Equivalence Relations for the Classical Capacity of Single-Mode Gaussian Quantum Channels

Joachim Schäfer,[1] Evgueni Karpov,[1] Raúl García-Patrón,[1,2] Oleg V. Pilyavets,[1] and Nicolas J. Cerf[1]

[1]*QuIC, Ecole Polytechnique de Bruxelles, CP 165, Université Libre de Bruxelles, 1050 Brussels, Belgium*

[2]*Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Straße 1, 85748 Garching, Germany*

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We prove the equivalence of an arbitrary single-mode Gaussian quantum channel and a newly defined fiducial channel preceded by a phase shift and followed by a Gaussian unitary operation. This equivalence implies that the energy-constrained classical capacity of any single-mode Gaussian channel can be calculated based on this fiducial channel, which is furthermore simply realizable with a beam splitter, two identical single-mode squeezers, and a two-mode squeezer. In a large domain of parameters, we also provide an analytical expression for the Gaussian classical capacity, exploiting its additivity, and prove that the classical capacity cannot exceed it by more than $1/\ln 2$ bits.

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Introduction.—Quantum channels play a key role in quantum information theory. In particular, bosonic Gaussian channels model most optical communication links, such as optical fibers or free space information transmission [1,2]. One of the central characteristics of quantum channels is their classical capacity. A lot of attention has already been devoted to the study of the classical capacity of Gaussian channels [3–22]. Since Gaussian encodings are more relevant for experimental implementations, easier to work with analytically, and conjectured to be optimal [11], the so-called Gaussian classical capacity was evaluated for specific Gaussian channels [14–22].

In this Letter, we greatly simplify the calculation of these capacities [23] for an arbitrary single-mode Gaussian channel. Namely, we show that any such channel is indistinguishable from a newly defined fiducial channel, preceded by a phase shift and followed by a general Gaussian unitary. Since neither the phase shift at the channel's input nor the Gaussian unitary at the channel's output affects the input energy constraint or changes the output entropy, the capacities of this channel are equal to those of the fiducial channel. This conclusion also holds for any cascade of Gaussian channels since the latter is equivalent to another Gaussian channel. Our results allow us to go beyond previous works on the Gaussian capacity [19–21] and provide its unified analytical expression valid for any Gaussian channel in some energy range, where it is additive. In this range, we prove that the capacity cannot exceed the Gaussian capacity by more than $1/\ln 2$ bits (generalizing [24]), the latter becoming the actual capacity if the minimum-output entropy conjecture for phase-insensitive Gaussian channels [11,25] is true.

Gaussian channel.—Let $\hat{\rho}^{\mathrm{G}}(\boldsymbol{\alpha}, \boldsymbol{V})$ be a single-mode Gaussian state, where the coherent vector $\boldsymbol{\alpha} \in \mathbb{R}^2$ and the covariance matrix (CM) $\boldsymbol{V} \in \mathbb{R}^2 \times \mathbb{R}^2$ are the first- and second-order moments of the 2 dimensionless quadratures, respectively, with $\hbar = 1$. Then, a single-mode Gaussian channel Φ is a completely positive trace-preserving map which is closed on the set of Gaussian states [11]. It transforms input states with moments $\{\boldsymbol{\alpha}_{\mathrm{in}}, \boldsymbol{V}_{\mathrm{in}}\}$ to output states with moments $\{\boldsymbol{\alpha}_{\mathrm{out}}, \boldsymbol{V}_{\mathrm{out}}\}$ according to

$$\boldsymbol{\alpha}_{\mathrm{out}} = \boldsymbol{X}\boldsymbol{\alpha}_{\mathrm{in}} + \boldsymbol{\delta}, \qquad \boldsymbol{V}_{\mathrm{out}} = \boldsymbol{X}\boldsymbol{V}_{\mathrm{in}}\boldsymbol{X}^{\mathsf{T}} + \boldsymbol{Y}, \tag{1}$$

where $\boldsymbol{\delta}$ is the displacement induced by the channel, $\boldsymbol{X}$ is a 2×2 real matrix, and $\boldsymbol{Y}$ is a 2×2 real, symmetric, and non-negative matrix. For simplicity, we choose $\boldsymbol{\delta} = 0$ in what follows (the capacity is not affected by $\boldsymbol{\delta}$), and focus on the action of the map Φ on second-order moments using the simplified notation $\Phi(\boldsymbol{V}_{\mathrm{in}}) = \boldsymbol{V}_{\mathrm{out}}$. Then, the map Φ is fully characterized by matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, which must satisfy $\boldsymbol{Y} + i(\boldsymbol{\Omega} - \boldsymbol{X}\boldsymbol{\Omega}\boldsymbol{X}^{\mathsf{T}})/2 \geq 0$ [11], where

$$\boldsymbol{\Omega} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2}$$

is the symplectic form [26]. In the following, we use the parameters

$$\tau = \det \boldsymbol{X}, \qquad y = \sqrt{\det \boldsymbol{Y}}, \tag{3}$$

where τ may be a channel transmissivity (if $0 \leq \tau \leq 1$) or amplification gain (if $\tau \geq 1$), while y characterizes the added noise. The map Φ describes a quantum channel if $y \geq |\tau - 1|/2$ [27]. Moreover, it is an entanglement breaking channel if $y \geq (|\tau| + 1)/2$ [28]. The single-mode Gaussian channels can therefore be conveniently represented in a (τ, y) plane, see Fig. 1.

Canonical decomposition.—Any single-mode Gaussian channel Φ can be decomposed as $\Phi = U_2 \circ \Phi^C \circ U_1$, where U_1 and U_2 are Gaussian unitaries, and Φ^C is a canonical channel characterized by the matrices $(\boldsymbol{X}_C, \boldsymbol{Y}_C)$ [29–31]. The action of a Gaussian unitary U on a Gaussian state can be completely specified by a symplectic transformation $\boldsymbol{M}$ acting on the second-order moments of the state (we ignore first-order moments), so that the canonical decomposition may be written as $(U_2 \circ \Phi^C \circ U_1)(\boldsymbol{V}_{\mathrm{in}}) = \boldsymbol{M}_2\Phi^C(\boldsymbol{M}_1\boldsymbol{V}_{\mathrm{in}}\boldsymbol{M}_1^{\mathsf{T}})\boldsymbol{M}_2^{\mathsf{T}}$. One can define seven classes of

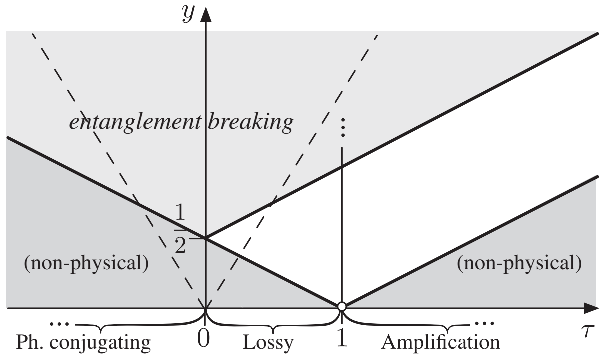

FIG. 1. Admissible regions in the parameter space (τ, y) for Gaussian quantum channels. Each thermal channel $\Phi^{\mathrm{TH}}_{(\tau,y)}$ is associated with a particular point (τ, y). The vertical line $\tau = 0$ corresponds to the zero-transmission channel as well as the classical signal channel Φ^{CS}. The vertical line $\tau = 1$ corresponds to the classical additive-noise channel. Both the perfect transmission channel and the single-quadrature classical noise channel Φ^{SQ} correspond to $(\tau = 1,\ y = 0)$. The Gaussian capacity of $\Phi^{F}_{(\tau,y,s)}$ is additive if $\bar{N} \geq \bar{N}_{\mathrm{thr}}$. This is equivalent to $y \leq y_{\mathrm{thr}} = |\tau|[e^{-2|s|}(1 + 2\bar{N}) - 1]/(1 - e^{-4|s|})$. An example of y_{thr} is given by the dashed line, where $\bar{N} = 0.5$ and $s = 0.12$.

canonical channels Φ^{C} (see Table I) [29–31]. The first five channels in Table I can be treated together, and we refer to them collectively as thermal channels, $\Phi^{\mathrm{TH}}_{(\tau,y)}$:

$$\boldsymbol{X}_{\mathrm{TH}} = \begin{pmatrix} \sqrt{|\tau|} & 0 \\ 0 & \mathrm{sgn}(\tau)\sqrt{|\tau|} \end{pmatrix}, \qquad \boldsymbol{Y}_{\mathrm{TH}} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}, \quad (4)$$

where $\mathrm{sgn}(\tau) = -1$ if $\tau < 0$ and $\mathrm{sgn}(\tau) = 1$ if $\tau \geq 0$. As shown in Fig. 2(a), any channel Φ^{TH} can be physically realized by a beam splitter with transmissivity T followed by a two-mode squeezer (TMS) with gain G [27]. For the zero-transmission ($\tau = 0$), lossy ($0 \leq \tau \leq 1$), amplification ($\tau \geq 1$), and classical additive-noise channel ($\tau = 1$), the output is given by the signal's output of the TMS, and these four canonical channels correspond to phase-insensitive channels. For the fifth canonical channel, i.e., the phase-conjugating channel ($\tau < 0$), the output is given by the idler's output of the TMS. These five channels map any thermal state to a thermal state, so we call them thermal channels. Each particular channel $\Phi^{\mathrm{TH}}_{(\tau,y)}$ corresponds to a single point in Fig. 1, where the relations between $(\tau,\ y)$ and $(T,\ G)$ are given in Table I. Finally, the sixth and seventh canonical channels are the classical signal (or quadrature erasing) channel Φ^{CS} and the single-quadrature classical noise channel Φ^{SQ}, which are not thermal channels (see [32]).

Fiducial channel.—Now, our central point is that the above canonical decomposition is not always useful for evaluating capacities of bosonic channels with input energy constraint (which is needed, otherwise the capacities are infinite). Indeed, the Gaussian unitary U_1 that precedes the canonical channel Φ^{C} affects, in general, the input energy. Therefore, we introduce a new decomposition in terms of a fiducial channel Φ^{F}, where the preceding unitary is passive and does not affect the input energy restriction. We show that this decomposition has the major advantage that the energy-constrained capacity of any Gaussian channel reduces to that of the fiducial channel Φ^{F}. The latter generalizes Φ^{TH} by introducing squeezing in the added noise

$$\boldsymbol{X}_{F} = \boldsymbol{X}_{\mathrm{TH}}, \qquad \boldsymbol{Y}_{F} = y\begin{pmatrix} e^{2s} & 0 \\ 0 & e^{-2s} \end{pmatrix}. \qquad (5)$$

Thus, it depends on three parameters $(\tau,\ y,\ s)$, and we denote it by $\Phi^{F}_{(\tau,y,s)}$. This channel can be physically realized by the setup depicted in Fig. 2(b), where the "idler" corresponds again to the output of the phase-conjugating channel and the "signal" to that of the other channels. In the case $0 \leq \tau \leq 1$, this channel corresponds to the mixing of the input state with an arbitrary squeezed thermal state on a beam splitter with transmissivity τ. The fiducial channel Φ^{F} can be used to decompose any Gaussian channel Φ (by taking proper limits, if necessary) [32].

Theorem 1. For a single-mode Gaussian channel Φ defined by matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ with $\tau \neq 0$ and $y > 0$, there exists a fiducial channel $\Phi^{F}_{(\tau,y,s)}$ defined by matrices $\boldsymbol{X}_F(\tau)$, $\boldsymbol{Y}_F(y,\ s)$ with τ and y obtained from Eq. (3), a symplectic transformation $\boldsymbol{M}$, and a rotation in phase space $\boldsymbol{\Theta}$ such that

TABLE I. Canonical channels Φ^{C} as defined in [29–31], and their new representation in terms of Φ^{TH}, Φ^{CS}, Φ^{SQ} and the corresponding matrices $(\boldsymbol{X}_C, \boldsymbol{Y}_C)$, where $\sigma_z = \mathrm{diag}(1, -1)$. The transmissivity $T \in [0, 1]$ of the beam splitter and the gain $G \geq 1$ of the two-mode squeezer correspond to the physical schemes in Fig. 2 and [32].

Channel	Symbol	Class	X_C	Y_C	τ	Domain of τ	Domain of y
Zero-transmission	$\mathcal{A}_1$		0	$(G - 1/2)\mathbb{1}$	0	0	$[1/2, \infty)$
Classical additive noise	$\mathcal{B}_2$		$\mathbb{1}$	$(G - 1)\mathbb{1}$	$TG = 1$	1	$[0, \infty)$
Lossy	$\mathcal{C}_L$	Φ^{TH}	$\sqrt{\tau}\mathbb{1}$	$[G(1 - T/2) - 1/2]\mathbb{1}$	TG	$[0, 1]$	$[(1 - \tau)/2, \infty)$
Amplification	$\mathcal{C}_A$		$\sqrt{\tau}\mathbb{1}$	$[G(1 - T/2) - 1/2]\mathbb{1}$	TG	$[1, \infty)$	$[(\tau - 1)/2, \infty)$
Phase-conjugating	$\mathcal{D}$		$\sqrt{\lvert\tau\rvert}\sigma_z$	$[(1 - T)(G - 1)/2 + G/2]\mathbb{1}$	$-T(G - 1)$	$(-\infty, 0]$	$[(1 - \tau)/2, \infty)$
Classical-signal	$\mathcal{A}_2$	Φ^{CS}	$(\mathbb{1} + \sigma_z)/2$	$(G - 1/2)\mathbb{1}$	0	0	$[1/2, \infty)$
Single-quad. cl. noise	$\mathcal{B}_1$	Φ^{SQ}	$\mathbb{1}$	$(\mathbb{1} - \sigma_z)/4$	1	1	0

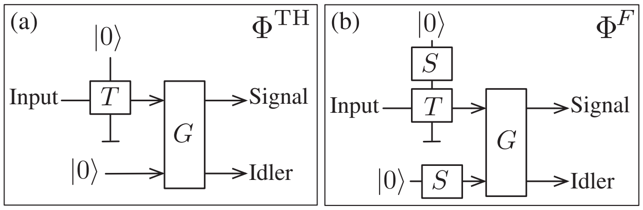

FIG. 2. Realization of (a) the thermal channel Φ^{TH} and (b) the fiducial channel Φ^F by a beam splitter with transmissivity T, a two-mode squeezer with gain G, and a single-mode squeezer S. Here $|0\rangle$ stands for the vacuum state, and "⊣" denotes "tracing out" the mode.

$$\boldsymbol{X} = \boldsymbol{M}\boldsymbol{X}_F(\tau)\boldsymbol{\Theta}, \qquad \boldsymbol{Y} = \boldsymbol{M}\boldsymbol{Y}_F(y, s)\boldsymbol{M}^{\mathsf{T}}, \tag{6}$$

where the explicit dependencies of $\boldsymbol{M}$, $\boldsymbol{\Theta}$, and s on the parameters of the channel Φ are presented in the Supplemental Material [32].

Proof. We only sketch the proof here (see [32] for the full proof). First, one finds matrices $\boldsymbol{\Theta}_Y$ and $\boldsymbol{S}_Y$ such that $\boldsymbol{S}_Y^{-1}\boldsymbol{\Theta}_Y^{\mathsf{T}}\boldsymbol{Y}\boldsymbol{\Theta}_Y\boldsymbol{S}_Y^{-1} = \mathrm{diag}(y, y)$, where $\boldsymbol{\Theta}_Y$ and $\boldsymbol{S}_Y$ denote matrices corresponding to a rotation and a squeezing operation, respectively. Second, one obtains the singular value decomposition $\boldsymbol{X} = \boldsymbol{\Theta}_{1X}\boldsymbol{S}_X\boldsymbol{X}_F\boldsymbol{\Theta}_{2X}$, where $\boldsymbol{X}_F$ reads as in Eq. (5). Then one defines $\boldsymbol{M} = \boldsymbol{\Theta}_{1X}\boldsymbol{S}_X\boldsymbol{\Theta}_F^{\mathsf{T}}$, where $\boldsymbol{\Theta}_F$ is found such that $\boldsymbol{M}^{\mathsf{T}}\boldsymbol{Y}\boldsymbol{M} = \boldsymbol{Y}_F = y\,\mathrm{diag}(e^{2s}, e^{-2s})$. The squeezing parameter s depends on all angles and squeezing operations $\boldsymbol{S}_X$, $\boldsymbol{S}_Y$. Finally, one introduces $\boldsymbol{\Theta}_F$ in $\boldsymbol{X}$, i.e., $\boldsymbol{X} = \boldsymbol{\Theta}_{1X}\boldsymbol{S}_X\boldsymbol{\Theta}_F^{\mathsf{T}}\boldsymbol{\Theta}_F\boldsymbol{X}_F\boldsymbol{\Theta}_{2X} = \boldsymbol{M}\boldsymbol{X}_F\boldsymbol{\Theta}$, where $\boldsymbol{\Theta}$ depends on $\boldsymbol{\Theta}_{2X}$, $\boldsymbol{\Theta}_F$, and the sign of τ. Despite Theorem 1 requires that $\tau \neq 0$ and $y > 0$, or, equivalently, that $\mathrm{rank}(\boldsymbol{X}) = \mathrm{rank}(\boldsymbol{Y}) = 2$, it can be extended to lower-rank cases with minor modifications [32]. □

Capacities.—The energy-constrained capacity C of the Gaussian channel Φ is defined as the maximal amount of bits that can be transmitted per use of the channel Φ given the mean photon number $\bar{N}$ at its input, i.e., [12,33]

$$C(\Phi, \bar{N}) = \lim_{n\to\infty}\frac{1}{n}C_\chi(\Phi^{\otimes n}, n\bar{N}), \tag{7}$$

where n is the number of channel uses, and C_χ is the one-shot capacity of the channel, i.e.,

$$\begin{aligned} C_\chi(\Phi, \bar{N}) &= \max_{\mu:\,\hat{\bar{\rho}}\in\mathcal{E}_{\bar{N}}}\chi(\Phi, \mu), \\ \chi(\Phi, \mu) &= S(\Phi[\hat{\bar{\rho}}]) - \int \mu(dx)S(\Phi[\hat{\rho}_x]). \end{aligned} \tag{8}$$

Here $S(\hat{\rho}) = -\mathrm{Tr}(\hat{\rho}\log_2\hat{\rho})$ is the von Neumann entropy. The maximum is taken over all probability measures $\mu(x)$ in the whole space $\mathcal{H}$ of pure symbol states $\hat{\rho}_x$ such that the average state $\hat{\bar{\rho}} = \int \mu(dx)\hat{\rho}_x$ belongs to the set $\mathcal{E}_{\bar{N}}$ of states which have a mean photon number not greater than $\bar{N}$. Since, in general, the one-shot capacity is not additive [34], one has to take the limit in Eq. (7), unless additivity is explicitly proven for the given channel. The decomposition stated in Theorem 1 implies:

Corollary 1. For a single-mode Gaussian channel Φ with parameters ($\tau \neq 0$, $y > 0$), there exists a fiducial channel Φ^F as defined in Theorem 1, such that

$$C(\Phi, \bar{N}) = C(\Phi^F, \bar{N}). \tag{9}$$

Proof. The symplectic transformation $\boldsymbol{M}$ that follows Φ^F in Theorem 1 does not change the entropies in χ and there is no energy constraint on the output of the channel. Hence, $\boldsymbol{M}$ can be omitted. Furthermore, the rotation $\boldsymbol{\Theta}$ preceding Φ^F in Theorem 1 may be regarded as a change of the reference phase that can be chosen arbitrarily; therefore, $\boldsymbol{\Theta}$ can be omitted as well. Thus, $C_\chi(\Phi, \bar{N}) = C_\chi(\Phi^F, \bar{N})$ holds. In order to evaluate the one-shot capacity of $\Phi^{\otimes n}$ we apply the same reasoning, where the preceding and following transformations are given by $\oplus_{i=1}^n\boldsymbol{M}$ and $\oplus_{i=1}^n\boldsymbol{\Theta}$, respectively. Hence, it follows that $C_\chi(\Phi^{\otimes n}, n\bar{N}) = C_\chi((\Phi^F)^{\otimes n}, n\bar{N})$ which together with Eq. (7) implies Eq. (9). Note that despite Eq. (9) requires $\tau \neq 0$ and $y > 0$, it can be easily extended to the general case [32]. □

We remark that if the corresponding fiducial channel $\Phi^F_{(\tau,y,s)}$ is entanglement breaking, then the one-shot capacities of both $\Phi^F_{(\tau,y,s)}$ and Φ are additive [35,36] and using Corollary 1 it follows that $C(\Phi, \bar{N}) = C_\chi(\Phi^F, \bar{N})$.

Gaussian capacities.—For experimental implementations and analytical calculations, it is convenient to focus on Gaussian encodings. We call the capacity restricted to Gaussian encodings the Gaussian capacity C^{G} [17–22]:

$$\begin{aligned} C^{\mathrm{G}}(\Phi, \bar{N}) &= \lim_{n\to\infty}\frac{1}{n}C_\chi^{\mathrm{G}}(\Phi^{\otimes n}, n\bar{N}), \\ C_\chi^{\mathrm{G}}(\Phi, \bar{N}) &= \max_{\mu^{\mathrm{G}}:\,\hat{\bar{\rho}}^{\mathrm{G}}\in\mathcal{E}_{\bar{N}}^{\mathrm{G}}}\chi(\Phi, \mu^{\mathrm{G}}), \end{aligned} \tag{10}$$

where $C_\chi^{\mathrm{G}}(\Phi, \bar{N})$ is the one-shot Gaussian capacity. The maximum is now taken over all probability measures $\mu^{\mathrm{G}}(\boldsymbol{\alpha}, \boldsymbol{V})$ on Gaussian symbol states $\hat{\rho}^{\mathrm{G}}(\boldsymbol{\alpha}, \boldsymbol{V})$ such that $\hat{\bar{\rho}}^{\mathrm{G}}(\bar{\boldsymbol{\alpha}}_{\mathrm{in}}, \bar{\boldsymbol{V}}_{\mathrm{in}}) = \int \mu^{\mathrm{G}}(d\boldsymbol{\alpha}, d\boldsymbol{V})\hat{\rho}^{\mathrm{G}}(\boldsymbol{\alpha}, \boldsymbol{V})$ is in the set $\mathcal{E}_{\bar{N}}^{\mathrm{G}}$ of Gaussian states with a mean photon number not greater than $\bar{N}$. Unlike previous works (e.g., [2,16]), we require the individual symbol states as well as the average state to be Gaussian. Then we prove that the one-shot Gaussian capacity of an arbitrary single-mode Gaussian channel Φ is given by the well-known expression [37] (see [32] for the proof)

$$\begin{aligned} C_\chi^{\mathrm{G}}(\Phi, \bar{N}) &= \max_{\boldsymbol{V}_{\mathrm{in}}, \boldsymbol{V}_{\mathrm{mod}}}\{\chi^{\mathrm{G}}(\bar{\nu}, \nu)|\mathrm{Tr}[\boldsymbol{V}_{\mathrm{in}} + \boldsymbol{V}_{\mathrm{mod}}] \le 2\bar{N} + 1\}, \\ \chi^{\mathrm{G}}(\bar{\nu}, \nu) &= g\left(\bar{\nu} - \frac{1}{2}\right) - g\left(\nu - \frac{1}{2}\right), \\ g(x) &= (x + 1)\log_2(x + 1) - x\log_2 x, \end{aligned} \tag{11}$$

where $\boldsymbol{V}_{\mathrm{in}}$ is the CM of a pure Gaussian input state $\hat{\rho}^{\mathrm{G}}(0, \boldsymbol{V}_{\mathrm{in}})$ satisfying $\det(2\boldsymbol{V}_{\mathrm{in}}) = 1$. Here $\boldsymbol{V}_{\mathrm{mod}}$ is the CM of a classical Gaussian distribution according to which the input state is displaced in order to generate the modulated input state $\hat{\bar{\rho}}^{\mathrm{G}}(0, \bar{\boldsymbol{V}}_{\mathrm{in}})$ with CM $\bar{\boldsymbol{V}}_{\mathrm{in}} = \boldsymbol{V}_{\mathrm{in}} + \boldsymbol{V}_{\mathrm{mod}}$ satisfying $\mathrm{Tr}[\bar{\boldsymbol{V}}_{\mathrm{in}}] \leq 2\bar{N} + 1$. Furthermore, $\nu = \sqrt{\det \boldsymbol{V}_{\mathrm{out}}}$ and $\bar{\nu} = \sqrt{\det \bar{\boldsymbol{V}}_{\mathrm{out}}}$ are the symplectic eigenvalues of the output and modulated output states with CM $\boldsymbol{V}_{\mathrm{out}} = \Phi(\boldsymbol{V}_{\mathrm{in}})$ and $\bar{\boldsymbol{V}}_{\mathrm{out}} = \Phi(\bar{\boldsymbol{V}}_{\mathrm{in}})$, respectively (see [32]).

The one-shot Gaussian capacity is equal to the Gaussian capacity, i.e., $C^{\mathrm{G}}(\Phi, \bar{N}) = C^{\mathrm{G}}_{\chi}(\Phi, \bar{N})$, provided it is additive. Interestingly, such an additivity can be proven if the input energy exceeds some threshold $\bar{N}_{\mathrm{thr}}$ (see [32]). Note that [16] also derives additivity but for a slightly different definition of C^{G}_{χ} and without respecting the energy constraint. In addition, an analog of Corollary 1 can easily be shown to hold for Gaussian capacities, namely $C^{\mathrm{G}}(\Phi, \bar{N}) = C^{\mathrm{G}}(\Phi^F, \bar{N})$. Therefore, using the fiducial channel Φ^F, we can analytically find the Gaussian capacity of any Gaussian channel in this high-energy regime:

Corollary 2. For a single-mode Gaussian channel Φ with parameters ($\tau \neq 0$, $y > 0$), there exists a fiducial channel Φ^F as defined in Theorem 1, such that

$$\begin{aligned} C^{\mathrm{G}}(\Phi, \bar{N}) &= C^{\mathrm{G}}(\Phi^F_{(\tau,y,s)}, \bar{N}) \\ &= g\left(|\tau|\bar{N} + y\cosh(2s) + \frac{|\tau| - 1}{2}\right) \\ &\quad - g\left(y + \frac{|\tau| - 1}{2}\right), \end{aligned} \tag{12}$$

$$\text{if } \bar{N} \geq \bar{N}_{\mathrm{thr}} = \frac{1}{2}\left(e^{2|s|} + \frac{2y}{|\tau|}\sinh(2|s|) - 1\right).$$

The proof is presented in [32]. Note that the energy threshold $\bar{N}_{\mathrm{thr}}$ depends on the parameter s characterizing the fiducial channel $\Phi^F_{(\tau,y,s)}$. For thermal channels $\Phi^{\mathrm{TH}} = \Phi^F_{(\tau,y,0)}$, the threshold $\bar{N}_{\mathrm{thr}} = 0$, so that additivity holds in the entire energy range. Then, Eq. (12) coincides with previously derived expressions for particular cases [4,11]. In Fig. 1, we illustrate an example of the domain where $\bar{N} \geq \bar{N}_{\mathrm{thr}}$, hence where Eq. (12) holds. Note, that Eq. (12) becomes the actual capacity $C(\Phi, \bar{N})$ (for $\bar{N} \geq \bar{N}_{\mathrm{thr}}$) of an arbitrary single-mode Gaussian channel Φ provided that the vacuum state is proven to minimize the output entropy of a single use of an ideal amplification channel [27,38].

Upper bounds.—Recently, upper bounds have been derived on the capacity of phase-insensitive channels, i.e., Φ^{TH} with $\tau \geq 0$ [24,39]. Using Corollary 2, we can generalize them to any Gaussian channel in the high-energy regime:

Corollary 3. For a single-mode Gaussian channel Φ with parameters ($\tau > 0$, $y > 0$) and $\bar{N} \geq \bar{N}_{\mathrm{thr}}$,

$$\begin{aligned} C^{\mathrm{G}}(\Phi, \bar{N}) \leq C(\Phi, \bar{N}) \leq \bar{C} \leq C^{\mathrm{G}}(\Phi, \bar{N}) + \frac{1}{\ln 2}, \\ \bar{C} = g\left(\frac{2\tau\bar{N} + (2y + 1 - \tau)\sinh^2 s}{2y + 1 + \tau}\right), \end{aligned} \tag{13}$$

where $C^{\mathrm{G}}(\Phi, \bar{N})$ is stated in Eq. (12).

Proof. The fiducial channel corresponding to Φ can be decomposed as $\Phi^F_{(\tau,y,s)} = \Phi^F_{(G,(G-1/2),s)} \circ \Phi^F_{(T,(1-T/2),s)}$ with $T = 2\tau/(2y + \tau + 1)$ (see Fig. 2 and Table I). Then, the capacity of $\Phi^F_{(\tau,y,s)}$ is upper bounded by the capacity of the first channel, i.e.,

$$C(\Phi, \bar{N}) = C(\Phi^F_{(\tau,y,s)}, \bar{N}) \leq C(\Phi^F_{(T,(1-T/2),s)}, \bar{N}) \leq \bar{C},$$

where $\bar{C} = g(T\bar{N} + (1 - T)\sinh^2 s)$ [17]. We define

$$\begin{aligned} \Delta(s) &\equiv \bar{C} - C^{\mathrm{G}} \\ &= g[A(B + 1)^{-1}] - g(A + B\cosh^2 s) + g(B), \end{aligned}$$

where $A = \tau\bar{N} + [y - (\tau - 1)/2]\sinh^2 s$ and $B = y + (\tau - 1/2)$. It was shown in [24] that $\Delta(0) < 1/\ln 2$. Since $\forall$ s: $\Delta(s) \leq \Delta(0)$, the corollary is proven. □

Note that for $\tau < 0$ we can state a similar upper bound on the capacity, $C(\Phi, \bar{N}) \leq \bar{C}$, where $\bar{C}$ is given by Eq. (13) with the replacement $y \to -y$. However, in this case the last inequality in Eq. (13) does not hold. In a similar fashion, we extend in [32] the bounds that were derived in [39].

Conclusions.—We have shown that an arbitrary single-mode Gaussian channel is either equivalent to a newly defined fiducial channel preceded by a phase shift and followed by a Gaussian unitary, or can be obtained in a proper limit of this combination. This equivalence was exploited to reduce the energy-constrained classical capacity of any single-mode Gaussian channel to that of the fiducial channel. We gave an analytical expression for the Gaussian capacity above the energy threshold, where additivity can be proven, and showed that in this case the classical capacity cannot exceed it by more than $1/\ln 2$ bits. We expect that our results will be useful for further studies on the capacities of Gaussian channels, especially for input energies below the energy threshold.

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