The Holy Grail of quantum optical communication

Cite as: AIP Conference Proceedings **1633**, 109 (2014); https://doi.org/10.1063/1.4903108
Published Online: 17 February 2015

Raúl García-Patrón, Carlos Navarrete-Benlloch, Seth Lloyd, et al.

The Holy Grail of Quantum Optical Communication

Raúl García-Patrón*, Carlos Navarrete-Benlloch*, Seth Lloyd†, Jeffrey H. Shapiro† and Nicolas J. Cerf**

*Max-Planck Institut für Quantenoptik, Hans-Kopfermann-Str. 1, D-85748 Garching, Germany
†Research Laboratory of Electronics, MIT, Cambridge, Massachusetts 02139, USA
**QuIC, Ecole Polytechnique de Bruxelles, CP 165, Université Libre de Bruxelles, 1050 Bruxelles, Belgium

Abstract. Optical parametric amplifiers together with phase-shifters and beamsplitters have certainly been the most studied objects in the field of quantum optics. Despite such an intensive study, optical parametric amplifiers still keep secrets from us. We will show how they hold the answer to one of the oldest problems in quantum communication theory, namely the calculation of the optimal communication rate of optical channels [1].

Keywords: Quantum Information, quantum communication, bosonic channels, classical capacity
PACS: 03.67.-a, 03.67.Hk, 42.50.-p, 89.70.-a, 89.70.Kn

Optical communication channels, such as optical fibers and amplifiers, are ubiquitous in today telecommunication networks. Therefore knowing their ultimate communication capacity is of crucial importance. Since quantum mechanics is currently the most accurate theory of the physical world, it is natural to seek the ultimate limits on communication set by quantum mechanics. Contrary to what happens in classical Shannon theory [2], a simple and universal formula for the capacity $C(\mathscr{M})$ of sending classical bits over a quantum channel $\mathscr{M}$ has not yet been found (nor disproved to exist) despite huge amounts of work by the quantum information community. Nevertheless, for some highly symmetric channels, such as depolarizing channels or unital qubit channels, people were able to obtain their capacity by showing it to be equal to the Holevo capacity

$$C_H(\mathscr{M}) = \max_{\mathscr{S}} \left(S(\mathscr{M}(\rho)) - \sum_a p_a S(\mathscr{M}(\rho_a)) \right) \tag{1}$$

where $\mathscr{S} = \{p_a, \rho_a\}$ is the coding source, $\rho = \sum_a p_a \rho_a$, and $S(\sigma)$ is the von Neumann entropy of the quantum state σ. For a long time it was strongly believed that the Holevo capacity $C_H(\mathscr{M})$ was additive and therefore gave the exact channel capacity for all quantum channels. This belief was proven to be wrong in 2009 by Hastings [3], so that the best definition of the classical capacity that we currently have requires the regularization

$$C(\mathscr{M}) = \lim_{n\to\infty} \frac{1}{n} C_H(\mathscr{M}^{\otimes n}). \tag{2}$$

where $\mathscr{M}^{\otimes n}$ stands for n uses of the channel.

An important step towards the elucidation of the classical capacity of an optical quantum channel was made in [4], where the authors showed that $C(\mathscr{L})$ of a pure-loss

Eleventh International Conference on Quantum Communication, Measurement and Computation (QCMC)
AIP Conf. Proc. 1633, 109-112 (2014); doi: 10.1063/1.4903108

channel $\mathscr{L}$—a good (but idealized) approximation of an optical fiber—is achieved by a random coding of coherent states using an isotropic Gaussian distribution. It had long been conjectured that such an encoding achieves $C(\mathscr{M})$ for the whole class of optical channels called single-mode *phase-insensitive* Gaussian bosonic channels, including noisy optical fibers and amplifiers [5, 6]. Despite multiple attempts, this conjecture has since then escaped a proof. In what follows we will show how this conjecture can be reduced to proving that the minimum output entropy for a single-use of the quantum-limited amplifier is achieved by the vacuum state.

QUANTUM MODEL OF OPTICAL CHANNELS

A quantum optical single-mode channel can be modeled as a single-mode Gaussian bosonic channel, where the the action of the channel on the mean vector d and covariance matrix γ of the input state is given by the relations

$$d \to Xd, \quad \gamma \to X\gamma X^T + Y. \tag{3}$$

For the map to be completely positive, X and Y must satisfy $Y \geq 0$, and $\det Y \geq (\det X - 1)^2$ [7], where the variance of the vacuum quadratures is normalized to 1 [5]. The map is called quantum-limited when the second inequality is saturated.

Phase-insensitive optical channels, such as optical fibers or amplifiers [8], correspond to $X = \mathrm{diag}(\sqrt{x}, \pm\sqrt{x})$ and $Y = \mathrm{diag}(y, y)$, with x being either the attenuation $0 \leq x \leq 1$ or the amplification $1 \leq x$ of the channel and y the added noise variance. As shown in [9] using the composition rule of Gaussian bosonic channels [7], it is easy to show that every

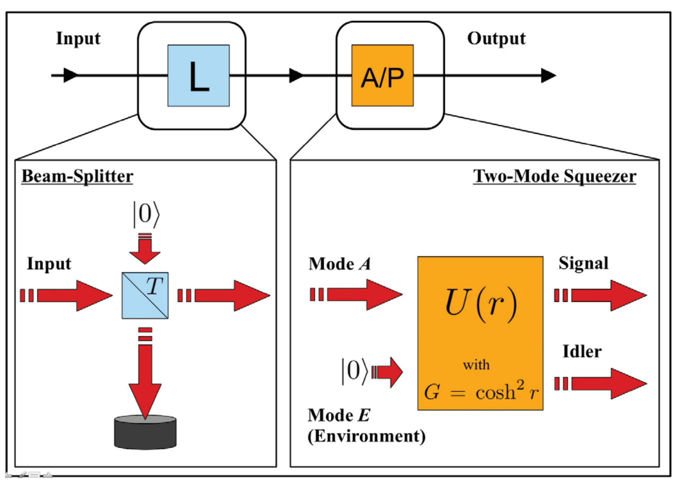

FIGURE 1. Any phase-insensitive Gaussian bosonic channel $\mathscr{M}$ with $\det X \geq 0$ ($\det X < 0$) is indistinguishable from a composed channel $\mathscr{A} \circ \mathscr{L}$ ($\mathscr{P} \circ \mathscr{L}$), where $\mathscr{L}$ is a pure-loss channel and $\mathscr{A}$ a quantum-limited amplifier ($\mathscr{P}$ a phase-conjugation channel). The Stinespring dilation of $\mathscr{L}$ is a beamsplitter of transmissivity T, while the amplifier $\mathscr{A}$ or phase-conjugation $\mathscr{P}$ becomes a two-mode squeezer of parameter r ($G = \cosh^2 r$), where the output of the channel corresponds to the signal (idler) output of the two-mode squeezer.

phase-insensitive channel $\mathscr{M}$, satisfying $\det X \geq 0$ ($\det X < 0$) is indistinguishable from the concatenation of a pure-loss channel $\mathscr{L}$ of transmissivity T with a quantum-limited amplifier $\mathscr{A}$ (phase-conjugation $\mathscr{P}$) of gain G (gain $G-1$), see Fig. 1. The Stinespring dilation of $\mathscr{L}$ is a beamsplitter of transmissivity T, while the amplifier $\mathscr{A}$ of gain G (phase-conjugation $\mathscr{P}$ of gain $x = G-1$) becomes a two-mode squeezer of parameter $r = \text{arcosh}\sqrt{G}$, where the output of the channel corresponds to the signal (idler) output of the two-mode squeezer. It is known that any channel $\mathscr{M}$ satisfying $\det X < 0$, i.e. $\mathscr{M} = \mathscr{P} \circ \mathscr{L}$, is entanglement-breaking [10].

THE REDUCTION

As stated earlier, our ultimate goal is to show that a random coding of coherent states using an isotropic Gaussian distribution achieves the capacity of any phase-insensitive Gaussian bosonic channel $\mathscr{M}$. In the single-letter variant of the problem, it is known, from the properties of the Holevo capacity, that one can achieve this goal by showing that vacuum $|0\rangle$ minimizes the output entropy of channel $\mathscr{M}$. However, the actual classical capacity is, not the Holevo capacity (1), but its regularized expression (2). Thus, to show that random coding of coherent states using an isotropic Gaussian distribution achieves the capacity of any phase-insensitive Gaussian bosonic channel $\mathscr{M}$, we need to prove a stronger result, namely that n vacuum inputs $|0\rangle^{\otimes n}$ minimize the output entropy of n uses of channel $\mathscr{M}$ ($\mathscr{M}^{\otimes n}$). We now sketch a proof, generalizing our work in [9], reducing the previous conjecture to proving that vacuum minimizes the output entropy of a single quantum-limited amplifier $\mathscr{A}$ [11].

Due to the concavity of the von Neumann entropy, the minimization can be reduced to the set of pure input states. Exploiting the decomposition $\mathscr{M} = \mathscr{A} \circ \mathscr{L}$ (when $\det X \geq 0$) it is easy to see, using the concavity of the von Neumann entropy, that the minimum output entropy of channel $\mathscr{M}^{\otimes n}$ is lower-bounded by that of channel $\mathscr{A}^{\otimes n}$, i.e., $\min_\phi S(\mathscr{M}^{\otimes n}(\phi)) \geq \min_\psi S(\mathscr{A}^{\otimes n}(\psi))$. Then using the fact the quantum-limited amplifier $\mathscr{A}$ is complementary to a phase-conjugation channel $\mathscr{P}$, i.e., they share the same Stinespring dilation and therefore $S(\mathscr{A}^{\otimes n}(\phi)) = S(\mathscr{P}^{\otimes n}(\phi))$, combined with the fact that $\mathscr{P}$ is entanglement breaking, we can shows that $\min_\phi S(\mathscr{M}^{\otimes n}(\phi)) \geq n \min_\psi S(\mathscr{A}(\psi))$ [12]. Then assuming that vacuum minimizes the output entropy of the single-letter channel $\mathscr{A}$ we obtain $\min_\phi S(\mathscr{M}^{\otimes n}(\phi)) \geq n S(\mathscr{A}(|0\rangle\langle 0|))$. Since the vacuum state is invariant under $\mathscr{L}$, the lower-bound is achievable, which finally proves

$$\min_\phi S(\mathscr{M}^{\otimes n}(\phi)) = n S(\mathscr{A}(|0\rangle\langle 0|)). \tag{4}$$

For the case of entanglement-breaking channels $\mathscr{M}$, i.e., $\det X < 0$, the proof is very similar. Using now the decomposition $\mathscr{M} = \mathscr{A} \circ \mathscr{P}$ and the fact the phase-conjugation channels $\mathscr{P}$ are entanglement breaking [12], we obtain $\min_\phi S(\mathscr{M}^{\otimes n}(\phi)) \geq n \min_\psi S(\mathscr{P}(|0\rangle\langle 0|))$. Then using the fact the quantum-limited amplifier $\mathscr{A}$ is complementary to a phase-conjugation channel $\mathscr{P}$, we can similarly reduce the conjecture to prove that vacuum minimizes the output entropy of $\mathscr{A}$.

The Stinespring dilation of a quantum-limited amplifier $\mathscr{A}$ being a two-mode squeezer of parameter r, with $G = \cosh^2 r$, the minimum output entropy conjecture for channel $\mathscr{A}$ is equivalent to prove that among all input states $|\phi\rangle_{AE} \equiv |\varphi\rangle \otimes |0\rangle$ of a two-mode squeezer

$$U(r) = \exp\left[r(a_A a_E - a_A^\dagger a_E^\dagger)/2\right], \tag{5}$$

the vacuum state $|0\rangle_{AE} \equiv |0\rangle \otimes |0\rangle$ minimizes the output entanglement. When restricting to the set of Gaussian states, the conjecture on the entangling properties of the two-mode squeezer is not difficult to prove (adapting the proof in [6]). In [9] it was shown to be true for the entire set of photon number states inputs. Finally, a detailed numerical analysis in [9] over arbitrary inputs states, suggest the conjecture being true in full generality.

CONCLUSION

Using the decomposition of phase-insensitive Gaussian bosonic channels into a pure-loss channel and a quantum-limited amplifier or phase-conjugation channel, we have reduced the open conjecture that random coding of coherent states achieves the capacity of phase-insensitive Gaussian bosonic channels to proving that vacuum minimizes the output entropy for a single-use of the quantum-limited amplifier channel. This brings a new perspective on one of the oldest open problems in quantum communication theory, which could potentially lead to its final solution by reducing it to a detailed study of the entangling properties of optical parametric amplifiers.

ACKNOWLEDGMENTS

R. G.-P., N.J.C., J.H.S., and S.L. acknowledge financial support from the W. M. Keck Foundation Center for Extreme Quantum Information Theory, R.G.-P. from the Humboldt foundation, C.N.-B. from the FPU program of the MICINN and the European Union FEDER through Project FIS2008-06024-C03-01, J.H.S. and S.L. from the ONR Basic Research Challenge Program, and N.J.C. from the F.R.S.-FNRS under project HIPERCOM.

REFERENCES

1. J. P. Gordon, IRE Proc. **50,** 1898 (1962).
2. C. E. Shannon, Bell Syst. Tech. J. **27,** 379 (1948).
3. M. B. Hastings, Nature Physics **5,** 255 (2009).
4. V. Giovannetti, *et al.*, Phys. Rev. Lett. **92,** 027902 (2004).
5. C. Weedbrook, *et al.*, Rev. Mod. Phys. **84,** 621 (2012).
6. V. Giovannetti, *et al.*, Phys. Rev. A **70,** 032315 (2004).
7. F. Caruso, V. Giovannetti, and A. S. Holevo, New. J. Phys. **8,** 310 (2004).
8. C. M. Caves and P. D. Drummond, Rev. Mod. Phys. **66,** 481 (1994).
9. R. García-Patrón, *et al.*, Phys. Rev. Lett. **108,** 110505 (2012).
10. A. S. Holevo, Probl. Inf. Trans. **44,** 3 (2008).
11. R. García-Patrón and N. J. Cerf, manuscript in preparation (2013).
12. P. Shor, J. Math. Phys. **43,** 4334 (2002).