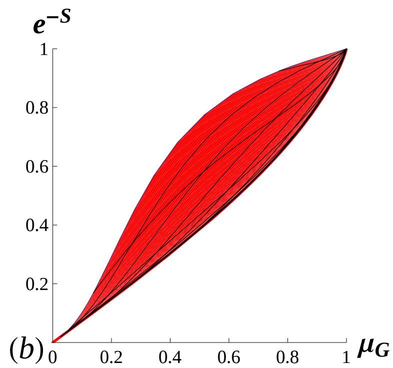


Figure 2. (a) Lower bound on the trace overlap $\mathrm{Tr}\,(\hat{\rho}\hat{\rho}_G)$ projected on the plane uncertainty-entropy ($\mu_G - e^{-S}$). This bound correpond to $B > 0$. (b) The upper bound on the trace overlap that is obtained for $B < 0$. The points have been obtained by numerical solving the Eq.(42) for the parameter F. We have used the first 100 terms of the series in Eqs.(42),(43) and (44).

imposed. For the reference Gaussian state, Eq.(40) to be normalizable we need $F < 0$, while there is no constraint on the parameter B. Since $B = 0$ simply gives the Gaussian solution i.e. the extreme line on the plane $S - \mu_G$ (von Neumann entropy- uncertainty), we expect that one $B-$branch corresponds to a maximum solution for the overlap $\mathrm{Tr}\,(\hat{\rho}\hat{\rho}_G)$ and the other to a minimum extremum solution.

Finally, one may observe that the form of the solution does not accommodate solutions which are pure states, apart from the trivial solution $|0\rangle$. On the other hand all the number states $|n\rangle$ are approachable in specific limits of the parameters A, B and F.

In Fig. 2 we present the bounds on the overlap $\mathrm{Tr}\,(\hat{\rho}\hat{\rho}_G)$ that we obtain by numerically solving the Eq.(42) for the parameter F. Then, with the help of Mathematica package we have created a projection of the bounds on the plane uncertainty-entropy using the indirect parametric plot relating the quantities S and μ_G, Eqs.(44),(42). We observe that the lower bound has very similar structure to the one obtained using the purity μ[12] instead of the entropy. The picks appearing on the $e^{-S} = 1$ axe (pure states) correspond, in the limit, to the number states as in.[12] In addition, as in[12] an area on the plane $\mu_G - e^{-S}$ remains uncovered implying the existence of a degenerate solution that cannot be identified by imposing variational methods on the density matrix.

As we have mentioned the Cauchy-Schwarz inequality gives us an upper estimation for the trace overlap that is $\sqrt{\mu_G \mu}$. This extremum value is not necessarily reachable since during the derivation of it the condition that $\hat{\rho}$ possess the same covariance matrix as $\hat{\rho}_G$ have not been taken into account. In Fig. 2(b) we depict the strict upper bound on $\mathrm{Tr}\,(\hat{\rho}\hat{\rho}_G)$ if the entropy and the uncertainty of the state are given. As is the case of the lower bound, the upper bound does not cover all the allowed in principle parametric area S and μ_G.

5. CONCLUSIONS

In conclusion we have derived tight upper and lower bounds on the trace overlap of a mixed state $\hat{\rho}$ with a reference Gaussian state $\hat{\rho}_G$ defined by the covariance matrix of $\hat{\rho}$, given the entropy S and the uncertainty of the state. These bounds are expressed in terms of series including three parameters that indirectly connect the three quantities of interest, $\mathrm{Tr}\,(\hat{\rho}\hat{\rho}_G)$, S and μ_G (inverse uncertainty). These bounds are of fundamental interest since they connect three basic characteristics of a mixed state. Given the need of construction of non-Gaussian states in the field of quantum information these bounds may be of practical interest as well.

In[12] we have accomplished a similar derivation characterizing mixedeness, by the purity μ. The bounds obtained in this work and in[12] have similar structure. In both cases the bounds are realized by states which are mixtures of the number states and as a consequence of this the only pure states saturating the extended uncertainty relations are the number states. By employing the purity we arrive to simpler expressions where only two parameters are involved. On the other hand the entropy extended uncertainty relation provides an upper bound on the trace overlap a task that cannot be achieved if the purity is used instead of the entropy.

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