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Weak-value representations of the electromagnetic field in phase space

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Abstract

The notion of weak measurement was introduced in 1988 by Aharonov. In this report, we will apply the weak value formalism to field of quantum optics. First, we will need to define the mathematical formulation of the measurement in quantum mechanics using the measurement postulate. We will also study two particular cases, namely the projective measurement and the POVM formalism. Then, we will detail the notion of weak measurement using the two-state vector formalism. The starting point of the weak measurement is to reduce the coupling between the measuring device and the system, so as to decrease the disturbance of the wavefunction. We will see that it naturally leads to the definition of the weak value which can take strange values. Then, we will give some applications of the weak measurement formalism. The transient density matrix which will be studied later will also be defined there. Afterwards, we will study the second quantization in order to introduce the coherent states which will be the main focus of the report. We will need to review the phase space distribution framework as well, including the Wigner quasi-probability distribution, the Q representation, and the P representation. Finally, we will study the transient density matrix using the tools defined earlier. We will see how we can extend the phase space distribution framework to the weak measurement formalism. The weird properties of the transient density will imply certain peculiar properties for the extended phase space distributions.
Keywords: weak-value measurement, quantum phase space, Wigner quasi-probability distribution, Q representation, P representation, coherent states, transient density matrix

Abstract

La notion de mesure faible a été pour la première fois définie par AHARONOV en 1988. Depuis lors, de nombreuses applications ont émergées. Dans ce rapport, nous nous intéresserons à l'étude de la mesure faible dans le cadre de l'optique quantique. D'abord, nous aurons besoin de définir certaines notions mathématiques pour formaliser la notion de mesure dans le cadre du postulat de la mesure. Nous étudierons notamment deux cas particuliers: les mesures projectives et les POVM. Ensuite, nous introduirons en détails la mesure faible à l'aide du "two-state vector formalism". Le point de départ de la mesure faible est de réduire le couplage entre l'appareil de mesure et le système afin de diminuer la perturbation de la fonction d'onde. Nous verrons que cela conduit naturellement à la notion des mesures faibles, qui peuvent prendre des valeurs étranges. Puis, nous donnerons quelques applications possibles de la mesure faible. La "transient density matrix" sera aussi introduite à ce moment, et sera étudier dans les chapitres suivant. Ensuite, nous étudierons la seconde quantification pour introduire les états cohérents qui seront l'objet d'étude central. Nous aurons aussi besoin de voir le formalisme des distributions dans l'espace de phase, notamment la fonction de WIGNER, la fonction Q et la fonction P. Finalement, nous étudierons la "transient density matrix" à l'aide des outils définis précédemment. Nous verrons aussi comment étendre les distributions dans l'espace de phase au formalisme de la mesure faible. Les propriétés étranges de la "transient density matrix" produiront des comportements inattendus des distributions dans l'espace de phase.
Mots-clés : mesure faible, distribution dans l'espace de phase, fonction de Wigner, fonction Q, fonction P, états cohérents

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Nomenclature

c	Speed of light, $c = 299792458\,\mathrm{m}\cdot\mathrm{s}^{-1}$.
ε_0	Vacuum permittivity, $\varepsilon_0 = 8.854187817 \times 10^{-12}\,\mathrm{F}\cdot\mathrm{m}^{-1}$.
h	Plank's constant, $h = 6.62606957(29) \times 10^{-34}\,\mathrm{J}\cdot\mathrm{s}$.
μ_0	Vacuum permeability, $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H}\cdot\mathrm{m}^{-1}$.
$\|0\rangle$	Vacuum state, see Eq. (3.2.4).
$(\boldsymbol{r}, t)$	Space-time point with coordinates $(\boldsymbol{r}, t)$.
$\left\langle \hat{A} \right\rangle$	Expected value of operator $\hat{A}$, see Eqs. (1.2.6) and (1.2.10).
$\left\langle \hat{A} \right\rangle_w$	Weak value of the observable $\hat{A}$, see Eq. (2.4.1).
c.c.	Complex conjugate.
$\mathrm{d}^2\alpha$	$\mathrm{d}^2\alpha \equiv \mathrm{d}\Re(\alpha)\,\mathrm{d}\Im(\alpha)$ if α is a complex number.
$\Im(\alpha)$	Imaginary part of α.
$\Re(\alpha)$	Real part of α.
$\|\alpha\rangle$, $\|\beta\rangle$, ...	Coherent states, see Eq. (3.3.1).
$\boldsymbol{\varepsilon}_{\boldsymbol{k}s}$	Polarization vector, see Eq. (3.1.17).
$\hat{\rho}$	Density matrix (or density operator), see Section 1.1.
ω_k	Angular frequency of mode k, see Eq. (3.1.18).
$\boldsymbol{A}(\boldsymbol{r}, t)$	Potential vector, see Eqs. (3.1.5) and (3.1.6).
$\boldsymbol{A}_{\boldsymbol{k}s}(t)$	Fourier coefficients of the potential vector, see Eq. (3.1.12).
$\hat{a}_{\boldsymbol{k}s}(t)$, $\hat{a}$	Creation operator, see Eq. .(3.1.41).
$\hat{a}^{\dagger}_{\boldsymbol{k}s}(t)$, $\hat{a}^{\dagger}$	Annihilation operator, see Eq. .(3.1.42).
$\boldsymbol{B}(\boldsymbol{r}, t)$	Magnetic field.

$\hat{D}(\alpha)$	Displacement operator, see Eq. (3.3.18).
$\boldsymbol{E}(\boldsymbol{r},t)$	Electric field.
$\hat{H}$	Hamiltonian operator.
H	Hamiltonian.
$\hat{1}$	Identity operator.
$\boldsymbol{k}$	Wave-number (or wave vector), see Eq. (3.1.12).
$\hat{M}_m$	Measurement operator associated with outcome m, see Section 1.2.
$\hat{n}_{\boldsymbol{k}s}(t)$, $\hat{n}$	Number operator of mode $(\boldsymbol{k},s)$, see Eq. (3.1.48).
$\hat{p}_{\boldsymbol{k}s}(t)$, $\hat{p}$	Field quadrature operator associated with the imaginary part of the complex amplitude represented by $\hat{a}$, see Eq. (3.3.4).
$\hat{P}_m$	Projector associated with the outcome m, see Section 1.3.
$P(\gamma)$	P representation, see Eq. (4.6.1).
$P_\alpha(\gamma)$	P representation of a coherent state, see Eq. (4.6.6).
$\hat{q}_{\boldsymbol{k}s}(t)$, $\hat{q}$	Field quadrature operator associated with the real part of the complex amplitude represented by $\hat{a}$, see Eq. (3.3.3).
$Q_{\lvert\alpha\rangle\langle\beta\rvert}(\gamma)$	Extend Q representation, see Eq. (5.2.2).
$Q(\gamma)$	Q representation, see Eq. (4.5.1).
$Q_\alpha(\gamma)$	Q representation of a coherent state, see Eq. (4.5.13).
$\hat{\sigma}_{\lvert\alpha\rangle\langle\beta\rvert}$	Transient density matrix, see Eq. (2.7.2).
$\hat{U}(\theta)$	Phase-shifting operator, see Eq. (3.3.25).
$W(q,p)$	Wigner quasi-probability distribution, see Eq. (4.4.3).
$W_\alpha(q,p)$	Wigner quasi-probability distribution of the coherent state $\lvert\alpha\rangle$, see Eq. (4.4.5).
$W_{\lvert\alpha\rangle\langle\beta\rvert}(q,p)$	Extended Wigner quasi-probability distribution, see Eq. (5.1.2).

Introduction

The field of quantum mechanics has been around for more than a century, yet there are still many things to discover. The strange concept of quantum teleportation first introduced in 1993 is one example. Furthermore, while we have made many advances, there are still fundamental problems that have not been solved. For instance, the measurement problem about how the wavefunction collapse occurs in the Copenhagen interpretation (if it does) is still an unresolved problem. Some avoid this problem by saying that we should only care if the predictions are correct or not. However, others have tried to do away with this collapse by re-interpreting quantum mechanics (such as Hugh Everett's many-worlds interpretation). Additionally, attempts have been made to derive Born's rule (which links the wavefunction to the outcome of an experiment) instead of postulating, but the results have been inconclusive.

Often, there comes a time when new and original mathematical tools are needed to go beyond the current state of knowledge. One such possible example is the weak measurement formalism introduced by Yakir Aharonov, David Z. Albert, and Lev Vaidman in 1988 [1]. The idea is simple: instead of considering the usual strong measurement which produces a collapse of the wavefunction, they defined the notion of weak measurement. This measurement is such that the coupling between the system and the measuring device is reduced, in order to decrease the disturbance of the wavefunction caused by the measurement. At first, many were skeptical that defining such a concept was rigorous or even useful. However, thirty years later, it is undeniable that this approach has been advantageous in numerous problems. It is a very promising tool in the field of quantum information theory where the measurement process and its impact on the system occupy a central role. As we will see, it is even possible to directly measure the wavefunction using weak values.

In this report, we will explore the weak measurement formalism applied to the field of quantum optics. More specifically, the pre- and post-selection scheme we will consider will always involve the coherent sates of light. There are several reasons to this choice. First, the coherent states of light hold a special place in quantum optics because of their similarity with classical states of light, and they possess interesting properties which might help us in our analysis. In addition, they are also important in the field of quantum information theory and computation when we use photons as qubits. Also, the phase space distribution formalism, while not specific to quantum optics, is a very powerful and advanced tool well adapted to this field. Finally, the weak measurement formalism has not been applied a great deal in the case of continuous variables. Our analysis will allow us to obtain a better insight of the weak measurement formalism as well as the phase space distributions.

Without much surprise, Chapter (1) will be dedicated to the theory of measurement in quantum mechanics. In it, we will give the mathematical framework needed to understand the measurement procedure. In Chapter (2), we will explain the weak measurement formalism in detail. The mathematical tools will be first introduced, then the implications and the applications will be explored.

Finally, the last section will introduce the notion of the *transient density matrix*, by analogy with the expected value, which we will try to characterize in the rest of the report. Chapter (3) will contain the reasoning behind the second quantization, and this will naturally lead us to the notion of Fock states and coherent states. Then, in Chapter (4), the phase space distributions will be defined, and we will prove how they can be used as effective tools in quantum mechanics. Finally, in Chapter (5), we will try to use all the notions developed earlier to study the transient density matrix of an experiment pre- and post-selected on coherent states.

Chapter 1

Measurements in quantum mechanics

In this chapter, we will introduce notions that are usually encountered when reading the literature about weak measurement, but are not specific to this formalism. We start by giving the definition of the density matrix and its properties because it is an essential tool in quantum mechanics. The subsequent sections deal with the measurement postulate (which tells us how the probability of an outcome can be link to the wavefunction or the density matrix), as well as the two common ways to implement this postulate using projection-valued measurement or the POVM formalism. Finally, the last section discusses the broader interpretation of the measurement postulate, in addition to its shortcomings. The main references are:

- General: [2, 3];
- Measurement: [4, Ch. 2], and [5];

1.1 Density matrix

The density matrix (or density operator) in quantum mechanics is an operator (usually written $\hat{\rho}$) which can be used to describe a system. It is Hermitian

$$\hat{\rho}^\dagger = \hat{\rho}, \tag{1.1.1}$$

so its eigenvalues are real. The interpretation of those eigenvalues [1] is that they represent the probabilities of having the corresponding eigenstates. Since the eigenvalues represent probabilities, they must add up to one so that the following property is always verified[2]:

$$\mathrm{Tr}\,\hat{\rho} = 1, \tag{1.1.2}$$

where the trace operator of an operator $\hat{A}$ is defined as

$$\mathrm{Tr}\left(\hat{A}\right) \equiv \sum_i \left\langle i \middle| \hat{A} \middle| i \right\rangle, \tag{1.1.3}$$

where $\{|i\rangle\}$ can be any orthonormal basis. Incidentally, in the case of a pure state, we also have

$$\mathrm{Tr}\,\hat{\rho}^2 = 1. \tag{1.1.4}$$

[1] The eigenvalues are the diagonal elements when the density matrix has been diagonalized, i.e. when it has been written in the basis formed by its eigenstates.

[2] The trace is simply the sum of the eigenvalues.

One important aspect is that the density matrix contains all the information that we can get from a state. Let us illustrate this concept with an example: consider a pure state $|\psi\rangle$. From quantum mechanics, we know that this state $|\psi\rangle$ is defined up to a phase factor of the form $e^{i\theta}$ ($\theta \in \mathbf{R}$) such that the system can equivalently be described by $e^{i\theta} |\psi\rangle$. However, this global phase θ cannot be determined experimentally, so it is a superfluous addition to the mathematical formulation. Essentially, it means that the description of the system using $|\psi\rangle$ contains more information than what we have access to in the laboratory. On the other hand, the corresponding density matrix is simply

$$\hat{\rho} = e^{i\theta} |\psi\rangle \langle\psi| e^{-i\theta} = |\psi\rangle \langle\psi| . \tag{1.1.5}$$

We see that the phase ambiguity disappears when the density matrix is used instead.

Furthermore, the density operator is a necessary tool to represent mixed states which are written as

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle \langle\psi_i| , \tag{1.1.6}$$

where $0 \leq p_i \leq 1$ such that $\sum_i p_i \langle\psi_i|\psi_i\rangle = 1$. If the $|\psi_i\rangle$ are all normalized, we can interpret the p_i as the probabilities of having prepared the corresponding states $|\psi_i\rangle \langle\psi_i|$. We should insist on the fact that p_i are probabilities in the classical sense. This has nothing to do with quantum mechanics, they simply model the fact that we do not have perfect control over the preparation process. As a matter of fact, the density operator formalism is indispensable when we want to describe reality because there is always an interaction between the system we want to study and the surrounding environment. This is why it constitutes a fundamental tool in quantum information theory because we have to take this (classical) uncertainty into account when studying quantum computers.

The (classical) statistical mixture of pure states should not be confused with the concept of superposition in quantum mechanics defined as

$$|\psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle . \tag{1.1.7}$$

Here, $|c_1|^2$ (resp. $|c_2|^2$) represents the probability of *measuring* the state $|\psi_1\rangle$ (resp. the state $|\psi_2\rangle$) if the system is in the state $|\psi\rangle$, and they add up to unity $|c_1|^2 + |c_2|^2 = 1$. This is a purely quantum mechanical effect, which has no classical analog. To further show the difference between superposition and statistical mixture, we can calculate the density operator $\hat{\rho} = |\psi\rangle \langle\psi|$

$$\begin{aligned} \hat{\rho} &= |\psi\rangle \langle\psi| \\ &= (c_1 |\psi_1\rangle + c_2 |\psi_2\rangle) (c_1^* \langle\psi_1| + c_2^* \langle\psi_2|) \\ &= |c_1|^2 |\psi_1\rangle \langle\psi_1| + |c_2|^2 |\psi_2\rangle \langle\psi_2| + c_1 c_2^* |\psi_1\rangle \langle\psi_2| + c_1^* c_2 |\psi_2\rangle \langle\psi_1| , \end{aligned} \tag{1.1.8}$$

which is obviously different from a statistical mixture

$$\hat{\rho}_{\text{mix}} = |c_1|^2 |\psi_1\rangle \langle\psi_1| + |c_2|^2 |\psi_2\rangle \langle\psi_2| . \tag{1.1.9}$$

1.2 The measurement postulate

According to [4, p. 84], the measurement postulate can be stated using a very general type of measurement. In that case, a quantum measurement of an arbitrary observable $\hat{A}$[3] can be described by a set of *measurement operators* $\left\{\hat{M}_m\right\}$ acting on the state space of the system being measured.

[3] An observable is any Hermitian operator associated with a physical measurement. All the observables have real eigenvalues.

Each index m corresponds to a specific possible outcome of the experiment. Those operators satisfy the completeness relation

$$\sum_m \hat{M}_m^\dagger \hat{M}_m = \hat{1}, \tag{1.2.1}$$

where $\hat{1}$ is the identity operator. The relation between the observable $\hat{A}$ and its measurement operators $\left\{\hat{M}_m\right\}$ is

$$\hat{A} = \sum_m \lambda_m \hat{M}_m^\dagger \hat{M}_m \tag{1.2.2}$$

Note that all possible measurement outcomes λ_m of a physical quantity described by an observable $\hat{A}$ are the eigenvalues of $\hat{A}$.

If the quantum system is in state $|\psi\rangle$ before the measurement, BORN's rule tells us that the probability $\mathbb{P}(m)$ of obtaining the result a_m is

$$\mathbb{P}(m) \equiv \left\langle \psi \left| \hat{M}_m^\dagger \hat{M}_m \right| \psi \right\rangle, \tag{1.2.3}$$

and the state $|\psi_m\rangle$ after the measurement is given by

$$|\psi_m\rangle \equiv \frac{\hat{M}_m |\psi\rangle}{\sqrt{\left\langle \psi \left| \hat{M}_m^\dagger \hat{M}_m \right| \psi \right\rangle}}. \tag{1.2.4}$$

The corresponding expected value $\left\langle \hat{A} \right\rangle$ is

$$\left\langle \hat{A} \right\rangle \equiv \sum_m \lambda_m \mathbb{P}(m) = \sum_m m \left\langle \psi \left| \hat{M}_m^\dagger \hat{M}_m \right| \psi \right\rangle = \left\langle \psi \left| \sum_m \lambda_m \hat{M}_m^\dagger \hat{M}_m \right| \psi \right\rangle, \tag{1.2.5}$$

or

$$\left\langle \hat{A} \right\rangle = \left\langle \psi \left| \hat{A} \right| \psi \right\rangle. \tag{1.2.6}$$

BORN's rule can also be stated in terms of density matrix $\hat{\rho} = |\psi\rangle \langle\psi|$. The probabilities are given by

$$\boxed{\mathbb{P}(m) \equiv \mathrm{Tr}\left(\hat{\rho}\hat{M}_m^\dagger \hat{M}_m\right) = \mathrm{Tr}\left(\hat{M}_m \hat{\rho} \hat{M}_m^\dagger\right) = \mathrm{Tr}\left(\hat{M}_m^\dagger \hat{M}_m \hat{\rho}\right),} \tag{1.2.7}$$

because the trace is invariant under cyclic permutations. The density matrix $\hat{\rho}_m = |\psi_m\rangle \langle\psi_m|$ after the measurement is

$$\boxed{\hat{\rho}_m \equiv \frac{\hat{M}_m \hat{\rho} \hat{M}_m^\dagger}{\mathrm{Tr}\left(\hat{M}_m \hat{\rho} \hat{M}_m^\dagger\right)}.} \tag{1.2.8}$$

Finally, the expected value $\left\langle \hat{A} \right\rangle$ is

$$\left\langle \hat{A} \right\rangle \equiv \sum_m \lambda_m \mathbb{P}(m) = \sum_m m \mathrm{Tr}\left(\hat{\rho}\hat{M}_m^\dagger \hat{M}_m\right) = \mathrm{Tr}\left(\hat{\rho} \sum_m \lambda_m \hat{M}_m^\dagger \hat{M}_m\right), \tag{1.2.9}$$

or

$$\boxed{\left\langle \hat{A} \right\rangle = \mathrm{Tr}\left(\hat{\rho}\hat{A}\right),} \tag{1.2.10}$$

where we used Eq. (1.2.2).

1.3 PVM formalism

In this section, we will see a particular case of the measurement operators defined in the measurement postulate called the projective measurement. It is a necessary detour since this is the type of measurement most people are familiar with in quantum mechanics. We will first review the associated mathematical formalism, then we will expose the shortcoming of this approach and why it is necessary to work with another type of measurement called the POVM formalism which provides a better framework for measurements in quantum mechanics.

The projective measurement (also called von Neumann measurement, projection-valued measurement, or simply PVM) is the type of measurement that is usually assumed in most introductory book on quantum mechanics because of its simplicity. The starting point is the spectral decomposition theorem of an operator which states that any arbitrary operator $\hat{A}$ can be decomposed as

$$\hat{A} = \sum_m \lambda_m \hat{P}_m, \tag{1.3.1}$$

where λ_m is the value of a measurement associated with the projector $\hat{P}_m$. Note that the values λ_m are simply the eigenvalues of the operator $\hat{A}$ associated with the projector $\hat{P}_m$ as stated in the measurement postulate. Additionally, each projector is orthogonal to all the others such that

$$\hat{P}_i \hat{P}_j = \delta_{ij} \hat{P}_i. \tag{1.3.2}$$

Now, in order to be called projectors, the set of operators $\left\{\hat{P}_m\right\}$ must obey two rules:

1. They must be Hermitian
$$\hat{P}_m^\dagger = \hat{P}_m, \tag{1.3.3}$$
2. They must also be idempotent (i.e. application of the operator more than once does not change the result):
$$\hat{P}_m \hat{P}_m = \hat{P}_m^2 = \hat{P}_m. \tag{1.3.4}$$

By comparing Eq. (1.3.1) with Eq. (1.2.2), we see that the projective measurement is a particular case where

$$\boxed{\hat{M}_m = \hat{P}_m.} \tag{1.3.5}$$

Indeed, in that case, we have

$$\hat{M}_m^\dagger \hat{M}_m = \hat{P}_m^\dagger \hat{P}_m = \hat{P}_m, \tag{1.3.6}$$

since the projectors are Hermitian (Eq. (1.3.3)). Note that we have implicitly considered that the operator $\hat{A}$ can be decomposed in terms of a discrete number of projectors $\left\{\hat{P}_m\right\}$. The generalization to the continuous case is, however, fairly straightforward because we only need to replace the sum by an integral over all possible values of $\lambda(x)$

$$\hat{A} = \int \lambda(x) \hat{P}(x) \,\mathrm{d}x, \tag{1.3.7}$$

where we used the variable $\lambda(x)$ here to distinguish it from the discrete case. Incidentally, the orthogonality condition becomes

$$\hat{P}(x) \hat{P}(y) = \delta(x-y) \hat{P}(x). \tag{1.3.8}$$

An important thing to notice is that the idempotence (Eq. (1.3.4)) of the projector implies the *repeatability* of a measurement, i.e. the measurement on a state that has just been measured will give the same result.

At first sight, the repeatability of the projective measurement seems to be a main advantage that is not necessarily found in the more general type of measurement. However, this is also where its weakness lies: not all measurement can be repeated. For instance, consider the (destructive) detection of a photon using a photo-detector. In this case, once the photon has been measured, it has been annihilated; obviously this prevents from detecting the photon a second time. This is one of the reasons why we need to define a more general framework. Another drawback presented by [4, p. 87] is that projective measurement can only effectively distinguish two states if they are orthogonal to each other. On the other hand, the POVM formalism that will be presented in the next section can be optimized to discriminate non-orthogonal states, if we allow that, some of the time, we cannot tell them apart. The important thing is that we are able to tell which measurements are inconclusive.

1.4 POVM formalism

The POVM (positive-operator valued measure) formalism is the most general type of measurement. First, we define the positive operators $\hat{E}_m$ as

$$\hat{E}_m \equiv \hat{M}_m^\dagger \hat{M}_m. \tag{1.4.1}$$

Each operator $\hat{E}_m$ is known as a POVM elements, and the complete set of POVM elements $\left\{\hat{E}_m\right\}$ forms what we call the POVM. Obviously, they respect the completeness relation

$$\sum_m \hat{E}_m = \sum_m \hat{M}_m^\dagger \hat{M}_m = \hat{1}, \tag{1.4.2}$$

and the probabilities of getting outcome λ_m of an arbitrary operator $\hat{A}$ associated with the measurement operators $\hat{M}_m$ (see Eq. (1.2.2)) if the state is in $\hat{\rho}$ before the measurement is simply

$$\mathbb{P}\left(\lambda_m\right) = \mathrm{Tr}\left(\hat{\rho}\hat{E}_m\right). \tag{1.4.3}$$

It is possible to see the projective measurement defined by the mutually orthogonal projectors $\left\{\hat{M}_m\right\} = \left\{\hat{P}_m\right\}$ as a particular case of the POVM formalism where all the POVM elements are equal to the measurement operators

$$\hat{E}_m = \hat{P}_m^\dagger \hat{P}_m = \hat{P}_m. \tag{1.4.4}$$

In reality, the POVM formalism can be seen as a projective measurement performed an extended (Neumark's dilation theorem), so the two frameworks turn out to be equivalent. However, the POVM formalism offers some advantages over the projective measurements such as [4, p. 91]

- It possesses a simpler structure because the measurement operators do not necessarily have to be mutually orthogonal projectors;
- It can be used to optimize the distinction of orthogonal (or not) states as will be shown below;
- It does not imply repeatability of the measurement.

Illustration

To better understand the advantage of the POVM formalism, let us consider that we want to distinguish two states given by (see [4, p. 92])

$$|\psi_1\rangle = |\phi_0\rangle, \tag{1.4.5}$$

$$|\psi_2\rangle = \frac{|\phi_0\rangle + |\phi_1\rangle}{\sqrt{2}}, \tag{1.4.6}$$

where $|\phi_0\rangle$ and $|\phi_1\rangle^a$ are two (normalized) states chosen such that

$$\langle\phi_1|\phi_0\rangle = 0. \tag{1.4.7}$$

Obviously, $|\psi_1\rangle$ and $|\psi_2\rangle$ are not orthogonal:

$$\begin{aligned}\langle\psi_2|\psi_1\rangle &= \frac{1}{\sqrt{2}}\left(\langle\phi_0| + \langle\phi_1|\right)|\phi_0\rangle \\ &= \frac{1}{\sqrt{2}}\left(\langle\phi_0|\phi_0\rangle + \langle\phi_1|\phi_0\rangle\right),\end{aligned} \tag{1.4.8}$$

so

$$\langle\psi_2|\psi_1\rangle = \frac{1}{\sqrt{2}} \neq 0. \tag{1.4.9}$$

As discussed in the previous section, the non-orthogonality implies that it is not possible to distinguish them using projective measurement. Now consider the POVM defined as

$$\hat{E}_1 \equiv \frac{\sqrt{2}}{1+\sqrt{2}}|\phi_1\rangle\langle\phi_1|, \tag{1.4.10}$$

$$\hat{E}_2 \equiv \frac{\sqrt{2}}{1+\sqrt{2}}\frac{\left(|\phi_0\rangle - |\phi_1\rangle\right)\left(\langle\phi_0| - \langle\phi_1|\right)}{2}, \tag{1.4.11}$$

$$\hat{E}_3 \equiv \hat{1} - \hat{E}_1 - \hat{E}_2. \tag{1.4.12}$$

First of all, is it trivial to verify that $\sum_m \hat{E}_m = \hat{1}$, so the positive operators $\hat{E}_m$ do form a POVM. Now, we see that

$$\hat{E}_1|\psi_1\rangle = 0, \qquad \hat{E}_1|\psi_2\rangle \neq 0, \tag{1.4.13}$$

$$\hat{E}_2|\psi_1\rangle \neq 0, \qquad \hat{E}_2|\psi_2\rangle = 0, \tag{1.4.14}$$

$$\hat{E}_3|\psi_1\rangle \neq 0, \qquad \hat{E}_3|\psi_2\rangle \neq 0. \tag{1.4.15}$$

Then

$$\mathbb{P}\left(\lambda_1, |\psi_1\rangle\right) = 0, \qquad \mathbb{P}\left(\lambda_1, |\psi_2\rangle\right) \neq 0, \tag{1.4.16}$$

$$\mathbb{P}\left(\lambda_2, |\psi_1\rangle\right) \neq 0, \qquad \mathbb{P}\left(\lambda_2, |\psi_2\rangle\right) = 0, \tag{1.4.17}$$

$$\mathbb{P}\left(\lambda_3, |\psi_1\rangle\right) \neq 0, \qquad \mathbb{P}\left(\lambda_3, |\psi_2\rangle\right) \neq 0, \tag{1.4.18}$$

where $\mathbb{P}\left(\lambda_m, |\psi_i\rangle\right)$ (given by Eq. (1.2.3)) is the probability to obtain the outcome λ_m associated with $\hat{E}_m$ when the system is in the state $|\psi_i\rangle$ before the measurement. From this, we deduce that if we get the outcome λ_1 (resp. λ_2), then the state was necessarily in $|\psi_2\rangle$ (resp. $|\psi_1\rangle$).

On the other hand, if λ_3 is the outcome, then nothing can be said. Therefore, from this simple reasoning, we see that it is possible to distinguish two non-orthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$ if we choose the POVM cleverly. Of course, some of the time, we will get λ_3, and the measurement will be inconclusive, but the point is that we know when the measurement must be discarded, and when we can use it to determine the discriminate the two states.

[a] Note that we have used a different notation from [4] to avoid confusion with the notation later on.

Nevertheless, there is one drawback to the POVM formalism. If we write the expression of the state after the measurement $\hat{\rho}_m$ (see Eq. (1.2.8)):

$$\boxed{\hat{\rho}_m \equiv \frac{\hat{M}_m \hat{\rho} \hat{M}_m^\dagger}{\mathrm{Tr}\left(\hat{M}_m \hat{\rho} \hat{M}_m^\dagger\right)},} \tag{1.4.19}$$

we see that $\hat{\rho}_m$ depends on the POVM element $\hat{M}_m$ which cannot be inferred from the POVM alone in general.

1.5 The measurement problem

Before moving on, it seems appropriate to briefly discuss about the measurement postulate and its consequence on the interpretation of quantum mechanics. Indeed, the process described by Eq. (1.2.4) (or Eq. (1.2.8)) known as the *collapse* of the wavefunction (or reduction of the wavepacket) has always been as source of much controversy in quantum mechanics. The main problem is the irreversibility of the collapse. Some have tried to remove this phenomenon, and attempted to derive Born's rule without assuming it, but the results have been inconclusive. The measurement problem is intrinsically linked to the role of the observer in quantum mechanics. Different interpretation exists besides the Copenhagen interpretation such as Everett's many-worlds interpretation or the de Broglie–Bohm theory, but there hasn't been conclusive experiment favoring one or the other as of right now. The weak measurement formalism described in the next chapter could perhaps change this in the future. The measurement problem is treated in great detail in [6].

1.6 Summary

This chapter helped us define all the mathematical notions needed to describe a state, as well as the outcome of a measurement or the state after the measurement for example. We started by stating the measurement postulate which tells us how we use the state $|\psi\rangle$ to calculate the probability density functions through the use of Born's rule. We also looked into the two most common types of mathematical implementation of the measurement operators: the projective measurement (which is the one that is usually presented in introductory books about quantum mechanics), and the POVM formalism. While the two can be shown to be equivalent, we saw that the POVM formalism possesses some advantages over the projective measurement because of the lower number of restrictions to define the POVM. Finally, we briefly evoked the measurement problem in quantum mechanics and its implications. In the next chapter, we will explore the concept of weak measurement, and how it can be used as a very powerful tool in quantum mechanics.

Chapter 2

Weak measurement

The idea of weak measurement was first introduced by Yakir Aharonov, David Z. Albert, and Lev Vaidman in their seminal paper entitled "How the Result of a Measurement of a Component of the Spin of a Spin–1/2 Particle Can Turn Out to be 100"[1]. In this chapter, we shall explain the train of thought which lead them to define the notion of weak value and weak measurement as presented in [7, 8] by Yakir Aharonov and Lev Vaidman which is very similar to the original approach, but slightly more complete. We will then focus on the seemingly paradoxical aspects of the weak values. A few examples of applications will be given to illustrate the importance of weak values in quantum mechanics. Finally, we will define the transient density matrix, and try to generalize the notion of expected value in the case of the weak measurement. The main references are:

- Weak measurement formalism: [9, 1, 7, 10];
- Weak measurement and experiments: [11, 12];
- Direct measurement of a wavefunction: [13];
- Transient density matrix: [14].

2.1 Two-state vector formalism

As discussed in the previous section, the "collapse" of the wavefunction after the measurement seems to be an irreversible process, and fundamentally time asymmetric. This is somewhat disturbing considering the fact the dynamical laws of quantum mechanics are time symmetric just like in classical mechanics (Hamilton's equation of motion). They believe that the time symmetry can be restored if we introduce a state evolving backwards in time. Normally, we consider that the state is represented only by a wavefunction evolving forward in time. After the measurement, the wavefunction is suddenly modified according to the measurement postulate, and then continues to move forward in time. However, this results from our conception of the arrow of time: as the authors put it, "we view the past as existing and future as nonexisting (yet)."

Their idea is to consider the description of a system between two (complete) measurements, that way, we can set boundary conditions in the past as well as in the future. Here, we have a wavefunction evolving forward in time coming from the past just as we did in the normal case. However, because of the time symmetry, we expect there also exists a wavefunction evolving backwards in time, coming from the future.

To illustrate this, let us consider an experiment where we perform the measurement of the observable $\hat{A}$ and $\hat{B}$ on a quantum system. We suppose that the outcome of the measurement at time t_1 (resp. t_2) of $\hat{A}$ (resp. $\hat{B}$) is the (non-degenerate) eigenvalue a (resp. b). Then, at an intermediate time, the system is characterized by a wavefunction evolving from the past $|\psi_1\rangle$, and another one coming from the future $\langle\psi_2|$:

$$|\psi_1\rangle = \exp\left\{-\mathrm{i}\int_{t_1}^{t} \hat{H}\,\mathrm{d}\tau\right\}|A=a\rangle\,, \tag{2.1.1}$$

$$\langle\psi_2| = \langle B=b|\exp\left\{-\mathrm{i}\int_{t}^{t_2} \hat{H}\,\mathrm{d}\tau\right\}, \tag{2.1.2}$$

with $t_1 < t < t_2$ (see Fig. 2.1.1 for an illustration).

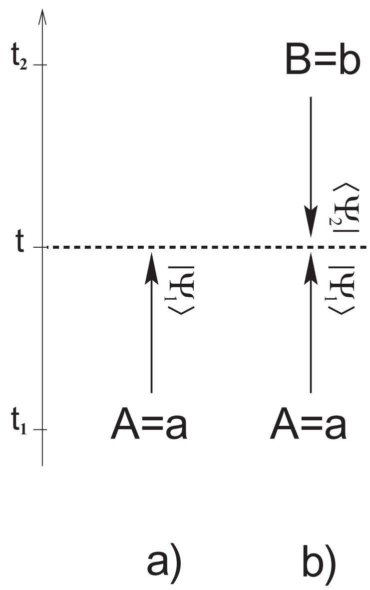


Figure 2.1.1: Description of quantum systems: (a) pre-selected, (b) post-selected. Source: [8, p. 4].

2.2 Disturbance between two non-commuting observables

It is a well-known fact that if we measure the outcomes of two non-commuting observables $\hat{A}$ and $\hat{B}$, the order in which we perform the measurement is going to affect the result. This is due to the fact that the measurement on $\hat{A}$ will disturb the wavefunction, so that the system is not described by the two wavefunctions $|\psi_1\rangle$ and $\langle\psi_2|$, and subsequently influence the outcome on $\hat{B}$, and *vice versa*.

Moreover, let $|\psi_1\rangle = |A=a\rangle$ be the initial state is and $\langle\psi_2| = \langle B=b|$ be the final state. If the free Hamiltonian is neglected, we might be inclined to say that the value of the observable $\hat{C} = \hat{A}+\hat{B}$ at an intermediate time will be the eigenvalue $c = a+b$. In reality, this is not the case because the measurement of $\hat{A}$ and $\hat{B}$ will still disturb each other. The idea of the *weak measurement* is to try to reduce the interaction between the system and the measuring device. In that case, we can consider that both $|\psi_1\rangle = |A=a\rangle$ and $\langle\psi_2| = \langle B=b|$ are not disturbed significantly. Therefore, we should measure both $A=a$ and $B=b$ at the intermediate time, such that $c=a+b$.

Note that the reduction of the coupling between the system and the measuring device has a negative consequence: it will increase the uncertainty on a single measurement. To remedy to this problem, it is necessary to perform this measurement on a large number of systems prepared in the same way, and then to compute the average value. That way, if we have N systems prepared in the same state, then the uncertainty on the average value will scale as $1/\sqrt{N}$.

2.3 Measurement process

So far, we have only described the measurement outcome and the state after the measurement; we have not discussed how the measurement procedure can be modeled using a Hamiltonian. According to von Neumann [5], we can view the measurement process as the result of the coupling between the system $|\psi\rangle$ and the pointer of a measurement device $|\psi_m\rangle$. Let us suppose we want to perform a measurement corresponding to an observable $\hat{A}$. Then, we can model this interaction by the following Hamiltonian [5]:

$$\hat{H} = g\left(t\right)\hat{p}\hat{A}, \tag{2.3.1}$$

where $g\left(t\right)$ is a normalized function with a compact support corresponding to the coupling between the system and the pointer, and $\hat{p}$ is the canonically conjugate variable to the pointer variable $\hat{q}$[1]. Typically, in an *ideal measurement*, the coupling described by $\hat{H}$ will only last a small amount of time (meaning the function $g\left(t\right)$ is non zero during a short period of time), allowing us to neglect the free Hamiltonian during the process of measurement.

2.4 Strange weak values

Now that we have defined all the mathematical tools to study the weak values, let us give its definition. If a system is described by the two-state vector $|\psi_1\rangle$ and $\langle\psi_2|$ (pre- and post-selection), then the *weak value* of an arbitrary observable $\hat{A}$ is

$$\boxed{\left\langle\hat{A}\right\rangle_w \equiv \frac{\left\langle\psi_2\left|\hat{A}\right|\psi_1\right\rangle}{\langle\psi_2|\psi_1\rangle}.} \tag{2.4.1}$$

Note that this weak value is generally a complex number because

$$\left(\frac{\left\langle\psi_2\left|\hat{A}\right|\psi_1\right\rangle}{\langle\psi_2|\psi_1\rangle}\right)^\dagger = \frac{\left\langle\psi_1\left|\hat{A}\right|\psi_2\right\rangle}{\langle\psi_1|\psi_2\rangle} \neq \frac{\left\langle\psi_2\left|\hat{A}\right|\psi_1\right\rangle}{\langle\psi_2|\psi_1\rangle}, \tag{2.4.2}$$

in general. When the state is only pre-selected, it reduces to

$$\left\langle\hat{A}\right\rangle_w = \frac{\left\langle\psi_1\left|\hat{A}\right|\psi_1\right\rangle}{\langle\psi_1|\psi_1\rangle} = \left\langle\psi_1\left|\hat{A}\right|\psi_1\right\rangle = \left\langle\hat{A}\right\rangle. \tag{2.4.3}$$

[1]The pointer variables $\hat{q}$ and $\hat{p}$ do not necessarily correspond to the position and the momentum, respectively; they can represent any observables of the pointer.

Illustration

It is interesting to consider a simple example where the $\left\langle \hat{A} \right\rangle_w$ can take seemingly weird values. Let us study the measurement of a spin–1/2 particle as proposed in [8, p. 14]. The experimental scheme is the following:

1. Pre-selection on the spin "up" on the x-axis, written $|\uparrow_x\rangle$ (eigenstate of $\hat{\sigma}_x$);
2. Weak measurement of the spin component in the ξ direction which is the bisector (45° angle) of x-axis and the y-axis. The corresponding operator can be written
$$\hat{\sigma}_\xi = \frac{\hat{\sigma}_x + \hat{\sigma}_y}{\sqrt{2}}. \tag{2.4.4}$$
3. Post-selection on the spin "up" on the y-axis, written $\langle\uparrow_y|$ (eigenstate of $\hat{\sigma}_y$) .

The corresponding weak value is

$$\langle \hat{\sigma}_\xi \rangle_w = \frac{\langle \uparrow_y | \hat{\sigma}_\xi | \uparrow_x \rangle}{\langle \uparrow_y | \uparrow_x \rangle} = \frac{1}{\sqrt{2}} \frac{\langle \uparrow_y | (\hat{\sigma}_x + \hat{\sigma}_y) | \uparrow_x \rangle}{\langle \uparrow_y | \uparrow_x \rangle} = \sqrt{2} > 1. \tag{2.4.5}$$

This might seem paradoxical at first because the usual expected value of a spin component along any arbitrary direction ξ is always bounded by

$$-1 \leq \langle \hat{\sigma}_\xi \rangle \leq 1, \tag{2.4.6}$$

since the eigenvalues are ± 1. However, we have to remember that the definition in Eq. (2.4.1) is clearly not a usual expected value. Therefore, it does not have to obey the usual rules governing expected values. This important value $\langle \hat{\sigma}_\xi \rangle_w$ is the result of the pre- and post-selection: the value of the spin component in-between has to be in accordance with both the measurement in the past and in the future.

Illustration

Another elegant example is given by [15]. We start by writing the Hamiltonian of a harmonic oscillator (mass $m = 1$)

$$\hat{H} = \frac{1}{2}\left[\hat{p}^2 + \omega^2 \hat{q}^2\right] = \frac{\hat{1}}{2} + \hat{n}, \tag{2.4.7}$$

where $\hat{n}$ is the number operator giving to the number of quanta in the harmonic oscillator, ω is the angular frequency and where $\hat{q}$ and $\hat{p}$ could be the position and the momentum operators, respectively. We can then write the number operator as

$$\hat{n} = \frac{1}{2}\left[\hat{p}^2 + \omega^2 \hat{q}^2\right] - \frac{\hat{1}}{2}. \tag{2.4.8}$$

We can then consider the follow experiment:

1. Pre-selection on $|q = 0\rangle$;

2. Weak measurement of the number operator;
3. Post-selection on $\langle p = 0|$.

The weak value is simple to calculate:

$$\langle \hat{n} \rangle_w = \frac{\left\langle p = 0 \middle| \frac{1}{2}\left[\hat{p}^2 + \omega^2 \hat{q}^2\right] - \frac{\hat{1}}{2} \middle| q = 0 \right\rangle}{\langle p = 0 | q = 0 \rangle}, \tag{2.4.9}$$

which yields

$$\langle \hat{n} \rangle_w = -\frac{1}{2}. \tag{2.4.10}$$

Once again, we get a weak value that is unexpected because we normally have

$$\langle \hat{n} \rangle \geq 0. \tag{2.4.11}$$

2.5 Measuring weak values

To have a better understanding of how weak values can be determined experimentally (especially since it is a complex number), we will explore two different cases: one where the system is only pre-selected and another where the system is pre- and post-selected.

2.5.1 Pre-selected state

This experiment will be repeated on a large number of systems, each being coupled to a measuring device. From Eq. (2.3.1), the interaction Hamiltonians are of the form

$$\hat{H} = g\left(t\right) \hat{q} \hat{A}. \tag{2.5.1}$$

For convenience and as a good approximation of reality, we will suppose that the wavefunction each measuring device is prepared in a Gaussian state (in the q basis) which can be written

$$\psi_i^{\text{MD}}\left(q\right) = \left(\Delta^2 \pi\right)^{-1/4} \mathrm{e}^{-\frac{q^2}{2\Delta^2}}, \tag{2.5.2}$$

where Δ represents the spread of the Gaussian. The corresponding probability density function is

$$\mathbb{P}_i^{\text{MD}}\left(q\right) = \left|\psi_i^{\text{MD}}\left(q\right)\right|^2 = \left(\Delta^2 \pi\right)^{-1/2} \mathrm{e}^{-\frac{q^2}{\Delta^2}} \tag{2.5.3}$$

If the initial state of the system is a superposition $|\psi_1\rangle = \sum_i \alpha_i |a_i\rangle$, then the wavefunction of the measuring device after the interaction resulting from (2.5.1) will be

$$\psi_f^{\text{MD}}\left(q\right) = \left(\Delta^2 \pi\right)^{-1/4} \sum_i \alpha_i |c_i\rangle \, \mathrm{e}^{-\frac{\left(q - a_i\right)^2}{2\Delta^2}}, \tag{2.5.4}$$

with a probability density function given by

$$\mathbb{P}_f^{\text{MD}}\left(q\right) = \left|\psi_f^{\text{MD}}\left(q\right)\right|^2 = \left(\Delta^2 \pi\right)^{-1/2} \sum_i |\alpha_i|^2 \, \mathrm{e}^{-\frac{\left(q - a_i\right)^2}{\Delta^2}}. \tag{2.5.5}$$

This probability density function is the sum of the initial density probability (see Eq. (2.5.3)), weighted by the coefficients $|\alpha_i|^2$, and centered on each eigenvalue a_i. As said in Section 2.2, when we reduce the coupling, the uncertainty, represented by Δ here, will increase. The weak measurement corresponds to the case where $\Delta \gg a_i$ for all eigenvalues a_i. Hence, we can perform the Taylor expansion of the exponential in Eq. (2.5.5) around $q = a_i$:

$$\mathbb{P}_f^{\mathrm{MD}}(q) \simeq \left(\Delta^2\pi\right)^{-1/2} \sum_i |\alpha_i|^2 \left(1 - \frac{(q-a_i)^2}{\Delta^2}\right), \tag{2.5.6}$$

which can be shown to be equivalent to [8, p. 12, (29)]

$$\mathbb{P}_f^{\mathrm{MD}}(q) \simeq \left(\Delta^2\pi\right)^{-1/2} \exp\left\{1 - \frac{\left(q - \sum_i |\alpha_i|^2 a_i\right)^2}{\Delta^2}\right\}. \tag{2.5.7}$$

This probability density function is identical to the initial one Eq. (2.5.2), except that it has been shifted by

$$\sum_i |\alpha_i|^2 a_i \equiv \left\langle \hat{A} \right\rangle = \left\langle \hat{A} \right\rangle_w. \tag{2.5.8}$$

Therefore, in the case of a pre-selected state, the weak value can be obtained by averaging the result of a weak measurement over a large number of identically prepared systems.

2.5.2 Pre- and post-selected state

Here, we consider the following experiment:

1. We pre-select on $|\psi_1\rangle$, i.e. we prepare each system in the same state, at time t_1;
2. We perform a weak measurement (i.e. the coupling between the system and pointer is very small) at time t ($t_1 < t < t_2$) of the observable $\hat{A}$;
3. We post-select on $\langle\psi_2|$ at time t_2. To implement this, we make a final measurement, and we discard the systems which do not yield a certain outcome.

As before, the wavefunction each measuring device is initially in a Gaussian state:

$$\psi_i^{\mathrm{MD}}(q) = \left(\Delta^2\pi\right)^{-1/4} \mathrm{e}^{-\frac{q^2}{2\Delta^2}}, \tag{2.5.9}$$

After the post-selection, wavefunction of each measuring device will be, up to a normalization factor, in the following state:

$$\psi_f^{\mathrm{MD}}(q) = \left\langle \psi_2 \middle| \mathrm{e}^{-\mathrm{i}\int \hat{H}\,\mathrm{d}t} \middle| \psi_1 \right\rangle \mathrm{e}^{-\frac{q^2}{2\Delta^2}} = \left\langle \psi_2 \middle| \mathrm{e}^{\mathrm{i}\hat{p}\hat{A}} \middle| \psi_1 \right\rangle \mathrm{e}^{-\frac{q^2}{2\Delta^2}}. \tag{2.5.10}$$

After some calculations, we can write in the p basis

$$\widetilde{\psi}_f^{\mathrm{MD}}(p) = \langle\psi_2|\psi_1\rangle\, \mathrm{e}^{-\mathrm{i}\langle\hat{A}\rangle_w p} \mathrm{e}^{-\Delta^2p^2/2} + \langle\psi_2|\psi_1\rangle \sum_{n=2}^{\infty} \frac{(\mathrm{i}p)^n}{n!} \left(\left\langle \hat{A}^n \right\rangle_w - \left\langle \hat{A} \right\rangle_w^n \right) \mathrm{e}^{-\Delta^2p^2/2}. \tag{2.5.11}$$

If the spread Δ is sufficiently large (which is the case when we reduce the coupling in Eq. (2.5.1)), the second term can be neglected when we go back to the q basis. Then, the wavefunction of the measuring device in the weak measurement formalism is

$$\psi_f^{\mathrm{MD}}(q) = \left(\Delta^2 \pi\right)^{-1/4} \mathrm{e}^{-\frac{\left(q - \langle \hat{A} \rangle_w\right)^2}{2\Delta^2}}. \tag{2.5.12}$$

The corresponding probability density function is a Gaussian centered on $q = \Re\left(\left\langle \hat{A} \right\rangle_w\right)$, so this measurement allows us to find the real part of the weak value. Similarly, we can show that the probability density function in the p basis is centered, up to a proportionality factor, the imaginary part of the weak value.

2.6 Applications

At first, the weak measurement formalism might seem like a mathematical curiosity. Moreover, because of the simplicity of the underlying principle, it would be easy to miss the importance and the powerfulness of this framework. For that reason, it will not be superfluous to present a few of the numerous and various applications of the weak measurement formalism. For example Johansen [16, 17], used this formalism to characterize the non-classicality of certain states. In [18], it has been found that any (strong) generalized measurement can be expressed as a sequence of weak measurement. Others have been able to resolve certain paradoxes in quantum mechanics such as HARDY's paradox [19, 20, 21]. Additionally, it has been applied to explore the very foundations of quantum mechanics. Indeed, in [22, 23], as the title says, weak measurement was used to probe the "average trajectories of single photons in a two-slit interferometer", and they have found that the results are in agreement with the Bohm trajectories predicted by the DE BROGLIE–BOHM interpretation of quantum mechanics. Hofmann has also employed this framework in a number of fundamental problems [24, 25] such as a re-interpretation of the collapse of the wavefunction as a Bayesian update of information. For example, he supports the idea that "Schrödinger's cat is already dead or alive before the measurement."

These are just a few, but certainly not all, areas where the weak measurement formalism has proven to be advantageous. In the rest of this section, we will investigate a very intriguing experiment that can be performed using the weak measurement formalism [13]. What is so fascinating is that they show that it is possible to directly measure the wavefunction of a state. Often, the wavefunction is only viewed as a mathematical tool used to do the calculations such as computing the probability density function (using BORN's rule) of finding a particle at a certain position. However, if this result is correct, it rather favors an interpretation of the wavefunction as a real physical object. Also, the simplicity of the principle makes it a very elegant application of weak measurement.

As indicated before, the weak value is defined as

$$\left\langle \hat{A} \right\rangle_w \equiv \frac{\left\langle \psi_2 \middle| \hat{A} \middle| \psi_1 \right\rangle}{\langle \psi_2 | \psi_1 \rangle}. \tag{2.6.1}$$

Even though this is a complex number, we have seen (see Section 2.5) that it is possible to determine experimentally both the real part and the imaginary part if we perform weak measurement on a pre- and post-selected system. For example, we could couple a pointer such that its position gives us $\Re\left\{\left\langle \hat{A} \right\rangle_w\right\}$ while the momentum kick of that pointer allows us to deduce $\Im\left\{\left\langle \hat{A} \right\rangle_w\right\}$. Also, it is necessary to repeat this experiment numerous times to attain the desired precision, since the uncertainty increases as the strength of the measurement decreases. Here, the pre- and post-selection scheme in this experiment is as follows

1. Pre-selection: all the systems are prepared in the state $|\psi_1\rangle = |\Psi\rangle$;
2. Weak measurement of the position $\hat{A} = |x\rangle\langle x|$;
3. Post selection: strong measurement of the momentum $\hat{p}$. In our final ensemble, we only keep the systems which yielded the outcome p_f, i.e. $\langle\psi_2| = \langle p = p_f|$.

In this case, the weak value of $\hat{A}$ is

$$\left\langle \hat{A} \right\rangle_w = \frac{\langle p = p_f|x\rangle\langle x|\Psi\rangle}{\langle p = p_f|\Psi\rangle} = \frac{\mathrm{e}^{\mathrm{i}px/\hbar}\Psi(x)}{\Phi(p_f)}, \tag{2.6.2}$$

where $\Phi(p)$ is the initial state $|\Psi\rangle$ in the momentum basis (the two are linked by a FOURIER transform). Now, if we choose $p_f = 0$, we simply get

$$\left\langle \hat{A} \right\rangle_w = \frac{\Psi(x)}{\Phi(0)}, \tag{2.6.3}$$

or

$$\boxed{\left\langle \hat{A} \right\rangle_w = k\Psi(x),} \tag{2.6.4}$$

where $k = 1/\Phi(0)$ is a constant (that we can eliminate by normalizing the wavefunction). As stated above, we see that the weak value directly gives us the value of the wavefunction at any arbitrary position x. The experimental realization (see Fig. 2.6.1). In order to compare the measured

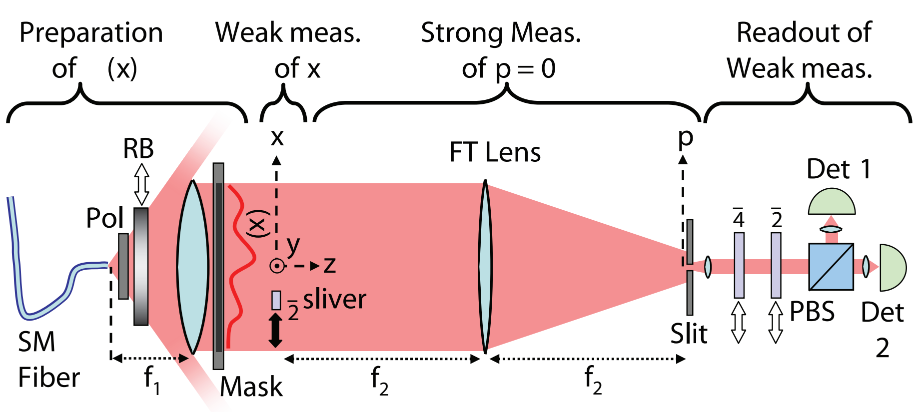


Figure 2.6.1: Experimental setup for the direct measurement of the photon transverse wavefunction. It can be divided into four parts: "preparation of the transverse wavefunction, weak measurement of the transverse position of the photon, post-selection of those photons with zero transverse momenta, and readout of the weak measurement." Source: [13].

wavefunction, they prepared all the photons in the same known states. The data deduced from the weak value seems to be in good agreement with the prepared state (see [13] for the graphics).

2.7 Transient density matrix

There is one more property of the weak value that we have not discussed yet. For convenience, we reproduce here the formula for the weak value of an arbitrary observable $\hat{A}$, where we pre-select on $|a\rangle$ and post-select on $\langle b|$:

$$\left\langle \hat{A} \right\rangle_w \equiv \frac{\left\langle b \middle| \hat{A} \middle| a \right\rangle}{\langle b|a\rangle} = \mathrm{Tr}\left(\frac{|a\rangle \langle b|}{\langle b|a\rangle} \hat{A} \right). \tag{2.7.1}$$

When we compare it to the formula for the expected value (see Eq. (1.2.10)), we might be inclined to define a new type of operator [26, p. 20, (77)] (see Fig. 2.7.1):

$$\hat{\sigma}_{|a\rangle\langle b|} \equiv \frac{|a\rangle \langle b|}{\langle b|a\rangle}, \tag{2.7.2}$$

so that

$$\left\langle \hat{A} \right\rangle_w = \mathrm{Tr}\left(\hat{\sigma}_{|a\rangle\langle b|} \hat{A} \right). \tag{2.7.3}$$

The rest of this report will be dedicated to studying this new operator which we will cal the *transient density matrix* because of the fact that it represents a sort of "density matrix" of the intermediate state between the pre- and post-selection.

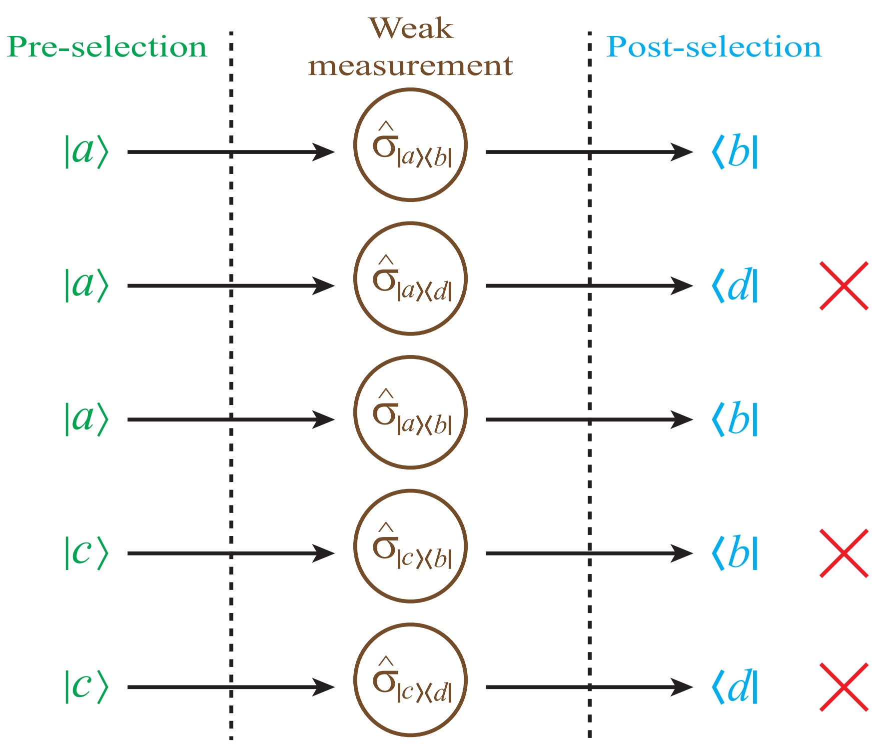


Figure 2.7.1: Pre- and post-section scheme for the definition of the transient density matrix $\hat{\sigma}_{|a\rangle\langle b|}$, this means that we want to keep the systems that are pre-selected on $|a\rangle$ and post-selected on $\langle b|$. Five different systems were represented. A cross on the right of a system indicates that it must be discarded because it does not respect the condition for the pre-selection and/or the post-section.

One interesting thing to note is that we have

$$\boxed{\mathrm{Tr}\left(\hat{\sigma}_{|a\rangle\langle b|} \right) = 1,} \tag{2.7.4}$$

like a usual density matrix (see Eq. (1.1.2)).

Proof

We want to calculate

$$\mathrm{Tr}\left(\hat{\sigma}_{|a\rangle\langle b|}\right) = \mathrm{Tr}\left(\frac{|a\rangle\langle b|}{\langle b|a\rangle}\right). \tag{2.7.5}$$

To compute the trace, we choose an arbitrary basis $\{|c\rangle\}$. Then

$$\mathrm{Tr}\left(\hat{\sigma}_{|a\rangle\langle b|}\right) = \sum_{|c\rangle}\frac{\langle c|a\rangle\langle b|c\rangle}{\langle b|a\rangle} = \sum_{|c\rangle}\frac{\langle b|c\rangle\langle c|a\rangle}{\langle b|a\rangle} = \frac{\left\langle b\left|\left(\sum_{|c\rangle}|c\rangle\langle c|\right)\right|a\right\rangle}{\langle b|a\rangle}. \tag{2.7.6}$$

Since we have the completeness relation

$$\sum_{|c\rangle}|c\rangle\langle c| = \hat{1}. \tag{2.7.7}$$

Therefore, we have

$$\mathrm{Tr}\left(\hat{\sigma}_{|a\rangle\langle b|}\right) = \frac{\langle b|a\rangle}{\langle b|a\rangle}, \tag{2.7.8}$$

or

$$\mathrm{Tr}\left(\hat{\sigma}_{|a\rangle\langle b|}\right) = 1, \tag{2.7.9}$$

Moreover, the transient density matrix stays invariant if we apply the following transformation

$$\boxed{|a\rangle \to |a'\rangle = \mathrm{e}^{\mathrm{i}\theta}|a\rangle \quad \Rightarrow \quad \hat{\sigma}_{|a'\rangle\langle b|} = \hat{\sigma}_{|a\rangle\langle b|}.} \tag{2.7.10}$$

This means the transient density matrix is unaffected by a global phase factor that could be present in the state $|a\rangle$ (or $|b\rangle$), a property also found in a density matrix (see Eq. (1.1.5)). We already know that this global phase factor will not have to be studied later on.

Proof

We find

$$\hat{\sigma}_{|a'\rangle\langle b|} = \frac{|a'\rangle\langle b|}{\langle b|a'\rangle} = \frac{\cancel{\mathrm{e}^{\mathrm{i}\theta}}|a\rangle\langle b|}{\langle b|a\rangle\cancel{\mathrm{e}^{\mathrm{i}\theta}}} = \frac{|a\rangle\langle b|}{\langle b|a\rangle} = \hat{\sigma}_{|a\rangle\langle b|}. \tag{2.7.11}$$

.

However, we should refrain from interpreting $\hat{\sigma}_{|\alpha\rangle\langle\beta|}$ as a usual density matrix. Indeed, the transient density matrix is not Hermitian:

$$\boxed{\hat{\sigma}^{\dagger}_{|a\rangle\langle b|} \neq \hat{\sigma}_{|a\rangle\langle b|},} \tag{2.7.12}$$

unlike the density matrix (see Eq. (1.1.1)). This implies that the eigenvalues of $\hat{\sigma}_{|a\rangle\langle b|}$ can be complex.

Proof

We have

$$\hat{\sigma}^{\dagger}_{|a\rangle\langle b|} = \left(\frac{|a\rangle\langle b|}{\langle b|a\rangle}\right)^{\dagger} = \frac{|b\rangle\langle a|}{\langle b|a\rangle} \neq \frac{|a\rangle\langle b|}{\langle b|a\rangle} = \hat{\sigma}_{|a\rangle\langle b|}, \tag{2.7.13}$$

in general.

2.7.1 Coherent states

In this report, we will study this transient density matrix when the pre- and post-selected states are both coherent states[2], i.e.

$$\boxed{\hat{\sigma}_{|\alpha\rangle\langle\beta|} \equiv \frac{|\alpha\rangle\langle\beta|}{\langle\beta|\alpha\rangle}.} \tag{2.7.14}$$

As we will see in the next chapter, the coherent states form an over-complete basis (see Eq. (3.3.11)) such that

$$\frac{1}{\pi}\iint_{\mathbf{R}^2} |\alpha\rangle\langle\alpha|\, \mathrm{d}^2\alpha = \frac{1}{\pi}\iint_{\mathbf{R}^2} |\beta\rangle\langle\beta|\, \mathrm{d}^2\beta = \pi. \tag{2.7.15}$$

Also, the probability of measuring the state $\langle\beta|$ if we prepare $|\alpha\rangle$ can be shown to be

$$\mathbb{P}(\beta|\alpha) = \mathbb{P}(\alpha|\beta) = \frac{|\langle\beta|\alpha\rangle|^2}{\pi} = \frac{\langle\beta|\alpha\rangle\langle\alpha|\beta\rangle}{\pi}. \tag{2.7.16}$$

Using those two expressions, we can test the consistency of the definition of the transient density matrix. Indeed, if we have no post-selection (we integrate over all possible final states $\langle\beta|$, then

$$\begin{aligned}\iint_{\mathbf{R}^2} \mathbb{P}(\beta|\alpha) \times \hat{\sigma}_{|\alpha\rangle\langle\beta|}\, \mathrm{d}^2\beta &= \iint_{\mathbf{R}^2} \frac{\cancel{\langle\beta|\alpha\rangle}\langle\alpha|\beta\rangle}{\pi} \times \frac{|\alpha\rangle\langle\beta|}{\cancel{\langle\beta|\alpha\rangle}}\, \mathrm{d}^2\beta \\ &= \iint_{\mathbf{R}^2} \frac{\langle\alpha|\beta\rangle}{\pi} |\alpha\rangle\langle\beta|\, \mathrm{d}^2\beta \\ &= \iint_{\mathbf{R}^2} |\alpha\rangle \frac{\langle\alpha|\beta\rangle}{\pi} \langle\beta|\, \mathrm{d}^2\beta \\ &= |\alpha\rangle\langle\alpha| \times \frac{1}{\pi}\iint_{\mathbf{R}^2} |\beta\rangle\langle\beta|\, \mathrm{d}^2\beta \\ &= |\alpha\rangle\langle\alpha|,\end{aligned} \tag{2.7.17}$$

[2]The notion of coherent state will be introduced in Chapter 3.

then we obtain the initial prepared state, which is what we would expect since the weak measurement does not disturb the wavefunction. Similarly, if we have no pre-selection

$$\begin{aligned}
\iint_{\mathbf{R}^2} \mathbb{P}(\alpha|\beta) \times \hat{\sigma}_{|\alpha\rangle\langle\beta|} \, \mathrm{d}^2\alpha &= \iint_{\mathbf{R}^2} \frac{\cancel{\langle\beta|\alpha\rangle} \langle\alpha|\beta\rangle}{\pi} \times \frac{|\alpha\rangle\langle\beta|}{\cancel{\langle\beta|\alpha\rangle}} \, \mathrm{d}^2\alpha \\
&= \iint_{\mathbf{R}^2} \frac{\langle\alpha|\beta\rangle}{\pi} |\alpha\rangle\langle\beta| \, \mathrm{d}^2\alpha \\
&= \iint_{\mathbf{R}^2} |\alpha\rangle \frac{\langle\alpha|\beta\rangle}{\pi} \langle\beta| \, \mathrm{d}^2\alpha \\
&= |\beta\rangle\langle\beta| \times \frac{1}{\pi} \iint_{\mathbf{R}^2} |\alpha\rangle\langle\alpha| \, \mathrm{d}^2\alpha \\
&= |\beta\rangle\langle\beta| , \qquad (2.7.18)
\end{aligned}$$

we get the final state.

2.8 Summary

In this chapter, we have introduced the notion of weak measurement. We have seen that it naturally leads to the definition of the weak value, which does not behave like the usual expected value: it can be complex, or out of the normal bounds. Then, we looked into the possible applications of the weak measurement formalism, and we focused our attention on the direct measurement of the wavefunction using weak values. Finally, we defined a new mathematical object called the transient density matrix by comparing the expression of the weak value to the expected value of an observable. While this transient density matrix is not a state (at least not in the usual sense), we will try to characterize this object in the next chapters. In order to do this, we will limit our analysis to the particular case where the pre-selected state (written $|\alpha\rangle$ from now on) and the post-selected state (written $\langle\beta|$ from now on) are both coherent states. The reason for this choice is twofold: first of all, the coherent states possess a lot of interesting properties which might help us in the characterization. Second of all, a lot of tools have been developed in the field of quantum optics to better represent normal density matrices using phase space distributions[3].

[3] Phase space distributions will be defined in Chapter 4.

Chapter 3

Quantum optics

In the first quantization, the particles are treated using quantum mechanics (e.g. fermionic particles respect PAULI's exclusion principle), but the surrounding environment (e.g. the electromagnetic field) is treated classically. The name itself comes from the quantization of the energy suggested by Max PLANCK when he studied the blackbody radiation [3, p. 10]. Indeed, he noticed that the energy contained in an electromagnetic wave of frequency ν can only be an integer multiple of $h\nu$ where $h = 6.62 \times 10^{-34}\,\mathrm{J \cdot s}$ is called the PLANCK's constant. In turn, in order to explain the photoelectric effect, this led Albert EINSTEIN to postulate the existence of the photon, a particle with an energy $h\nu$, and which corresponds to a *quantum* of electromagnetic energy[1].

The first quantization described above is a fundamental principle of quantum mechanics. The second quantization (which refers to the quantization of the electromagnetic field), on the other hand, can be avoided for the description of a large range of phenomena. However, certain properties of the electromagnetic field (such as the ones presented in this report) cannot be explored without introducing the second quantization. For this reason, we will now describe the train of thought used to quantize the electromagnetic field.

The beginning of this section explains the reasoning behind the second quantization of the free non-interacting electromagnetic field to help us understand where the quantization of the field arises. It should be emphasized that the goal of this section is not to present a thorough derivation of this quantization (which would be too lengthy), but to provide an overview of this process. For this reason, we will skip the lengthy calculations (or place them in the appendices) when possible and focus instead on the physics behind the quantization.

Once that is done, it will allow us to introduce the creation and the annihilation operators, as well as Fock states which have a simple physical interpretation. Then, for the rest of the chapter, we will focus our attention on the coherent states of light and review their main properties.

- The main references are: [27, Ch. 10–11, 21], [28, Ch. 1], [29, Ch. 7], [30, 31, 32], [33, Ch. 2–3], [34, Ch. 1–2].

[1] Interestingly enough, while the notion of quantization was first introduced for the energy contained in the electromagnetic field, the electromagnetic field itself was not formulated immediately.

3.1 Second quantization

3.1.1 Classical fields

The starting point of this development is to determine the form of the electromagnetic field in the classical picture. Then, we will show how it is possible to obtain the quantized form of the electromagnetic field.

First, we write Maxwell's equations in empty space and in the absence of sources (e.g. charges or currents):

$$\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r}, t) = -\frac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{r}, t), \tag{3.1.1}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B}(\boldsymbol{r}, t) = \frac{1}{c^2}\frac{\partial}{\partial t}\boldsymbol{E}(\boldsymbol{r}, t), \tag{3.1.2}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{r}, t) = 0, \tag{3.1.3}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B}(\boldsymbol{r}, t) = 0, \tag{3.1.4}$$

where $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ are respectively the electric field and the magnetic field at the space-time point $(\boldsymbol{r}, t)$. Let us now introduce the vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$ defined by the two relations

$$\boldsymbol{B}(\boldsymbol{r}, t) \equiv \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r}, t), \tag{3.1.5}$$

$$\boldsymbol{E}(\boldsymbol{r}, t) \equiv -\frac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{r}, t), \tag{3.1.6}$$

and is chosen to satisfy the Coulomb gauge, i.e.

$$\boldsymbol{\nabla} \cdot \boldsymbol{A}(\boldsymbol{r}, t) = 0, \quad \forall (\boldsymbol{r}, t). \tag{3.1.7}$$

Injecting Eqs. (3.1.5) and (3.1.6) in Eq. (3.1.2):

$$\boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r}, t)] = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{A}(\boldsymbol{r}, t), \tag{3.1.8}$$

$$\Leftrightarrow \quad \boldsymbol{\nabla} \cdot [\boldsymbol{\nabla} \cdot \boldsymbol{A}(\boldsymbol{r}, t)] - \Delta \boldsymbol{A}(\boldsymbol{r}, t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{A}(\boldsymbol{r}, t), \tag{3.1.9}$$

where we have used the following vectorial identity in the last equation:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{C}) = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{C} - \Delta \boldsymbol{C}. \tag{3.1.10}$$

Using the Coulomb gauge (Eq. (3.1.7)), Eq. (3.1.9) yields the wave equation for the vector potential:

$$\boxed{\Delta \boldsymbol{A}(\boldsymbol{r}, t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{A}(\boldsymbol{r}, t) = 0.} \tag{3.1.11}$$

From the Eqs. (3.1.5) and (3.1.6), we see that solving this equation will immediately give us the form of the electromagnetic field.

3.1.2 Plane-wave decomposition

To find the general form of the solution to this wave equation, it is customary to write the Fourier expansion of $\boldsymbol{A}(\boldsymbol{r}, t)$. While there is no fundamental problem in directly using the Fourier transform, it is also possible to decompose the vector potential using the Fourier series[2], and introduce the continuous case later on if need be.

In order to write the Fourier expansion associated with the vector potential, it is necessary to postulate that the electromagnetic field (and therefore the vector potential) is contained in a cube of side L. We then require $\boldsymbol{A}(\boldsymbol{r},t)$ to be periodic at the boundaries of that cube[3]. We can then write

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{1}{\varepsilon_0^{1/2} L^{3/2}} \sum_{\boldsymbol{k},s} \boldsymbol{A}_{\boldsymbol{k}s}(t)\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{3.1.12}$$

where ε_0 is the vacuum permittivity, $\boldsymbol{A}_{\boldsymbol{k}s}(t)$ are the Fourier coefficients , $\boldsymbol{k}$ is called the wave-number vector (or wave vector for short), and where s corresponds to the polarization of the vector potential (see for example [27, Ch. 10.2.2] for an extensive discussion on the polarization of the vector potential). This allows us to interpret the vector potential as the superposition of vector potentials for each mode defined by $(\boldsymbol{k},s)$.

Since the vector potential must have the periodicity of the box, the wave vector $\boldsymbol{k} = (k_1, k_2, k_3)$ must respect the following constraints:

$$\begin{cases} k_1 = \dfrac{2\pi}{L} n_1, & n_1 \in \mathbf{Z}, \\ k_2 = \dfrac{2\pi}{L} n_2, & n_2 \in \mathbf{Z}, \\ k_3 = \dfrac{2\pi}{L} n_3, & n_3 \in \mathbf{Z}. \end{cases} \tag{3.1.13}$$

As for the polarization, we would normally need three linearly independent polarization vectors ($s = 1,2,3$) to form a basis. However, the Coulomb gauge (Eq. (3.1.7)) applied to the Fourier expansion (Eq. (3.1.12)) leads to

$$\boxed{\boldsymbol{k}\cdot\boldsymbol{A}_{\boldsymbol{k}s}(t) = 0.} \tag{3.1.14}$$

Therefore, we only need two linearly independent polarization vectors ($s = 1,2$) to describe the vector potential. Note that their actual form is not important here; they can be chosen in different ways (two circular polarizations or two linear polarization for example) depending on the problem we want to solve.

Furthermore, to insure that the vector potential remains real, we have to impose

$$\boldsymbol{A}^*_{-\boldsymbol{k}s}(t) = \boldsymbol{A}_{\boldsymbol{k}s}(t) \tag{3.1.15}$$

Now, if we inject the expression given by Eq. (3.1.12) into the wave equation (Eq. (3.1.11)), we get

$$\left(-k^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{A}_{\boldsymbol{k}s}(t) = 0, \tag{3.1.16}$$

yielding the solution

$$\boldsymbol{A}_{\boldsymbol{k}s}(t) = c_{\boldsymbol{k}s}\mathrm{e}^{-\mathrm{i}\omega_k t}\boldsymbol{\varepsilon}_{\boldsymbol{k}s} + d_{\boldsymbol{k}s}\mathrm{e}^{\mathrm{i}\omega_k t}\boldsymbol{\varepsilon}_{\boldsymbol{k}s}, \tag{3.1.17}$$

where $c_{\boldsymbol{k}s}$, $d_{\boldsymbol{k}s}$ are coefficients to be determined, $\boldsymbol{\varepsilon}_{\boldsymbol{k}s}$ is the polarization vector and where ω_k are the angular frequencies for each mode, defined by the dispersion relation

$$\boxed{\omega_k \equiv |\boldsymbol{k}|\, c.} \tag{3.1.18}$$

[2]Using the Fourier series will lead to a discrete (but infinite) number of modes instead of a continuous (also infinite) number of modes if we were using the Fourier transform.

[3]Note that the continuous case is simply obtained by making the cube infinitely large.

It can even be simplified to

$$\boxed{\boldsymbol{A}_{\boldsymbol{k}s}(t) = c_{\boldsymbol{k}s}\mathrm{e}^{-\mathrm{i}\omega_k t}\boldsymbol{\varepsilon}_{\boldsymbol{k}s} + c^*_{-\boldsymbol{k}s}\mathrm{e}^{\mathrm{i}\omega_k t}\boldsymbol{\varepsilon}^*_{-\boldsymbol{k}s},} \tag{3.1.19}$$

by using the relation

$$d_{\boldsymbol{k}s}\boldsymbol{\varepsilon}_{\boldsymbol{k}s} = c^*_{-\boldsymbol{k}s}\boldsymbol{\varepsilon}^*_{-\boldsymbol{k}s}, \tag{3.1.20}$$

which simply derives from Eq. (3.1.15). We substitute this result in Eq. (3.1.12):

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{1}{\varepsilon_0^{1/2}L^{3/2}}\sum_{\boldsymbol{k},s}\left[c_{\boldsymbol{k}s}\mathrm{e}^{-\mathrm{i}\omega_k t}\boldsymbol{\varepsilon}_{\boldsymbol{k}s} + c^*_{-\boldsymbol{k}s}\mathrm{e}^{\mathrm{i}\omega_k t}\boldsymbol{\varepsilon}^*_{-\boldsymbol{k}s}\right]\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{3.1.21}$$

or

$$\boxed{\boldsymbol{A}(\boldsymbol{r},t) = \frac{1}{\varepsilon_0^{1/2}L^{3/2}}\sum_{\boldsymbol{k},s}\left[c_{\boldsymbol{k}s}\mathrm{e}^{(\boldsymbol{k}\cdot\boldsymbol{r}-\mathrm{i}\omega_k t)}\boldsymbol{\varepsilon}_{\boldsymbol{k}s} + c^*_{\boldsymbol{k}s}\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\right].} \tag{3.1.22}$$

Finally, from Eqs. (3.1.5) and (3.1.6), we obtain the expression of the corresponding electric field

$$\boxed{\boldsymbol{E}(\boldsymbol{r},t) = \frac{\mathrm{i}}{\varepsilon_0^{1/2}L^{3/2}}\sum_s\sum_{\boldsymbol{k}}\omega_k\left[c_{\boldsymbol{k}s}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}_{\boldsymbol{k}s} - c^*_{\boldsymbol{k}s}(t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\right],} \tag{3.1.23}$$

and magnetic field

$$\boxed{\boldsymbol{B}(\boldsymbol{r},t) = \frac{\mathrm{i}}{\varepsilon_0^{1/2}L^{3/2}}\sum_s\sum_{\boldsymbol{k}}\left[c_{\boldsymbol{k}s}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{k}\times\boldsymbol{\varepsilon}_{\boldsymbol{k}s}) - c^*_{\boldsymbol{k}s}(t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{k}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s})\right].} \tag{3.1.24}$$

3.1.3 Hamiltonian of the (classical) electromagnetic field

From classical electrodynamics, we know that the Hamiltonian of the electromagnetic field is [27, p. 472, (10.2–25)]

$$H = \frac{1}{2}\int_{\mathbf{R}^3}\left[\varepsilon_0\boldsymbol{E}^2(\boldsymbol{r},t) + \frac{1}{\mu_0}\boldsymbol{B}^2(\boldsymbol{r},t)\right]\mathrm{d}^3r, \tag{3.1.25}$$

which can be written, after some calculations (see Appendix A.1), as

$$H = 2\sum_{\boldsymbol{k},s}\omega_k^2\left|c_{\boldsymbol{k}s}(t)\right|^2, \tag{3.1.26}$$

or

$$\boxed{H = \frac{1}{2}\sum_{\boldsymbol{k},s}\left[p^2_{\boldsymbol{k}s}(t) + \omega_k^2 q^2_{\boldsymbol{k}s}(t)\right],} \tag{3.1.27}$$

where $q_{\boldsymbol{k}s}(t)$ and $p_{\boldsymbol{k}s}(t)$ are a pair of (real) canonical variables defined as

$$q_{\boldsymbol{k}s}(t) = \left[c_{\boldsymbol{k}s}(t) + c^*_{\boldsymbol{k}s}(t)\right] = 2\Re\left[c_{\boldsymbol{k}s}(t)\right], \tag{3.1.28}$$

$$p_{\boldsymbol{k}s}(t) = -\mathrm{i}\omega_k\left[c_{\boldsymbol{k}s}(t) - c^*_{\boldsymbol{k}s}(t)\right] = 2\omega_k\Im\left[c_{\boldsymbol{k}s}(t)\right]. \tag{3.1.29}$$

This last form of the Hamiltonian can be easily interpreted as the sum of an infinite (but discrete) set of independent (i.e. uncoupled) harmonic oscillators, one for each mode defined by $(\boldsymbol{k}, s)$.

Moreover, it is important to note that, in our case, $q_{\boldsymbol{k}s}(t)$ and $p_{\boldsymbol{k}s}(t)$ are canonical coordinates and *not* (resp.) the position and the momentum of a particle. They verify the Poisson brackets relations:

$$\left\{q_{\boldsymbol{k}s}(t), q_{\boldsymbol{m}u}(t)\right\}_{\mathrm{P}} = 0, \tag{3.1.30}$$

$$\left\{p_{\boldsymbol{k}s}(t), p_{\boldsymbol{m}u}(t)\right\}_{\mathrm{P}} = 0, \tag{3.1.31}$$

$$\left\{q_{\boldsymbol{k}s}(t), p_{\boldsymbol{m}u}(t)\right\}_{\mathrm{P}} = \delta^3_{\boldsymbol{k}\boldsymbol{m}}\delta_{su}, \tag{3.1.32}$$

where

$$\{f, g\}_{\mathrm{P}} = \sum_{\boldsymbol{k}s} \left(\frac{\partial f}{\partial q_{\boldsymbol{k}s}(t)} \frac{\partial g}{\partial p_{\boldsymbol{k}s}(t)} - \frac{\partial f}{\partial p_{\boldsymbol{k}s}(t)} \frac{\partial g}{\partial q_{\boldsymbol{k}s}(t)} \right), \tag{3.1.33}$$

where $f\left(q_{\boldsymbol{k}s}(t), p_{\boldsymbol{k}s}(t), t\right)$ and $g\left(q_{\boldsymbol{k}s}(t), p_{\boldsymbol{k}s}(t), t\right)$. The canonical variables also obey Hamilton's equations of motion:

$$\begin{cases} \dot{q}_{\boldsymbol{k}s}(t) = \dfrac{\partial H}{\partial p_{\boldsymbol{k}s}(t)} = \{q_{\boldsymbol{k}s}(t), H\}_{\mathrm{P}} \\ \dot{p}_{\boldsymbol{k}s}(t) = -\dfrac{\partial H}{\partial q_{\boldsymbol{k}s}(t)} = \{p_{\boldsymbol{k}s}(t), H\}_{\mathrm{P}} \end{cases} \tag{3.1.34}$$

There is one last remark we can make about this Hamiltonian: if $q_{\boldsymbol{k}s}(t) = 0, \quad \forall \boldsymbol{k}, s, t$ and $p_{\boldsymbol{k}s}(t) = 0, \quad \forall \boldsymbol{k}, s, t$, then the Hamiltonian is

$$H = 0. \tag{3.1.35}$$

This is what we would normally expect since it means the amplitudes of each mode $(\boldsymbol{k}, s)$ of the vector potential (related to the canonical variables by Eqs. (3.1.28) and (3.1.29)) are all equal to zero, i.e. there is no electromagnetic field. We will see that in the quantum world, this intuition is lost.

3.1.4 Quantization

This is where the quantization takes place: it consists of replacing the canonical variables $q_{\boldsymbol{k}s}(t)$ and $p_{\boldsymbol{k}s}(t)$ by operators $\hat{q}_{\boldsymbol{k}s}(t)$ and $\hat{p}_{\boldsymbol{k}s}(t)$ where

$$\boxed{\hat{p}_{\boldsymbol{k}s}(t) \equiv -\mathrm{i}\hbar \frac{\hat{\partial}}{\partial q_{\boldsymbol{k}s}(t)}.} \tag{3.1.36}$$

This process is called the correspondence principle. It is easy to demonstrate (see Appendix A.2) that in this case, the Poisson brackets are replaced by commutators as follows[4]

$$[\hat{q}_{\boldsymbol{k}s}(t), \hat{q}_{\boldsymbol{m}u}(t)] = 0, \tag{3.1.37}$$

$$[\hat{p}_{\boldsymbol{k}s}(t), \hat{p}_{\boldsymbol{m}u}(t)] = 0, \tag{3.1.38}$$

$$[\hat{q}_{\boldsymbol{k}s}(t), \hat{p}_{\boldsymbol{m}u}(t)] = \mathrm{i}\hbar\delta^3_{\boldsymbol{k}\boldsymbol{m}}\delta_{su}. \tag{3.1.39}$$

Thus, the Hamiltonian operator is

$$\hat{H} = \frac{1}{2} \sum_{\boldsymbol{k},s} \left[\hat{p}^2_{\boldsymbol{k}s}(t) + \omega_k^2 \hat{q}^2_{\boldsymbol{k}s}(t)\right]. \tag{3.1.40}$$

If we define the *annihilation* $\hat{a}_{\boldsymbol{k}s}(t)$ and *creation* $\hat{a}^\dagger_{\boldsymbol{k}s}(t)$ operators (we will see why they are called that way later on) as [27, p.474–475, (10.3–5), (10.3–6)]

$$\hat{a}_{\boldsymbol{k}s}(t) \equiv \frac{1}{(2\hbar\omega_k)^{1/2}} \left[\omega_k \hat{q}_{\boldsymbol{k}s}(t) + \mathrm{i}\hat{p}_{\boldsymbol{k}s}(t)\right], \tag{3.1.41}$$

$$\hat{a}^\dagger_{\boldsymbol{k}s}(t) \equiv \frac{1}{(2\hbar\omega_k)^{1/2}} \left[\omega_k \hat{q}_{\boldsymbol{k}s}(t) - \mathrm{i}\hat{p}_{\boldsymbol{k}s}(t)\right], \tag{3.1.42}$$

[4]There exists also another convention for this commutation where $[\hat{q}_{\boldsymbol{k}s}(t), \hat{p}_{\boldsymbol{m}u}(t)] = 2\mathrm{i}\hbar\delta^3_{\boldsymbol{k}\boldsymbol{m}}\delta_{su}$. One should always be careful when consulting the literature because this can lead to slightly different expressions.

the commutation relations can be written as

$$\left[\hat{a}_{\boldsymbol{ks}}(t), \hat{a}_{\boldsymbol{mu}}(t)\right] = 0, \tag{3.1.43}$$

$$\left[\hat{a}^{\dagger}_{\boldsymbol{ks}}(t), \hat{a}^{\dagger}_{\boldsymbol{mu}}(t)\right] = 0, \tag{3.1.44}$$

$$\left[\hat{a}_{\boldsymbol{ks}}(t), \hat{a}^{\dagger}_{\boldsymbol{mu}}(t)\right] = \delta^3_{\boldsymbol{km}}\delta_{su}. \tag{3.1.45}$$

Replacing the operators $\hat{q}_{\boldsymbol{ks}}(t)$ and $\hat{p}_{\boldsymbol{ks}}(t)$by the newly defined operators $\hat{a}_{\boldsymbol{ks}}(t)$ and $\hat{a}^{\dagger}_{\boldsymbol{ks}}(t)$, and using the commutation rules (Eq. (3.1.45)), we obtain

$$\hat{H} = \sum_{\boldsymbol{k},s} \hbar\omega_k \left[\hat{a}^{\dagger}_{\boldsymbol{ks}}(t)\,\hat{a}_{\boldsymbol{ks}}(t) + \frac{1}{2}\right], \tag{3.1.46}$$

or alternatively

$$\boxed{\hat{H} = \sum_{\boldsymbol{k},s} \hbar\omega_k \left[\hat{n}_{\boldsymbol{ks}}(t) + \frac{1}{2}\right],} \tag{3.1.47}$$

where

$$\hat{n}_{\boldsymbol{ks}}(t) \equiv \hat{a}^{\dagger}_{\boldsymbol{ks}}(t)\,\hat{a}_{\boldsymbol{ks}}(t), \tag{3.1.48}$$

is called the *number operator* because it correspond to the number of quanta $\hbar\omega_k$ in each mode $(\boldsymbol{k}, s)$. As before, this Hamiltonian has the same form as the Hamiltonian for an infinite sum of uncoupled harmonic oscillator. We can decompose this sum into two parts. The first one $\hat{H} = \sum_{\boldsymbol{k},s} \hbar\omega_k \hat{N}_{\boldsymbol{ks}}(t)$ is the sum of quanta of energy which are called *photons*. This is where we can see the meaning and the impact of the quantization of the electromagnetic field. Indeed, for a mode $(\boldsymbol{k}, s)$ the energy in that mode can only vary by increments of $\hbar\omega_k$ instead of varying continuously in between.

The second part $\hat{H} = \sum_{\boldsymbol{k},s} \hbar\omega_k/2$ is a bit more mysterious. It means that, even if there are no photons, the free non-interacting electromagnetic field possesses a residual energy which we call the quantum vacuum zero-point energy. This is due to HEISENBERG's uncertainty principle: a quantum-mechanical harmonic oscillator can never be at rest (i.e. the ground level energy is not zero). Conceptually, this is problematic because the sum is infinite, and that would mean the amount of energy in the vacuum is also infinite. In practice, this is generally avoided because we usually calculate differences of energy and this zero-point contribution disappears. This process is one aspect of the *renormalization* process commonly used in quantum electrodynamics for example.

However, this is one of the reasons why we have trouble reconciling quantum mechanics with General Relativity. In General Relativity, gravity is the consequence of the geometrical curvature of space-time. The problem is that curvature depends on the (*absolute*, not relative) distribution of energy in space-time. If the amount of energy was infinite everywhere, the curvature would also be infinite, and so would be the force of gravity exerted on our bodies, which is fortunately not the case.

The only way to completely avoid the infinity is to truncate the sum to a certain $k_{\max}$. One possible justification would be to say that the correspondence principle is a helpful tool to guide us to the correct answer, but it does not always immediately give it to us. Nevertheless, there exists no satisfactory explanation yet on how to treat this term.

3.2 Fock states

From now on, we will omit the $(\boldsymbol{k}, s)$ indices corresponding to the mode so as to simplify the notation, and because we will only consider one mode. For the same reason, we will also remove the explicit time dependence in our notation, with the implicit assumption that all operators are evaluated at the same time.

The Fock states, also called number states,[5] represent states with a fixed number of quanta (photons in our case). They are written as $|n\rangle$, where n is the number of quanta in the mode we consider. Mathematically, they correspond to the eigenstates of the number operator $\hat{n}$ defined earlier (see Eq. (3.1.48)) associated with the eigenvalues n (called the *occupation number*) [27, p. 476, (10.4–3)]:

$$\hat{n}\,|n\rangle \equiv n\,|n\rangle\,, \tag{3.2.1}$$

In the last section, we also defined the annihilation and creation operators (Eqs. (3.1.41) and (3.1.42)) without explaining this nomenclature. It is possible to show that those operators act on the Fock states accordingly:

$$\hat{a}^{\dagger}\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \tag{3.2.2}$$

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,. \tag{3.2.3}$$

It is now easy to understand their denomination: the creation operator $\hat{a}^{\dagger}$ *adds* a photon in the mode, whereas the annihilation operator $\hat{a}$ *destroys* a photon in the mode. If we define vacuum state as the state where the number of quanta is equal to zero:

$$|0\rangle \equiv |n=0\rangle\,, \tag{3.2.4}$$

then, we can write any Fock state $|n\rangle$ as

$$|n\rangle = \frac{\left(\hat{a}^{\dagger}\right)^{n}}{\sqrt{n!}}\,|0\rangle\,. \tag{3.2.5}$$

Moreover, it is interesting to notice that they form an orthonormal basis:

$$\langle n|k\rangle = \delta_{nk}, \tag{3.2.6}$$

and verify the completeness relation

$$\sum_{n=0}^{\infty} |n\rangle\,\langle n| = \hat{1}, \tag{3.2.7}$$

where $\hat{1}$ is the identity operator.

3.3 Coherent states

While the Fock states are easy to define and interpret, they are not easy to manipulate in practice. This is due to the fact that the electromagnetic is field not accurately portrayed by a single Fock state, but more by a superposition of those states. To this end, we will introduce the *coherent states* $|\alpha\rangle$ which are particularly adapted to describe the state of the electromagnetic field coming from a coherent source of light such as a laser. They were first discovered by Erwin SCHRÖDINGER, but he discarded them as mathematical curiosities. It is Roy J. GLAUBER who showed that they were considerably appropriate in the field of quantum optics.

[5] Rigorously, the Fock states correspond to the states resulting from the tensor product of all the eigenstates $|n_{\boldsymbol{k}s}\rangle$ associated with each mode $(\boldsymbol{k}, s)$. However, since we only consider one mode here, we will also use this denomination for the eigenstates $|n\rangle$.

3.3.1 Definition

These states are defined as the right eigenstates of the annihilation operator $\hat{a}$ associated with the eigenvalues α [27, p. 523, (11.2–1)]:

$$\boxed{\hat{a}\,|\alpha\rangle \equiv \alpha\,|\alpha\rangle\,, \quad \alpha \in \mathbf{C}.} \tag{3.3.1}$$

It is trivial to remark that the corresponding states $\langle\alpha|$ are the left eigenstates of the creation operator $\hat{a}^\dagger$. The coherent states can be expressed in the Fock basis by

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,, \quad \alpha \in \mathbf{C}, \tag{3.3.2}$$

where we can write $\alpha = q+\mathrm{i}p, \quad q,p \in \mathbf{R}$, q being the real amplitude and p the imaginary amplitude (they are also collectively referred to as the field quadratures). The corresponding operators are the ones obtained by inverting Eqs. (3.1.41) and (3.1.42):

$$\hat{q} \equiv \left(\frac{\hbar}{2\omega}\right)^{1/2}\left[\hat{a}+\hat{a}^\dagger\right], \tag{3.3.3}$$

$$\hat{p} \equiv \mathrm{i}\left(\frac{\hbar\omega}{2}\right)^{1/2}\left[\hat{a}^\dagger-\hat{a}\right]. \tag{3.3.4}$$

Since they commute according to Eq. (3.1.39), it is possible to show that there exists the following uncertainty principle between $\hat{q}$ and $\hat{p}$

$$\Delta\hat{q}\Delta\hat{p} \geq \frac{\hbar}{2}. \tag{3.3.5}$$

What is remarkable about the coherent states is that they saturate this inequality, namely,

$$\boxed{\Delta\hat{q}\Delta\hat{p} = \frac{\hbar}{2}, \quad \text{with} \quad \Delta\hat{q} = \Delta\hat{p} = \left(\frac{\hbar}{2}\right)^{1/2}.} \tag{3.3.6}$$

This is the reason why the coherent states are sometimes labeled as quasi-classical states because they represent the closest approximation of a quantum state to a classical one where $\Delta q = \Delta p = 0$. Note that there also exists so-called *squeezed state of light* which also saturate the uncertainty principle, but where $\Delta\hat{q}$ is allowed to be different from $\Delta\hat{p}$. Schematically, we can represent coherent states by a circle of diameter $\Delta\hat{q} = \Delta\hat{p}$ in phase space (q, s). Similarly, squeezed states correspond to an ellipse of dimension $\Delta\hat{q}$ and $\Delta\hat{p}$ as shown on Fig. 3.3.1.

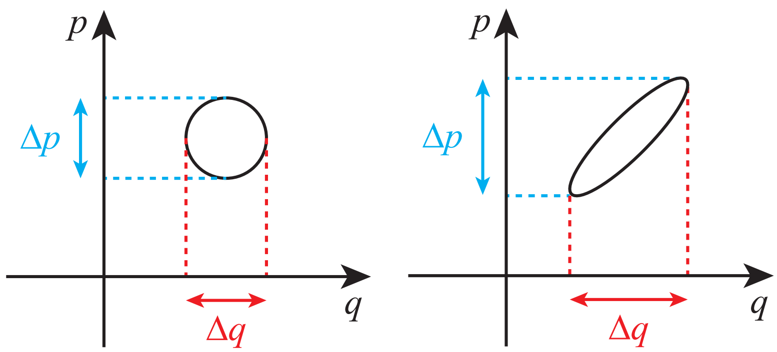
3.3.2 Properties

Since they are physical states, they must be normalized:

$$\langle\alpha|\alpha\rangle = 1, \quad \forall\alpha \in \mathbf{C}, \tag{3.3.7}$$

but they do not form an orthonormal basis like the Fock states (in fact, they represent an over-complete basis), because these states are not mutually orthogonal:

$$\langle\alpha|\beta\rangle \neq \delta^2(\alpha-\beta) \equiv \delta\left(\Re\left[\alpha-\beta\right]\right)\delta\left(\Im\left[\alpha-\beta\right]\right). \tag{3.3.8}$$



(a) Coherent state. (b) Squeezed state.

Figure 3.3.1: Schematic representation of states in phase space (q, s).

Proof

Indeed, using the orthonormality (Eq. (3.2.6)) of the Fock states

$$\begin{aligned}
\langle\beta|\alpha\rangle &= \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{-|\beta|^2/2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\beta^*)^n}{\sqrt{n!}}\frac{\alpha^m}{\sqrt{m!}}\overbrace{\langle n|m\rangle}^{\delta_{mn}} \\
&= \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{-|\beta|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha\beta^*)^n}{n!} \\
&= \mathrm{e}^{-\left(|\alpha|^2+|\beta|^2-2\alpha\beta^*\right)/2} \\
&\neq \delta^2(\alpha-\beta)\,. \qquad (3.3.9)
\end{aligned}$$

This quantity, called the *overlap*, is generally a complex number. Its magnitude can be thought of as a measure of how “close” the two states are. It can also be written as

$$\boxed{\langle\beta|\alpha\rangle = \exp\left\{-\frac{|\alpha-\beta|^2}{2}\right\}\exp\left\{\frac{\beta^*\alpha-\beta\alpha^*}{2}\right\},} \qquad (3.3.10)$$

This is the formula we will use during the calculations. Since $|\langle\beta|\alpha\rangle|^2 = \exp\left(-|\alpha-\beta|^2\right)$, we see that the overlap will decrease very rapidly to zero as the distance between α and β grows; this means $|\alpha\rangle$ and $|\beta\rangle$ become more and more orthogonal. While the coherent states clearly do not obey the completeness relation, it can be shown that they verify a similar property

$$\frac{1}{\pi}\iint_{\mathbf{R}^2}|\alpha\rangle\langle\alpha|\,\mathrm{d}\alpha^2 = 1, \qquad (3.3.11)$$

where

$$\mathrm{d}\alpha^2 \equiv \mathrm{dRe}\,(\alpha)\,\mathrm{dIm}\,(\alpha)\,. \qquad (3.3.12)$$

3.3.3 Displacement operator

By virtue of the expansion of the Fock states on the vacuum state (Eq. (3.2.5)), the coherent state can also be expressed as

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}} |0\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\left(\alpha\hat{a}^\dagger\right)^n}{n!} |0\rangle \tag{3.3.13}$$

or

$$\boxed{|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \mathrm{e}^{\alpha\hat{a}^\dagger} |0\rangle .} \tag{3.3.14}$$

It is interesting to see that the coherent state $|\alpha\rangle = |0\rangle$ is the only one that coincides with a Fock state (the vacuum state in this case). With this last expression, we can interpret the coherent state as a displaced vacuum state [27, p. 526, (11.3–7)]:

$$|\alpha\rangle = \hat{D}(\alpha) |0\rangle , \tag{3.3.15}$$

where $\hat{D}(\alpha)$ is called the *displacement operator* defined by

$$\hat{D}(\alpha) \equiv \mathrm{e}^{-|\alpha|^2/2} \mathrm{e}^{\alpha\hat{a}^\dagger} \mathrm{e}^{-\alpha^*\hat{a}}, \tag{3.3.16}$$

because

$$\mathrm{e}^{-\alpha^*\hat{a}} |0\rangle = |0\rangle . \tag{3.3.17}$$

Using the Campbell–Baker–Hausdorff theorem (see Appendix B.1) with $\hat{A} = \alpha\hat{a}^\dagger$, $\hat{B} = -\alpha^*\hat{a}$, $x = 1$ and because $\left[\hat{a}, \hat{a}^\dagger\right] = 1$, we can express it in a compact form

$$\boxed{\hat{D}(\alpha) \equiv \mathrm{e}^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}.} \tag{3.3.18}$$

It is a unitary operator and possesses the following interesting properties

$$\hat{D}^\dagger(\alpha) \hat{D}(\alpha) = \hat{1}, \quad \forall\alpha \in \mathbf{C}, \tag{3.3.19}$$

$$\hat{D}^\dagger(\alpha) = \hat{D}(-\alpha), \quad \forall\alpha \in \mathbf{C}, \tag{3.3.20}$$

$$\hat{D}(\gamma) \hat{D}(\alpha) = \mathrm{e}^{(\gamma\alpha^* - \gamma^*\alpha)/2} \hat{D}(\alpha + \gamma), \quad \forall\alpha, \gamma \in \mathbf{C}. \tag{3.3.21}$$

This last relation allows us to write (after some minor steps)

$$\hat{D}(\gamma) \hat{D}(\alpha) |0\rangle = \mathrm{e}^{(\gamma\alpha^* - \gamma^*\alpha)/2} \hat{D}(\alpha + \gamma) |0\rangle , \tag{3.3.22}$$

i.e.

$$\hat{D}(\gamma) |\alpha\rangle = \mathrm{e}^{(\gamma\alpha^* - \gamma^*\alpha)/2} |\alpha + \gamma\rangle . \tag{3.3.23}$$

Therefore, the translation[6] of a coherent state $|\alpha\rangle \to |\alpha + \gamma\rangle$ is

$$\boxed{|\alpha + \gamma\rangle = \mathrm{e}^{(\gamma^*\alpha - \gamma\alpha^*)/2} \hat{D}(\gamma) |\alpha\rangle .} \tag{3.3.24}$$

[6] Here, we refer to the translation of the α in the complex plane by a value of γ.

3.3.4 Phase-shifting operator

The phase-shifting operator is defined as [35, p. 18, (2.6)]

$$\boxed{\hat{U}(\theta) \equiv \mathrm{e}^{-\mathrm{i}\theta\hat{n}}.} \tag{3.3.25}$$

It can be used to define the rotation[7] of a coherent state $|\alpha\rangle \to \left|\alpha \mathrm{e}^{-\mathrm{i}\theta}\right\rangle$ as [35, p. 26, (2.47)]

$$\boxed{\left|\alpha \mathrm{e}^{-\mathrm{i}\theta}\right\rangle = \hat{U}(\theta)|\alpha\rangle.} \tag{3.3.26}$$

The Hermitian conjugate of the phase-shifting operator is

$$\hat{U}^{\dagger}(\theta) = \mathrm{e}^{+\mathrm{i}\theta\hat{n}^{\dagger}}. \tag{3.3.27}$$

Since[8]

$$\hat{n}^{\dagger} = \left(\hat{a}^{\dagger}\hat{a}\right)^{\dagger} = (\hat{a})^{\dagger}\left(\hat{a}^{\dagger}\right)^{\dagger} = \hat{a}^{\dagger}\hat{a} = \hat{n}, \tag{3.3.28}$$

we finally have

$$\boxed{\hat{U}^{\dagger}(\theta) = \mathrm{e}^{\mathrm{i}\theta\hat{n}} \equiv \hat{U}(-\theta).} \tag{3.3.29}$$

This naturally leads to the unitarity of the phase-shifting operator

$$\boxed{\hat{U}^{\dagger}(\theta)\hat{U}(\theta) = \hat{1}.} \tag{3.3.30}$$

3.4 Summary

In this chapter, we have introduced the quantization of the electromagnetic field which is an essential tool to study the states of light in quantum mechanics. In order to do this, we started from classical electrodynamics, and then added the quantization by imposing the canonical commutation relations. After, we defined the Fock states (or number states); they correspond to states with a fixed number of quanta (photons here). While they are easy to interpret, they are not easy to manipulate and they do not accurately represent the state coming from the usual sources of light encountered in laboratory (such as laser). This led us to the concept of coherent states of light. We studied them in detail because we will use them later. In the next chapter, we will see how we can efficiently characterize the coherent states.

[7] Here, we refer to the rotation of the α in the complex plane by an angle $-\theta$.

[8] One should not forget to permute the terms when taking the Hermitian conjugate of the product of operators.

Chapter 4

Phase space distributions

The phase space distributions (such as the WIGNER quasi-probability distribution, the P representation and the Q representation) are mathematical tools that can be used to characterize a state more efficiently because they possess some interesting properties. It is important to note that there exists a one-to-one correspondence between the density matrix and each of its corresponding phase space distributions. This means that it is equivalent to describe the state using either the density matrix or any of the available phase space distributions (which is not limited to the three distributions cited above).

The shortcomings of using the density matrix $\hat{\rho}$ directly might not be apparent at first. Indeed, a coherent state can be written fairly simply

$$\hat{\rho} = |\alpha\rangle \langle\alpha| \,. \tag{4.0.1}$$

However, if we want to represent it more precisely, we have to choose a basis. Naturally, we could use the Fock basis for that purpose, since we have seen there exists a relation between the Fock states and the coherent states (Eq. (3.3.2)). This is where the problems arise: in order to represent $\hat{\rho}$, it is necessary to write a matrix of infinite dimension since we sum over all Fock states (which are countably infinite) in Eq. (3.3.2). Obviously, this method is not a very intuitive or efficient way to characterize an arbitrary state.

On the other hand, the phase space distribution can be represented graphically, so the visualization is much easier. For instance, the WIGNER quasi-probability distribution (that we will describe below) of a coherent state is simply a Gaussian. As a result, it provides an alternative way to characterize states instead of the usual density matrix formalism which is not always easy to interpret/represent. Additionally, phase space distribution makes it possible to use one's intuition much more easily.

Furthermore, each distribution possesses different properties which offer more insight about the state in question. For instance, we know there exists certain bounds; if those bounds are exceeded, then the density matrix does not represent a state (at least not in the conventional sense). This can be used as a diagnostic tool to check if the calculations are correct. Moreover, the two marginal distributions of the WIGNER quasi-probability distribution $W(q,p)$ directly give the probability density of the state with respect to either q or p. Therefore, the form of the $W(q,p)$ informs us about the corresponding probabilities. These are just a few examples where the phase space distributions offer an advantage over the traditional matrix representation, but their benefits is certainly not limited to those cases.

In the first section, we will formalize the definition of a phase space distribution using the expected value of an arbitrary operator. We will see that there does not exist a single definition of a phase space distribution. Then, we will present a systematic way to introduce the different phase space distributions, and how this diversity can be used to our advantage. Finally, we will review the main phase space distributions that are used in practice, and that will be used later one as characterization tools: the WIGNER quasi-probability distribution, the Q representation and the Glauber-Sudarshan P representation.

Once again, we will not look into the time dependence of those phase space distributions, so we will omit the time parameter in the equations, while keeping in mind that in all generality, the phase space distributions depend on time. The main references are:

- General (and Q representation): [36, 37], [34, Ch. 3], [27, Ch. 11], [33, Ch. 4];
- WIGNER: [38, 39, 40, 41, 42];
- P representation:[43, 44]

4.1 Motivation

The interest in the phase space formulation of quantum mechanics can be summarized by expressing the expected value of an operator in phase space. Let $\hat{\rho}(\hat{q},\hat{p})$ be the density matrix of a certain state, let $\hat{A}(\hat{q},\hat{p})$ be an arbitrary operator and let $F^f(q,p)$ and $f(\xi,\eta)$ be a certain distribution function. Then, we have

$$\left\langle \hat{A}(\hat{q},\hat{p}) \right\rangle = \mathrm{Tr}\left[\hat{\rho}(\hat{q},\hat{p})\,\hat{A}(\hat{q},\hat{p})\right] = \iint_{\mathbf{R}^2} A(q,p)\,F(q,p)\;\mathrm{d}q\mathrm{d}p, \tag{4.1.1}$$

where the scalar function $A(q,p)$ is obtained by replacing the *operators* $\hat{q}$ and $\hat{p}$ (left-hand side) by the *variables* q and p (right-hand side). In essence, this equation means that the expected value of an operator $\hat{A}(\hat{q},\hat{p})$ (right-hand side) can be calculated using the scalar function associated with this operator and a distribution function (left-hand side). We will see that this is a very powerful mathematical framework of quantum mechanics because it allows to speed up the calculations.

From Eq. (4.1.1), we realize that the central tool that remains to be defined is the phase space distribution function. In reality, it turns out that there does not exist a single definition of this distribution function.

One way to illustrate the fact that the phase space distribution function is not unique is to use Eq. (4.1.1) for two similar, but different operators. First, if $\hat{A}_1(\hat{q},\hat{p}) = \exp(\mathrm{i}\xi\hat{q} + \mathrm{i}\eta\hat{p})$, then

$$\mathrm{Tr}\left[\hat{\rho}(\hat{q},\hat{p})\,\mathrm{e}^{\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}}\right] = \iint_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}\xi q+\mathrm{i}\eta p}F_1(q,p)\;\mathrm{d}q\mathrm{d}p, \tag{4.1.2}$$

and if $\hat{A}_2(\hat{q},\hat{p}) = \exp(\mathrm{i}\xi\hat{q})\exp(\mathrm{i}\eta\hat{p})$, then

$$\mathrm{Tr}\left[\hat{\rho}(\hat{q},\hat{p})\,\mathrm{e}^{\mathrm{i}\xi\hat{q}}\mathrm{e}^{\mathrm{i}\eta\hat{p}}\right] = \iint_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}\xi q+\mathrm{i}\eta p}F_2(q,p)\;\mathrm{d}q\mathrm{d}p. \tag{4.1.3}$$

However, we see that the phase space distributions must be different:

$$F_1(q,p) \neq F_2(q,p), \tag{4.1.4}$$

since the operators are not equal:

$$\mathrm{e}^{\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}} = \mathrm{e}^{\mathrm{i}\eta\hat{p}}\mathrm{e}^{\mathrm{i}\xi\hat{q}}\mathrm{e}^{-\mathrm{i}\xi\eta\hbar/2} \neq \mathrm{e}^{\mathrm{i}\xi\hat{q}}\mathrm{e}^{\mathrm{i}\eta\hat{p}}, \tag{4.1.5}$$

where the first equality was obtained by applying Campbell–Baker–Hausdorff theorem (Eq. (B.1.2)) with $x = \mathrm{i}$, $\hat{A} = \hat{q}$, $\hat{B} = \hat{p}$ and $[\hat{q}, \hat{p}] = \mathrm{i}\hbar$. This result comes from the fact that the operators $\hat{q}$ and $\hat{p}$ do not commute. Therefore, in order to uniquely define a phase space distribution, it is necessary to choose a rule of associating the *operators* $\hat{q}$ and $\hat{p}$ to their scalar counterparts q and p.

According to Cohen [36], all the quantum phase space distributions $F^f(q,p)$ that will be studied can be expressed in the following form:

$$\boxed{\mathrm{Tr}\left[\hat{\rho}(\hat{q},\hat{p})\, f(\xi,\eta)\, \mathrm{e}^{\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}}\right] = \iint_{\mathbf{R}^2} F^f(q,p)\, \mathrm{e}^{\mathrm{i}\xi q+\mathrm{i}\eta p}\, \mathrm{d}q\mathrm{d}p,} \tag{4.1.6}$$

This is also equivalent to (see Appendix A.3)

$$\boxed{F^f(q,p) = \frac{1}{4\pi^2}\iiint_{\mathbf{R}^3}\left\langle q' + \frac{1}{2}\eta\hbar\left|\hat{\rho}\right| q' - \frac{1}{2}\eta\hbar\right\rangle f(\xi,\eta)\, \mathrm{e}^{\mathrm{i}\xi(q'-q)}\mathrm{e}^{-\mathrm{i}\eta p}\, \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}q'.} \tag{4.1.7}$$

The function $f(\xi,\eta)$is the mathematical objects that models the rule of association that is chosen between the *operators* $\hat{q}$ and $\hat{p}$ and the *scalar variables* q and p.

4.2 Distributions in complex phase space

So far, we have used the operators $\hat{q}$ and $\hat{p}$ to describe the phase space distributions. However, some phase space distributions are usually defined for the operators $\hat{a}$ and $\hat{a}^\dagger$ (and their corresponding scalar variables α and α^*). For this reason, we will show the correspondence of the phase space distributions in terms of $\hat{q}$ and $\hat{p}$ or in terms of $\hat{a}$ and $\hat{a}^\dagger$.

From Eqs. (3.1.41) and (3.1.42), we deduce

$$\alpha = \frac{1}{(2\hbar\omega)^{1/2}}\left[\omega q + \mathrm{i}p\right], \tag{4.2.1}$$

$$\alpha^* = \frac{1}{(2\hbar\omega)^{1/2}}\left[\omega q - \mathrm{i}p\right], \tag{4.2.2}$$

or from Eqs. (3.3.3) and (3.3.4), we write

$$q = \left(\frac{\hbar}{2\omega}\right)^{1/2}\left[\alpha + \alpha^*\right] = \left(\frac{2\hbar}{\omega}\right)^{1/2}\Re(\alpha), \tag{4.2.3}$$

$$p = \mathrm{i}\left(\frac{\hbar\omega}{2}\right)^{1/2}\left[\alpha^* - \alpha\right] = (2\hbar\omega)^{1/2}\,\Im(\alpha). \tag{4.2.4}$$

The two phase space distributions $F^f(q,p)$ and $F^f(\alpha,\alpha^*)$ can be linked using the normalization condition:

$$\iint_{\mathbf{R}^2} F^f(q,p)\, \mathrm{d}q\mathrm{d}p = 1 = \iint_{\mathbf{R}^2} F^f(\alpha,\alpha^*)\, \mathrm{d}^2\alpha, \tag{4.2.5}$$

where

$$\mathrm{d}\alpha^2 \equiv \mathrm{d}\Re(\alpha)\, \mathrm{d}\Im(\alpha) = \frac{1}{2\hbar}\mathrm{d}q\mathrm{d}p. \tag{4.2.6}$$

Therefore,

$$\boxed{F^f(\alpha,\alpha^*) = 2\hbar F^f(q,p).} \tag{4.2.7}$$

Also, the variables ξ and η will be replaced by z and z^* defined by

$$z \equiv \mathrm{i}\xi\sqrt{\frac{\hbar}{2\omega}} - \eta\sqrt{\frac{\hbar\omega}{2}}. \tag{4.2.8}$$

In that case, the definition of the phase space distribution (Eq. (4.1.7)) transforms as follows

$$\boxed{F^f\left(\alpha,\alpha^*\right) \equiv \frac{1}{\pi^2}\iint_{\mathbf{R}^2} \mathrm{Tr}\left\{\hat{\rho}\left(\hat{a},\hat{a}^\dagger\right)\mathrm{e}^{z\hat{a}^\dagger - z^*\hat{a}} f\left(z,z^*\right)\right\}\mathrm{e}^{z^*\alpha - z\alpha^*}\,\mathrm{d}^2 z.} \tag{4.2.9}$$

4.3 Rule of association and operator ordering

Before studying the common phase space distribution, let us make one last remark about the functions $f\left(\xi,\eta\right)$. In the literature, it is common to find references to the concept of *operator ordering* associated with a choice of $f\left(\xi,\eta\right)$. Actually, it refers to the order of the operators $\hat{q}$ and $\hat{p}$ in the exponential factor $\exp\left(\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}\right)$. For example, if we choose

$$f^{\mathrm{W}}\left(\xi,\eta\right) = 1, \tag{4.3.1}$$

then

$$f^{\mathrm{W}}\left(\xi,\eta\right)\mathrm{e}^{\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}} = \mathrm{e}^{\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}} = \mathrm{e}^{\mathrm{i}(\xi\hat{q}+\eta\hat{p})/2+\mathrm{i}(\eta\hat{p}+\xi\hat{q})/2}. \tag{4.3.2}$$

In that case, we see that the operators $\hat{q}$ and $\hat{p}$ are expressed in a symmetric order, and this order is known as the Weyl *order*. In reality, this choice of $f^{\mathrm{W}}\left(\xi,\eta\right)$ yields the Wigner quasi-probability distribution that will presented in the next section.

A less trivial example can be obtained easily. Indeed, using Campbell–Baker–Hausdorff theorem (Eq. (B.1.2)) with $x=\mathrm{i}$, $\hat{A}=\hat{q}$, $\hat{B}=\hat{p}$ and $[\hat{q},\hat{p}]=\mathrm{i}\hbar$, we can write

$$\mathrm{e}^{\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}} = \mathrm{e}^{\mathrm{i}\eta\hat{p}}\mathrm{e}^{\mathrm{i}\xi\hat{q}}\mathrm{e}^{-\mathrm{i}\xi\eta\hbar/2}, \tag{4.3.3}$$

$$\mathrm{e}^{\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}} = \mathrm{e}^{\mathrm{i}\xi\hat{q}}\mathrm{e}^{\mathrm{i}\eta\hat{p}}\mathrm{e}^{\mathrm{i}\xi\eta\hbar/2}, \tag{4.3.4}$$

or

$$\mathrm{e}^{\mathrm{i}\xi\eta\hbar/2}\mathrm{e}^{\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}} = \mathrm{e}^{\mathrm{i}\eta\hat{p}}\mathrm{e}^{\mathrm{i}\xi\hat{q}} = \left(\sum_{m=0}^{\infty}\frac{(\eta\hat{p})^m}{m!}\right)\left(\sum_{n=0}^{\infty}\frac{(\xi\hat{q})^n}{n!}\right), \tag{4.3.5}$$

$$\mathrm{e}^{-\mathrm{i}\xi\eta\hbar/2}\mathrm{e}^{\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}} = \mathrm{e}^{\mathrm{i}\xi\hat{q}}\mathrm{e}^{\mathrm{i}\eta\hat{p}} = \left(\sum_{m=0}^{\infty}\frac{(\xi\hat{q})^m}{m!}\right)\left(\sum_{n=0}^{\infty}\frac{(\eta\hat{p})^n}{n!}\right),. \tag{4.3.6}$$

Therefore, if we choose

$$f^{\mathrm{S}}\left(\xi,\eta\right) = \mathrm{e}^{-\mathrm{i}\xi\eta\hbar/2}, \tag{4.3.7}$$

the operators will be arranged such that all the powers of the operator $\hat{q}$ are on the left of all the powers of the operator $\hat{p}$: this is called the *standard order*. Similarly, the choice

$$f^{\mathrm{AS}}\left(\xi,\eta\right) = \mathrm{e}^{\mathrm{i}\xi\eta\hbar/2}, \tag{4.3.8}$$

does exactly the opposite (all the powers of the operator $\hat{q}$ are now on the right of all the powers of the operator $\hat{p}$), and it is named the *anti-standard-order*.

This is where the strength (or weakness) of a phase space distribution stems. To illustrate this point, let Ω be an arbitrary operator ordering, $\bar{\Omega}$ be the reciprocal operator ordering, and $\hat{A}\left(\hat{a},\hat{a}^\dagger\right)$ be an arbitrary operator. Then, it can be shown that for the Ω-ordered operator $\hat{A}^{(\Omega)}\left(\hat{a},\hat{a}^\dagger\right)$ [27, p. 559, (11.10–2)]

$$\left\langle \hat{A}^{(\Omega)}\left(\hat{a},\hat{a}^\dagger\right)\right\rangle = \frac{1}{\pi}\iint_{\mathbf{R}^2}\rho^{(\bar{\Omega})}\left(\alpha,\alpha^*\right)A^{(\Omega)}\left(\alpha,\alpha^*\right)\mathrm{d}^2\alpha. \tag{4.3.9}$$

Essentially, it means the following: if the representation is such that the phase space distribution can be obtained by putting the operator $\hat{\rho}$ in the $\bar{\Omega}$ order, and then replacing the operators $\hat{a}$ and $\hat{a}^\dagger$ by α and α^*, then the representation of an arbitrary operator $\hat{A}\left(\hat{a}, \hat{a}^\dagger\right)$ is obtained by putting it in the Ω order, and then replacing the operators $\hat{a}$ and $\hat{a}^\dagger$ by α and α^*. This is important because, depending on the problem, we might have an abundance of operators in a specific order Ω. Consequently, we could take advantage of this fact by working in the representation which is Ω-ordered because it will be easy to determine the corresponding scalar function $A^{(\Omega)}\left(\alpha, \alpha^*\right)$.

Now that we have clarified the motivation behind the quantum phase space formalism and how the operator ordering plays a special role in the choice of a phase space distribution, we can go on to present the various representations we will use later on in a systematic way. First, we will give the definition, then expose some of its properties and give an example. Note that the properties will be sometimes be proven by writing the state $\hat{\rho}$ as the (classical) statistical mixture of pure states as defined by Eq. (1.1.6):

$$\hat{\rho} = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| . \tag{4.3.10}$$

4.4 WIGNER quasi-probability distribution

4.4.1 Definition

The WIGNER quasi-probability distribution $W\left(q, p\right)$ is undoubtedly the most famous example of a phase space distribution because it is used in many other fields, not just in quantum optics. As said previously, it corresponds to a simple choice for $f^{\mathrm{W}}\left(\xi, \eta\right)$ because

$$f^W\left(\xi, \eta\right) = 1. \tag{4.4.1}$$

In that case, Eq. (4.1.7) becomes

$$W\left(q, p\right) = \frac{1}{4\pi^2} \iiint_{\mathbf{R}^3} \left\langle q' + \frac{1}{2}\eta\hbar \right| \hat{\rho} \left| q' - \frac{1}{2}\eta\hbar \right\rangle \mathrm{e}^{\mathrm{i}\xi\left(q'-q\right)} \mathrm{e}^{-\mathrm{i}\eta p}\, \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}q'. \tag{4.4.2}$$

This reduces to

$$\boxed{W\left(q, p\right) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\langle q + \frac{1}{2}\eta\hbar \right| \hat{\rho} \left| q - \frac{1}{2}\eta\hbar \right\rangle \mathrm{e}^{-\mathrm{i}\eta p}\, \mathrm{d}\eta,} \tag{4.4.3}$$

because

$$\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\xi\left(q'-q\right)}\, \mathrm{d}\xi = 2\pi\delta\left(q' - q\right). \tag{4.4.4}$$

In the case of a coherent state $\hat{\rho} = \left|\alpha\right\rangle\left\langle\alpha\right|$, it simply becomes

$$W_\alpha\left(q, p\right) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\langle q + \frac{1}{2}\eta\hbar \middle| \alpha \right\rangle \left\langle \alpha \middle| q - \frac{1}{2}\eta\hbar \right\rangle \mathrm{e}^{-\mathrm{i}\eta p}\, \mathrm{d}\eta, \tag{4.4.5}$$

or

$$W_\alpha\left(q, p\right) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} \psi_\alpha\left(q + \frac{1}{2}\eta\hbar\right) \psi_\alpha^*\left(q - \frac{1}{2}\eta\hbar\right) \mathrm{e}^{-\mathrm{i}\eta p}\, \mathrm{d}\eta, \tag{4.4.6}$$

where

$$\psi_\alpha\left(q + \frac{1}{2}\eta\hbar\right) \equiv \left\langle q + \frac{1}{2}\eta\hbar \middle| \alpha \right\rangle. \tag{4.4.7}$$

The result is a 2D Gaussian (see Section (4.7) for the derivation):

$$\boxed{W_{\alpha}(q,p)=\frac{1}{\pi\hbar}\exp\left\{-2\left[\left(\frac{\omega}{2\hbar}\right)^{1/2}q-\Re(\alpha)\right]^{2}-2\left[\frac{1}{(2\hbar\omega)^{1/2}}p-\Im(\alpha)\right]^{2}\right\}.} \tag{4.4.8}$$

or

4.4.2 Properties

1. Quasi-probability distribution in which operators are described in **Weyl order** ($\hat{q}$ and $\hat{p}$ are arranged symmetrically).

$$\boxed{f^{\mathrm{W}}(\xi,\eta)=1.} \tag{4.4.9}$$

2. **Normalization to unity**

$$\boxed{\iint_{\mathbf{R}^2} W(q,p)\,\mathrm{d}q\mathrm{d}p=1.} \tag{4.4.10}$$

Proof

We have

$$\begin{aligned}W(q,p)&=\frac{1}{2\pi}\iiint_{\mathbf{R}^3}\left\langle q+\frac{1}{2}\eta\hbar\left|\hat{\rho}\right|q-\frac{1}{2}\eta\hbar\right\rangle\mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta\mathrm{d}q\mathrm{d}p\\&=\frac{1}{\cancel{2\pi}}\iint_{\mathbf{R}^2}\left\langle q+\frac{1}{2}\eta\hbar\left|\hat{\rho}\right|q-\frac{1}{2}\eta\hbar\right\rangle\cancel{2\pi}\delta(\eta)\,\mathrm{d}\eta\mathrm{d}q\\&=\int_{-\infty}^{+\infty}\langle q\left|\hat{\rho}\right|q\rangle\,\mathrm{d}q\\&\equiv\mathrm{Tr}\,\hat{\rho}\\&=1,\end{aligned} \tag{4.4.11}$$

since the state is normalized.

3. **Upper and lower bounds**

$$\boxed{|W(q,p)|\leq\frac{1}{\pi\hbar}.} \tag{4.4.12}$$

Proof

We have

$$\begin{aligned}|W(q,p)|^2&=\left|\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\langle q+\frac{1}{2}\eta\hbar\left|\sum_i p_i\left|\psi_i\right\rangle\left\langle\psi_i\right|\right|q-\frac{1}{2}\eta\hbar\right\rangle\mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta\right|^2\\&=\left|\frac{\sum_i p_i}{2\pi}\int_{-\infty}^{+\infty}\psi_i\left(q+\frac{1}{2}\eta\hbar\right)\psi_i^*\left(q-\frac{1}{2}\eta\hbar\right)\mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta\right|^2.\end{aligned} \tag{4.4.13}$$

We apply the CAUCHY–SCHWARZ inequality:

$$|W(q,p)|^2 \leq \frac{\sum_i p_i}{(2\pi)^2} \int_{-\infty}^{+\infty} \left|\psi_i\left(q + \frac{1}{2}\eta\hbar\right)\right|^2 \mathrm{d}\eta \times \int_{-\infty}^{+\infty} \left|\psi_i\left(q - \frac{1}{2}\eta\hbar\right)\right|^2 \mathrm{d}\eta. \quad (4.4.14)$$

We then perform the substitutions

$$u = q + \frac{1}{2}\eta\hbar, \quad (4.4.15)$$

$$v = q - \frac{1}{2}\eta\hbar, \quad (4.4.16)$$

which leads to

$$\mathrm{d}\eta = \frac{2}{\hbar}\,\mathrm{d}u, \quad (4.4.17)$$

$$\mathrm{d}\eta = -\frac{2}{\hbar}\,\mathrm{d}v, \quad (4.4.18)$$

with

$$\lim_{\eta\to-\infty} u = -\infty, \qquad \lim_{\eta\to+\infty} u = +\infty, \quad (4.4.19)$$

$$\lim_{\eta\to-\infty} v = +\infty, \qquad \lim_{\eta\to+\infty} v = -\infty. \quad (4.4.20)$$

Therefore

$$\int_{-\infty}^{+\infty} \left|\psi_i\left(q + \frac{1}{2}\eta\hbar\right)\right|^2 \mathrm{d}\eta = \frac{2}{\hbar}\int_{-\infty}^{+\infty} |\psi_i(u)|^2 \,\mathrm{d}u = \frac{2}{\hbar},$$

and

$$\begin{aligned}\int_{-\infty}^{+\infty} \left|\psi_i\left(q - \frac{1}{2}\eta\hbar\right)\right|^2 \mathrm{d}\eta &= -\frac{2}{\hbar}\int_{+\infty}^{-\infty} |\psi_i(v)|^2 \,\mathrm{d}v \\ &= +\frac{2}{\hbar}\int_{-\infty}^{+\infty} |\psi_i(v)|^2 \,\mathrm{d}v \\ &= \frac{2}{\hbar}. \end{aligned} \quad (4.4.21)$$

Eq. (4.4.14) becomes

$$|W(q,p)|^2 \leq \frac{\sum_i p_i}{4\pi^2} \times \frac{2}{\hbar} \times \frac{2}{\hbar} = \frac{1}{\pi^2\hbar^2}, \quad (4.4.22)$$

or

$$|W(q,p)| \leq \frac{1}{\pi\hbar}. \quad (4.4.23)$$

This is the best bound we can find, and it allows the WIGNER quasi-probability distribution to become slightly negative, but not too much. That is why we cannot interpret it directly as a joint probability distribution of observing the state in (q,p) because negative probabilities have no clear interpretation. This strange property should be not taken treated as a failure of the formalism. Indeed, the phase space distribution is only a mathematical tool to make the calculations easier. We can use any quasi-probability distributions as long as it yields the

correct expected values according to Eq. (4.1.6)).
In reality, this property is the result of the HEISENBERG's uncertainty principle: we cannot attribute a signification to a single point in phase space, we need to integrate over a small region in order to obtain any meaningful information. In the case of the WIGNER quasi-probability distribution, it can be shown that if we integrate it over a sub-region of at least a few $\hbar$, then the result is always positive [39], so the interpretation problem is gone. Also, some [45] have proposed to use this negativity as a criterion for characterizing the degree of non-classicality of a quantum state.

4. **Reality**

$$\boxed{W^*(q,p) = W^*(q,p)\,.} \tag{4.4.24}$$

Proof

We have

$$\begin{aligned} W^*(q,p) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left\langle q + \frac{1}{2}\eta\hbar \left|\hat{\rho}\right| q - \frac{1}{2}\eta\hbar \right\rangle^* \mathrm{e}^{+\mathrm{i}\eta p}\,\mathrm{d}\eta \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left\langle q - \frac{1}{2}\eta\hbar \left|\hat{\rho}^\dagger\right| q + \frac{1}{2}\eta\hbar \right\rangle \mathrm{e}^{\mathrm{i}\eta p}\,\mathrm{d}\eta \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left\langle q - \frac{1}{2}\eta\hbar \left|\hat{\rho}\right| q + \frac{1}{2}\eta\hbar \right\rangle \mathrm{e}^{\mathrm{i}\eta p}\,\mathrm{d}\eta, \end{aligned} \tag{4.4.25}$$

because $\hat{\rho}^\dagger = \hat{\rho}$. Now if we make the substitution $\eta \to -\eta$, we obtain

$$\begin{aligned} W^*(q,p) &= -\frac{1}{2\pi}\int_{+\infty}^{-\infty} \left\langle q + \frac{1}{2}\eta\hbar \left|\hat{\rho}\right| q - \frac{1}{2}\eta\hbar \right\rangle \mathrm{e}^{\mathrm{i}\eta p}\,\mathrm{d}\eta \\ &= +\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left\langle q + \frac{1}{2}\eta\hbar \left|\hat{\rho}\right| q - \frac{1}{2}\eta\hbar \right\rangle \mathrm{e}^{\mathrm{i}\eta p}\,\mathrm{d}\eta \\ &\equiv W(q,p)\,. \end{aligned} \tag{4.4.26}$$

5. **Marginal distributions (or reduced distributions)**

$$\boxed{\int_{-\infty}^{\infty} W(q,p)\ \mathrm{d}p = \langle q\left|\hat{\rho}\right|q\rangle = \mathrm{Tr}\,(\hat{\rho}\left|q\right\rangle\left\langle q\right|) = \langle |q\rangle\langle q|\rangle\,,} \tag{4.4.27}$$

and

$$\boxed{\int_{-\infty}^{\infty} W(q,p)\ \mathrm{d}x = \langle p\left|\hat{\rho}\right|p\rangle = \mathrm{Tr}\,(\hat{\rho}\left|p\right\rangle\left\langle p\right|) = \langle |p\rangle\langle p|\rangle\,.} \tag{4.4.28}$$

Proof

We have

$$\int_{-\infty}^{\infty} W(q,p)\ \mathrm{d}p = \frac{1}{2\pi}\iint_{\mathbf{R}^2}\left\langle q+\frac{1}{2}\eta\hbar\left|\hat{\rho}\right|q-\frac{1}{2}\eta\hbar\right\rangle \mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta\mathrm{d}p$$

$$= \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty}\left\langle q+\frac{1}{2}\eta\hbar\left|\hat{\rho}\right|q-\frac{1}{2}\eta\hbar\right\rangle \cancel{2\pi}\delta(\eta)\ \mathrm{d}\eta$$

$$= \langle q|\hat{\rho}|q\rangle \tag{4.4.29}$$

$$= \mathrm{Tr}\ (\hat{\rho}\,|q\rangle\langle q|)\,. \tag{4.4.30}$$

The other relation can be obtained similarly by using the completeness relation of the momentum basis $\{|p\rangle\}$ (this method will be used in Section 5.1.3.4).

In the case of a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$ this leads to

$$\int_{-\infty}^{\infty} W(q,p)\ \mathrm{d}p = |\langle q|\psi\rangle|^2\,, \tag{4.4.31}$$

and

$$\int_{-\infty}^{\infty} W(q,p)\ \mathrm{d}q = |\langle p|\psi\rangle|^2\,. \tag{4.4.32}$$

Those two properties are very interesting because it allows us to have easy access to the probability densities. Also, just by looking at the shape of the $W(q,p)$, we can potentially get an idea of the shape of these probability densities. In addition, this is one way to reconstruct the WIGNER quasi-probability distribution in a lab. Scientist have access to the marginal distribution, and, by measuring them for different angles, they can recover the shape of $W(q,p)$ from the data in a process identical to medical tomography.

Illustration

Using Eq. (4.1.1), we can express expected values using the WIGNER quasi-probability distribution. Let $\hat{A}(\hat{q},\hat{p})$ be an arbitrary operator and $\hat{\rho}$ be the density matrix of a state. If

$$A(q,p) \equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\langle q+\frac{1}{2}\eta\hbar\left|\hat{A}(\hat{q},\hat{p})\right|q-\frac{1}{2}\eta\hbar\right\rangle \mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta, \tag{4.4.33}$$

and

$$W(q,p) \equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\langle q+\frac{1}{2}\eta\hbar\left|\hat{\rho}\right|q-\frac{1}{2}\eta\hbar\right\rangle \mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta, \tag{4.4.34}$$

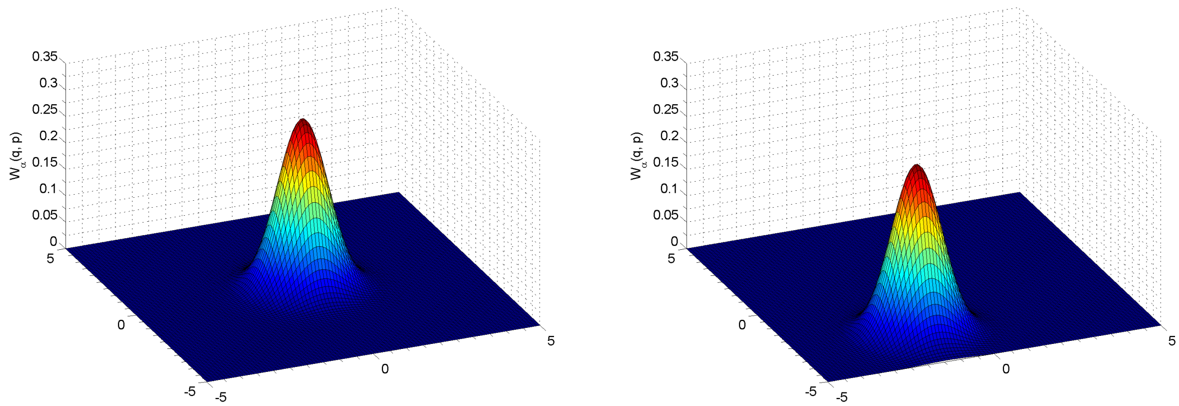
then

$$\left\langle \hat{A}\right\rangle = \mathrm{Tr}\ \left[\hat{\rho}(\hat{q},\hat{p})\,\hat{A}(\hat{q},\hat{p})\right] = \iint_{\mathbf{R}^2} A(q,p)\,W(q,p)\ \mathrm{d}q\mathrm{d}p. \tag{4.4.35}$$

4.4.3 Example

The WIGNER quasi-probability distribution of a coherent is simply a Gaussian as presented on Fig. 4.4.1 (see Section 4.7 for the derivation, Eqs. (4.4.8) and (4.7.15) for the final expressions). We also notice that the WIGNER quasi-probability distribution of a coherent state $|\alpha = 1+2\mathrm{i}\rangle$ is identical to the vacuum state one, except it has been translated. This mirrors what we said in Section 3.3.3 where the coherent state can be viewed as a displaced vacuum state.

We should insist here on the fact that the WIGNER quasi-probability distribution also plays a special role in quantum information theory. Indeed, the states that are the most used in quantum information using photons as qubits are the *Gaussian* states. They actually correspond to the states that have a *Gaussian* WIGNER quasi-probability distribution.



(a) Vacuum state $|\alpha\rangle = |0\rangle$. (b) Coherent state $|\alpha = -1 - 2\mathrm{i}\rangle$.

Figure 4.4.1: WIGNER quasi-probability distribution of two coherent states with $\hbar = \omega = 1$.

4.5 Q representation

4.5.1 Definition

The Q representation[1] is a quasi-probability distribution defined as [33, p. 65, (4.50)]

$$\boxed{Q(\gamma) \equiv \frac{\langle \gamma | \hat{\rho} | \gamma \rangle}{\pi} = \frac{1}{\pi} \mathrm{Tr}\left(\hat{\rho} |\gamma\rangle \langle \gamma|\right).} \tag{4.5.1}$$

It is proportional to the expected value of the projector $|\gamma\rangle\langle\gamma|$ for the state defined by $\hat{\rho}$, i.e. to the probability that the state $|\gamma\rangle\langle\gamma|$ is measured when the system is prepared in the state $\hat{\rho}$.

4.5.2 Properties

1. Quasi-probability distribution in which operators are described in **anti-normal order** (creation operators $\hat{a}^\dagger$ on the right of the annihilation operators $\hat{a}$):

$$\boxed{f^Q(z, z^*) = \exp\left[\frac{|z|^2}{2}\right].} \tag{4.5.2}$$

[1] It is sometimes called the HUSIMI quasi-probability distribution, but it is a particular case of the latter according to [37].

2. **Normalization to unity**

$$\boxed{\iint_{\mathbf{R}^2} Q(\gamma)\ \mathrm{d}\gamma^2 = 1.} \tag{4.5.3}$$

Proof

We have

$$\begin{aligned}
\iint_{\mathbf{R}^2} Q(\gamma)\ \mathrm{d}\gamma^2 &= \iint_{\mathbf{R}^2} \frac{1}{\pi}\sum_i p_i \langle\gamma|\psi_i\rangle \langle\psi_i|\gamma\rangle\ \mathrm{d}\gamma^2 \\
&= \frac{1}{\pi}\sum_i p_i \iint_{\mathbf{R}^2} \langle\psi_i|\gamma\rangle \langle\gamma|\psi_i\rangle\ \mathrm{d}\gamma^2 \\
&= \sum_i p_i \left\langle \psi_i \middle| \left(\frac{1}{\pi}\iint_{\mathbf{R}^2} |\gamma\rangle\langle\gamma|\ \mathrm{d}\gamma^2\right) \middle| \psi_i \right\rangle \\
&= \sum_i p_i \langle\psi_i|\psi_i\rangle \\
&= \mathrm{Tr}\ (\hat{\rho}) \\
&= 1,
\end{aligned} \tag{4.5.4}$$

where we used the over-completeness of the coherent states (Eq. (3.3.11)), the fact that the states are normalized $\langle\psi_i|\psi_i\rangle = 1$ and that the probabilities must add up to unity.

3. **Upper and lower bounds**

$$\boxed{0 \le Q(\gamma) \le \frac{1}{\pi}.} \tag{4.5.5}$$

Proof

We have

$$Q(\gamma) = \frac{1}{\pi}\sum_i p_i \langle\gamma|\psi_i\rangle \langle\psi_i|\gamma\rangle = \frac{1}{\pi}\sum_i p_i \left|\langle\gamma|\psi_i\rangle\right|^2. \tag{4.5.6}$$

Since $\left|\langle\gamma|\psi_i\rangle\right|^2$ corresponds to the probability of measuring the state $|\gamma\rangle\langle\gamma|$ when system is prepared in $|\psi_i\rangle\langle\psi_i|$ so

$$0 \le \left|\langle\gamma|\psi_i\rangle\right|^2 \le 1. \tag{4.5.7}$$

Then,

$$0 \le \frac{1}{\pi}\sum_i p_i \left|\langle\gamma|\psi_i\rangle\right|^2 \le \frac{1}{\pi}\sum_i p_i = \frac{1}{\pi}, \tag{4.5.8}$$

or

$$0 \le Q(\gamma) \le \frac{1}{\pi}. \tag{4.5.9}$$

Of all the quasi-probability distribution that we will study, this is the only one that is always positive or equal to zero, no matter the state $\hat{\rho}$.

4. **Reality**

$$\boxed{Q^*(\gamma) = Q(\gamma).} \tag{4.5.10}$$

Proof

It is pretty easy to see

$$\begin{aligned} Q^*(\gamma) &= \left(\frac{1}{\pi}\sum_i p_i \langle\gamma|\psi_i\rangle\langle\psi_i|\gamma\rangle\right)^* \\ &= \frac{1}{\pi}\sum_i p_i \langle\gamma|\psi_i\rangle^*\langle\psi_i|\gamma\rangle^* \\ &= \frac{1}{\pi}\sum_i p_i \langle\psi_i|\gamma\rangle\langle\gamma|\psi_i\rangle \\ &= \frac{1}{\pi}\sum_i p_i \langle\gamma|\psi_i\rangle\langle\psi_i|\gamma\rangle \\ &= Q(\gamma). \end{aligned} \tag{4.5.11}$$

4.5.3 Example

The Q representation of a coherent state $\hat{\rho} = |\alpha\rangle\langle\alpha|$ is very easy to calculate using the formula for the overlap of two coherent states (Eq. 3.3.10):

$$Q_\alpha(\gamma) \equiv \frac{\langle\gamma|\hat{\rho}|\gamma\rangle}{\pi} == \frac{\langle\gamma|\alpha\rangle\langle\alpha|\gamma\rangle}{\pi} = \frac{|\langle\gamma|\alpha\rangle|^2}{\pi}, \tag{4.5.12}$$

or

$$\boxed{Q_\alpha(\gamma) = \frac{\mathrm{e}^{-|\alpha-\gamma|^2}}{\pi}.} \tag{4.5.13}$$

Evidently, we see that this function is indeed positive, upper bounded by $1/\pi$ and always real.

4.6 Glauber--Sudarshan *P* representation

For completeness, we introduce here the Glauber--Sudarshan ***P*** representation because of its importance in quantum optics. However, we will not be able to study it in details because its definition make it hard to calculate.

4.6.1 Definition

The Glauber--Sudarshan P representation is a quasi-probability distribution defined as [33, p. 58, (4.8)]

$$\boxed{\hat{\rho} \equiv \iint_{\mathbf{R}^2} P(\gamma)\,|\gamma\rangle\langle\gamma|\,\mathrm{d}^2\gamma,} \tag{4.6.1}$$

where

$$\mathrm{d}\gamma^2 \equiv \mathrm{d}\Re(\gamma)\, \mathrm{d}\Im(\gamma)\,. \tag{4.6.2}$$

While the P representation does not always exist as a well-behaved function, it has been demonstrated that it always exists as a distribution with singularities [46]. Also, according to [27], if the system has a classical analog (e.g. a coherent state), it is non-negative everywhere like an ordinary probability distribution. If the system has no classical analog (e.g. entangled system, incoherent Fock state), P will be negative somewhere. Even if the P representation stays positive everywhere, we should be careful about interpreting it as a true probability density. Indeed, since the coherent states are not mutually orthogonal, "it would not describe probability of mutually exclusive states" [27, p. 541].

An alternative definition can be given (here for a single mode) [44]

$$P(\gamma) = \sum_{n=0}^{+\infty}\sum_{n'=0}^{+\infty} \langle n\,|\hat{\rho}|\,k\rangle \frac{\sqrt{n!n'!}}{(n+n')!\,(2\pi r)} \mathrm{e}^{r^2+\mathrm{i}(n-n')\theta} \left[\left(-\frac{\partial}{\partial r}\right)^{n+n'} \delta(r)\right], \tag{4.6.3}$$

where r and θ are respectively the amplitude and the phase of γ. Once again, we see that this is certainly not an easy object to calculate, especially since there is an infinite sum of derivatives of Dirac's delta function. It's not surprising that the P representation can become highly singular by looking at this definition.

Incidentally, in this quasi-probability distribution, the operators are described in **normal order** (creation operators $\hat{a}^\dagger$ on the left of the annihilation operators $\hat{a}$):

$$\boxed{f^P(z, z^*) = \exp\left[-\frac{|z|^2}{2}\right],} \tag{4.6.4}$$

In fact, it arises naturally in quantum optics because a lot of operators are in the normal order.

4.6.2 Example

The P representation is not easy to manipulate because of its definition. However, if we take the case of a coherent where

$$\hat{\rho} = |\alpha\rangle\langle\alpha|\,, \tag{4.6.5}$$

then, it is pretty obvious that

$$\boxed{P_\alpha(\gamma) = \delta^2(\alpha-\gamma)\,,} \tag{4.6.6}$$

because

$$\int \delta^2(\alpha-\gamma)\,|\gamma\rangle\langle\gamma|\,\mathrm{d}^2\gamma = |\alpha\rangle\langle\alpha| = \hat{\rho}. \tag{4.6.7}$$

4.6.3 Generalization

As suggested by [43], it is also possible to generalize the P representation using the following definition:

$$\hat{\rho} = \int_D \Lambda(\gamma,\zeta)\, P(\gamma,\zeta)\, \mathrm{d}\mu(\gamma,\zeta)\,, \tag{4.6.8}$$

where

$$\Lambda(\gamma,\zeta) = \frac{|\gamma\rangle\langle\zeta^*|}{\langle\zeta^*|\gamma\rangle}, \tag{4.6.9}$$

$\mathrm{d}\mu(\gamma,\zeta)$ is the integration measure and D is the domain of integration. The choice of $\mathrm{d}\mu(\gamma,\zeta)$ will define different P representations:

1. Glauber–Sudarshan P representation

$$\mathrm{d}\mu(\gamma,\zeta) = \delta^2(\gamma^* - \zeta)\,\mathrm{d}^2\gamma\,\mathrm{d}^2\zeta. \tag{4.6.10}$$

2. Complex P representation

$$\mathrm{d}\mu(\gamma,\zeta) = \mathrm{d}\gamma\,\mathrm{d}\zeta. \tag{4.6.11}$$

3. Positive P representation

$$\mathrm{d}\mu(\gamma,\zeta) = \mathrm{d}^2\gamma\,\mathrm{d}^2\zeta. \tag{4.6.12}$$

4.7 Relations between quantum phase space distributions

As we have seen each quantum phase space distribution constitutes an equivalent of a state described by $\hat{\rho}$. Also, depending on the ordering of the operators (see Section 4.3), some distributions might be easier to evaluate than others. For that reason, it could prove to be useful to find relations between the distributions because it could make the calculations easier. If we take two arbitrary phase space distributions $F^1(\alpha,\alpha^*)$ and $F^2(\alpha',\alpha'^*)$, then they are related by [37, p. 168]:

$$\boxed{F^1(\alpha,\alpha^*) = \iint_{\mathbf{R}^2} g\left(\alpha'-\alpha,\alpha'^*-\alpha^*\right) F^2\left(\alpha',\alpha'^*\right)\,\mathrm{d}^2\alpha',} \tag{4.7.1}$$

where

$$\boxed{g(\alpha,\alpha^*) \equiv \frac{1}{\pi^2}\iint_{\mathbf{R}^2} \mathrm{e}^{z\alpha^*-z^*\alpha}\frac{f^1(z,z^*)}{f^2(z,z^*)}\,\mathrm{d}^2 z.} \tag{4.7.2}$$

For simplicity, we will use the superscript Q, P or W for the Q representation, the Glauber–Sudarshan P representation and the Wigner quasi-probability distribution, respectively. After some calculations (see Appendix A.4), we obtain the following relations (see also [37, p. 169]:

$$F^Q(\alpha,\alpha^*) = \frac{2}{\pi}\int_{\mathbf{R}^2} \mathrm{e}^{-2|\alpha'-\alpha|^2} F^W\left(\alpha',\alpha'^*\right)\,\mathrm{d}^2\alpha', \tag{4.7.3}$$

$$F^W(\alpha,\alpha^*) = \frac{2}{\pi}\int_{\mathbf{R}^2} \mathrm{e}^{-2|\alpha'-\alpha|^2} F^P\left(\alpha',\alpha'^*\right)\,\mathrm{d}^2\alpha', \tag{4.7.4}$$

$$F^Q(\alpha,\alpha^*) = \frac{1}{\pi}\int_{\mathbf{R}^2} \mathrm{e}^{-|\alpha'-\alpha|^2} F^P\left(\alpha',\alpha'^*\right)\,\mathrm{d}^2\alpha'. \tag{4.7.5}$$

One interesting thing to remark about the Q representation is equal to the convolution of the Wigner quasi-probability distribution with a Gaussian, which is itself equal to the convolution of the P representation with the same Gaussian; this convolution can be viewed as a smoothing of the function with a Gaussian filter. We can therefore expect the Q representation to be the smoothest of the three, while the P representation will contain the most irregularities. In fact, the P representation can even become more singular than Dirac's delta function [27, p. 541].

Since the definition of the P representation (Eq. (4.6.1)) is not easy to manipulate, it might be interesting to express the P representation with respect to the other phase space distributions. Unfortunately, when trying to compute a relation expressing the P representation in terms of Q representation or the Wigner quasi-probability distribution, the integration on $\mathrm{d}^2 z$ in $g(\alpha,\alpha^*)$ does not converge, so in that case, the best we can say when trying to write the P representation in terms of Q representation is

$$F^P(\alpha,\alpha^*) = \iint_{\mathbf{R}^2} g\left(\alpha'-\alpha,\alpha'^*-\alpha^*\right) F^Q\left(\alpha',\alpha'^*\right)\,\mathrm{d}^2\alpha', \tag{4.7.6}$$

with

$$g(\alpha,\alpha^*) \equiv \frac{1}{4\pi^2} \iint_{\mathbf{R}^2} \mathrm{e}^{z\alpha^* - z^*\alpha + z_r^2 + z_i^2}\, \mathrm{d}^2 z. \tag{4.7.7}$$

It is most likely necessary to integrate over α' before integrating over z which is not possible without postulating a form for $F^Q(\alpha',\alpha'^*)$ or $F^W(\alpha',\alpha'^*)$ (see Appendix A.4). There also exists a formal differential relation [37, p. 169]:

$$F^P(\alpha,\alpha^*) = \exp\left(-\frac{1}{2}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right) F^W(\alpha,\alpha^*) = \exp\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right) F^Q(\alpha,\alpha^*), \tag{4.7.8}$$

but this expression is not necessarily simpler to use because of the exponential.

Illustration

As an example of the usefulness of these relations, we can calculate the WIGNER quasi-probability distribution of a coherent state $|\alpha\rangle\langle\alpha|$. In Section 4.6.2, we have seen that the corresponding P representation is simply

$$F^P(\gamma,\gamma^*) \equiv P_\alpha(\gamma) = \delta^2(\alpha-\gamma). \tag{4.7.9}$$

Now, if we use Eq. (4.7.4), we have

$$F^W(\gamma,\gamma^*) = \frac{2}{\pi}\int_{\mathbf{R}^2} \mathrm{e}^{-2|\gamma'-\gamma|^2} F^P(\gamma',\gamma'^*)\, \mathrm{d}^2\gamma' = \frac{2}{\pi}\int_{\mathbf{R}^2} \mathrm{e}^{-2|\gamma'-\gamma|^2} \delta^2(\alpha-\gamma')\, \mathrm{d}^2\gamma', \tag{4.7.10}$$

or

$$\boxed{F^W(\gamma,\gamma^*) = \frac{2}{\pi}\mathrm{e}^{-2|\alpha-\gamma|^2}.} \tag{4.7.11}$$

Equivalently, from Eq. (4.2.7), we can write the WIGNER quasi-probability distribution in terms of variables q and p (which are related to γ and γ^* through Eqs. (4.2.3) and (4.2.4)):

$$q = \left(\frac{\hbar}{2\omega}\right)^{1/2}[\alpha+\alpha^*] = \left(\frac{2\hbar}{\omega}\right)^{1/2}\Re(\alpha), \tag{4.7.12}$$

$$p = \mathrm{i}\left(\frac{\hbar\omega}{2}\right)^{1/2}[\alpha^*-\alpha] = \left(\frac{\hbar\omega}{2}\right)^{1/2}\frac{[\alpha-\alpha^*]}{\mathrm{i}} = (2\hbar\omega)^{1/2}\Im(\alpha). \tag{4.7.13}$$

$$\boxed{W_\alpha(q,p) = \frac{1}{\pi\hbar}\exp\left\{-2\left[\left(\frac{\omega}{2\hbar}\right)^{1/2} q - \Re(\alpha)\right]^2 - 2\left[\frac{1}{(2\hbar\omega)^{1/2}}p - \Im(\alpha)\right]^2\right\}.} \tag{4.7.14}$$

If we choose $\hbar = \omega = 1$ as is often done in quantum optics, and consider the vacuum state $|\alpha\rangle = |0\rangle$, then it reduces to a very compact form[a]

$$\boxed{W_\alpha(q,p) = \frac{1}{\pi}\mathrm{e}^{-q^2-p^2}.} \tag{4.7.15}$$

Once again, we should insist that these relations can be helpful to save some time in the calculations. In this example, if we had to do the calculations

[a]Sometimes, another form is found for the WIGNER quasi-probability distribution in terms of q and p variables (see [31] for example). This is simply due to the convention that is used for the commutator as mentioned in footnote 4, p. 26).

4.8 Summary

In this chapter, we reviewed the notion of phase space distributions, and we showed that it is completely equivalent to the density matrix operator. However, the phase space distributions possesses different properties which can prove to be useful to interpret a state. We started by describing the motivation behind this framework, and then we gave the mathematical formulation. We demonstrated that the definition of the phase space distribution associated with a specific density matrix operator is not unique, and each definition correspond to a certain operator ordering. We then studied systematically each of the three most common phase space distributions used in the field of quantum optics: the WIGNER quasi-probability distribution, the Q representation and the GLAUBER--SUDARSHAN $\boldsymbol{P}$ representation. Their main properties were investigated, and we gave the expression of the phase space distribution of a coherent state in each case. Finally, we derived the main relations there exists between phase space distribution because they can turn out to be very useful in practice. Now that we have all the tools we need, we will now be able to characterize the transient density matrix that we defined in Chapter 2.

Chapter 5

Extension of the phase space distributions

As we said at the end of Chapter 2, we are interested in characterizing the transient density matrix, when the pre- and post-selected are both coherent state. The transient density matrix which was defined earlier (Eq. (2.7.14)) can be written as follows (using the formula for the overlap Eq. (3.3.10))

$$\boxed{\hat{\sigma}_{|\alpha\rangle\langle\beta|} \equiv \frac{|\alpha\rangle\langle\beta|}{\langle\beta|\alpha\rangle} = |\alpha\rangle\langle\beta| \times \exp\left\{\frac{|\alpha-\beta|^2}{2}\right\}\exp\left\{\frac{\beta\alpha^*-\beta^*\alpha}{2}\right\}.} \tag{5.0.1}$$

As we have seen in Chapter 2, we have defined this transient density matrix $\hat{\sigma}_{|\alpha\rangle\langle\beta|}$ by making comparing between the weak value (Eq. (2.4.1)) and the expected value (Eq. (1.2.10)). We could pursue this analogy by looking at the definition of the expected value in terms of phase space distribution (Eq. (4.1.1))

$$\left\langle \hat{A}(\hat{q},\hat{p})\right\rangle = \mathrm{Tr}\left[\hat{\rho}(\hat{q},\hat{p})\,\hat{A}(\hat{q},\hat{p})\right] = \iint_{\mathbf{R}^2} A(q,p)\,F(q,p)\,\mathrm{d}q\mathrm{d}p. \tag{5.0.2}$$

We could then decide to extend it using the transient density matrix as such

$$\left\langle \hat{A}(\hat{q},\hat{p})\right\rangle_w = \mathrm{Tr}\left(\hat{\sigma}_{|\alpha\rangle\langle\beta|}(\hat{q},\hat{p})\,\hat{A}(\hat{q},\hat{p})\right) = \iint_{\mathbf{R}^2} A(q,p)\,F(q,p)\,\mathrm{d}q\mathrm{d}p, \tag{5.0.3}$$

where the scalar function $F(q,p)$ is given by (see Eq. (4.1.7))

$$\boxed{F^f(q,p) = \frac{1}{4\pi^2}\iiint_{\mathbf{R}^3}\left\langle q'+\frac{1}{2}\eta\hbar\left|\hat{\sigma}_{|\alpha\rangle\langle\beta|}\right|q'-\frac{1}{2}\eta\hbar\right\rangle f(\xi,\eta)\,\mathrm{e}^{\mathrm{i}\xi(q'-q)}\mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}q',} \tag{5.0.4}$$

or in the complex plane by (see Eq. (4.2.9))

$$\boxed{F^f(\alpha,\alpha^*) \equiv \frac{1}{\pi^2}\iint_{\mathbf{R}^2}\mathrm{Tr}\left\{\hat{\sigma}_{|\alpha\rangle\langle\beta|}\left(\hat{a},\hat{a}^\dagger\right)\mathrm{e}^{z\hat{a}^\dagger-z^*\hat{a}}f(z,z^*)\right\}\mathrm{e}^{z^*\alpha-z\alpha^*}\,\mathrm{d}^2z.} \tag{5.0.5}$$

As discussed in Chapter 4, the functions $f(\xi,\eta)$ (or $f(z,z^*)$) will uniquely define the phase space distribution. In this chapter, we will study the different phase space distributions associated with the transient density matrix. To do this, we will explore the different properties of those distributions. Indeed, we will see that they can behave differently (they can be complex for example) because of certain properties of the transient density matrix. We will use the term *extended* for qualifying the phase space distributions related to the transient density matrix to emphasize that we study an operator which is not a density matrix.

Note that we will use the usual convention $\hbar = \omega = 1$ to make the calculations a bit easier to follow. This is not a problem because we are mostly interested in the shape of the different phase space distributions, and not the actual values. Besides, a simple dimensional analysis of the result will allows us to obtain the correct value if we decide to calculate it. The drawback being that we will not be able to verify the validity of our expressions using a dimensional analysis.

5.1 Extended WIGNER quasi-probability distribution

In the last section, we have defined a new kind of mathematical object called the transient density matrix $\hat{\sigma}_{|\alpha\rangle\langle\beta|}$. In order to study it, we will first calculate the WIGNER quasi-probability distribution of this element, instead the usual density matrix $\hat{\rho}$. The corresponding distribution will be referred to as the extended WIGNER quasi-probability distribution subsequently and is given by (from Eq. (4.4.3)):

$$W_{|\alpha\rangle\langle\beta|}(q,p) \equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\langle q+\frac{1}{2}\eta\left|\hat{\sigma}_{|\alpha\rangle\langle\beta|}\right|q-\frac{1}{2}\eta\right\rangle \mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta, \tag{5.1.1}$$

or

$$\boxed{W_{|\alpha\rangle\langle\beta|}(q,p) \equiv \frac{1}{2\pi}\frac{1}{\langle\beta|\alpha\rangle}\int_{-\infty}^{+\infty}\left\langle q+\frac{1}{2}\eta|\alpha\right\rangle\left\langle\beta|q-\frac{1}{2}\eta\right\rangle \mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta,} \tag{5.1.2}$$

where we used the definition of the transient density matrix $\hat{\sigma}_{|\alpha\rangle\langle\beta|}$ (see Eq. (2.7.14)).

5.1.1 Derivation of the extended WIGNER quasi-probability distribution

5.1.1.1 Calculation of the wavefunction of a coherent state

First of all, let us recall a few definitions which we will need for the calculations. A coherent state $|\alpha\rangle$ is defined by (see Eq. (3.3.1)):

$$\hat{a}\,|\alpha\rangle \equiv \alpha\,|\alpha\rangle\,, \tag{5.1.3}$$

where (see Eq. (3.1.41))

$$\hat{a} \equiv \frac{1}{\sqrt{2}}\left[\hat{q}+\mathrm{i}\hat{p}\right], \tag{5.1.4}$$

where $\hbar = \omega = 1$ for the same reason stated above. We also have (see Eq. (3.1.36))

$$\hat{p} \equiv -\mathrm{i}\frac{\hat{\partial}}{\partial q}. \tag{5.1.5}$$

To find the wavefunction $\psi_\alpha(q)$, we may notice that

$$\langle q\,|\hat{a}|\,\alpha\rangle = \alpha\,\langle q|\alpha\rangle \equiv \alpha\psi_\alpha(q)\,. \tag{5.1.6}$$

Now, let us inject (5.1.4) into (5.1.6):

$$\begin{aligned}
&\left\langle q \left| \frac{1}{\sqrt{2}} \left[\hat{q} + \frac{\hat{\partial}}{\partial q} \right] \right| \alpha \right\rangle = \alpha \psi_\alpha (q) \\
\Leftrightarrow \quad & \frac{1}{\sqrt{2}} \left[q + \frac{\partial}{\partial q} \right] \psi_\alpha (q) = \alpha \psi_\alpha (q) \\
\Leftrightarrow \quad & \frac{\partial \psi_\alpha (q)}{\partial q} = \left[\sqrt{2}\alpha - q \right] \psi_\alpha (q) \\
\Leftrightarrow \quad & \int_{\psi_\alpha(0)}^{\psi_\alpha(q)} \frac{\mathrm{d}\psi_\alpha (q')}{\psi_\alpha (q')} = \int_0^q \left[\sqrt{2}\alpha - q' \right] \mathrm{d}q' \\
\Leftrightarrow \quad & \ln \left[\frac{\psi_\alpha (q)}{\psi_\alpha (0)} \right] = \sqrt{2}\alpha q - \frac{q^2}{2},
\end{aligned} \tag{5.1.7}$$

which yields

$$\psi_\alpha (q) = \psi_\alpha (0) \times \exp \left[-\frac{1}{2} \left(q^2 - 2\sqrt{2}\alpha q \right) \right]. \tag{5.1.8}$$

In order to find the constant $\psi_\alpha (0)$, we have to use the fact that the wavefunction of a physical state must be normalized, i.e.

$$\int_{-\infty}^{+\infty} |\psi_\alpha (q)|^2 \, \mathrm{d}q = 1. \tag{5.1.9}$$

Let us perform the integration:

$$\begin{aligned}
\int_{-\infty}^{+\infty} |\psi_\alpha (q)|^2 \, \mathrm{d}q &= \int_{-\infty}^{+\infty} \psi_\alpha (q) \, \psi_\alpha^* (q) \, \mathrm{d}q \\
&= \int_{-\infty}^{+\infty} |\psi_\alpha (0)|^2 \times \exp \left[-\frac{1}{2} \left(q^2 - 2\sqrt{2}\alpha q + q^2 - 2\sqrt{2}\alpha^* q \right) \right] \mathrm{d}q \\
&= \int_{-\infty}^{+\infty} |\psi_\alpha (0)|^2 \times \exp \left[- \left(q^2 - \sqrt{2} \left(\alpha + \alpha^* \right) q \right) \right] \mathrm{d}q.
\end{aligned} \tag{5.1.10}$$

Using the formula (A.5.1) with

$$\begin{cases} a = 1, \\ b = -\sqrt{2} \left(\alpha + \alpha^* \right), \\ c = 0, \end{cases} \tag{5.1.11}$$

we obtain

$$\int_{-\infty}^{+\infty} |\psi_\alpha (q)|^2 \, \mathrm{d}q = |\psi_\alpha (0)|^2 \sqrt{\pi} \exp \left(\frac{2 \left(\alpha + \alpha^* \right)^2}{4} \right) = |\psi_\alpha (0)|^2 \sqrt{\pi} \exp \left(\frac{\left(\alpha + \alpha^* \right)^2}{2} \right), \tag{5.1.12}$$

Since

$$2\Re \left(\alpha \right) = \left(\alpha + \alpha^* \right), \tag{5.1.13}$$

and using the result (5.1.12) in (5.1.9), we find

$$\psi_\alpha (0) = \frac{1}{\pi^{1/4}} \exp \left[-\Re^2 \left(\alpha \right) + \mathrm{i}\phi \right], \tag{5.1.14}$$

where $\phi \in \mathbf{R}$ is an arbitrary phase factor which remains to be chosen. Since

$$q^2 - 2\sqrt{2}\alpha q = q^2 - 2\sqrt{2}\alpha q + \left[\frac{-2\sqrt{2}\alpha q}{2q}\right]^2 - \left[\frac{-2\sqrt{2}\alpha q}{2q}\right]^2 = \left[q - \sqrt{2}\alpha\right]^2 - 2\alpha^2, \tag{5.1.15}$$

we can re-write (5.1.8) as

$$\psi_\alpha(q) = \frac{1}{\pi^{1/4}} \exp\left[-\Re^2(\alpha) + \mathrm{i}\phi\right] \times \exp\left\{-\frac{1}{2}\left[q - \sqrt{2}\alpha\right]^2 + \alpha^2\right\}. \tag{5.1.16}$$

Finally, to make this expression a bit simpler, we just have to observe that

$$\alpha^2 = \left[\Re(\alpha) + \mathrm{i}\Im(\alpha)\right]^2 = \Re^2(\alpha) - \Im^2(\alpha) + 2\mathrm{i}\Re(\alpha)\Im(\alpha), \tag{5.1.17}$$

so that

$$\alpha^2 - \Re^2(\alpha) = -\Im^2(\alpha) + 2\mathrm{i}\Re(\alpha)\Im(\alpha). \tag{5.1.18}$$

We then choose the global phase factor ϕ such that

$$\mathrm{i}\phi + 2\mathrm{i}\Re(\alpha)\Im(\alpha) = 0. \tag{5.1.19}$$

This finally gives us a compact expression for the wavefunction of a coherent state:

$$\boxed{\psi_\alpha(q) = \frac{1}{\pi^{1/4}} \exp\left[-\Im^2(\alpha)\right] \times \exp\left\{-\frac{1}{2}\left[q - \sqrt{2}\alpha\right]^2\right\}.} \tag{5.1.20}$$

Please note that there is a sign error before the term $\Im^2(\alpha)$ in [27, p. 529 (11.4–5)].

5.1.1.2 Calculations

Overlap We will express the different terms separately. Using the formula for the overlap of two coherent states (see Eq. (3.3.10)), the first term is

$$\boxed{\frac{1}{\langle\beta|\alpha\rangle} = \exp\left\{+\frac{|\alpha-\beta|^2}{2}\right\} \exp\left\{\frac{\beta\alpha^* - \beta^*\alpha}{2}\right\}.} \tag{5.1.21}$$

Wavefunction of $|\alpha\rangle$ The second term can be expressed using Eq. (5.1.20):

$$\begin{aligned}\left\langle q + \frac{1}{2}\eta|\alpha\right\rangle &\equiv \psi_\alpha\left(q + \frac{1}{2}\eta\right) \\ &= \frac{1}{\pi^{1/4}} \mathrm{e}^{-\Im^2(\alpha)} \exp\left\{-\frac{1}{2}\left[\left(q + \frac{1}{2}\eta\right) - \sqrt{2}\alpha\right]^2\right\} \\ &= \frac{1}{\pi^{1/4}} \mathrm{e}^{-\Im^2(\alpha)} \exp\left\{-\frac{1}{2}\left[\left(q + \frac{1}{2}\eta\right)^2 + 2\alpha^2 - 2\left(q + \frac{1}{2}\eta\right)\sqrt{2}\alpha\right]\right\},\end{aligned} \tag{5.1.22}$$

which yields

$$\boxed{\begin{aligned}\left\langle q + \frac{1}{2}\eta|\alpha\right\rangle = &\frac{1}{\pi^{1/4}} \mathrm{e}^{-\Im^2(\alpha)} \exp\left\{-\frac{1}{2}\left[q^2 + \frac{1}{4}\eta^2 + q\eta\right]\right\} \\ &\times \exp\left\{-\frac{1}{2}\left[2\alpha^2 - 2\left(q + \frac{1}{2}\eta\right)\sqrt{2}\alpha\right]\right\}.\end{aligned}} \tag{5.1.23}$$

Note that $\alpha^2 = \Re^2(\alpha) - \Im^2(\alpha) + 2\mathrm{i}\Re(\alpha)\Im(\alpha) \neq \Re^2(\alpha) + \Im^2(\alpha) = |\alpha|^2$.

Wavefunction of $\langle\beta|$ Similarly, the third term can be written as

$$\begin{aligned}\left\langle\beta|q-\frac{1}{2}\eta\right\rangle &\equiv \psi_\beta^*\left(q-\frac{1}{2}\eta\right)\\ &= \frac{1}{\pi^{1/4}}e^{-\Im^2(\beta)}\exp\left\{-\frac{1}{2}\left[\left(q-\frac{1}{2}\eta\right)-\sqrt{2}\beta^*\right]^2\right\}\\ &= \frac{1}{\pi^{1/4}}e^{-\Im^2(\beta)}\exp\left\{-\frac{1}{2}\left[\left(q-\frac{1}{2}\eta\right)^2+2\beta^{*2}-2\left(q-\frac{1}{2}\eta\right)\sqrt{2}\beta^*\right]\right\},\end{aligned} \tag{5.1.24}$$

such that

$$\boxed{\begin{aligned}\left\langle\beta|q-\frac{1}{2}\eta\right\rangle = \frac{1}{\pi^{1/4}}e^{-\Im^2(\beta)}\exp\left\{-\frac{1}{2}\left[q^2+\frac{1}{4}\eta^2-q\eta\right]\right\}\\ \times\exp\left\{-\frac{1}{2}\left[2\beta^{*2}-2\left(q-\frac{1}{2}\eta\right)\sqrt{2}\beta^*\right]\right\}.\end{aligned}} \tag{5.1.25}$$

Integration Now, we want to evaluate the integral over η in Eq. (5.1.2), so let us separate the which contain η and the other terms in the integrand:

$$\begin{aligned}\left\langle q+\frac{1}{2}\eta|\alpha\right\rangle\left\langle\beta|q-\frac{1}{2}\eta\right\rangle e^{-i\eta p} = \frac{1}{\sqrt{\pi}}\exp\left\{-\Im^2(\alpha)-\Im^2(\beta)\right\}\\ \times\exp\left\{-\frac{1}{2}\left[q^2+\frac{1}{4}\eta^2+\cancel{q\eta}-2\left(q+\frac{1}{2}\eta\hbar\right)\sqrt{2}\alpha\right.\right.\\ q^2+\frac{1}{4}\eta^2-\cancel{q\eta}-2\left(q-\frac{1}{2}\eta\hbar\right)\sqrt{2}\beta^*\\ \left.\left.2\left(\alpha^2+\beta^{*2}\right)\right]-i\eta p\right\}\end{aligned} \tag{5.1.26}$$

then

$$\begin{aligned}\left\langle q+\frac{1}{2}\eta|\alpha\right\rangle\left\langle\beta|q-\frac{1}{2}\eta\right\rangle e^{-i\eta p} = \frac{1}{\sqrt{\pi}}\exp\left\{-\Im^2(\alpha)-\Im^2(\beta)\right\}\\ \times\exp\left\{-\left[q^2+\left(\alpha^2+\beta^{*2}\right)-q\sqrt{2}\left(\alpha+\beta^*\right)\right]\right\}\\ \times\exp\left\{-\left[\frac{1}{4}\eta^2+\frac{1}{2}\sqrt{2}\left(\beta^*-\alpha\right)\eta\right]-i\eta p\right\}.\end{aligned} \tag{5.1.27}$$

From Eq. (5.1.2), we see that to continue, we need to calculate the following integral

$$\begin{aligned}\int_{-\infty}^{+\infty}\left\langle q+\frac{1}{2}\eta|\alpha\right\rangle\left\langle\beta|q-\frac{1}{2}\eta\right\rangle e^{-i\eta p}\,d\eta = \frac{1}{\sqrt{\pi}}\exp\left\{-\Im^2(\alpha)-\Im^2(\beta)\right\}\\ \times\exp\left\{-\left[q^2+\left(\alpha^2+\beta^{*2}\right)-q\sqrt{2}\left(\alpha+\beta^*\right)\right]\right\}\\ \times I_\eta,\end{aligned} \tag{5.1.28}$$

where we have defined

$$I_\eta \equiv \int_{-\infty}^{+\infty}\exp\left\{-\left[\frac{1}{4}\eta^2+\left[ip+\frac{1}{\sqrt{2}}\left(\beta^*-\alpha\right)\right]\eta\right]\right\}d\eta. \tag{5.1.29}$$

It is easy to notice that I_η has exactly the form as the integral given in Eq. (A.5.1) if we take

$$\begin{cases} a = & \frac{1}{4}, \\ b = & \mathrm{i}p + \frac{1}{\sqrt{2}}\left(\beta^* - \alpha\right), \\ c = & 0. \end{cases} \tag{5.1.30}$$

Using the result (A.5.1), we thus obtain

$$I_\eta = \sqrt{\frac{\pi}{\frac{1}{4}}}\exp\left\{\frac{\left[\mathrm{i}p + \frac{1}{\sqrt{2}}\left(\beta^* - \alpha\right)\right]^2}{\cancel{4} \times \frac{1}{\cancel{4}}}\right\} = 2\sqrt{\pi}\exp\left\{\left[\mathrm{i}p + \frac{1}{\sqrt{2}}\left(\beta^* - \alpha\right)\right]^2\right\}. \tag{5.1.31}$$

Extended WIGNER quasi-probability distribution From the Eq. (5.1.31) we can deduce the result of Eq. (5.1.28). Then, we inject this and Eq. (5.1.21) in the definition of the extended WIGNER quasi-probability distribution (Eq. (5.1.2)), we obtain

$$\begin{aligned} W_{|\alpha\rangle\langle\beta|}\left(q,p\right) = & \frac{1}{\cancel{2}\pi}\exp\left\{\frac{|\alpha-\beta|^2}{2} + \right\}\exp\left\{\frac{\beta\alpha^* - \beta^*\alpha}{2}\right\} \\ & \times \frac{1}{\cancel{\sqrt{\pi}}}\exp\left\{-\Im^2\left(\alpha\right) - \Im^2\left(\beta\right)\right\} \\ & \times \exp\left\{-\left[q^2 + \left(\alpha^2 + \beta^{*2}\right) - q\sqrt{2}\left(\alpha + \beta^*\right)\right]\right\} \\ & \times \cancel{2}\cancel{\sqrt{\pi}}\exp\left\{\left[\frac{1}{\sqrt{2}}\left(\beta^* - \alpha\right) + \mathrm{i}p\right]^2\right\}, \end{aligned} \tag{5.1.32}$$

Since

$$\begin{aligned} q^2 - q\sqrt{2}\left(\alpha + \beta^*\right) &= q^2 - 2q \times \frac{1}{2}\sqrt{2}\left(\alpha + \beta^*\right) + \left[\frac{1}{2}\sqrt{2}\left(\alpha + \beta^*\right)\right]^2 - \left[\frac{1}{2}\sqrt{2}\left(\alpha + \beta^*\right)\right]^2 \\ &= \left[q - \frac{1}{\sqrt{2}}\left(\alpha + \beta^*\right)\right]^2 - \frac{1}{2}\left(\alpha + \beta^*\right)^2, \end{aligned} \tag{5.1.33}$$

and

$$\begin{aligned} \left(\alpha^2 + \beta^{*2}\right) - \frac{1}{2}\left(\alpha + \beta^*\right)^2 &= \frac{1}{2}\left(\cancel{2}\alpha^2 + \cancel{2}\beta^{*2} - \cancel{\alpha^2} - \cancel{\beta^{*2}} - 2\alpha\beta^*\right) \\ &= \frac{1}{2}\left(\alpha^2 + \beta^{*2} - 2\alpha\beta^*\right) \\ &= \frac{1}{2}\left(\alpha - \beta^*\right)^2, \end{aligned}$$

we finally have a compact expression for the extended WIGNER quasi-probability distribution:

$$\boxed{\begin{aligned} W_{|\alpha\rangle\langle\beta|}\left(q,p\right) = \frac{1}{\pi}\exp&\left\{\frac{|\alpha-\beta|^2}{2} + \frac{\beta\alpha^* - \beta^*\alpha}{2} - \Im^2\left(\alpha\right) - \Im^2\left(\beta\right) - \frac{\left(\alpha - \beta^*\right)^2}{2}\right\} \\ \times \exp&\left\{-\left[q + \frac{1}{\sqrt{2}}\left(\alpha + \beta^*\right)\right]^2 + \left[\mathrm{i}p + \frac{1}{\sqrt{2}}\left(\beta^* - \alpha\right)\right]^2\right\}. \end{aligned}} \tag{5.1.34}$$

5.1.2 Transformation

Technically, if we wanted to study the newly defined extended WIGNER quasi-probability distribution, we would have to vary 4 parameters (because α and β are complex numbers). For this reason, it is useful to determine the set of transformations which leave this distribution invariant (or very similar). That way, it will decrease the number of parameters that must be studied to characterize this distribution.

5.1.2.1 Translation

Before verifying if the that WIGNER quasi-probability distribution is invariant or similar under a translation of α and β, it is useful to first prove the following relation:

$$\boxed{\hat{D}(\zeta)|q\rangle = \mathrm{e}^{\frac{(\zeta-\zeta^*)}{\sqrt{2}}q}\mathrm{e}^{\frac{(\zeta^2-\zeta^{*2})}{4}}\left|q+\frac{(\zeta+\zeta^*)}{\sqrt{2}}\right\rangle,} \tag{5.1.35}$$

or equivalently

$$\boxed{\langle q|\hat{D}^\dagger(\zeta) = \left\langle q+\frac{(\zeta+\zeta^*)}{\sqrt{2}}\right|\mathrm{e}^{\frac{(\zeta^*-\zeta)}{\sqrt{2}}q}\mathrm{e}^{\frac{(\zeta^{*2}-\zeta^2)}{4}}.} \tag{5.1.36}$$

Proof

From the definition of the displacement operator (Eq. (3.3.18)), we recall that

$$\hat{D}(\zeta) \equiv \mathrm{e}^{\zeta\hat{a}^\dagger-\zeta^*\hat{a}}. \tag{5.1.37}$$

Moreover, the definitions of the creation and annihilation operators (Eqs. (3.1.41) and (3.1.42)) allow us to write

$$\hat{a} \equiv \frac{1}{\sqrt{2}}(\hat{q}+\mathrm{i}\hat{p}), \tag{5.1.38}$$

$$\hat{a}^\dagger \equiv \frac{1}{\sqrt{2}}(\hat{q}-\mathrm{i}\hat{p}). \tag{5.1.39}$$

We thus have

$$\zeta\hat{a}^\dagger-\zeta^*\hat{a} = \delta\frac{1}{\sqrt{2}}(\hat{q}-\mathrm{i}\hat{p})-\zeta^*\frac{1}{\sqrt{2}}(\hat{q}+\mathrm{i}\hat{p}) = \frac{(\zeta-\zeta^*)}{\sqrt{2}}\hat{q}-\frac{(\zeta+\zeta^*)}{\sqrt{2}}\mathrm{i}\hat{p}. \tag{5.1.40}$$

The displacement operator can then be expressed as

$$\hat{D}(\zeta) = \exp\left\{\frac{(\zeta-\zeta^*)}{\sqrt{2}}\hat{q}-\frac{(\zeta+\zeta^*)}{\sqrt{2}}\mathrm{i}\hat{p}\right\}. \tag{5.1.41}$$

It is possible to re-write the exponential using the CAMPBELL–BAKER–HAUSDORFF theorem with

$$\hat{A} = \frac{(\zeta - \zeta^*)}{\sqrt{2}} \hat{q}, \tag{5.1.42}$$

$$\hat{B} = -\frac{(\zeta + \zeta^*)}{\sqrt{2}} \mathrm{i}\hat{p}, \tag{5.1.43}$$

$$x = 1. \tag{5.1.44}$$

In that case

$$\frac{\left[\hat{A}, \hat{B}\right]}{2} = \frac{1}{2}\left[\frac{(\zeta - \zeta^*)}{\sqrt{2}}\hat{q}, -\frac{(\zeta + \zeta^*)}{\sqrt{2}}\mathrm{i}\hat{p}\right] = -\frac{1}{4}(\zeta - \zeta^*)(\zeta + \zeta^*)\,\mathrm{i}\underbrace{[\hat{q}, \hat{p}]}_{=\mathrm{i}} = +\frac{\left(\zeta^2 - \zeta^{*2}\right)}{4}, \tag{5.1.45}$$

or

$$\frac{\left[\hat{A}, \hat{B}\right]}{2} = \frac{\left(\zeta^2 - \zeta^{*2}\right)}{4}. \tag{5.1.46}$$

Using Eq. (B.1.2), we can write

$$\hat{D}(\zeta) = \mathrm{e}^{-\frac{(\zeta+\zeta^*)}{\sqrt{2}}\mathrm{i}\hat{p}}\mathrm{e}^{\frac{(\zeta-\zeta^*)}{\sqrt{2}}\hat{q}}\mathrm{e}^{\frac{\left(\zeta^2-\zeta^{*2}\right)}{4}}. \tag{5.1.47}$$

Since

$$\mathrm{e}^{a\hat{q}}|q\rangle = \mathrm{e}^{aq}|q\rangle, \quad a \in \mathbf{C}, \tag{5.1.48}$$

$$\mathrm{e}^{\mathrm{i}b\hat{p}}|q\rangle = |q - b\rangle, \quad b \in \mathbf{R}, \tag{5.1.49}$$

we obtain

$$\hat{D}(\zeta)|q\rangle = \mathrm{e}^{-\frac{(\zeta+\zeta^*)}{\sqrt{2}}\mathrm{i}\hat{p}}\mathrm{e}^{\frac{(\zeta-\zeta^*)}{\sqrt{2}}\hat{q}}\mathrm{e}^{\frac{\left(\zeta^2-\zeta^{*2}\right)}{4}}|q\rangle, \tag{5.1.50}$$

i.e.

$$\hat{D}(\zeta)|q\rangle = \mathrm{e}^{\frac{(\zeta-\zeta^*)}{\sqrt{2}}q}\mathrm{e}^{\frac{\left(\zeta^2-\zeta^{*2}\right)}{4}}\left|q + \frac{(\zeta+\zeta^*)}{\sqrt{2}}\right\rangle. \tag{5.1.51}$$

Moreover

$$\langle q|\hat{D}^\dagger(\zeta) = \left[\hat{D}(\zeta)|q\rangle\right]^\dagger = \left\langle q + \frac{(\zeta+\zeta^*)}{\sqrt{2}}\right|\mathrm{e}^{\frac{(\zeta^*-\zeta)}{\sqrt{2}}q}\mathrm{e}^{\frac{\left(\zeta^{*2}-\zeta^2\right)}{4}}. \tag{5.1.52}$$

Now, we will prove that

$$\boxed{W_{|\alpha+\delta\rangle\langle\beta+\delta|}(q, p) = W_{|\alpha\rangle\langle\beta|}\left(q - \frac{(\zeta+\zeta^*)}{\sqrt{2}}, p - \frac{(\zeta-\zeta^*)}{\sqrt{2}\mathrm{i}}\right),} \tag{5.1.53}$$

or

$$\boxed{W_{|\alpha+\delta\rangle\langle\beta+\delta|}(q, p) = W_{|\alpha\rangle\langle\beta|}\left(q - \frac{\Re(\zeta)}{\sqrt{2}}, p - \frac{\Im(\zeta)}{\sqrt{2}\mathrm{i}}\right).} \tag{5.1.54}$$

This means that the effect of a translation of α and β by a complex number ζ is just to translate the extended WIGNER quasi-probability distribution, without changing the shape. Therefore, we can limit our analysis to

$$\boxed{\alpha \equiv 0 + 0\mathrm{i}} \tag{5.1.55}$$

This decrease the number of parameters to the two found in β.

Proof

We will consider the following transformation:

$$\begin{cases} |\alpha\rangle \to |\alpha'\rangle = |\alpha+\zeta\rangle\,, \\ |\beta\rangle \to |\beta'\rangle = |\beta+\zeta\rangle\,. \end{cases} \tag{5.1.56}$$

Using the formula for the translation of a coherent state (Eq. (3.3.24)), we can write

$$|\alpha+\zeta\rangle = \exp\left\{\frac{\zeta^*\alpha-\zeta\alpha^*}{2}\right\}\hat{D}(\zeta)|\alpha\rangle\,, \tag{5.1.57}$$

$$|\beta+\zeta\rangle = \exp\left\{\frac{\zeta^*\beta-\zeta\beta^*}{2}\right\}\hat{D}(\zeta)|\beta\rangle\,. \tag{5.1.58}$$

To make it more readable, we will use the notation

$$\phi_\alpha \equiv \frac{\zeta^*\alpha-\zeta\alpha^*}{2}, \tag{5.1.59}$$

$$\phi_\beta \equiv \frac{\zeta^*\beta-\zeta\beta^*}{2}, \tag{5.1.60}$$

so that

$$|\alpha+\zeta\rangle = \mathrm{e}^{\phi_\alpha}\hat{D}(\zeta)|\alpha\rangle\,, \tag{5.1.61}$$

$$|\beta+\zeta\rangle = \mathrm{e}^{\phi_\beta}\hat{D}(\zeta)|\beta\rangle\,. \tag{5.1.62}$$

In that case, the extended WIGNER quasi-probability distribution becomes

$$\begin{aligned} W_{|\alpha'\rangle\langle\beta'|}(q,p) &= \frac{1}{2\pi}\frac{1}{\langle\beta'|\alpha'\rangle}\int_{-\infty}^{+\infty}\left\langle q+\frac{1}{2}\eta|\alpha'\right\rangle\left\langle\beta'|q-\frac{1}{2}\eta\right\rangle\mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta \\ &= \frac{1}{2\pi}\frac{1}{\langle\beta+\zeta|\alpha+\zeta\rangle}\int_{-\infty}^{+\infty}\left\langle q+\frac{1}{2}\eta|\alpha+\zeta\right\rangle\left\langle\beta+\zeta|q-\frac{1}{2}\eta\right\rangle\mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta \\ &= \frac{1}{2\pi}\frac{1}{\left\langle\beta\left|\cancel{\mathrm{e}^{\phi_\beta}}\hat{D}^\dagger(\zeta)\hat{D}(\zeta)\cancel{\mathrm{e}^{\phi_\alpha}}\right|\alpha\right\rangle} \\ &\quad\times\int_{-\infty}^{+\infty}\left\langle q+\frac{1}{2}\eta\left|\hat{D}(\zeta)\cancel{\mathrm{e}^{\phi_\alpha}}\right|\alpha+\zeta\right\rangle\left\langle\beta+\zeta\left|\cancel{\mathrm{e}^{\phi_\beta}}\hat{D}^\dagger(\zeta)\right|q-\frac{1}{2}\eta\right\rangle\mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta \\ &= \frac{1}{2\pi}\frac{1}{\langle\beta|\alpha\rangle}\int_{-\infty}^{+\infty}\left\langle q+\frac{1}{2}\eta\left|\hat{D}(\zeta)\right|\alpha+\zeta\right\rangle\left\langle\beta+\zeta\left|\hat{D}^\dagger(\zeta)\right|q-\frac{1}{2}\eta\right\rangle\mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta, \end{aligned} \tag{5.1.63}$$

where we used the fact that the displacement operator is unitary (Eq. (3.3.19)). Using the Hermitian conjugation of the displacement operator (Eq. (3.3.20)):

$$W_{|\alpha'\rangle\langle\beta'|}(q,p) = \frac{1}{2\pi}\frac{1}{\langle\beta|\alpha\rangle}\int_{-\infty}^{+\infty}\left\langle q+\frac{1}{2}\eta\left|\hat{D}^{\dagger}(-\zeta)\right|\alpha+\zeta\right\rangle\left\langle\beta+\zeta\left|\hat{D}(-\zeta)\right|q-\frac{1}{2}\eta\right\rangle \mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta. \tag{5.1.64}$$

Let us simplify the integrand

$$I = \left\langle q+\frac{1}{2}\eta\left|\hat{D}^{\dagger}(-\zeta)\right|\alpha+\zeta\right\rangle\left\langle\beta+\zeta\left|\hat{D}(-\zeta)\right|q-\frac{1}{2}\eta\right\rangle \mathrm{e}^{-\mathrm{i}\eta p}. \tag{5.1.65}$$

Using Eq. (5.1.36), we know that

$$\left\langle q+\frac{1}{2}\eta\right|\hat{D}^{\dagger}(-\zeta) = \langle q|\,\hat{D}^{\dagger}(\zeta) = \left\langle\left(q+\frac{1}{2}\eta\right)+\frac{((-\zeta)+(-\zeta)^*)}{\sqrt{2}}\right| \mathrm{e}^{\frac{((-\zeta)^*-(-\zeta))}{\sqrt{2}}\left(q+\frac{1}{2}\eta\right)}\mathrm{e}^{\frac{\left((-\zeta)^{*2}-(-\zeta)^2\right)}{4}} \tag{5.1.66}$$

or

$$\left\langle q+\frac{1}{2}\eta\right|\hat{D}^{\dagger}(-\zeta) = \left\langle q-\frac{(\zeta+\zeta^*)}{\sqrt{2}}+\frac{1}{2}\eta\right| \mathrm{e}^{-\frac{(\zeta^*-\zeta)}{\sqrt{2}}\left(q+\frac{1}{2}\eta\right)}\mathrm{e}^{-\frac{\left(\zeta^{*2}-\zeta^2\right)}{4}}. \tag{5.1.67}$$

Similarly, with Eq. (5.1.35), we can write

$$\hat{D}(-\zeta)\left|q-\frac{1}{2}\eta\right\rangle = \mathrm{e}^{\frac{((-\zeta)-(-\zeta)^*)}{\sqrt{2}}\left(q-\frac{1}{2}\eta\right)}\mathrm{e}^{\frac{\left((-\zeta)^2-(-\zeta)^{*2}\right)}{4}}\left|\left(q-\frac{1}{2}\eta\right)+\frac{((-\zeta)+(-\zeta)^*)}{\sqrt{2}}\right\rangle, \tag{5.1.68}$$

or

$$\hat{D}(-\zeta)\left|q-\frac{1}{2}\eta\right\rangle = \mathrm{e}^{-\frac{(\zeta-\zeta^*)}{\sqrt{2}}\left(q-\frac{1}{2}\eta\right)}\mathrm{e}^{-\frac{\left(\zeta^2-\zeta^{*2}\right)}{4}}\left|q-\frac{(\zeta+\zeta^*)}{\sqrt{2}}-\frac{1}{2}\eta\right\rangle. \tag{5.1.69}$$

Therefore

$$\begin{aligned}
I &= \left\langle q+\frac{1}{2}\eta\left|\hat{D}^{\dagger}(-\zeta)\right|\alpha+\zeta\right\rangle\left\langle\beta+\zeta\left|\hat{D}(-\zeta)\right|q-\frac{1}{2}\eta\right\rangle \mathrm{e}^{-\mathrm{i}\eta p}\\
&= \left\langle q-\frac{(\zeta+\zeta^*)}{\sqrt{2}}+\frac{1}{2}\eta\left|\mathrm{e}^{-\frac{(\zeta^*-\zeta)}{\sqrt{2}}\left(q+\frac{1}{2}\eta\right)}\cancel{\mathrm{e}^{-\frac{\left(\zeta^{*2}-\zeta^2\right)}{4}}}\right|\alpha+\zeta\right\rangle\\
&\times\left\langle\beta+\zeta\left|\mathrm{e}^{-\frac{(\zeta-\zeta^*)}{\sqrt{2}}\left(q-\frac{1}{2}\eta\right)}\cancel{\mathrm{e}^{-\frac{\left(\zeta^2-\zeta^{*2}\right)}{4}}}\right|q-\frac{(\zeta+\zeta^*)}{\sqrt{2}}-\frac{1}{2}\eta\right\rangle \mathrm{e}^{-\mathrm{i}\eta p}\\
&= \left\langle q-\frac{(\zeta+\zeta^*)}{\sqrt{2}}+\frac{1}{2}\eta\left|\mathrm{e}^{-\frac{(\zeta^*-\zeta)}{\sqrt{2}}\left(+\frac{1}{2}\eta\right)}\right|\alpha+\zeta\right\rangle\left\langle\beta+\zeta\left|\mathrm{e}^{-\frac{(\zeta-\zeta^*)}{\sqrt{2}}\left(-\frac{1}{2}\eta\right)}\right|q-\frac{(\zeta+\zeta^*)}{\sqrt{2}}-\frac{1}{2}\eta\right\rangle \mathrm{e}^{-\mathrm{i}\eta p}.
\end{aligned} \tag{5.1.70}$$

Injecting Eqs. (5.1.67) and (5.1.69) into Eq. (5.1.64), we obtain

$$\begin{aligned} W_{|\alpha'\rangle\langle\beta'|}(q,p) &= \frac{1}{2\pi}\frac{1}{\langle\beta|\alpha\rangle}\int_{-\infty}^{+\infty}\left\langle q-\frac{(\zeta+\zeta^*)}{\sqrt{2}}+\frac{1}{2}\eta|\alpha+\zeta\right\rangle\left\langle\beta+\zeta|q-\frac{(\zeta+\zeta^*)}{\sqrt{2}}-\frac{1}{2}\eta\right\rangle \mathrm{e}^{-\left(\mathrm{i}p-\frac{(\zeta-\zeta^*)}{\sqrt{2}}\right)\eta}\,\mathrm{d}\eta,\\ &= W_{|\alpha\rangle\langle\beta|}\left(q-\frac{(\zeta+\zeta^*)}{\sqrt{2}},p-\frac{(\zeta-\zeta^*)}{\sqrt{2}\mathrm{i}}\right)\\ &= W_{|\alpha\rangle\langle\beta|}\left(q-\frac{\Re(\zeta)}{\sqrt{2}},p-\frac{\Im(\zeta)}{\sqrt{2}\mathrm{i}}\right), \end{aligned} \tag{5.1.71}$$

because

$$(\zeta+\zeta^*) = 2\Re(\zeta), \tag{5.1.72}$$

$$\frac{(\zeta-\zeta^*)}{\mathrm{i}} = 2\Im(\zeta). \tag{5.1.73}$$

5.1.2.2 Rotation

While no formal proof is given (see Appendix (A.6)), it seems that the shape of the extended WIGNER quasi-probability distribution does not change significantly (but it is rotated) when plotting it. Therefore, we will limit our study to

$$\beta \in \mathbf{R}_+. \tag{5.1.74}$$

5.1.3 Properties

5.1.3.1 Normalization

The extended WIGNER quasi-probability distribution is still normalized to unity

$$\iint_{\mathbf{R}^2} W_{\hat{\sigma}_{|i\rangle\langle f|}}(x,p)\,\mathrm{d}x\mathrm{d}p = 1. \tag{5.1.75}$$

Proof

We have

$$\begin{aligned}\iint_{\mathbf{R}^2} W_{|\alpha\rangle\langle\beta|}(q,p)\,\mathrm{d}q\mathrm{d}p &= \frac{1}{2\pi}\frac{1}{\langle\beta|\alpha\rangle}\iiint_{\mathbf{R}^3}\left\langle q+\frac{1}{2}\eta|\alpha\right\rangle\left\langle\beta|q-\frac{1}{2}\eta\right\rangle \mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta\mathrm{d}q\mathrm{d}p\\ &= \frac{1}{\cancel{2\pi}}\frac{1}{\langle\beta|\alpha\rangle}\iint_{\mathbf{R}^2}\left\langle q+\frac{1}{2}\eta|\alpha\right\rangle\left\langle\beta|q-\frac{1}{2}\eta\right\rangle \cancel{2\pi}\delta(\eta)\,\mathrm{d}\eta\mathrm{d}q\\ &= \frac{1}{\langle\beta|\alpha\rangle}\int_{-\infty}^{+\infty}\langle q|\alpha\rangle\langle\beta|q\rangle\,\mathrm{d}q\\ &= \frac{1}{\langle\beta|\alpha\rangle}\int_{-\infty}^{+\infty}\langle\beta|q\rangle\langle q|\alpha\rangle\,\mathrm{d}q\\ &= \frac{1}{\langle\beta|\alpha\rangle}\left\langle\beta\left|\left(\int_{-\infty}^{+\infty}|q\rangle\langle q|\,\mathrm{d}q\right)\right|\alpha\right\rangle\\ &= \frac{\langle\beta|\alpha\rangle}{\langle\beta|\alpha\rangle}\\ &= 1,\end{aligned}\tag{5.1.76}$$

where we have used the fact that the states $\{|q\rangle\}$ form an orthonormal basis, so they verify the completeness relation:

$$\int_{-\infty}^{+\infty}|q\rangle\langle q|\,\mathrm{d}q = \hat{1}.\tag{5.1.77}$$

5.1.3.2 Bounds

$$\boxed{\left|W_{|\alpha\rangle\langle\beta|}(q,p)\right| \leq \frac{1}{\pi\left|\langle\beta|\alpha\rangle\right|} = \frac{\mathrm{e}^{|\alpha-\beta|^2/2}}{\pi}.}\tag{5.1.78}$$

Unfortunately, this gives us a bound for the absolute value of the extended Wigner quasi-probability distribution. Indeed, we will see below that the values can become complex. This means that the (magnitude of the) maximum values of the extended Wigner quasi-probability distribution can potentially become very large if the distance between α and β is important. This makes sense because when the separation increases, the pre-selection state $|\alpha\rangle$ and the post-selection state $\langle\beta|$ become more and more orthogonal (the overlap decreases). Physically, this means that the two events become more and more incompatible[1], so it is not surprising that values of the extended Wigner quasi-probability distribution becomes stranger as the separation increases. On the other hand, when the two states are close to each other, the $W_{|\alpha\rangle\langle\beta|}(q,p)$ behaves potentially like the usual Wigner quasi-probability distribution. In the limit where $\beta\to\alpha$, we obviously obtain the normal bounds found in Section 4.4.

It is possible to obtain a bound for the real and imaginary part of the extended Wigner quasi-probability distribution, even though it might be possible to get a better bounds using another method

$$\boxed{-\frac{\mathrm{e}^{|\alpha-\beta|^2/2}}{\pi} \leq \Re\left(W_{|\alpha\rangle\langle\beta|}(q,p)\right) \leq \frac{\mathrm{e}^{|\alpha-\beta|^2/2}}{\pi} \quad\text{and}\quad -\frac{\mathrm{e}^{|\alpha-\beta|^2/2}}{\pi} \leq \Im\left(W_{|\alpha\rangle\langle\beta|}(q,p)\right) \leq \frac{\mathrm{e}^{|\alpha-\beta|^2/2}}{\pi}.}\tag{5.1.79}$$

[1] Indeed, the conditional probability of measuring the state $\langle\beta|$ if we prepared $|\alpha\rangle$ is $\mathbb{P}(\beta|\alpha) = \left|\langle\beta|\alpha\rangle\right|^2/\pi$.

Proof

We have

$$\left|W_{|\alpha\rangle\langle\beta|}\left(q,p\right)\right|^2 = \left|\frac{1}{2\pi}\frac{1}{\langle\beta|\alpha\rangle}\int_{-\infty}^{+\infty}\left\langle q+\frac{1}{2}\eta|\alpha\right\rangle\left\langle\beta|q-\frac{1}{2}\eta\right\rangle \mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta\right|^2$$
$$= \left|\frac{\sum_i p_i}{2\pi}\int_{-\infty}^{+\infty}\psi_\alpha\left(q+\frac{1}{2}\eta\hbar\right)\psi_\beta^*\left(q-\frac{1}{2}\eta\hbar\right)\mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta\right|^2. \tag{5.1.80}$$

We apply the CAUCHY–SCHWARZ inequality:

$$|W\left(q,p\right)|^2 \leq \frac{1}{\left(2\pi\right)^2}\frac{1}{|\langle\beta|\alpha\rangle|^2}\int_{-\infty}^{+\infty}\left|\psi_\alpha\left(q+\frac{1}{2}\eta\right)\right|^2\mathrm{d}\eta\times\int_{-\infty}^{+\infty}\left|\psi_\beta\left(q-\frac{1}{2}\eta\right)\right|^2\mathrm{d}\eta. \tag{5.1.81}$$

We then perform the substitutions

$$u = q+\frac{1}{2}\eta, \tag{5.1.82}$$
$$v = q-\frac{1}{2}\eta, \tag{5.1.83}$$

which leads to

$$\mathrm{d}\eta = 2\,\mathrm{d}u, \tag{5.1.84}$$
$$\mathrm{d}\eta = -2\,\mathrm{d}v, \tag{5.1.85}$$

with

$$\lim_{\eta\to-\infty} u = -\infty, \qquad \lim_{\eta\to+\infty} u = +\infty, \tag{5.1.86}$$
$$\lim_{\eta\to-\infty} v = +\infty, \qquad \lim_{\eta\to+\infty} v = -\infty. \tag{5.1.87}$$

Therefore

$$\int_{-\infty}^{+\infty}\left|\psi_\alpha\left(q+\frac{1}{2}\eta\right)\right|^2\mathrm{d}\eta = 2\int_{-\infty}^{+\infty}|\psi_\alpha\left(u\right)|^2\,\mathrm{d}u = 2,$$

and

$$\int_{-\infty}^{+\infty}\left|\psi_\beta\left(q-\frac{1}{2}\eta\right)\right|^2\mathrm{d}\eta = -2\int_{+\infty}^{-\infty}|\psi_\beta\left(v\right)|^2\,\mathrm{d}v = +2\int_{-\infty}^{+\infty}|\psi_\beta\left(v\right)|^2\,\mathrm{d}v = 2 \tag{5.1.88}$$

Eq. (5.1.81) becomes

$$|W\left(q,p\right)|^2 \leq \frac{1}{4\pi^2}\frac{1}{|\langle\beta|\alpha\rangle|^2}\times 2\times 2 = \frac{1}{\pi^2\,|\langle\beta|\alpha\rangle|^2}, \tag{5.1.89}$$

or

$$|W\left(q,p\right)| \leq \frac{1}{\pi\,|\langle\beta|\alpha\rangle|} = \frac{\mathrm{e}^{|\alpha-\beta|^2/2}}{\pi}, \tag{5.1.90}$$

where we used the overlap of the coherent states (Eq. (3.3.10)).

5.1.3.3 Reality

The extended WIGNER quasi-probability distribution is not real:

$$\boxed{W^*_{|\alpha\rangle\langle\beta|}(q,p) \neq W_{|\alpha\rangle\langle\beta|}(q,p),} \tag{5.1.91}$$

this is due to the fact that the transient density matrix is not Hermitian in general. If it is, however, then the WIGNER quasi-probability distribution is real. In that case, $\hat{\sigma}_{|\alpha\rangle\langle\beta|}$ could represent

Proof

We have

$$\begin{aligned} W^*_{|\alpha\rangle\langle\beta|}(q,p) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\left\langle q+\frac{1}{2}\eta \left|\hat{\sigma}_{|\alpha\rangle\langle\beta|}\right| q-\frac{1}{2}\eta\right\rangle\right)^* \mathrm{e}^{+\mathrm{i}\eta p}\,\mathrm{d}\eta \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\left\langle q-\frac{1}{2}\eta \left|\hat{\sigma}^\dagger_{|\alpha\rangle\langle\delta|}\right| q+\frac{1}{2}\eta\right\rangle\right) \mathrm{e}^{\mathrm{i}\eta p}\,\mathrm{d}\eta \\ &= -\frac{1}{2\pi}\int_{+\infty}^{-\infty} \left(\left\langle q+\frac{1}{2}\eta' \left|\hat{\sigma}^\dagger_{|\alpha\rangle\langle\delta|}\right| q-\frac{1}{2}\eta'\right\rangle\right) \mathrm{e}^{-\mathrm{i}\eta' p}\,\mathrm{d}\eta' \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\left\langle q+\frac{1}{2}\eta' \left|\hat{\sigma}^\dagger_{|\alpha\rangle\langle\delta|}\right| q-\frac{1}{2}\eta'\right\rangle\right) \mathrm{e}^{-\mathrm{i}\eta' p}\,\mathrm{d}\eta' \\ &\neq W_{|\alpha\rangle\langle\beta|}(q,p), \end{aligned} \tag{5.1.92}$$

because, in general, the transient density matrix is not Hermitian (see Eq. (2.7.12)):

$$\hat{\sigma}^\dagger_{|\alpha\rangle\langle\beta|} \neq \hat{\sigma}_{|\alpha\rangle\langle\beta|}. \tag{5.1.93}$$

5.1.3.4 Marginal distributions

We want to prove

$$\boxed{\int_{-\infty}^{\infty} W_{|\alpha\rangle\langle\beta|}(q,p)\,\mathrm{d}p = \left\langle q\left|\hat{\sigma}_{|\alpha\rangle\langle\beta|}\right| q\right\rangle = \mathrm{Tr}\left(\hat{\sigma}_{|\alpha\rangle\langle\beta|}\left|q\right\rangle\left\langle q\right|\right) = \left\langle\left|q\right\rangle\left\langle q\right|\right\rangle_w,} \tag{5.1.94}$$

and

$$\boxed{\int_{-\infty}^{\infty} W_{|\alpha\rangle\langle\beta|}(q,p)\,\mathrm{d}q = \left\langle p\left|\hat{\sigma}_{|\alpha\rangle\langle\beta|}\right| p\right\rangle = \mathrm{Tr}\left(\hat{\sigma}_{|\alpha\rangle\langle\beta|}\left|q\right\rangle\left\langle q\right|\right) = \left\langle\left|p\right\rangle\left\langle p\right|\right\rangle_w.} \tag{5.1.95}$$

Proof

We have

$$\int_{-\infty}^{\infty} W_{|\alpha\rangle\langle\beta|}(q,p)\,\mathrm{d}p = \frac{1}{2\pi}\iint_{\mathbf{R}^2}\left\langle q+\frac{1}{2}\eta\left|\hat{\sigma}_{|\alpha\rangle\langle\beta|}\right|q-\frac{1}{2}\eta\right\rangle \mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta\mathrm{d}p$$
$$= \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty}\left\langle q+\frac{1}{2}\eta\left|\hat{\sigma}_{|\alpha\rangle\langle\beta|}\right|q-\frac{1}{2}\eta\right\rangle \cancel{2\pi}\delta(\eta)\,\mathrm{d}\eta$$
$$= \left\langle q\left|\hat{\sigma}_{|\alpha\rangle\langle\beta|}\right|q\right\rangle \tag{5.1.96}$$
$$= \langle|q\rangle\langle q|\rangle_w\,. \tag{5.1.97}$$

For the other marginal distribution, we need to calculate

$$\int_{-\infty}^{\infty} W_{|\alpha\rangle\langle\beta|}(q,p)\,\mathrm{d}q = \frac{1}{2\pi}\iint_{\mathbf{R}^2}\left\langle q+\frac{1}{2}\eta\left|\hat{\sigma}_{|\alpha\rangle\langle\beta|}\right|q-\frac{1}{2}\eta\right\rangle \mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta\mathrm{d}q. \tag{5.1.98}$$

We use the completeness of the $\{|p\rangle\}$ basis:

$$\int_{-\infty}^{\infty} W_{|\alpha\rangle\langle\beta|}(q,p)\,\mathrm{d}q = \frac{1}{2\pi}\iint_{\mathbf{R}^4}\left\langle q+\frac{1}{2}\eta|p\right\rangle\left\langle p\left|\hat{\sigma}_{|\alpha\rangle\langle\beta|}\right|p'\right\rangle\left\langle p'|q-\frac{1}{2}\eta\right\rangle \mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta\mathrm{d}q\mathrm{d}p\mathrm{d}p'. \tag{5.1.99}$$

We know the overlap between the $\{|p\rangle\}$ basis and the $\{|q\rangle\}$:

$$\left\langle q+\frac{1}{2}\eta|p\right\rangle = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}p\left(q+\frac{1}{2}\eta\right)}, \tag{5.1.100}$$
$$\left\langle p'|q-\frac{1}{2}\eta\right\rangle = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}p'\left(q-\frac{1}{2}\eta\right)}, \tag{5.1.101}$$

so

$$\int_{-\infty}^{\infty} W_{|\alpha\rangle\langle\beta|}(q,p)\,\mathrm{d}q = \frac{1}{(2\pi)^2}\iint_{\mathbf{R}^4}\mathrm{e}^{-\mathrm{i}p\left(q+\frac{1}{2}\eta\right)}\left\langle p\left|\hat{\sigma}_{|\alpha\rangle\langle\beta|}\right|p'\right\rangle \mathrm{e}^{\mathrm{i}p'\left(q-\frac{1}{2}\eta\right)}\mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta\mathrm{d}q\mathrm{d}p\mathrm{d}p'$$
$$= \frac{1}{(2\pi)^2}\iint_{\mathbf{R}^4}\left\langle p\left|\hat{\sigma}_{|\alpha\rangle\langle\beta|}\right|p'\right\rangle \mathrm{e}^{\mathrm{i}q(p'-p)}\mathrm{e}^{-\mathrm{i}\eta(p+p')/2}\mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta\mathrm{d}q\mathrm{d}p\mathrm{d}p'$$
$$= \frac{1}{(2\pi)^{\cancel{2}}}\iint_{\mathbf{R}^3}\left\langle p\left|\hat{\sigma}_{|\alpha\rangle\langle\beta|}\right|p'\right\rangle \cancel{2\pi}\delta\left(p'-p\right)\mathrm{e}^{-\mathrm{i}\eta(p+p')/2}\mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta\mathrm{d}p\mathrm{d}p'$$
$$= \frac{1}{2\pi}\iint_{\mathbf{R}^2}\left\langle p\left|\hat{\sigma}_{|\alpha\rangle\langle\beta|}\right|p\right\rangle \mathrm{e}^{-\mathrm{i}\eta(p+p)/2}\mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\eta\mathrm{d}p$$
$$= \frac{1}{\cancel{2\pi}}\iint_{\mathbf{R}^2}\left\langle p\left|\hat{\sigma}_{|\alpha\rangle\langle\beta|}\right|p\right\rangle \cancel{2\pi}\delta(p)\,\mathrm{d}\eta$$
$$= \int_{-\infty}^{+\infty}\left\langle p\left|\hat{\sigma}_{|\alpha\rangle\langle\beta|}\right|p\right\rangle \mathrm{d}\eta$$
$$= \mathrm{Tr}\left(\hat{\sigma}_{|\alpha\rangle\langle\beta|}|p\rangle\langle p|\right) \tag{5.1.102}$$
$$= \langle|p\rangle\langle p|\rangle_w\,. \tag{5.1.103}$$

We see that the marginal distributions are related to the weak values (instead of the expected values in the case of the normal WIGNER quasi-probability distribution). This fact is very interesting because it gives an experimental method to determine the extended WIGNER quasi-probability distribution using quantum tomography, except that we have to measure the weak values here.

However, we have seen in Chapter 2, that the weak values can be determined, even though they are complex in general using the pre- and post-selection.

5.1.4 Graphical representation

It is sufficient to limit our study to

$$\boxed{\begin{cases}\alpha \equiv 0 + 0\mathrm{i}, \\ \beta \in \mathbf{R}_{+}.\end{cases}} \tag{5.1.104}$$

For reference, the plots are given in the Appendix C.

5.2 Extended Q representation

5.2.1 Derivation of the extended Q representation

The Q function is defined as

$$\boxed{Q_{\rho}\left(\gamma\right) \equiv \frac{1}{\pi}\left\langle \gamma\left|\hat{\rho}\right|\gamma\right\rangle = \frac{1}{\pi}\mathrm{Tr}\,\left(\hat{\rho}\left|\gamma\right\rangle\left\langle\gamma\right|\right),} \tag{5.2.1}$$

In that case, we define the extended Q representation by replacing $\hat{\rho}$ by $\hat{\sigma}_{|\alpha\rangle\langle\beta|}$

$$\boxed{Q_{|\alpha\rangle\langle\beta|}\left(\gamma\right) \equiv \frac{1}{\pi}\frac{\langle\gamma|\alpha\rangle\langle\beta|\gamma\rangle}{\langle\beta|\alpha\rangle}.} \tag{5.2.2}$$

The calculations can be done fairly easily

$$\begin{aligned}
Q_{|\alpha\rangle\langle\beta|}\left(\gamma\right) &= \frac{1}{\pi}\frac{1}{\langle\beta|\alpha\rangle}\langle\gamma|\alpha\rangle\langle\beta|\gamma\rangle \\
&= \frac{1}{\pi}\exp\left[\frac{1}{2}\left(|\beta|^2 + |\alpha|^2 - 2\beta^*\alpha\right)\right] \times \exp\left[-\frac{1}{2}\left(|\gamma|^2 + |\alpha|^2 - 2\gamma^*\alpha\right)\right] \\
&\quad \times \exp\left[-\frac{1}{2}\left(|\beta|^2 + |\gamma|^2 - 2\beta^*\gamma\right)\right] \\
&= \frac{1}{\pi}\exp\left[-\left|\gamma\right|^2 - \beta^*\alpha + \gamma^*\alpha + \beta^*\gamma\right],
\end{aligned} \tag{5.2.3}$$

which can also be written as

$$\boxed{Q_{|\alpha\rangle\langle\beta|}\left(\gamma\right) \equiv \frac{\exp\left[-\left(\gamma-\alpha\right)\left(\gamma-\beta\right)^*\right]}{\pi},} \tag{5.2.4}$$

or

$$Q_{|\alpha\rangle\langle\beta|}\left(\gamma\right) = \frac{1}{\pi}\mathrm{Tr}\,\left[\left(\frac{|\alpha\rangle\langle\beta|}{\langle\beta|\alpha\rangle}\right)\left|\gamma\right\rangle\left\langle\gamma\right|\right], \tag{5.2.5}$$

that we could compare to the Q representation of a coherent state $|\alpha\rangle\langle\alpha|$ (see Eq. (4.5.13)):

$$Q_{\alpha}\left(\gamma\right) = \frac{\exp\left[-\left|\gamma-\alpha\right|^2\right]}{\pi}, \tag{5.2.6}$$

which is simply proportional to the overlap squared of the two coherent states. We will see that $Q_{|\alpha\rangle\langle\beta|}\left(\gamma\right)$ can be negative, whereas $Q_{\rho}\left(\gamma\right)$ is always positive and bounded by $1/\pi$.

5.2.2 Transformation

Just like in the case of the extended WIGNER quasi-probability distribution, let us find the transformations which leave the extended Q representation unchanged (or almost identical), in order to simplify the analysis (by reducing the number of parameters). Essentially, we will prove that it is sufficient to vary the distance between α and β to reproduce all the possible shapes for the extended Q representation. In the subsequent sections, it will be enough to study the cases defined by

$$\boxed{\begin{cases}\alpha \equiv 0 + 0\mathrm{i}, \\ \beta \in \mathbf{R}_+.\end{cases}} \tag{5.2.7}$$

5.2.2.1 Translation

Here, we will demonstrate that

$$\boxed{Q_{|\alpha+\zeta\rangle\langle\beta+\zeta|}\left(\gamma\right) = Q_{|\alpha\rangle\langle\beta|}\left(\gamma-\zeta\right).} \tag{5.2.8}$$

This means that the effect of a translation in the complex plane of α and β by a complex number ζ is equivalent to the translation of the whole Q representation by a complex number $-\zeta$. This property allows us to study the particular case

$$\boxed{\alpha \equiv 0 + 0\mathrm{i}} \tag{5.2.9}$$

This reduces our analysis to only two parameters found in β.

Proof

We will consider the following transformation:

$$\begin{cases}|\alpha\rangle \to |\alpha'\rangle = |\alpha+\zeta\rangle, \\ |\beta\rangle \to |\beta'\rangle = |\beta+\zeta\rangle.\end{cases} \tag{5.2.10}$$

Using the formula for the translation of a coherent state (Eq. (3.3.24)), we can write

$$|\alpha+\zeta\rangle = \exp\left\{\frac{\zeta^*\alpha - \zeta\alpha^*}{2}\right\}\hat{D}\left(\zeta\right)|\alpha\rangle, \tag{5.2.11}$$

$$|\beta+\zeta\rangle = \exp\left\{\frac{\zeta^*\beta - \zeta\beta^*}{2}\right\}\hat{D}\left(\zeta\right)|\beta\rangle. \tag{5.2.12}$$

To make it more readable, we will use the notation

$$\phi_\alpha \equiv \frac{\zeta^*\alpha - \zeta\alpha^*}{2}, \tag{5.2.13}$$

$$\phi_\beta \equiv \frac{\zeta^*\beta - \zeta\beta^*}{2}, \tag{5.2.14}$$

so that

$$|\alpha+\zeta\rangle = \mathrm{e}^{\phi_\alpha}\hat{D}\left(\zeta\right)|\alpha\rangle, \tag{5.2.15}$$

$$|\beta+\zeta\rangle = \mathrm{e}^{\phi_\beta}\hat{D}\left(\zeta\right)|\beta\rangle. \tag{5.2.16}$$

In that case, the extended Q representation becomes

$$
\begin{aligned}
Q_{|\alpha'\rangle\langle\beta'|}(\gamma) &= \frac{1}{\pi}\frac{\langle\gamma|\alpha'\rangle\langle\beta'|\gamma\rangle}{\langle\alpha'|\beta'\rangle} \\
&= \frac{1}{\pi}\frac{\langle\gamma|\alpha+\zeta\rangle\langle\beta+\zeta|\gamma\rangle}{\langle\alpha+\zeta|\beta+\zeta\rangle} \\
&= \frac{1}{\pi}\frac{\left\langle\gamma\left|\cancel{\mathrm{e}^{\phi_\alpha}}\hat{D}(\zeta)\right|\alpha\right\rangle\left\langle\beta\left|\hat{D}^\dagger(\zeta)\cancel{\mathrm{e}^{\phi_\beta^*}}\right|\gamma\right\rangle}{\left\langle\beta\left|\hat{D}^\dagger(\zeta)\cancel{\mathrm{e}^{\phi_\beta^*}}\cancel{\mathrm{e}^{\phi_\alpha}}\hat{D}(\zeta)\right|\alpha\right\rangle} \\
&= \frac{1}{\pi}\frac{\left\langle\gamma\left|\hat{D}(\zeta)\right|\alpha\right\rangle\left\langle\beta\left|\hat{D}^\dagger(\zeta)\right|\gamma\right\rangle}{\left\langle\beta\left|\hat{D}^\dagger(\zeta)\hat{D}(\zeta)\right|\alpha\right\rangle} \\
&= \frac{1}{\pi}\frac{1}{\langle\beta|\alpha\rangle}\left\langle\gamma\left|\hat{D}(\zeta)\right|\alpha\right\rangle\left\langle\beta\left|\hat{D}^\dagger(\zeta)\right|\gamma\right\rangle,
\end{aligned}
\tag{5.2.17}
$$

where we used the unitarity of the displacement operator (Eq. (3.3.19)) in the last equation. The property of the Hermitian conjugate of the displacement operator (Eq. (3.3.20)) gives us

$$
Q_{|\alpha'\rangle\langle\beta'|}(\gamma) = \frac{1}{\pi}\frac{\left\langle\gamma\left|\hat{D}^\dagger(-\zeta)\right|\alpha\right\rangle\left\langle\beta\left|\hat{D}(-\zeta)\right|\gamma\right\rangle}{\langle\beta|\alpha\rangle}. \tag{5.2.18}
$$

Now, using the definition of the displacement operator (Eq. (3.3.18)), we can apply it to the states $|\gamma\rangle$ the following way

$$
Q_{|\alpha'\rangle\langle\beta'|}(\gamma) = \frac{1}{\pi}\frac{\left\langle\gamma-\zeta\left|\mathrm{e}^{\phi_\gamma^*}\right|\alpha\right\rangle\left\langle\beta\left|\mathrm{e}^{\phi_\gamma}\right|\gamma-\zeta\right\rangle}{\langle\beta|\alpha\rangle} \tag{5.2.19}
$$

where

$$
\phi_\gamma \equiv -\frac{(-\zeta^*)\gamma-(-\zeta)\gamma^*}{2} = \frac{\zeta^*\gamma-\zeta\gamma^*}{2}. \tag{5.2.20}
$$

Since

$$
\phi_\gamma + \phi_\gamma^* = \frac{\zeta^*\gamma-\zeta\gamma^*}{2} + \frac{\zeta\gamma^*-\zeta^*\gamma}{2} = 0, \tag{5.2.21}
$$

we finally get

$$
Q_{|\alpha'\rangle\langle\beta'|}(\gamma) = \frac{1}{\pi}\frac{\langle\gamma-\zeta|\alpha\rangle\langle\beta|\gamma-\zeta\rangle}{\langle\beta|\alpha\rangle},
$$

or

$$
Q_{|\alpha'\rangle\langle\beta'|}(\gamma) = Q_{|\alpha\rangle\langle\beta|}(\gamma-\zeta). \tag{5.2.22}
$$

5.2.2.2 Rotation

Here, we will demonstrate that

$$
\boxed{Q_{\left|\alpha\mathrm{e}^{-\mathrm{i}\theta}\right\rangle\left\langle\beta\mathrm{e}^{-\mathrm{i}\theta}\right|}(\gamma) = Q_{|\alpha\rangle\langle\beta|}\left(\gamma\mathrm{e}^{\mathrm{i}\theta}\right).} \tag{5.2.23}
$$

This means that the rotation in the complex plane of α and β by an angle $-\theta$ will simply rotate (by an angle $+\theta$) the extended Q representation, without changing its shape. Taking into account the information from the last section, we can reduce our analysis to one parameter, namely the distance between α and β, without a loss of generality. From now on, we will suppose that θ is chosen such that β is positive and on the real axis:

$$\boxed{\beta \in \mathbf{R}_+.} \tag{5.2.24}$$

Proof

We will consider the the following transformation:

$$\begin{cases} |\alpha\rangle \to |\alpha'\rangle = \left|\alpha e^{-i\theta}\right\rangle, \\ |\beta\rangle \to |\beta'\rangle = \left|\beta e^{-i\theta}\right\rangle. \end{cases} \tag{5.2.25}$$

Using the formula for the rotation of a coherent state (Eq. (3.3.26)), we can write

$$\left|\alpha e^{-i\theta}\right\rangle = \hat{U}(\theta)|\alpha\rangle, \tag{5.2.26}$$

$$\left|\beta e^{-i\theta}\right\rangle = \hat{U}(\theta)|\beta\rangle. \tag{5.2.27}$$

The extended Q representation becomes

$$\begin{aligned} Q_{|\alpha'\rangle\langle\beta'|}(\gamma) &= \frac{1}{\pi}\frac{\langle\gamma|\alpha'\rangle\left\langle\beta e^{-i\theta}|\gamma\right\rangle}{\left\langle\beta'|\alpha e^{-i\theta}\right\rangle} \\ &= \frac{1}{\pi}\frac{\left\langle\gamma|\alpha e^{-i\theta}\right\rangle\left\langle\beta e^{-i\theta}|\gamma\right\rangle}{\left\langle\beta e^{-i\theta}|\alpha e^{-i\theta}\right\rangle} \\ &= \frac{1}{\pi}\frac{\left\langle\gamma\left|\hat{U}(\theta)\right|\alpha\right\rangle\left\langle\beta\left|\hat{U}^\dagger(\theta)\right|\gamma\right\rangle}{\left\langle\beta\left|\hat{U}^\dagger(\theta)\hat{U}(\theta)\right|\alpha\right\rangle}, \\ &= \frac{1}{\pi}\frac{\left\langle\gamma\left|\hat{U}(\theta)\right|\alpha\right\rangle\left\langle\beta\left|\hat{U}^\dagger(\theta)\right|\gamma\right\rangle}{\langle\beta|\alpha\rangle e^{i\theta}}, \end{aligned} \tag{5.2.28}$$

where we used the unitary of the phase-shifting operator (Eq. (3.3.30)). The Hermitian conjugation of the phase-shifting operator is given by Eq. (3.3.29), so we have

$$Q_{|\alpha'\rangle\langle\beta'|}(\gamma) = \frac{1}{\pi}\frac{\left\langle\gamma\left|\hat{U}^\dagger(-\theta)\right|\alpha\right\rangle\left\langle\beta\left|\hat{U}(-\theta)\right|\gamma\right\rangle}{\langle\beta|\alpha\rangle}. \tag{5.2.29}$$

We now apply the phase-shifting operator on the states $|\gamma\rangle$ as per Eq. (3.3.25):

$$Q_{|\alpha'\rangle\langle\beta'|}(\gamma) = \frac{1}{\pi}\frac{\left\langle\gamma e^{i\theta}|\alpha\right\rangle\left\langle\beta|\gamma e^{i\theta}\right\rangle}{\langle\beta|\alpha\rangle}, \tag{5.2.30}$$

or

$$Q_{|\alpha'\rangle\langle\beta'|}(\gamma) = e^{i\theta} Q_{|\alpha\rangle\langle\beta|}\left(\gamma e^{i\theta}\right). \tag{5.2.31}$$

5.2.2.3 Conjugation conjugation

It is interesting to note that

$$\boxed{Q^*_{|\alpha\rangle\langle\beta|}(\gamma) = Q_{|\beta\rangle\langle\alpha|}(\gamma).} \tag{5.2.32}$$

Proof

We have

$$\begin{aligned} Q_{|\alpha\rangle\langle\beta|}(\gamma) &= \frac{1}{\pi}\left(\frac{\langle\gamma|\alpha\rangle\langle\beta|\gamma\rangle}{\langle\beta|\alpha\rangle}\right)^* \\ &= \frac{1}{\pi}\frac{\langle\gamma|\alpha\rangle^*\langle\beta|\gamma\rangle^*}{\langle\beta|\alpha\rangle^*} \\ &= \frac{1}{\pi}\frac{\langle\alpha|\gamma\rangle\langle\gamma|\beta\rangle}{\langle\alpha|\beta\rangle} \\ &= \frac{1}{\pi}\frac{\langle\gamma|\beta\rangle\langle\alpha|\gamma\rangle}{\langle\alpha|\beta\rangle} \\ &\equiv Q_{|\beta\rangle\langle\alpha|}(\gamma). \end{aligned} \tag{5.2.33}$$

5.2.3 Properties

5.2.3.1 Normalization

The extended Q representation is still normalized

$$\boxed{\int_{\mathbf{R}^2} Q_{|\alpha\rangle\langle\beta|}(\gamma)\,\mathrm{d}^2\gamma = 1.} \tag{5.2.34}$$

Proof

$$\begin{aligned} \int_{\mathbf{R}^2} Q_{|\alpha\rangle\langle\beta|}(\gamma)\,\mathrm{d}^2\gamma &= \frac{1}{\pi}\int_{\mathbf{R}^2}\frac{\langle\gamma|\alpha\rangle\langle\beta|\gamma\rangle}{\langle\beta|\alpha\rangle}\,\mathrm{d}^2\gamma \\ &= \frac{1}{\pi}\int_{\mathbf{R}^2}\frac{\langle\beta|\gamma\rangle\langle\gamma|\alpha\rangle}{\langle\beta|\alpha\rangle}\,\mathrm{d}^2\gamma \\ &= \frac{\left\langle\beta\middle|\left(\frac{1}{\pi}\int_{\mathbf{R}^2}|\gamma\rangle\langle\gamma|\,\mathrm{d}^2\gamma\right)\middle|\alpha\right\rangle}{\langle\beta|\alpha\rangle} \\ &= \frac{\langle\beta|\alpha\rangle}{\langle\beta|\alpha\rangle} \\ &= 1, \end{aligned} \tag{5.2.35}$$

where we used the over-completeness of the coherent states (Eq. (3.3.11)).

5.2.3.2 Bounds

Even though the $Q_{|\alpha\rangle\langle\beta|}(\gamma)$ can reach arbitrary large values by taking specific α and β, it still remains bounded for a fixed couple (α, β):

$$\boxed{\left|Q_{|\alpha\rangle\langle\beta|}(\gamma)\right| \leq \frac{\exp\left\{-\Re\left[(\gamma-\alpha)(\gamma-\beta)^*\right]/2\right\}}{\pi}.} \tag{5.2.36}$$

.

Proof

Using Eq. (5.2.4), we have

$$\left|Q_{|\alpha\rangle\langle\beta|}(\gamma)\right|^2 = \frac{\left|\exp\left[-(\gamma-\alpha)(\gamma-\beta)^*\right]\right|^2}{\pi^2} = \frac{\exp\left[-(\gamma-\alpha)(\gamma-\beta)^* - (\gamma-\alpha)^*(\gamma-\beta)\right]}{\pi^2}, \tag{5.2.37}$$

which can also be written as

$$\left|Q_{|\alpha\rangle\langle\beta|}(\gamma)\right|^2 = \frac{\exp\left\{-\Re\left[(\gamma-\alpha)(\gamma-\beta)^*\right]\right\}}{\pi^2}, \tag{5.2.38}$$

or

$$\left|Q_{|\alpha\rangle\langle\beta|}(\gamma)\right| \leq \frac{\exp\left\{-\Re\left[(\gamma-\alpha)(\gamma-\beta)^*\right]/2\right\}}{\pi}.$$

5.2.3.3 Reality

The extended Q representation can be complex

$$\boxed{Q^*_{|\alpha\rangle\langle\beta|}(\gamma) \neq Q_{|\alpha\rangle\langle\beta|}(\gamma).} \tag{5.2.39}$$

5.2.4 Graphical representation

As we saw before, the extended Q representation of the transient density matrix is

$$Q_{|\alpha\rangle\langle\beta|}(\gamma) \equiv \frac{\exp\left[-(\gamma-\alpha)(\gamma-\beta)^*\right]}{\pi}. \tag{5.2.40}$$

Moreover, it is sufficient to limit our study to

$$\boxed{\begin{cases} \alpha \equiv 0 + 0\mathrm{i}, \\ \beta \in \mathbf{R}_+. \end{cases}} \tag{5.2.41}$$

For reference, the plots are given in the Appendix C.

5.3 Extended P representation

5.3.1 Possibility of derivation

As we saw in Chapter (4), the P representation is defined as

$$\hat{\rho} \equiv \iint_{\mathbf{R}^2} P(\gamma) |\gamma\rangle \langle\gamma| \, \mathrm{d}\gamma^2. \tag{5.3.1}$$

Since the P representation exists for any mixture of coherent states, the extended P representation defined as

$$\boxed{\hat{\sigma}_{|\alpha\rangle\langle\beta|} \equiv \iint_{\mathbf{R}^2} P_{|\alpha\rangle\langle\beta|}(\gamma) |\gamma\rangle \langle\gamma| \, \mathrm{d}\gamma^2,} \tag{5.3.2}$$

might exist as well. As always, this definition is not easy to implement. For this reason, we could try to calculate the extended P representation using the relation between the Q representation and the P representation defined by (see Eq. (A.4.2))

$$F^P(\gamma, \gamma^*) = \iint_{\mathbf{R}^2} g\left(\zeta' - \gamma, \zeta'^* - \gamma^*\right) F^Q(\zeta, \zeta^*) \, \mathrm{d}^2\zeta', \tag{5.3.3}$$

with

$$g(\gamma, \gamma^*) \equiv \frac{1}{4\pi^2} \iint_{\mathbf{R}^2} \mathrm{e}^{z\gamma^* - z^*\gamma + z_r^2 + z_i^2} \, \mathrm{d}^2 z, \tag{5.3.4}$$

where

$$z_r \equiv \Re(z), \tag{5.3.5}$$

$$z_i \equiv \Im(z). \tag{5.3.6}$$

Note that this relation was defined for normal phase space distributions. Therefore, it is not certain that they remain valid in the case of the extended phase distributions. However, we could verify the validity of the result by using the definition of the extended P representation (Eq. (5.3.2)). The calculation is not given here since the P is usually ill-behaved.

5.3.2 Remark

If we take the generalized positive P representation defined in Section 4.6.3 by

$$\hat{\rho} \equiv \iint_{\mathbf{R}^2} \Lambda(\gamma, \zeta) P(\gamma, \zeta) \, \mathrm{d}^2\gamma \, \mathrm{d}^2\zeta, \tag{5.3.7}$$

where

$$\Lambda(\gamma, \zeta) = \frac{|\gamma\rangle \langle\zeta^*|}{\langle\zeta^*|\gamma\rangle}, \tag{5.3.8}$$

then we could define

$$\hat{\sigma}_{|\alpha\rangle\langle\beta|} \equiv \iint_{\mathbf{R}^2} \Lambda_{|\alpha\rangle\langle\beta|}(\gamma, \zeta) P(\gamma, \zeta) \, \mathrm{d}^2\gamma \, \mathrm{d}^2\zeta. \tag{5.3.9}$$

In that case, it is trivial to see that

$$\boxed{\Lambda_{|\alpha\rangle\langle\beta|}(\gamma, \zeta) = \delta^2(\gamma - \alpha) \, \delta^2(\zeta^* - \beta).} \tag{5.3.10}$$

While the derivation is simple and intriguing, it does not offer a lot of information about the state unfortunately because it's .

5.4 Summary

The results that we found are given in Table (5.4.1). We saw that the properties of the transient density matrix (especially) lead to a strange behavior of the phase space distribution which is exacerbated when the overlap of the pre- and post-selected states are reduced.

Name	$W\left(q,p\right)$	$W_{\|\alpha\rangle\langle\beta\|}\left(q,p\right)$	$Q\left(\gamma\right)$	$Q_{\|\alpha\rangle\langle\beta\|}\left(\gamma\right)$
Normalized	Yes	Yes	Yes	Yes
Real	Yes	No	Yes	No
Non-negative	No	No	Yes	No
Bounds	$\left\|W\left(q,p\right)\right\| \leq 1/\pi$	$\left\|W_{\|\alpha\rangle\langle\beta\|}\left(q,p\right)\right\| = \mathrm{e}^{\|\alpha-\beta\|^2/2}/\pi$	$\left\|Q\left(\gamma\right)\right\| \leq 1/\pi$	$\frac{\mathrm{e}^{-\Re\left[(\gamma-\alpha)(\gamma-\beta)^*\right]/2}}{\pi}$
Marginal distributions	Yes	Yes	No	No

Table 5.4.1: Summary of the properties of the extended phase space distributions.

Conclusion

In this report, we tried to give a thorough overview of the weak value formalism applied to the field of quantum optics. We started with a description of the measurement in quantum mechanics. After reviewing the density matrix formulation, we stated the measurement postulate, as well as BORN's rule for obtaining probabilities in quantum mechanics. Then, we saw that there are multiple possible implementations of the measurement postulate. Two of them were explored: the more familiar projective measurement, and also the POVM formalism, an important concept in quantum information theory.

The next step was to review the notion weak measurement as defined by AHARONOV *et al* using the two-state vector formalism. We saw that the reduced interaction between the measuring device and the system implies a smaller disturbance, but also a bigger uncertainty on the measurement. This drawback can be overcome by repeating the experiment on a larger number of system. This led us to explore the strange concept of weak value. They are similar to the expected value, except they have weird properties: they can be out the normal bounds for an expected value and, in some cases, they can become complex. To emphasize the importance of the weak value, we gave a few applications were it is used. Finally, we extended the density matrix by a transient density matrix which was explored subsequently in the case where both the pre- and post-selected states are coherent states of light.

Before studying this transient density light, we therefore needed to study the field of quantum optics. We explained how the quantization of the electromagnetic field (second quantization) takes place in quantum mechanics. The Fock states (or number states) emerged naturally from the quantization. However, since they are not good approximation to real states of light by themselves, we had to define the coherent states which are a superposition of the Fock states. We then inspected some of the main properties that would be useful later on.

The next chapter was dedicated to the phase space distributions. We motivated the reason for this mathematical framework by showing how the expected value can be calculated more efficiently. We also saw that the phase space distributions possess a one-to-one correspondence with the density operator; we could therefore use them as a characterization tool. However, we demonstrated that the phase space distributions are not uniquely defined, so there are several ones. We presented the three main distributions used in the field of quantum optics: the WIGNER quasi-probability distribution, the Q representation and the P representation. Each have their own advantages and disadvantages based on their properties. For example, each of them correspond to a certain operator ordering, so it might prove to be easier to employ one or the other depending on this fact.

Afterwards, by comparing the weak value with the expected, we extended the phase space distribution framework to the weak measurement formalism using the transient density matrix defined earlier. The reason for this was two-fold: first of all, it allowed us to inspect the nature of the transient density matrix in greater detail; second of all, we wanted to see if the phase space distribution could be indeed extended to the notion of weak value. This way, we could use this framework as a tool for calculating weak values, mirroring the method that is already used for the calculations of expected values. We then studied the properties of those extended phase space distributions. While the extended P representation could not be calculated, we saw that the other two are in general complex and can contain values exceeding the normal bounds for these distributions. This was partly linked to the non-Hermitian nature of the transient density matrix. If we tried to interpret the transient density matrix as a state, it would mean that the probabilities are negative or higher than one. We also observed that the abnormal behavior was strongly correlated with the choice of the pre- and post-selected states. Additionally, we noticed that the phase space distribution framework could therefore be used to calculate weak values.

While we have tried to be thorough as possible, there still remains several avenues worth exploring. One could perhaps extend the complete framework of phase space distribution to the weak value. It would be interesting to study the influence of the operator ordering in this extended framework. Another path is to try to directly link the shape of the phase space distributions with certain properties of the weak value experiment, as well as to interpret the complex nature of the transient density matrix. Weak values could also perhaps be used to give a better understanding of the phase space distributions. Needless to say, the weak measurement formalism still has a lot to offer.

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Appendix A

Additional calculations

A.1 Hamiltonian of the (classical) electromagnetic field

We will show that the Hamiltonian of the Hamiltonian of the (classical) electromagnetic field can be written as

$$\boxed{H = 2\sum_{\boldsymbol{k},s} \omega_k^2 \left|c_{\boldsymbol{k}s}(t)\right|^2} \tag{A.1.1}$$

where

$$c_{\boldsymbol{k}s}(t) \equiv c_{\boldsymbol{k}s} \mathrm{e}^{-\mathrm{i}\omega_k t} \tag{A.1.2}$$

is introduced to simplify the notations.

Proof

We start from the definition of the Hamiltonian for the (classical) electromagnetic field (Eq. (3.1.25)):

$$H = \frac{1}{2}\int_{\mathbf{R}^3} \left[\varepsilon_0 \boldsymbol{E}^2(\boldsymbol{r},t) + \frac{1}{\mu_0}\boldsymbol{B}^2(\boldsymbol{r},t)\right] \mathrm{d}^3 r, \tag{A.1.3}$$

where the electric and magnetic fields are given by (Eqs. (3.1.23) and (3.1.24))

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{\mathrm{i}}{\varepsilon_0^{1/2} L^{3/2}} \sum_s \sum_{\boldsymbol{k}} \omega_k \left[c_{\boldsymbol{k}s}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \boldsymbol{\varepsilon}_{\boldsymbol{k}s} - c^*_{\boldsymbol{k}s}(t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\right], \tag{A.1.4}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{\mathrm{i}}{\varepsilon_0^{1/2} L^{3/2}} \sum_s \sum_{\boldsymbol{k}} \left[c_{\boldsymbol{k}s}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} (\boldsymbol{k}\times\boldsymbol{\varepsilon}_{\boldsymbol{k}s}) - c^*_{\boldsymbol{k}s}(t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} (\boldsymbol{k}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s})\right]. \tag{A.1.5}$$

Moreover, during the calculations, we will make use of the transversality condition

$$\boldsymbol{k}\cdot\boldsymbol{\varepsilon}_{\boldsymbol{k}s} = 0, \tag{A.1.6}$$

the orthonormality of the polarization vectors

$$\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}_{\boldsymbol{k}q} = 0, \tag{A.1.7}$$

as well as the properties of the Fourier transform

$$\int_{L^3} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{r}}\,\mathrm{d}^3\boldsymbol{r} = L^3\delta^3_{\boldsymbol{kp}}, \tag{A.1.8}$$

and

$$(\boldsymbol{k}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s})\cdot(\boldsymbol{k}\times\boldsymbol{\varepsilon}_{\boldsymbol{k}q}) = k^2\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}_{\boldsymbol{k}q} = k^2\delta_{sq}, \tag{A.1.9}$$

where we have used the orthonormality of the polarization vectors in the vectorial identity

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{a}\cdot\boldsymbol{c})\times(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{b}\cdot\boldsymbol{c})\times(\boldsymbol{a}\cdot\boldsymbol{d})\,. \tag{A.1.10}$$

Let us first calculate the contribution due to the electric field:

$$\begin{aligned}
\int_{\mathbf{R}^3}\varepsilon_0\boldsymbol{E}^2(\boldsymbol{r},t)\,\mathrm{d}^3r &= \int_{\mathbf{R}^3}\cancel{\varepsilon_0}\left\{\frac{\mathrm{i}}{\cancel{\varepsilon_0^{1/2}}L^{3/2}}\sum_s\sum_{\boldsymbol{k}}\omega_k\left[c_{\boldsymbol{k}s}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}_{\boldsymbol{k}s} - c^*_{\boldsymbol{k}s}(t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\right]\right.\\
&\qquad\left.\times\frac{\mathrm{i}}{\cancel{\varepsilon_0^{1/2}}L^{3/2}}\sum_q\sum_{\boldsymbol{p}}\omega_p\left[c_{\boldsymbol{p}q}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}_{\boldsymbol{p}q} - c^*_{\boldsymbol{p}q}(t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}^*_{\boldsymbol{p}q}\right]\right\}\mathrm{d}^3r\\
&= -\frac{1}{L^3}\int_{\mathbf{R}^3}\left\{\sum_{s,q}\sum_{\boldsymbol{k},\boldsymbol{p}}\omega_k\omega_p\left[c_{\boldsymbol{k}s}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}_{\boldsymbol{k}s} - c^*_{\boldsymbol{k}s}(t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\right]\right.\\
&\qquad\left.\times\left[c_{\boldsymbol{p}q}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}_{\boldsymbol{p}q} - c^*_{\boldsymbol{p}q}(t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}^*_{\boldsymbol{p}q}\right]\right\}\mathrm{d}^3r
\end{aligned} \tag{A.1.11}$$

The integral

$$I_E = \int_{\mathbf{R}^3}\left[c_{\boldsymbol{k}s}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}_{\boldsymbol{k}s} - c^*_{\boldsymbol{k}s}(t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\right]\times\left[c_{\boldsymbol{p}q}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}_{\boldsymbol{p}q} - c^*_{\boldsymbol{p}q}(t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}^*_{\boldsymbol{p}q}\right]\mathrm{d}^3r, \tag{A.1.12}$$

can be computed easily:

$$\begin{aligned}
I_E &= \int_{\mathbf{R}^3}c_{\boldsymbol{k}s}(t)\,c_{\boldsymbol{p}q}(t)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}+\boldsymbol{p})\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}_{\boldsymbol{p}q}\,\mathrm{d}^3r - \int_{\mathbf{R}^3}c_{\boldsymbol{k}s}(t)\,c^*_{\boldsymbol{p}q}(t)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}^*_{\boldsymbol{p}q}\,\mathrm{d}^3r\\
&\quad - \int_{\mathbf{R}^3}c^*_{\boldsymbol{k}s}(t)\,c_{\boldsymbol{p}q}(t)\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}_{\boldsymbol{p}q}\,\mathrm{d}^3r + \int_{\mathbf{R}^3}c^*_{\boldsymbol{k}s}(t)\,c^*_{\boldsymbol{p}q}(t)\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}+\boldsymbol{p})\cdot\boldsymbol{r}}\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}^*_{\boldsymbol{p}q}\,\mathrm{d}^3r\\
&= c_{\boldsymbol{k}s}(t)\,c_{\boldsymbol{p}q}(t)\,L^3\delta^3_{-\boldsymbol{kp}}\boldsymbol{\varepsilon}_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}_{\boldsymbol{p}q} - c_{\boldsymbol{k}s}(t)\,c^*_{\boldsymbol{p}q}(t)\,L^3\delta^3_{\boldsymbol{kp}}\boldsymbol{\varepsilon}_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}^*_{\boldsymbol{p}q}\\
&\quad - c^*_{\boldsymbol{k}s}(t)\,c_{\boldsymbol{p}q}(t)\,L^3\delta^3_{\boldsymbol{kp}}\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}_{\boldsymbol{p}q} + c^*_{\boldsymbol{k}s}(t)\,c^*_{\boldsymbol{p}q}(t)\,L^3\delta^3_{-\boldsymbol{kp}}\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}^*_{\boldsymbol{p}q}.
\end{aligned} \tag{A.1.13}$$

Plugging this result into Eq. (A.1.11):

$$\begin{aligned}\int_{\mathbf{R}^3} \varepsilon_0 \boldsymbol{E}^2(\boldsymbol{r},t)\,\mathrm{d}^3r = -\frac{1}{\cancel{L^3}}\sum_{s,q}\sum_{\boldsymbol{k},\boldsymbol{p}}\omega_k\omega_p\Big\{&c_{\boldsymbol{k}s}(t)\,c_{\boldsymbol{p}q}(t)\cancel{L^3}\delta^3_{-\boldsymbol{k}\boldsymbol{p}}\boldsymbol{\varepsilon}_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}_{\boldsymbol{p}q}\\ &-c_{\boldsymbol{k}s}(t)\,c^*_{\boldsymbol{p}q}(t)\cancel{L^3}\delta_{\boldsymbol{k}\boldsymbol{p}}\boldsymbol{\varepsilon}_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}^*_{\boldsymbol{p}q}\\ &-c^*_{\boldsymbol{k}s}(t)\,c_{\boldsymbol{p}q}(t)\cancel{L^3}\delta_{\boldsymbol{k}\boldsymbol{p}}\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}_{\boldsymbol{p}q}.\\ &+c^*_{\boldsymbol{k}s}(t)\,c^*_{\boldsymbol{p}q}(t)\cancel{L^3}\delta^3_{-\boldsymbol{k}\boldsymbol{p}}\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}^*_{\boldsymbol{p}q}\Big\}.\\ =-\sum_{s,q}\sum_{\boldsymbol{k}}\omega_k^2\Big\{&c_{\boldsymbol{k}s}(t)\,c_{-\boldsymbol{k}q}(t)\,\boldsymbol{\varepsilon}_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}_{-\boldsymbol{k}q}-c_{\boldsymbol{k}s}(t)\,c^*_{\boldsymbol{k}q}(t)\,\boldsymbol{\varepsilon}_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}^*_{\boldsymbol{k}q}\\ &-c^*_{\boldsymbol{k}s}(t)\,c_{\boldsymbol{k}q}(t)\,\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}_{\boldsymbol{k}q}+c^*_{\boldsymbol{k}s}(t)\,c^*_{-\boldsymbol{k}q}(t)\,\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}^*_{-\boldsymbol{k}q}\Big\}\\ =-\sum_{s,q}\sum_{\boldsymbol{k}}\omega_k^2\Big\{&c_{\boldsymbol{k}s}(t)\,c_{-\boldsymbol{k}q}(t)\,\boldsymbol{\varepsilon}_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}_{-\boldsymbol{k}q}-c_{\boldsymbol{k}s}(t)\,c^*_{\boldsymbol{k}q}(t)\,\delta_{sq}\\ &-c^*_{\boldsymbol{k}s}(t)\,c_{\boldsymbol{k}q}(t)\,\delta_{sq}+c^*_{\boldsymbol{k}s}(t)\,c^*_{-\boldsymbol{k}q}(t)\,\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}^*_{-\boldsymbol{k}q}\Big\},\end{aligned} \tag{A.1.14}$$

which yields

$$\boxed{\int_{\mathbf{R}^3}\varepsilon_0\boldsymbol{E}^2(\boldsymbol{r},t)\,\mathrm{d}^3r=\sum_s\sum_{\boldsymbol{k}}\omega_k^2\left\{2\,|c_{\boldsymbol{k}s}(t)|^2-\sum_q\left[c_{\boldsymbol{k}s}(t)\,c_{-\boldsymbol{k}q}(t)\,\boldsymbol{\varepsilon}_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}_{-\boldsymbol{k}q}+\text{c.c.}\right]\right\}}. \tag{A.1.15}$$

We can apply the same reasoning to obtain the contribution due to the magnetic field:

$$\begin{aligned}\int_{\mathbf{R}^3}\frac{1}{\mu_0}\boldsymbol{B}^2(\boldsymbol{r},t)\,\mathrm{d}^3r=\int_{\mathbf{R}^3}\frac{1}{\mu_0}\Bigg\{&\frac{\mathrm{i}}{\varepsilon_0^{1/2}L^{3/2}}\times\sum_s\sum_{\boldsymbol{k}}\left[c_{\boldsymbol{k}s}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{k}\times\boldsymbol{\varepsilon}_{\boldsymbol{k}s})-c^*_{\boldsymbol{k}s}(t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{k}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s})\right]\\ &\times\frac{\mathrm{i}}{\varepsilon_0^{1/2}L^{3/2}}\sum_q\sum_{\boldsymbol{p}}\left[c_{\boldsymbol{p}q}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}(\boldsymbol{p}\times\boldsymbol{\varepsilon}_{\boldsymbol{p}q})-c^*_{\boldsymbol{p}q}(t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\left(\boldsymbol{p}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{p}q}\right)\right]\Bigg\}\,\mathrm{d}^3r\\ =-\frac{1}{L^3c^2}\sum_{s,q}\sum_{\boldsymbol{k},\boldsymbol{p}}\Bigg\{&\int_{\mathbf{R}^3}\left[c_{\boldsymbol{k}s}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{k}\times\boldsymbol{\varepsilon}_{\boldsymbol{k}s})-c^*_{\boldsymbol{k}s}(t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{k}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s})\right]\\ &\times\left[c_{\boldsymbol{p}q}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}(\boldsymbol{p}\times\boldsymbol{\varepsilon}_{\boldsymbol{p}q})-c^*_{\boldsymbol{p}q}(t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\left(\boldsymbol{p}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{p}q}\right)\right]\,\mathrm{d}^3r\Bigg\},\end{aligned} \tag{A.1.16}$$

where we have used the relation.

$$c^2=\varepsilon_0\mu_0. \tag{A.1.17}$$

We start by computing the integration over the variable r:

$$\begin{aligned}I_B=\int_{\mathbf{R}^3}&\left[c_{\boldsymbol{k}s}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{k}\times\boldsymbol{\varepsilon}_{\boldsymbol{k}s})-c^*_{\boldsymbol{k}s}(t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{k}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s})\right]\\ &\times\left[c_{\boldsymbol{p}q}(t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}(\boldsymbol{p}\times\boldsymbol{\varepsilon}_{\boldsymbol{p}q})-c^*_{\boldsymbol{p}q}(t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\left(\boldsymbol{p}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{p}q}\right)\right]\,\mathrm{d}^3r.\end{aligned} \tag{A.1.18}$$

The calculation is slightly lengthier than in the case of the electric field, but not too complicated:

$$\begin{aligned} I_B = & \int_{\mathbf{R}^3} c_{\boldsymbol{k}s}(t)\, c_{\boldsymbol{p}q}(t)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}+\boldsymbol{p})\cdot\boldsymbol{r}} \left(\boldsymbol{k}\times\boldsymbol{\varepsilon}_{\boldsymbol{k}s}\right)\cdot\left(\boldsymbol{p}\times\boldsymbol{\varepsilon}_{\boldsymbol{p}q}\right)\, \mathrm{d}^3 r \\ & - \int_{\mathbf{R}^3} c_{\boldsymbol{k}s}(t)\, c^*_{\boldsymbol{p}q}(t)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{r}} \left(\boldsymbol{k}\times\boldsymbol{\varepsilon}_{\boldsymbol{k}s}\right)\cdot\left(\boldsymbol{p}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{p}q}\right)\, \mathrm{d}^3 r \\ & - \int_{\mathbf{R}^3} c^*_{\boldsymbol{k}s}(t)\, c_{\boldsymbol{p}q}(t)\, \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{r}} \left(\boldsymbol{k}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\right)\cdot\left(\boldsymbol{p}\times\boldsymbol{\varepsilon}_{\boldsymbol{p}q}\right)\, \mathrm{d}^3 r \\ & + \int_{\mathbf{R}^3} c^*_{\boldsymbol{k}s}(t)\, c^*_{\boldsymbol{p}q}(t)\, \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}+\boldsymbol{p})\cdot\boldsymbol{r}} \left(\boldsymbol{k}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\right)\cdot\left(\boldsymbol{p}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{p}q}\right) \mathrm{d}^3 r, \end{aligned} \tag{A.1.19}$$

i.e.

$$\begin{aligned} I_B = & c_{\boldsymbol{k}s}(t)\, c_{\boldsymbol{p}q}(t)\, L^3 \delta^3_{-\boldsymbol{k}\boldsymbol{p}} \left(\boldsymbol{k}\times\boldsymbol{\varepsilon}_{\boldsymbol{k}s}\right)\cdot\left(\boldsymbol{p}\times\boldsymbol{\varepsilon}_{\boldsymbol{p}q}\right) \\ & - c_{\boldsymbol{k}s}(t)\, c^*_{\boldsymbol{p}q}(t)\, L^3 \delta^3_{\boldsymbol{k}\boldsymbol{p}} \left(\boldsymbol{k}\times\boldsymbol{\varepsilon}_{\boldsymbol{k}s}\right)\cdot\left(\boldsymbol{p}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{p}q}\right) \\ & - c^*_{\boldsymbol{k}s}(t)\, c_{\boldsymbol{p}q}(t)\, L^3 \delta^3_{\boldsymbol{k}\boldsymbol{p}} \left(\boldsymbol{k}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\right)\cdot\left(\boldsymbol{p}\times\boldsymbol{\varepsilon}_{\boldsymbol{p}q}\right) \\ & + c^*_{\boldsymbol{k}s}(t)\, c^*_{\boldsymbol{p}q}(t)\, L^3 \delta^3_{-\boldsymbol{k}\boldsymbol{p}} \left(\boldsymbol{k}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\right)\cdot\left(\boldsymbol{p}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{p}q}\right). \end{aligned} \tag{A.1.20}$$

We inject this result into Eq. (A.1.16):

$$\begin{aligned} \int_{\mathbf{R}^3} \frac{1}{\mu_0} \boldsymbol{B}^2(\boldsymbol{r},t)\, \mathrm{d}^3 r = & -\frac{1}{\cancel{L^3} c^2} \sum_{s,q} \sum_{\boldsymbol{k},\boldsymbol{p}} \Big\{ c_{\boldsymbol{k}s}(t)\, c_{\boldsymbol{p}q}(t) \cancel{L^3} \delta^3_{-\boldsymbol{k}\boldsymbol{p}} \left(\boldsymbol{k}\times\boldsymbol{\varepsilon}_{\boldsymbol{k}s}\right)\cdot\left(\boldsymbol{p}\times\boldsymbol{\varepsilon}_{\boldsymbol{p}q}\right) \\ & - c_{\boldsymbol{k}s}(t)\, c^*_{\boldsymbol{p}q}(t) \cancel{L^3} \delta^3_{\boldsymbol{k}\boldsymbol{p}} \left(\boldsymbol{k}\times\boldsymbol{\varepsilon}_{\boldsymbol{k}s}\right)\cdot\left(\boldsymbol{p}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{p}q}\right) \\ & + c^*_{\boldsymbol{k}s}(t)\, c_{\boldsymbol{p}q}(t) \cancel{L^3} \delta^3_{\boldsymbol{k}\boldsymbol{p}} \left(\boldsymbol{k}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\right)\cdot\left(\boldsymbol{p}\times\boldsymbol{\varepsilon}_{\boldsymbol{p}q}\right) \\ & - c^*_{\boldsymbol{k}s}(t)\, c^*_{\boldsymbol{p}q}(t) \cancel{L^3} \delta^3_{-\boldsymbol{k}\boldsymbol{p}} \left(\boldsymbol{k}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\right)\cdot\left(\boldsymbol{p}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{p}q}\right) \Big\} \\ = & -\frac{1}{c^2} \sum_{s,q} \sum_{\boldsymbol{k}} \{ c_{\boldsymbol{k}s}(t)\, c_{-\boldsymbol{k}q}(t) \left(\boldsymbol{k}\times\boldsymbol{\varepsilon}_{\boldsymbol{k}s}\right)\cdot\left(-\boldsymbol{k}\times\boldsymbol{\varepsilon}_{-\boldsymbol{k}q}\right) \\ & - c_{\boldsymbol{k}s}(t)\, c^*_{\boldsymbol{k}q}(t) \left(\boldsymbol{k}\times\boldsymbol{\varepsilon}_{\boldsymbol{k}s}\right)\cdot\left(\boldsymbol{k}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{k}q}\right) \\ & - c^*_{\boldsymbol{k}s}(t)\, c_{\boldsymbol{k}q}(t) \left(\boldsymbol{k}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\right)\cdot\left(\boldsymbol{k}\times\boldsymbol{\varepsilon}_{\boldsymbol{k}q}\right) \\ & + c^*_{\boldsymbol{k}s}(t)\, c^*_{-\boldsymbol{k}q}(t) \left(\boldsymbol{k}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\right)\cdot\left(-\boldsymbol{k}\times\boldsymbol{\varepsilon}^*_{\boldsymbol{k}q}\right) \}. \end{aligned} \tag{A.1.21}$$

By applying the formula given by Eq. (A.1.9) as well as the vectorial identity in Eq. (A.1.10), this can be transformed into

$$\begin{aligned} \int_{\mathbf{R}^3} \frac{1}{\mu_0} \boldsymbol{B}^2(\boldsymbol{r},t)\, \mathrm{d}^3 r = & -\frac{1}{c^2} \sum_{s,q} \sum_{\boldsymbol{k}} \Big\{ -k^2 c_{\boldsymbol{k}s}(t)\, c_{-\boldsymbol{k}q}(t)\, \boldsymbol{\varepsilon}_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}_{-\boldsymbol{k}q} - k^2 c_{\boldsymbol{k}s}(t)\, c^*_{\boldsymbol{k}q}(t)\, \delta_{sq} \\ & + k^2 c^*_{\boldsymbol{k}s}(t)\, c_{\boldsymbol{k}q}(t)\, \delta_{sq} - k^2 c^*_{\boldsymbol{k}s}(t)\, c^*_{-\boldsymbol{k}q}(t)\, \boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}^*_{\boldsymbol{k}q} \Big\}\, \mathrm{d}^3 k. \\ = & +\sum_{s,q} \sum_{\boldsymbol{k}} \left(\frac{k}{c}\right)^2 \Big\{ c_{\boldsymbol{k}s}(t)\, c_{-\boldsymbol{k}q}(t)\, \boldsymbol{\varepsilon}_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}_{-\boldsymbol{k}q} + c_{\boldsymbol{k}s}(t)\, c^*_{\boldsymbol{k}q}(t)\, \delta_{sq} \\ & + c^*_{\boldsymbol{k}s}(t)\, c_{\boldsymbol{k}q}(t)\, \delta_{sq} - k^2 c^*_{\boldsymbol{k}s}(t)\, c^*_{-\boldsymbol{k}q}(t)\, \boldsymbol{\varepsilon}^*_{\boldsymbol{k}s}\cdot\boldsymbol{\varepsilon}^*_{\boldsymbol{k}q} \Big\}\, \mathrm{d}^3 k, \end{aligned} \tag{A.1.22}$$

which can be written as

$$\boxed{\int_{\mathbf{R}^3} \frac{1}{\mu_0} \boldsymbol{B}^2(\boldsymbol{r}, t)\, \mathrm{d}^3 r = \sum_{s,q} \sum_{\boldsymbol{k}} \omega_k^2 \left\{ 2\left|c_{\boldsymbol{k}s}(t)\right|^2 + \sum_q \left[c_{\boldsymbol{k}s}(t)\, c_{-\boldsymbol{k}q}(t)\, \boldsymbol{\varepsilon}_{\boldsymbol{k}s} \cdot \boldsymbol{\varepsilon}_{-\boldsymbol{k}q} + \text{c.c.}\right] \right\}.} \tag{A.1.23}$$

because (see Eq. (3.1.18))

$$\left(\frac{k}{c}\right)^2 = \omega_k^2. \tag{A.1.24}$$

We can now determine the form of the Hamiltonian:

$$\begin{aligned} H &= \int_{\mathbf{R}^3} \varepsilon_0 \boldsymbol{E}^2(\boldsymbol{r}, t)\, \mathrm{d}^3 r + \int_{\mathbf{R}^3} \frac{1}{\mu_0} \boldsymbol{B}^2(\boldsymbol{r}, t)\, \mathrm{d}^3 r \\ &= \frac{1}{2} \sum_s \sum_{\boldsymbol{k}} \omega_k^2 \left\{ 2\left|c_{\boldsymbol{k}s}(t)\right|^2 + 2\left|c_{\boldsymbol{k}s}(t)\right|^2 \right. \\ &\quad - \sum_q \left[c_{\boldsymbol{k}s}(t)\, c_{-\boldsymbol{k}q}(t)\, \boldsymbol{\varepsilon}_{\boldsymbol{k}s} \cdot \boldsymbol{\varepsilon}_{-\boldsymbol{k}q} + \text{c.c.}\right] \\ &\quad \left. + \sum_q \left[c_{\boldsymbol{k}s}(t)\, c_{-\boldsymbol{k}q}(t)\, \boldsymbol{\varepsilon}_{\boldsymbol{k}s} \cdot \boldsymbol{\varepsilon}_{-\boldsymbol{k}q} + \text{c.c.}\right] \right\}, \end{aligned} \tag{A.1.25}$$

which yields the expected result

$$\boxed{H = 2 \sum_{\boldsymbol{k},s} \omega_k^2 \left|c_{\boldsymbol{k}s}(t)\right|^2.} \tag{A.1.26}$$

A.2 Correspondence rule

If we use the following correspondence rule

$$p_{\boldsymbol{k}s}(t) \to \hat{p}_{\boldsymbol{k}s}(t) = -\mathrm{i}\hbar \frac{\hat{\partial}}{\partial q_{\boldsymbol{k}s}(t)}, \tag{A.2.1}$$

then

$$\boxed{\left[\hat{q}_{\boldsymbol{k}s}(t), \hat{p}_{\boldsymbol{m}u}(t)\right] = \mathrm{i}\hbar \delta^3_{\boldsymbol{k}\boldsymbol{m}} \delta_{su}.} \tag{A.2.2}$$

Proof

Let us first consider the case where $\boldsymbol{m} = \boldsymbol{k}$ and $u = s$. Indeed, for any function $f\left(q_{\boldsymbol{k}s}(t)\right)$, we have

$$\begin{aligned}\left[\hat{q}_{\boldsymbol{k}s}(t), \hat{p}_{\boldsymbol{k}s}(t)\right] f\left(q_{\boldsymbol{k}s}(t)\right) &= \left(\hat{q}_{\boldsymbol{k}s}(t)\,\hat{p}_{\boldsymbol{k}s}(t) - \hat{p}_{\boldsymbol{k}s}(t)\,\hat{q}_{\boldsymbol{k}s}(t)\right) f\left(q_{\boldsymbol{k}s}(t)\right) \\ &= -\mathrm{i}\hbar\left(\hat{q}\frac{\hat{\partial}}{\partial q}\left[f\left(q_{\boldsymbol{k}s}(t)\right)\right] - \frac{\hat{\partial}}{\partial q}\left[\hat{q}_{\boldsymbol{k}s}(t)\, f\left(q_{\boldsymbol{k}s}(t)\right)\right]\right) \\ &= -\mathrm{i}\hbar\left(\hat{q}_{\boldsymbol{k}s}(t)\frac{\hat{\partial}}{\partial q}\left[f\left(q_{\boldsymbol{k}s}(t)\right)\right] - \hat{q}_{\boldsymbol{k}s}(t)\frac{\hat{\partial}}{\partial q}\left[f\left(q_{\boldsymbol{k}s}(t)\right)\right] - f\left(q_{\boldsymbol{k}s}(t)\right)\right) \\ &= +\mathrm{i}\hbar f\left(q_{\boldsymbol{k}s}(t)\right), \end{aligned} \tag{A.2.3}$$

which leads to the expected commutation rule

$$\left[\hat{q}_{\boldsymbol{k}s}(t), \hat{p}_{\boldsymbol{k}s}(t)\right] = \mathrm{i}\hbar. \tag{A.2.4}$$

The case where $\boldsymbol{m} \neq \boldsymbol{k}$ and/or $u \neq s$ is trivial to demonstrate if we consider that two modes $(\boldsymbol{k}, s)$ and $(\boldsymbol{m}, u)$ are uncoupled.

A.3 Inverting the relation given by Eq. (4.1.6)

$$\mathrm{Tr}\left[\hat{\rho}(\hat{q},\hat{p})\,\mathrm{e}^{\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}} f(\xi,\eta)\right] = \iint_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}\xi q+\mathrm{i}\eta p} F^f(q,p)\,\mathrm{d}q\mathrm{d}p, \tag{A.3.1}$$

can be written as

$$F^f(q,p,t) = \frac{1}{4\pi^2}\iiint_{\mathbf{R}^3}\left\langle q' + \frac{1}{2}\eta\hbar\left|\rho\right| q' - \frac{1}{2}\eta\hbar\right\rangle f(\xi,\eta)\,\mathrm{e}^{\mathrm{i}\xi(q'-q)}\mathrm{e}^{-\mathrm{i}\eta p}\,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}q'.$$

Proof

First, we must recall that (see Eq. (1.1.3))

$$\begin{aligned}\mathrm{Tr}\left[\hat{\rho}(\hat{q},\hat{p})\,\mathrm{e}^{\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}} f(\xi,\eta)\right] &= \int_{-\infty}^{+\infty}\left\langle q''\left|\hat{\rho}(\hat{q},\hat{p})\,\mathrm{e}^{\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}} f(\xi,\eta)\right| q''\right\rangle \mathrm{d}q'' \\ &= \int_{-\infty}^{+\infty}\left\langle q''\left|\hat{\rho}(\hat{q},\hat{p})\,\mathrm{e}^{\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}}\right| q''\right\rangle f(\xi,\eta)\,\mathrm{d}q''. \end{aligned} \tag{A.3.2}$$

Using the Campbell–Baker–Hausdorff theorem (Eq. (B.1.2)) with $x = \mathrm{i}$, $\hat{A} = \hat{q}$, $\hat{B} = \hat{p}$ and $[\hat{q},\hat{p}] = \mathrm{i}\hbar$, we can write:

$$\mathrm{e}^{\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}} = \mathrm{e}^{\mathrm{i}\eta\hat{p}}\mathrm{e}^{\mathrm{i}\xi\hat{q}}\mathrm{e}^{-\mathrm{i}\xi\eta\hbar/2}. \tag{A.3.3}$$

Since

$$\mathrm{e}^{\mathrm{i}\xi\hat{q}}\left|q''\right\rangle = \left|q''\right\rangle \mathrm{e}^{\mathrm{i}\xi q''}, \tag{A.3.4}$$

and

$$\mathrm{e}^{\mathrm{i}\eta\hat{p}}\left|q''\right\rangle = \left|q'' - \eta\hbar\right\rangle, \tag{A.3.5}$$

we have

$$\mathrm{Tr}\left[\hat{\rho}(\hat{q},\hat{p})\,\mathrm{e}^{\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}}f(\xi,\eta)\right]=\int_{-\infty}^{+\infty}\left\langle q''\left|\hat{\rho}(\hat{q},\hat{p})\,\mathrm{e}^{\mathrm{i}\eta\hat{p}}\mathrm{e}^{\mathrm{i}\xi\hat{q}}\mathrm{e}^{-\mathrm{i}\xi\eta\hbar/2}\right|q''\right\rangle f(\xi,\eta)\;\mathrm{d}q''$$
$$=\int_{-\infty}^{+\infty}\left\langle q''\left|\hat{\rho}(\hat{q},\hat{p})\right|q''-\eta\hbar\right\rangle \mathrm{e}^{\mathrm{i}\xi q''}\mathrm{e}^{-\mathrm{i}\xi\eta\hbar/2}f(\xi,\eta)\;\mathrm{d}q''. \quad \text{(A.3.6)}$$

If we make the substitution $q''=q'+\eta\hbar/2$, then

$$\mathrm{Tr}\left[\hat{\rho}(\hat{q},\hat{p})\,\mathrm{e}^{\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}}f(\xi,\eta)\right]=\int_{-\infty}^{+\infty}\left\langle q'+\frac{1}{2}\eta\hbar\left|\hat{\rho}(\hat{q},\hat{p})\right|q'-\frac{1}{2}\eta\hbar\right\rangle \mathrm{e}^{\mathrm{i}\xi q'}\cancel{\mathrm{e}^{\mathrm{i}\eta\hbar/2}\mathrm{e}^{-\mathrm{i}\eta\hbar/2}}f(\xi,\eta)\;\mathrm{d}q''. \quad \text{(A.3.7)}$$

Finally, we take the inverse Fourier transform (see (B.2.2)) of Eq. (A.3.7), we obtain

$$\boxed{F^{f}(q,p,t)=\frac{1}{4\pi^2}\iiint_{\mathbf{R}^3}\left\langle q'+\frac{1}{2}\eta\hbar\left|\rho\right|q'-\frac{1}{2}\eta\hbar\right\rangle f(\xi,\eta)\,\mathrm{e}^{\mathrm{i}\xi(q'-q)}\mathrm{e}^{-\mathrm{i}\eta p}\;\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}q'.} \quad \text{(A.3.8)}$$

A.4 Relations between phase space distributions

If we take two arbitrary phase space distributions $F^1(\alpha,\alpha^*)$ and $F^2(\alpha',\alpha'^*)$, then they are related by [37]:

$$F^1(\alpha,\alpha^*)=\iint_{\mathbf{R}^2} g\left(\alpha'-\alpha,\alpha'^*-\alpha^*\right)F^2\left(\alpha',\alpha'^*\right)\,\mathrm{d}^2\alpha, \quad \text{(A.4.1)}$$

where

$$g(\alpha,\alpha^*)\equiv\frac{1}{4\pi^2}\iint_{\mathbf{R}^2}\mathrm{e}^{z\alpha^*-z^*\alpha}\frac{f^1(z,z^*)}{f^2(z,z^*)}\,\mathrm{d}^2z. \quad \text{(A.4.2)}$$

For simplicity, we use the superscripts Q, P or W for the Q representation, the Glauber–Sudarshan P representation and the Wigner quasi-probability distribution, respectively. Also, we will use the indices i and r to represent the real part and the imaginary part of a complex number as such

$$z_r=\Re(z), \quad \text{(A.4.3)}$$
$$z_i=\Im(z). \quad \text{(A.4.4)}$$

As a reminder, we have

$$f^{\mathrm{W}}(z,z^*)=1, \quad \text{(A.4.5)}$$
$$f^{\mathrm{Q}}(z,z^*)=\mathrm{e}^{-|z|^2/2}=\mathrm{e}^{-z_r^2/2-z_i^2/2}, \quad \text{(A.4.6)}$$
$$f^{\mathrm{P}}(z,z^*)=\mathrm{e}^{|z|^2/2}=\mathrm{e}^{z_r^2/2+z_i^2/2}. \quad \text{(A.4.7)}$$

We will try to find the relations in two cases, since the calculations are almost identical to determine the other relations: one example where the integration on z in $g(\alpha,\alpha^*)$ is possible (W in terms of P), and another where it is not (P in terms of Q).

A.4.1 Distribution W in terms of P

We want to show

$$\boxed{F^W(\alpha,\alpha^*) = \frac{2}{\pi}\int_{\mathbf{R}^2} \mathrm{e}^{-2|\alpha'-\alpha|^2} F^P(\alpha',\alpha'^*)\,\mathrm{d}^2\alpha'.} \tag{A.4.8}$$

Proof

We have

$$\frac{f^W(z,z^*)}{f^P(z,z^*)} = \exp\left\{-\frac{(z_r^2+z_i^2)}{2}\right\}. \tag{A.4.9}$$

Since

$$z\alpha^* - z^*\alpha = 2\mathrm{i}\Im(z\alpha^*) = 2\mathrm{i}(z_i\alpha_r - z_r\alpha_i) \tag{A.4.10}$$

we have

$$g(\alpha,\alpha^*) = \frac{1}{4\pi^2}\iint_{\mathbf{R}^2} \exp\left\{-\left(\frac{z_r^2}{2} - 2\mathrm{i}\alpha_i z_r\right) - \left(\frac{z_i^2}{2} + 2\mathrm{i}\alpha_r z_i\right)\right\}\,\mathrm{d}^2 z. \tag{A.4.11}$$

The integration on z_r can be performed using Eq. (A.5.1) with

$$\begin{cases} a = \frac{1}{2} \\ b = -2\mathrm{i}\alpha_i, \\ c = 0, \end{cases} \tag{A.4.12}$$

and the integration on z_i using

$$\begin{cases} a = \frac{1}{2}, \\ b = 2\mathrm{i}\alpha_r, \\ c = 0, \end{cases} \tag{A.4.13}$$

and it yields

$$g(\alpha,\alpha^*) = \frac{2\pi}{\pi^2}\exp\left\{\frac{(-2\mathrm{i}\alpha_i)^2}{4\times\frac{1}{2}} + \frac{(2\mathrm{i}\alpha_r)^2}{4\times\frac{1}{2}}\right\} = \frac{2}{\pi}\exp\left\{-\frac{\alpha_r^2+\alpha_i^2}{2}\right\}, \tag{A.4.14}$$

or

$$g(\alpha,\alpha^*) = \frac{2}{\pi}\mathrm{e}^{-|\alpha|^2/2}, \tag{A.4.15}$$

which yields the correct expression

$$F^W(\alpha,\alpha^*) = \frac{2}{\pi}\int_{\mathbf{R}^2} \mathrm{e}^{-2|\alpha'-\alpha|^2} F^P(\alpha',\alpha'^*)\,\mathrm{d}^2\alpha. \tag{A.4.16}$$

A.4.2 Distribution P in terms of Q

We want to show that it is not possible to find a simpler relation than

$$\boxed{F^P(\alpha,\alpha^*) = \iint_{\mathbf{R}^2} g\left(\alpha'-\alpha,\alpha'^*-\alpha^*\right) F^Q\left(\alpha',\alpha'^*\right)\,\mathrm{d}^2\alpha'.} \tag{A.4.17}$$

Proof

We have

$$\frac{f^P(z,z^*)}{f^Q(z,z^*)} = \exp\left\{z_r^2+z_i^2\right\}, \tag{A.4.18}$$

then

$$\mathrm{e}^{z\alpha^*-z^*\alpha}\frac{f^1(z,z^*)}{f^2(z,z^*)} = \exp\left\{\left(z_r^2+2\mathrm{i}\alpha_i z_r\right)+\left(z_i^2-2\mathrm{i}\alpha_r z_i\right)\right\}. \tag{A.4.19}$$

We can re-write the argument of the exponential as

$$\left(z_r^2+2\mathrm{i}\alpha_i z_r\right)+\left(z_i^2-2\mathrm{i}\alpha_r z_i\right) = \left(z_r+\mathrm{i}\alpha_i\right)^2-\alpha_i^2+\left(z_i+\mathrm{i}\alpha_r\right)^2+\alpha_r^2 \tag{A.4.20}$$

Clearly, we see that the integration on either z_r or z_i in $g(\alpha,\alpha^*)$ will not converge because

$$\int_{-\infty}^{+\infty}\mathrm{e}^{u^2}\mathrm{d}u^2 \tag{A.4.21}$$

diverges. The integration on α in Eq. (A.4.1) must thus be performed before the integration on z to have a chance of having a relation between the two. Since the integration on α depends on the actual form of $F^Q(\alpha',\alpha'^*)$, the following relation is the simplest form we can write

$$F^P(\alpha,\alpha^*) = \iint_{\mathbf{R}^2} g\left(\alpha'-\alpha,\alpha'^*-\alpha^*\right) F^Q\left(\alpha',\alpha'^*\right)\,\mathrm{d}^2\alpha', \tag{A.4.22}$$

with

$$g(\alpha,\alpha^*) \equiv \frac{1}{4\pi^2}\iint_{\mathbf{R}^2}\mathrm{e}^{z\alpha^*-z^*\alpha+z_r^2+z_i^2}\,\mathrm{d}^2 z. \tag{A.4.23}$$

A.5 Integration of a Gaussian

We will show that

$$\boxed{\int_{-\infty}^{+\infty}\mathrm{e}^{-\left(ax^2+bx\right)+c}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}+c\right),\quad a\in\mathbf{R}_+,\ b,c\in\mathbf{C}.} \tag{A.5.1}$$

Proof

First, note that

$$\begin{aligned}
-\left(ax^2+bx\right)+c &= -\left[\left(\sqrt{a}x\right)^2+2\times\left(\sqrt{a}x\right)\times\frac{b}{2\sqrt{a}}\right]+c \\
&= -\left[\left(\sqrt{a}x\right)^2+2\times\left(\sqrt{a}x\right)\times\frac{b}{2\sqrt{a}}+\frac{b^2}{4a}-\frac{b^2}{4a}\right]+c \\
&= -\left[\left(\sqrt{a}x\right)^2+2\times\left(\sqrt{a}x\right)\times\frac{b}{2\sqrt{a}}+\frac{b^2}{4a}\right]+\frac{b^2}{4a}+c, \qquad \text{(A.5.2)}
\end{aligned}$$

which allows us to write

$$-\left(ax^2+bx\right)+c = -\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)^2+\frac{b^2}{4a}+c. \qquad \text{(A.5.3)}$$

Then, we want to evaluate the integral

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-\left(ax^2+bx\right)+c}\,\mathrm{d}x = \exp\left(\frac{b^2}{4a}+c\right)\int_{-\infty}^{+\infty}\exp\left[-\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)^2\right]\mathrm{d}x. \qquad \text{(A.5.4)}$$

In order to perform the integration on the right hand side, we will apply two successive substitutions. First, we use the following substitution:

$$y \equiv \sqrt{a}x+\frac{b}{2\sqrt{a}}. \qquad \text{(A.5.5)}$$

In that case,

$$\mathrm{d}y = \sqrt{a}\,\mathrm{d}x, \qquad \text{(A.5.6)}$$

which means that

$$\mathrm{d}x = \frac{1}{\sqrt{a}}\,\mathrm{d}y. \qquad \text{(A.5.7)}$$

Moreover

$$\begin{cases}\lim_{x\to+\infty}\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right) = +\infty \\ \lim_{x\to-\infty}\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right) = -\infty\end{cases}. \qquad \text{(A.5.8)}$$

The integral then becomes

$$\begin{aligned}
\exp\left(\frac{b^2}{4a}+c\right)\int_{-\infty}^{+\infty}\exp\left[-\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)^2\right]\mathrm{d}x &= \exp\left(\frac{b^2}{4a}+c\right)\frac{1}{\sqrt{a}}\int_{-\infty}^{+\infty}\mathrm{e}^{-y^2}\,\mathrm{d}y \\
&= \exp\left(\frac{b^2}{4a}+c\right)\frac{2}{\sqrt{a}}\int_{0}^{+\infty}\mathrm{e}^{-y^2}\,\mathrm{d}y, \qquad \text{(A.5.9)}
\end{aligned}$$

since the function we want to integrate is symmetric with respect to the vertical axis. We now introduce a new variable of integration u defined by

$$u \equiv y^2. \qquad \text{(A.5.10)}$$

In that case,

$$\mathrm{d}u = 2y\,\mathrm{d}y = 2u^{1/2}\,\mathrm{d}y, \qquad \text{(A.5.11)}$$

which is equivalent to

$$\mathrm{d}y = \frac{u^{-1/2}}{2}\,\mathrm{d}u. \tag{A.5.12}$$

Furthermore

$$\begin{cases} \lim_{y\to+\infty} y^2 = +\infty, \\ \lim_{y\to 0} y^2 = 0. \end{cases} \tag{A.5.13}$$

We thus have

$$\exp\left(\frac{b^2}{4a}+c\right)\frac{2}{\sqrt{a}}\int_0^{+\infty}\mathrm{e}^{-y^2}\,\mathrm{d}y = \exp\left(\frac{b^2}{4a}+c\right)\frac{\cancel{2}}{\sqrt{a}}\int_0^{+\infty}\mathrm{e}^{-u}\,\frac{u^{-1/2}}{\cancel{2}}\,\mathrm{d}u.$$

Here, we can recognize the Gamma function $\Gamma(z)$ defined by

$$\boxed{\Gamma(z) \equiv \int_0^{+\infty} t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t.} \tag{A.5.14}$$

In our case, the integral equal to

$$\int_0^{+\infty}\mathrm{e}^{-u}\,u^{-1/2}\,\mathrm{d}u = \Gamma(1/2), \tag{A.5.15}$$

where $\Gamma(1/2) = \sqrt{\pi}$ (see [47] for example). Finally, we find the result

$$\int_{-\infty}^{+\infty}\mathrm{e}^{-\left(ax^2+bx\right)+c}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}+c\right). \tag{A.5.16}$$

A.6 Attempt: application of the phase-shifting operator of the extended Wigner quasi-probability distribution

Here, we present the attempt made to determine the effect of the phase-shifting operator on a state of the form $|q\rangle$. If this effect can be determined, it should be fairly easy to see how the extended Wigner quasi-probability distribution is transformed when α and β are rotated by an angle of θ in the complex plane.

From the definition of the phase-shifting operator (Eq. (3.3.25)),

$$\hat{U}(\theta) \equiv \mathrm{e}^{-\mathrm{i}\theta\hat{n}} = \mathrm{e}^{-\mathrm{i}\theta\hat{a}^\dagger\hat{a}}. \tag{A.6.1}$$

As before, we have

$$\hat{a} \equiv \frac{1}{\sqrt{2}}\left(\hat{q}+\mathrm{i}\hat{p}\right), \tag{A.6.2}$$

$$\hat{a}^\dagger \equiv \frac{1}{\sqrt{2}}\left(\hat{q}-\mathrm{i}\hat{p}\right), \tag{A.6.3}$$

so that

$$\begin{aligned}
-\mathrm{i}\theta\hat{a}^{\dagger}\hat{a} &= -\mathrm{i}\theta\frac{1}{2}\left(\hat{q}+\mathrm{i}\hat{p}\right)\left(\hat{q}-\mathrm{i}\hat{p}\right) \\
&= -\mathrm{i}\frac{\theta}{2}\left(\hat{q}^2-\mathrm{i}\hat{q}\hat{p}+\mathrm{i}\hat{p}\hat{q}+\hat{p}^2\right) \\
&= -\mathrm{i}\frac{\theta}{2}\left(\hat{q}^2-\mathrm{i}\left(\hat{q}\hat{p}-\hat{p}\hat{q}\right)+\hat{p}^2\right) \\
&= -\mathrm{i}\frac{\theta}{2}\left(\hat{q}^2-\mathrm{i}\underbrace{\left[\hat{q},\hat{p}\right]}_{=\mathrm{i}}+\hat{p}^2\right) \\
&= -\mathrm{i}\frac{\theta}{2}\left(1+\hat{q}^2+\hat{p}^2\right).
\end{aligned} \tag{A.6.4}$$

Then

$$\hat{U}\left(\theta\right) = \mathrm{e}^{-\mathrm{i}\theta/2}\exp\left\{-\mathrm{i}\frac{\theta}{2}\left(\hat{q}^2+\hat{p}^2\right)\right\}. \tag{A.6.5}$$

We will use CAMPBELL–BAKER–HAUSDORFF theorem (Eq. (B.1.2)) with

$$\hat{A} = \hat{q}^2, \tag{A.6.6}$$

$$\hat{B} = \hat{p}^2, \tag{A.6.7}$$

$$x = -\mathrm{i}\frac{\theta}{2}. \tag{A.6.8}$$

In that case

$$\begin{aligned}
\frac{\left[\hat{A},\hat{B}\right]}{2} &= \frac{\left[\hat{q}^2,\hat{p}^2\right]}{2} \\
&= \frac{1}{2}\left(\hat{q}\left[\hat{q},\hat{p}^2\right]+\left[\hat{q},\hat{p}^2\right]\hat{q}\right) \\
&= \frac{1}{2}\left(\hat{p}\hat{q}\left[\hat{q},\hat{p}\right]+\hat{q}\left[\hat{q},\hat{p}\right]\hat{p}+\hat{p}\left[\hat{q},\hat{p}\right]\hat{q}+\left[\hat{q},\hat{p}\right]\hat{q}\hat{p}\right) \\
&= \frac{\mathrm{i}}{2}\left(\hat{p}\hat{q}+\hat{q}\hat{p}+\hat{p}\hat{q}+\hat{q}\hat{p}\right) \\
&= \mathrm{i}\left(\hat{p}\hat{q}+\hat{q}\hat{p}\right) \\
&= \mathrm{i}\left(2\hat{p}\hat{q}+\mathrm{i}\right) \\
&= 2\mathrm{i}\hat{p}\hat{q}-1.
\end{aligned} \tag{A.6.9}$$

Therefore, the phase-shifting operator can be written

$$\hat{U}\left(\theta\right) = \mathrm{e}^{-\mathrm{i}\theta/2}\exp\left(\hat{p}^2\right)\exp\left(\hat{q}^2\right)\exp\left\{-\mathrm{i}\frac{\theta}{2}\left(2\mathrm{i}\hat{p}\hat{q}-1\right)\right\}, \tag{A.6.10}$$

or

$$\hat{U}\left(\theta\right) = \exp\left(\hat{p}^2\right)\exp\left(\hat{q}^2\right)\exp\left\{\theta\hat{p}\hat{q}\right\}. \tag{A.6.11}$$

Unfortunately, it is not clear how the term $\exp\left\{\theta\hat{p}\hat{q}\right\}$ acts a state of the form $|q\rangle$.

Appendix B

Theorems and definitions

B.1 Campbell–Baker–Hausdorff theorem

Here, we reproduce the statement of this theorem given in [27, Ch. 10.11.5, p. 519]. Let $\hat{A}$, $\hat{B}$ be two operators that do not necessarily commute, but whose commutator $\left[\hat{A},\hat{B}\right]$ commutes with both $\hat{A}$ and $\hat{B}$, so that

$$\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]=0=\left[\hat{B},\left[\hat{A},\hat{B}\right]\right]. \tag{B.1.1}$$

Then

$$\exp\left[x\left(\hat{A}+\hat{B}\right)\right]=\exp\left(x\hat{A}\right)\exp\left(x\hat{B}\right)\exp\left(-x^2\left[\hat{A},\hat{B}\right]/2\right)=\exp\left(x\hat{B}\right)\exp\left(x\hat{A}\right)\exp\left(x^2\left[\hat{A},\hat{B}\right]/2\right) \tag{B.1.2}$$

B.2 Fourier transform

Let $F(q,p)$ be a function of two scalar variables q and p, and let $\widetilde{F}(\xi,\eta)$ be its Fourier transform where ξ and η are respectively the canonically conjugate variables of q and p. Then, the Fourier transform is defined as

$$\boxed{\widetilde{F}(\xi,\eta)\equiv\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F(q,p)\;\mathrm{e}^{\mathrm{i}\xi q+\mathrm{i}\eta p}\,\mathrm{d}q\mathrm{d}p,} \tag{B.2.1}$$

and the inverse Fourier transform is

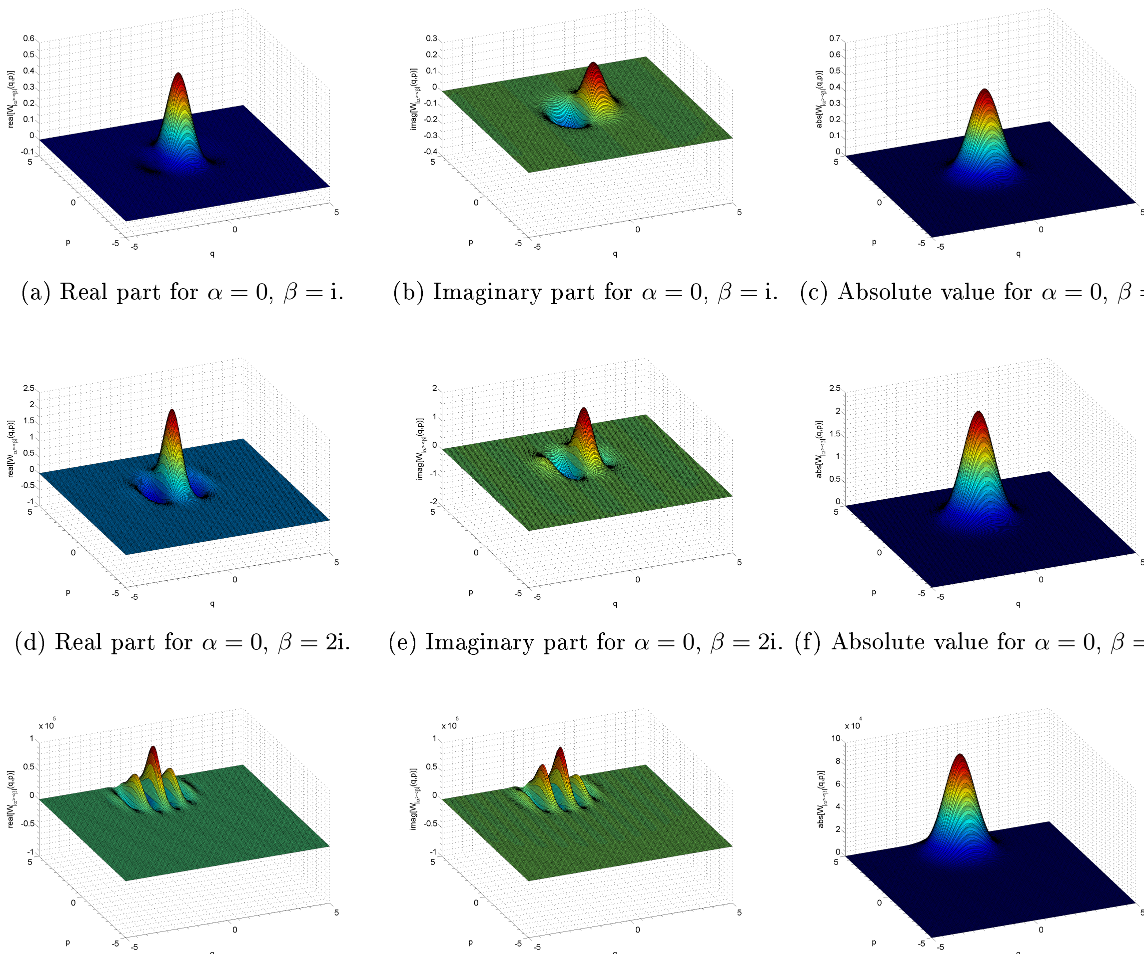
$$\boxed{F(q,p)\equiv\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\widetilde{F}(\xi,\eta)\;\mathrm{e}^{-\mathrm{i}\xi q-\mathrm{i}\eta p}\,\mathrm{d}\xi\mathrm{d}\eta.} \tag{B.2.2}$$

Appendix C

Plots

C.1 Wigner quasi-probability distribution

C.2 Q representation

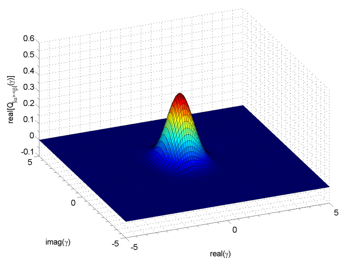


(a) Real part for $\alpha = 0$, $\beta = \mathrm{i}$. (b) Imaginary part for $\alpha = 0$, $\beta = \mathrm{i}$. (c) Absolute value for $\alpha = 0$, $\beta = \mathrm{i}$.

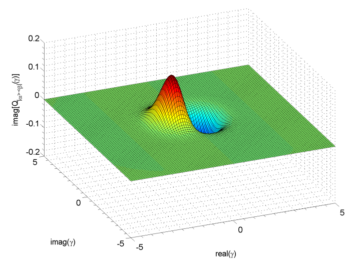
(d) Real part for $\alpha = 0$, $\beta = 2\mathrm{i}$. (e) Imaginary part for $\alpha = 0$, $\beta = 2\mathrm{i}$. (f) Absolute value for $\alpha = 0$, $\beta = 2\mathrm{i}$.

(g) Real part for $\alpha = 0$, $\beta = 5\mathrm{i}$. (h) Imaginary part for $\alpha = 0$, $\beta = 5\mathrm{i}$. (i) Absolute value for $\alpha = 0$, $\beta = 5\mathrm{i}$.

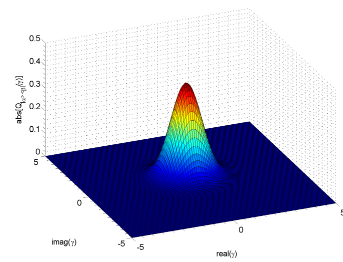
Figure C.1.1: Plots of the extended WIGNER quasi-probability distribution representation.



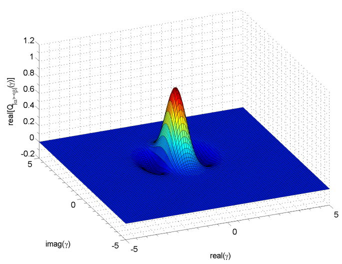
(a) Real part for $\alpha = 0$, $\beta = \mathrm{i}$.



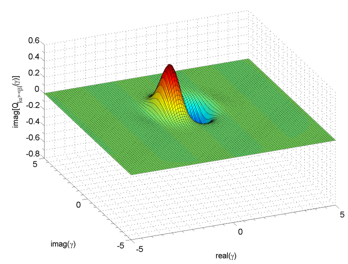
(b) Imaginary part for $\alpha = 0$, $\beta = \mathrm{i}$.



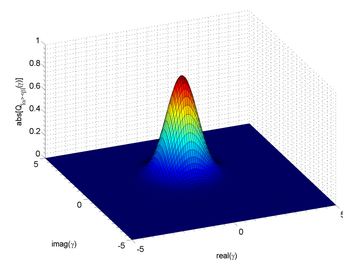
(c) Absolute value for $\alpha = 0$, $\beta = \mathrm{i}$.



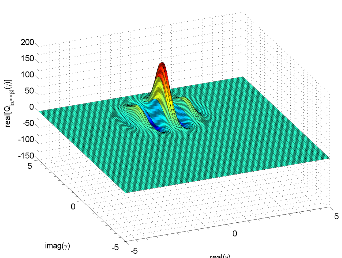
(d) Real part for $\alpha = 0$, $\beta = 2\mathrm{i}$.



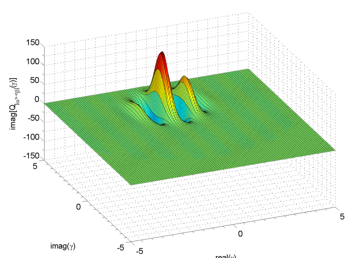
(e) Imaginary part for $\alpha = 0$, $\beta = 2\mathrm{i}$.



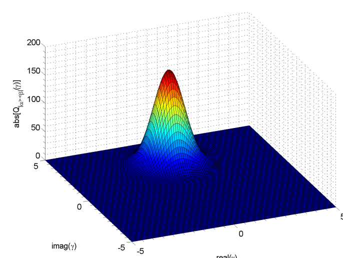
(f) Absolute value for $\alpha = 0$, $\beta = 2\mathrm{i}$.



(g) Real part for $\alpha = 0$, $\beta = 5\mathrm{i}$.



(h) Imaginary part for $\alpha = 0$, $\beta = 5\mathrm{i}$.



(i) Absolute value for $\alpha = 0$, $\beta = 5\mathrm{i}$.

Figure C.2.1: Plots of the extended $\boldsymbol{Q}$ representation.

Appendix D

Matlab code

D.1 Wigner quasi-probability distribution

```
%% Config
format long;
LineWidth = 1; %Taille de la ligne
FontSize = 12; %Taille du texte
MarkerSize = 12; %Taille des points
winSize = 0.95; %Window size proportion

nbPts = 200; %Number of points along one direction
q = linspace(-5, 5, nbPts);
p = linspace(-5, 5, nbPts);
[qmesh, pmesh] = meshgrid(q, p);

X = q;
Y = p;

alpha = 0 + 0*1i;
beta = 0 + 5i;

%% Wigner function of coherent state rho = |alpha><alpha|
% % W_a = (1/pi)*exp( -2*(qmesh/sqrt(2) - real(alpha)).^2 - 2*(pmesh/sqrt(2) - imag(
    alpha)).^2 );
% W_a = @(alpha) (1/pi)*exp( -2*(qmesh/sqrt(2) - real(alpha)).^2 - 2*(pmesh/sqrt(2) -
     imag(alpha)).^2 );
%
% Za = W_a(alpha);
% Zb = W_a(beta);
% title_name = '';
% figure_name = 'Wigner function of coherent state rho = |\alpha><\alpha|';
% xlabel_name = 'q';
% ylabel_name = 'p';
% zlabel_name = 'W_\alpha(q, p)';
% file_name = 'Wigner-coherent';
%
% h = figure('units', 'normalized', 'position', [0 0 1 winSize], 'name', figure_name)
    ;
% hold on; %Figure maximisee
% par = surf(X,Y,Z);
```

```
% set(get(par, 'Parent'), 'FontSize', FontSize); grid on; grid minor;
% title(title_name, 'Fontsize', FontSize);
% xlabel(xlabel_name, 'Fontsize', FontSize);
% ylabel(ylabel_name, 'Fontsize', FontSize);
% zlabel(zlabel_name, 'Fontsize', FontSize);
% view(-23,38); %Orientation for plot
% % print(h,'-dpng', '-r200', [file_name, '_', num2str(alpha)]);

%%
%% W function of transient density matrix sigma = |alpha><beta|
%%
Cst = @(alpha, beta)  (1/pi)*exp( abs(alpha - beta)^2/2 + (beta*conj(alpha) - conj(
    beta)*alpha)/2 - imag(alpha)^2 - imag(beta)^2 - (alpha - conj(beta))^2/2 );
exp_qp = @(alpha, beta) exp( -(qmesh + (alpha + conj(beta))/sqrt(2)).^2 + (1i*pmesh +
     (conj(beta) - alpha)/sqrt(2)).^2 );

W_ab = @(alpha, beta) Cst(alpha, beta).*exp_qp(alpha, beta);

Z1 = real(W_ab(alpha, beta));
Z2 = imag(W_ab(alpha, beta));
Z3 = abs(W_ab(alpha, beta));
title_name = '';
figure_name = 'W function of coherent state rho = |\alpha><\beta|';
xlabel_name = 'q';
ylabel_name = 'p';
file_name = 'W-transient';

%% Real part of W_|alpha><beta|(q,p)
zlabel_name = 'real[W_{|\alpha><\beta|}(q,p)]';

h = figure('units', 'normalized', 'position', [0 0 1 winSize], 'name', [figure_name,
    ' (real)']);
hold on; %Figure maximisee
par = surf(X,Y,Z1);
set(get(par, 'Parent'), 'FontSize', FontSize); grid on; grid minor;
title(title_name, 'Fontsize', FontSize);
xlabel(xlabel_name, 'Fontsize', FontSize);
ylabel(ylabel_name, 'Fontsize', FontSize);
zlabel(zlabel_name, 'Fontsize', FontSize);
view(-23,38); %Orientation for plot
print(h,'-dpng', '-r200', [file_name, '-real_', num2str(alpha), '_', num2str(beta)]);

%% Imaginary part of W_|alpha><beta|(q,p)
zlabel_name = 'imag[W_{|\alpha><\beta|}(q,p)]';

h = figure('units', 'normalized', 'position', [0 0 1 winSize], 'name', [figure_name,
    ' (conj)']);
hold on; %Figure maximisee
par = surf(X,Y,Z2);
set(get(par, 'Parent'), 'FontSize', FontSize); grid on; grid minor;
title(title_name, 'Fontsize', FontSize);
xlabel(xlabel_name, 'Fontsize', FontSize);
ylabel(ylabel_name, 'Fontsize', FontSize);
zlabel(zlabel_name, 'Fontsize', FontSize);
view(-23,38); %Orientation for plot
print(h,'-dpng', '-r200', [file_name, '-imag_', num2str(alpha), '_', num2str(beta)]);

%% Absolute value of W_|alpha><beta|(q,p)
zlabel_name = 'abs[W_{|\alpha><\beta|}(q,p)]';
```

```

h = figure('units', 'normalized', 'position', [0 0 1 winSize], 'name', [figure_name,
    ' (abs)']);
hold on; %Figure maximisee
par = surf(X,Y,Z3);
set(get(par, 'Parent'), 'FontSize', FontSize); grid on; grid minor;
title(title_name, 'Fontsize', FontSize);
xlabel(xlabel_name, 'Fontsize', FontSize);
ylabel(ylabel_name, 'Fontsize', FontSize);
zlabel(zlabel_name, 'Fontsize', FontSize);
view(-23,38); %Orientation for plot
print(h,'-dpng', '-r200', [file_name, '-abs_', num2str(alpha), '_', num2str(beta)]);
```

D.2 Q representation

```
%% Config
format long;
LineWidth = 1; %Taille de la ligne
FontSize = 12; %Taille du texte
MarkerSize = 12; %Taille des points
winSize = 0.95; %Window size proportion

nbPts = 100; %Number of points along one direction
q = linspace(-5, 5, nbPts);
p = linspace(-5, 5, nbPts);
[qmesh, pmesh] = meshgrid(q, p);

X = q;
Y = p;

gamma = qmesh + 1i*pmesh;

%% Q function of coherent state rho = |alpha><alpha|
% alpha = 0 + 0*1i;
%
% Q_a = (1/pi)*exp(- abs(alpha - gamma).^2);
%
% Z = Q_a;
% title_name = '';
% figure_name = 'Wigner function of coherent state rho = |\alpha><\alpha|';
% xlabel_name = '';
% ylabel_name = '';
% zlabel_name = 'Q_\alpha(\gamma)';
% file_name = 'Q-coherent';
%
% h = figure('units', 'normalized', 'position', [0 0 1 winSize], 'name', figure_name)
    ;
% hold on; %Figure maximisee
% par = surf(X,Y,Z);
% set(get(par, 'Parent'), 'FontSize', FontSize); grid on; grid minor;
% title(title_name, 'Fontsize', FontSize);
% xlabel(xlabel_name, 'Fontsize', FontSize);
% ylabel(ylabel_name, 'Fontsize', FontSize);
% zlabel(zlabel_name, 'Fontsize', FontSize);
% view(-23,38); %Orientation for plot
% % print(h,'-dpng', '-r200', [file_name, '_', num2str(alpha)]);

```

```
%%
%% Q function of coherent state rho = |alpha><beta|
alpha = 0 + 0*1i;
beta = 0 + 2*1i;

Q_ab = (1/pi)*exp( -(gamma - alpha).*conj(gamma - beta));

Z1 = real(Q_ab);
Z2 = imag(Q_ab);
Z3 = abs(Q_ab);
title_name = '';
figure_name = 'Q function of coherent state rho = |\alpha><\beta|';
xlabel_name = 'real(\gamma)';
ylabel_name = 'imag(\gamma)';
file_name = 'Q-transient';

%% Real part of Q_|alpha><beta|(gamma)
zlabel_name = 'real[Q_{|\alpha><\beta|}(\gamma)]';

h = figure('units', 'normalized', 'position', [0 0 1 winSize], 'name', [figure_name,
    ' (real)']);
hold on; %Figure maximisee
par = surf(X,Y,Z1);
set(get(par, 'Parent'), 'FontSize', FontSize); grid on; grid minor;
title(title_name, 'Fontsize', FontSize);
xlabel(xlabel_name, 'Fontsize', FontSize);
ylabel(ylabel_name, 'Fontsize', FontSize);
zlabel(zlabel_name, 'Fontsize', FontSize);
view(-23,38); %Orientation for plot
print(h,'-dpng', '-r200', [file_name, '-real_', num2str(alpha), '_', num2str(beta)]);

%% Imaginary part of Q_|alpha><beta|(gamma)
zlabel_name = 'imag[Q_{|\alpha><\beta|}(\gamma)]';

h = figure('units', 'normalized', 'position', [0 0 1 winSize], 'name', [figure_name,
    ' (conj)']);
hold on; %Figure maximisee
par = surf(X,Y,Z2);
set(get(par, 'Parent'), 'FontSize', FontSize); grid on; grid minor;
title(title_name, 'Fontsize', FontSize);
xlabel(xlabel_name, 'Fontsize', FontSize);
ylabel(ylabel_name, 'Fontsize', FontSize);
zlabel(zlabel_name, 'Fontsize', FontSize);
view(-23,38); %Orientation for plot
print(h,'-dpng', '-r200', [file_name, '-imag_', num2str(alpha), '_', num2str(beta)]);

%% Absolute value of Q_|alpha><beta|(gamma)
zlabel_name = 'abs[Q_{|\alpha><\beta|}(\gamma)]';

h = figure('units', 'normalized', 'position', [0 0 1 winSize], 'name', [figure_name,
    ' (abs)']);
hold on; %Figure maximisee
par = surf(X,Y,Z3);
set(get(par, 'Parent'), 'FontSize', FontSize); grid on; grid minor;
title(title_name, 'Fontsize', FontSize);
xlabel(xlabel_name, 'Fontsize', FontSize);
ylabel(ylabel_name, 'Fontsize', FontSize);
zlabel(zlabel_name, 'Fontsize', FontSize);
view(-23,38); %Orientation for plot
```

```
print(h,'-dpng', '-r200', [file_name, '-abs_', num2str(alpha), '_', num2str(beta)]);
```