Gaussian deterministic and probabilistic transformations of bosonic quantum fields

Squeezing and entanglement generation

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Ithaka

As you set out for Ithaka
hope the voyage is a long one,
full of adventure, full of discovery.
Laistrygonians and Cyclops,
angry Poseidon - don't be afraid of them:
you'll never find things like that on your way
as long as you keep your thoughts raised high,
as long as a rare excitement
stirs your spirit and your body.
Laistrygonians and Cyclops,
wild Poseidon - you won't encounter them
unless you bring them along inside your soul,
unless your soul sets them up in front of you.

Hope the voyage is a long one.
May there be many a summer morning when,
with what pleasure, what joy,
you come into harbors seen for the first time;
may you stop at Phoenician trading stations
to buy fine things,
mother of pearl and coral, amber and ebony,
sensual perfume of every kind-
as many sensual perfumes as you can;
and may you visit many Egyptian cities
to gather stores of knowledge from their scholars.

Keep Ithaka always in your mind.
Arriving there is what you are destined for.
But do not hurry the journey at all.
Better if it lasts for years,
so you are old by the time you reach the island,
wealthy with all you have gained on the way,
not expecting Ithaka to make you rich.

Ithaka gave you the marvelous journey.
Without her you would not have set out.
She has nothing left to give you now.

And if you find her poor, Ithaka won't have fooled you.

Wise as you will have become, so full of experience,

you will have understood by then what these Ithakas mean.

C.P. Cavafy, Collected Poems.
Translated by Edmund Keeley and Philip Sherrard.
Edited by George Savidis. Revised Edition.

Princeton University Press, 1992

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Dedicated to my mother Maria, father Nikolaos, brother Panayotis, and to my constant source of joyfulness Efthymia. My only homeland.

Philosophical statement that motivated this work

It is a common saying that nobody understands quantum mechanics. I would rather say that nobody understands quantum *and* classical physics. This is because when it comes to science one has to rethink and redefine the meaning of the word *understand*. The author's opinion is that science, and in particular physics, is a reverse engineering problem with only tools mathematics and experiments. Nature has no laws, humans map Nature to their own perspective with their reverse engineering tools and they code Nature into rules.

This work was motived from personal curiosity of how *things* work; what are the rules. By *things* I mean physical entities that have some importance on technology or are of fundamental interest. So, what more fundamental and fascinating than quantum light and its manipulation?

Abstract

The processing of information based on the generation of common quantum optical states (e.g., coherent states) and the measurement of the quadrature components of the light field (e.g., homodyne detection) is often referred to as continuous-variable quantum information processing. It is a very fertile field of investigation, at a crossroads between quantum optics and information theory, with notable successes such as unconditional continuous-variable quantum teleportation or Gaussian quantum key distribution. In quantum optics, the states of the light field are conveniently characterized using a phase-space representation (e.g., Wigner function), and the common optical components effect simple affine transformations in phase space (e.g., rotations). In quantum information theory, one often needs to determine entropic characteristics of quantum states and operations, since the von Neuman entropy is the quantity at the heart of entanglement measures or channel capacities. Computing entropies of quantum optical states requires instead turning to a state-space representation of the light field, which formally is the Fock space of a bosonic mode.

This interplay between phase-space and state-space representations does not represent a particular problem as long as Gaussian states (e.g., coherent, squeezed, or thermal states) and Gaussian operations (e.g., beam splitters or squeezers) are concerned. Indeed, Gaussian states are fully characterized by the first- and second-order moments of mode operators, while Gaussian operations are defined via their actions on these moments. The so-called symplectic formalism can be used to treat all Gaussian transformations on Gaussian states, including mixed states of an arbitrary number of modes, and the entropies of Gaussian states are directly linked to their symplectic eigenvalues.

This thesis is concerned with the Gaussian transformations applied onto arbitrary states of light, in which case the symplectic formalism is inapplicable and this phase-to-state space interplay becomes highly non trivial. A first motivation to consider arbitrary (non-Gaussian) states of light results from various Gaussian no-go theorems in continuous-variable quantum information theory. For instance, universal quantum computing, quantum entanglement concentration, or quantum error correction are known to be impossible when restricted to the Gaussian realm. A second motivation comes from the fact that several fundamental quantities, such as the entanglement of formation of a Gaussian state or the communication capacity of a Gaussian channel, rely on an optimization over all states, including non-Gaussian states even though the considered state or channel is Gaussian. This thesis is therefore devoted to developing new tools in order to compute state-space properties (e.g., entropies) of transformations defined in phase-space or conversely to computing phase-space properties (e.g., mean-field amplitudes) of transformations defined in state space. Remarkably, even some basic questions such as the

entanglement generation of optical squeezers or beam splitters were unsolved, which gave us a nice work-bench to investigate this interplay.

In the first part of this thesis (Chapter 3), we considered a recently discovered Gaussian probabilistic transformation called the noiseless optical amplifier. More specifically, this is a process enabling the amplification of a quantum state without introducing noise. As it has long been known, when amplifying a quantum signal, the arising of noise is inevitable due to the unitary evolution that governs quantum mechanics. It was recently realized, however, that one can drop the unitarity of the amplification procedure and trade it for a noiseless, albeit probabilistic (heralded) transformation. The fact that the transformation is probabilistic is mathematically reflected in the fact that it is non trace-preserving. This quantum device has gained much interest during the last years because it can be used to compensate losses in a quantum channel, for entanglement distillation, probabilistic quantum cloning, or quantum error correction. Several experimental demonstrations of this device have already been carried out. Our contribution to this topic has been to derive the action of this device on squeezed states and to prove that it acts quite surprisingly as a universal (phase-insensitive) optical squeezer, conserving the signal-to-noise ratio just as a phase-sensitive optical amplifier but for all quadratures at the same time. This also brought into surface a paradoxical effect, namely that such a device could seemingly lead to instantaneous signaling by circumventing the quantum no-cloning theorem. This paradox was discussed and resolved in our work.

In a second step, the action of the noiseless optical amplifier and it dual operation (i.e., heralded noiseless attenuator) on non-Gaussian states has been examined. We have observed that the mean-field amplitude may decrease in the process of noiseless amplification (or may increase in the process of noiseless attenuation), a very counterintuitive effect that Gaussian states cannot exhibit. This work illustrates the above-mentioned phase-to-state space interplay since these devices are defined as simple filtering operations in state space but inferring their action on phase-space quantities such as the mean-field amplitude is not straightforward. It also illustrates the difficulty of dealing with non-Gaussian states in Gaussian transformations (these noiseless devices are probabilistic but Gaussian). Furthermore, we have exhibited an experimental proposal that could be used to test this counterintuitive feature. The proposed set-up is feasible with current technology and robust against usual inefficiencies that occur in optical experiment. Noiseless amplification and attenuation represent new important tools, which may offer interesting perspectives in quantum optical communications. Therefore, further understanding of these transformations is both of fundamental interest and important for the development and analysis of protocols exploiting these tools. Our work provides a better understanding of these transformations and reveals that the intuition based on ordinary (deterministic phase-insensitive) amplifiers and losses is not always applicable to the noiseless amplifiers and attenuators. In the last part of this thesis, we have considered the entropic characterization of some of the most fundamental Gaussian transformations in quantum optics, namely a beam splitter and two-mode squeezer. A beam splitter effects a simple rotation in phase space, while a two-mode squeezer produces a Bogoliubov transformation. Thus, there is a well-known phase-space characterization in terms of symplectic transformations, but the difficulty originates from that one must return to state space in order to access quantum entropies or entanglement. This is again a hard problem, linked to the above-mentioned interplay in the reverse direction this time. As soon as non-Gaussian states are concerned, there is no way of calculating the entropy produced by such Gaussian transformations. We have investigated two novel tools in order to treat non-Gaussian states under Gaussian transformations, namely majorization theory and the replica method.

In Chapter 4, we have started by analyzing the entanglement generated by a beam splitter that is fed with a photon-number state, and have shown that the entanglement monotones can be neatly combined with majorization theory in this context. Majorization theory provides a preorder relation between bipartite pure quantum states, and gives a necessary and sufficient condition for the existence of a deterministic LOCC (local operations and classical communication) transformation from one state to another. We have shown that the state resulting from n photons impinging on a beam splitter majorizes the corresponding state with any larger photon number $n' > n$, implying that the entanglement monotonically grows with n, as expected. In contrast, we have proven that such a seemingly simple optical component may have a rather surprising behavior when it comes to majorization theory: it does not necessarily lead to states that obey a majorization relation if one varies the transmittance (moving towards a balanced beam splitter). These results are significant for entanglement manipulation, giving rise in particular to a catalysis effect. Moving forward, in Chapter 5, we took the step of introducing the replica method in quantum optics, with the goal of achieving an entropic characterization of general Gaussian operations on a bosonic quantum field. The replica method, a tool borrowed from statistical physics, can also be used to calculate the von Neumann entropy and is the last line of defense when the usual definition is not practical, which is often the case in quantum optics since the definition involves calculating the eigenvalues of some (infinite-dimensional) density matrix. With this method, the entropy produced by a two-mode squeezer (or parametric optical amplifier) with non-trivial input states has been studied. As an application, we have determined the entropy generated by amplifying a binary superposition of the vacuum and an arbitrary Fock state, which yields a surprisingly simple, previously analytical expression. Finally, we have turned to the replica method in the context of field theory, and have examined the behavior of a bosonic field with finite temperature when the temperature decreases. To this end, information theoretical tools were used, such as the geometric entropy and the mutual information, and interesting connection between phase transitions and informational quantities were found. More specifically, dividing the field in two spatial regions and calculating the mutual information between these two regions, it turns out that the mutual information is non-differentiable exactly at the critical temperature for the formation of the Bose-Einstein condensate.

The replica method provides a new angle of attack to access quantum entropies in fundamental Gaussian bosonic transformations, that is quadratic interactions between bosonic mode operators such as Bogoliubov transformations. The difficulty of accessing entropies produced when transforming non-Gaussian states is also linked to several currently unproven entropic conjectures on Gaussian optimality in the context of bosonic channels. Notably, determining the capacity of a multiple-access or broadcast Gaussian bosonic channel is pending on being able to access entropies. We anticipate that the replica method may become an invaluable tool in order to reach a complete entropic characterization of Gaussian bosonic transformations, or perhaps even solve some of these pending conjectures on Gaussian bosonic channels.

Publications

- *Quantum entropic characterization of Gaussian optical transformations using the replica method* C. N. Gagatsos, A. I. Karanikas, G. Kordas, and N. J. Cerf. arXiv: 1408.5062

- *Heralded noiseless amplification and attenuation of non-gaussian states of light* C. N. Gagatsos, J. Fiurasek, A. Zavatta, M. Bellini, and N. J. Cerf. Phys. Rev. A 89, 062311 (2014)

- *Majorization relations and entanglement generation in a beam splitter* C. N. Gagatsos,O. Oreshkov, and N. J. Cerf. Phys. Rev. A 87, 042307 (2013)

- *Mutual information and Bose-Einstein condensation* C. N. Gagatsos, A. I. Karanikas and G. I. Kordas. Open Syst. and Inf. Dyn. (World Scientific) Vol. 20, Issue 02 (2013)

- *Probabilistic phase-insensitive optical squeezer in compliance with causality* C. N. Gagatsos, E. Karpov, and N. J. Cerf. Phys. Rev. A 86, 012324 (2012)

Chapter 1

Quantum mechanics

1.1 The very basics

1.1.1 Pure states and linear operators

In quantum mechanics the most fundamental concept is the pure state, i.e. a mathematical entity that encodes the physical system and can provide any information about the system under investigation. More precisely, to any system that does not interact with anything, i.e. a closed system, we associate a complex vector space with inner product which is more commonly known as Hilbert space [Zet01]. A pure state is a complex vector on Hilbert space normalized to unity. That means that if $\{|k\rangle\}$ is set of vectors that forms a basis with the properties of completeness,

$$\sum_k |k\rangle\langle k| = \mathbb{I} \tag{1.1}$$

and orthonormality,

$$\langle m|k\rangle = \delta_{mk}, \tag{1.2}$$

where $\mathbb{I}$ is the unit matrix and δ_{mk} is the Kronecker delta. The most important of the above properties is the one that refers to completeness.

Any pure state $|\psi\rangle$ can therefore be expressed as a linear superposition of the basis' vectors,

$$|\psi\rangle = \sum_k c_k|k\rangle \tag{1.3}$$

where c_k are complex numbers. The absolute square $|c_j|^2$ is interpreted as the probability that the physical system is corresponding to the vector $|j\rangle$. All the probabilities should always satisfy the condition of normalization,

$$\sum_k |c_k|^2 = 1. \tag{1.4}$$

The matrix representation of a pure state reads,

$$|\psi\rangle = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \end{pmatrix}. \tag{1.5}$$

Note that a pure state does not have a unique representation on Hilbert space. In fact the eigenvectors $\{|k\rangle\}$ of any operator $\hat{A}$ form a complete and orthogonal basis as long as $\hat{A}$ is Hermitian (in infinite dimension however, the eigenvectors of a Hermitian operator do not necessarily form a basis) [Sak94], i.e. it satisfies $\hat{A}^\dagger = \hat{A}$. Also we may describe many systems at the same time by taking the tensor product of single-system pure states,

$$|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots = |\psi_1 \psi_2 \ldots\rangle. \tag{1.6}$$

Let us consider $|\Psi_{AB}\rangle$ to be a pure state describing a bipartite system AB. Then there exist two orthogonal and complete bases for systems A and B respectively, $\{|i_A\rangle\}$ and $\{|i_B\rangle\}$, such that,

$$|\Psi_{AB}\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle. \tag{1.7}$$

This representation is called Schmidt decomposition [NC00]. The non-negative real numbers λ_i satisfy $\sum_i \lambda_i^2 = 1$.

An operation on a quantum mechanical system, that is in a separable state, is described as the action of some operator $\hat{O}$ on $|\Psi\rangle$. If the operator involves all states $|\psi_j\rangle$ of the tensor product, then the operator describes a global operation. If it involves only some of them, then the operator describes a local operation. In quantum information processing it is common to have several local operations and moreover to allow classical communication between the isolated subsystems. That case is referred to as an LOCC (Local Operation and Classical Communication) protocol. The classical communication gives information about the outcome of some previous local measurement and serves as to what local operation should one use next in order to achieve some desired outcome.

The operators that act on pure states should be unitary, $\hat{O}^{\dagger}\hat{O} = \mathbb{I}$ so that condition (1.4) is satisfied. Another important remark is that in general when two (or more) operators act on a pure state, the order of action plays a role,

$$\hat{O}_1\hat{O}_2|\Psi\rangle \neq \hat{O}_2\hat{O}_1|\Psi\rangle, \tag{1.8}$$

therefore it is logical to define the so-called commutator,

$$\left[\hat{O}_1, \hat{O}_2\right] = \hat{O}_1\hat{O}_2 - \hat{O}_2\hat{O}_1. \tag{1.9}$$

An important question is how a pure state evolves in time. It can be argued that the time evolution of a pure state is given by the Schrödinger equation (see for example a modern approach [HR02]),

$$i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle \tag{1.10}$$

where $\hat{H}$ is the Hamiltonian of the system. For two time instants $t_2 > t_1$ from equation (1.10) we can write,

$$|\psi(t_2)\rangle = \hat{U}(t_1, t_2)|\psi(t_1)\rangle \tag{1.11}$$

where,

$$\hat{U}(t_1, t_2) = \exp\left[-\frac{i}{\hbar}(t_2 - t_1)\hat{H}\right]|\psi(t_1)\rangle \tag{1.12}$$

is a unitary operator. Note that the Hamiltonian in equation (1.12) is assumed to be time independent. The Hamiltonian is a Hermitian operator and corresponds to the energy of the system in the sense that the eigenvalues of $\hat{H}$ gives the accessible energy levels. Let the eigenvalues-eigenvectors problem for the Hamiltonian be,

$$\hat{H}|e_k\rangle = e_k|e_k\rangle \tag{1.13}$$

then the mean value of the energy for a system in pure state $|\psi\rangle$ reads,

$$\langle\hat{H}\rangle = \langle\psi|\hat{H}|\psi\rangle = \sum_k |c_k|^2 e_k. \tag{1.14}$$

In fact all physical observables correspond to Hermitian operators $\hat{O}$ [Zet01]. If the system is described by a pure state $|\psi\rangle$ then the mean value reads,

$$\langle\hat{O}\rangle = \langle\psi|\hat{O}|\psi\rangle. \tag{1.15}$$

The standard deviation of an observable is given by [Zet01],

$$\Delta\hat{O} = \sqrt{\langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2}. \tag{1.16}$$

In general for two observables $\hat{A}$ and $\hat{B}$, it can be proven that,

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle| \tag{1.17}$$

which is the Heisenberg's uncertainty principle.

1.1.2 Mixed states and density operator

Given a pure state $|\psi\rangle$, we can construct the density operator, that is another useful object and is found by taking the outer product,

$$\hat{\rho} = |\psi\rangle\langle\psi| \tag{1.18}$$

so if the system is described by a the pure state $|\psi\rangle = \sum_k c_k|k\rangle$, then the matrix representation of the density operator reads,

$$\hat{\rho} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \end{pmatrix} \begin{pmatrix} c_0^* & c_1^* & \dots \end{pmatrix} = \begin{pmatrix} |c_0|^2 & c_0c_1^* & \dots \\ c_1c_0^* & |c_1|^2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{1.19}$$

where now the normalization condition (1.4) involves the trace of the density matrix, namely,

$$\mathrm{tr}(\hat{\rho}) = 1. \tag{1.20}$$

The density operator becomes handy when the physical system cannot be described by a vector on Hilbert space, that is when we deal with statistical mixtures of pure states [Sak94], i.e. mixed states,

$$\hat{\rho} = \sum_i p_i|\psi_i\rangle\langle\psi_i| \tag{1.21}$$

where $\sum_i p_i = 1$. Since we can describe both pure and mixed states with the density operator formalism, henceforth when we refer to a state we will mean the density operator. In general the conditions that an operator $\hat{\rho}$ should satisfy in order to be considered as a valid density operator are,

$$\mathrm{tr}\hat{\rho} = 1 \tag{1.22}$$

$$\hat{\rho} \geq 0 \tag{1.23}$$

$$\hat{\rho} = \hat{\rho}^{\dagger} \tag{1.24}$$

where the second condition should be interpreted as that the density operator should be positive semidefinite, i.e. to have non-negative eigenvalues. From the last property, i.e that $\hat{\rho}$ is Hermitian, and the simple observation that the state describes something that exists, we can argue that the density operator corresponds to a physical quantity.

Let us consider a composite system, i.e. a system that we divide arbitrary in several subsystems. If the subsystems are independent, then the density operator of the system is the tensor product of the density operators of the subsystems. If the system is described by a mixed state of possibly correlated subsystems, then in order to find the density matrix that corresponds to one of the subsystems, i.e. the reduced density operator, we have to trace out the degrees of freedom that do not belong to subsystem under consideration. Namely, if the density matrix $\hat{\rho}_s$ of the system has a representation on some basis,

$$\hat{\rho}_s = \sum_{k_1,l_1,k_2,l_2,\ldots} c_{k_1,l_1,k_2,l_2\ldots} |k_1, k_2, \ldots\rangle\langle l_1, l_2, \ldots| \tag{1.25}$$

then, for example, by tracing out the degrees of freedom $j \geq 2$, we find a reduced density operator,

$$\hat{\rho}_{sub} = \sum_{k_1,l_1,k_2,l_2,\ldots} c_{k_1,l_1,k_2,l_2,\ldots} \langle k_2, \ldots | l_2, \ldots\rangle |k_1\rangle\langle l_1|. \tag{1.26}$$

If the subsystem under consideration is named A while the traced out subsystems are referred as B then the trace out procedure, called the partial trace, is denoted as,

$$\hat{\rho}_{sub} = \mathrm{tr}_B(\hat{\rho}_s). \tag{1.27}$$

When we consider the density operator of some subsystem with density operator $\hat{\rho}_{sub}$, the mean value of some Hermitian operator $\hat{A}$ reads [Sak94],

$$\langle \hat{A} \rangle = \mathrm{tr}(\hat{A}\hat{\rho}_{sub}). \tag{1.28}$$

Let us now introduce the dynamics, that is how the state of system evolves in time. Any quantum operation $\mathcal{T}(\hat{\rho})$ on some state $\hat{\rho}$, is the result of the composition of elementary operations. More precisely, the evolution of a state can break down to the following fundamental actions [BL07],

unitary dynamics

$$\hat{\rho}' = \hat{U}\hat{\rho}\hat{U}^\dagger, \tag{1.29}$$

composition of systems that do not interact

$$\hat{\rho}' = \hat{\rho} \otimes \hat{\sigma}, \tag{1.30}$$

partial trace

see (1.27)

and

von-Neumann measurements

that is orthogonal projections that we will introduce in section 1.1.3.

The transformation $\mathcal{T}(.)$ must map density operators to density operators, first of all that means that the map $\mathcal{T}(.)$ must be trace preserving, i.e. the output operator must have $\mathrm{tr}\rho' = 1$. The other condition for $\mathcal{T}(.)$ comes from considering that the map $\mathcal{T}(.)$ may act only in one of the subsystems, while in the other part of the system nothing happens or equivalently the identity matrix is applied. So the map now reads $\mathcal{T} \otimes \mathbb{I}(.)$ and we require that $\hat{\rho}' = \mathcal{T} \otimes \mathbb{I}(\hat{\rho})$ is a positive operator. That imposes the condition that the map $\mathcal{T}(.)$ is completely positive.

In this way, all legitimate maps are called completely positive trace preserving maps, abbreviated CPTP maps. As we will see in section 2.1.4 and in chapter 3 the trace preservation property may be interpreted as the map being deterministic. Therefore if we consider completely positive but trace decreasing maps, abbreviated CPTD maps, the operation becomes probabilistic.

Completely positive maps can assume the form [BL07],

$$\mathcal{T}(\hat{\rho}) = \sum_i \hat{A}_i \hat{\rho} \hat{A}_i^\dagger \tag{1.31}$$

which is referred to as Krauss decomposition while $\hat{A}_i$ are the Krauss operators. If moreover $\sum_i \hat{A}_i^\dagger \hat{A}_i = \mathbb{I}$ then the map is trace preserving while if $\sum_i \hat{A}_i^\dagger \hat{A}_i < \mathbb{I}$ it is trace decreasing. If $\sum_i \hat{A}_i \hat{A}_i^\dagger = \mathbb{I}$ then the map is unital, i.e. it maps the identity matrix to the identity matrix.

Another way to express the completely positive maps comes naturally if we consider what really happens in a generic quantum procedure. Namely, let us consider as system of interest $\hat{\rho}$ (the open system) and some environment $\hat{\rho}_e$ that we do not have access to or we do not care to have access to. The initial state $\hat{\rho} \otimes \hat{\rho}_e$ is transformed by some global unitary operator as $\hat{U}(\hat{\rho} \otimes \hat{\rho}_e)\hat{U}^\dagger$. We want to know what happens to the open system. To this end we trace out the degrees of freedom of the environment,

$$\mathcal{T}(\hat{\rho}) = \mathrm{tr}_e \left(\hat{U}(\hat{\rho} \otimes \hat{\rho}_e)\hat{U}^\dagger \right) \tag{1.32}$$

where the last formula is equivalent to equation (1.31) [BL07].

The maps $\mathcal{T}(.)$ represent quantum procedures that usually are desired to result to some target state $\hat{\sigma}$. In many cases, mainly because of the presence of some environment, the output state $\hat{\rho}$ is not the target state. An important quantity would be how close the output state is to the target state. Such a figure of merit is the fidelity defined as,

$$F(\hat{\rho}, \hat{\sigma}) = \mathrm{tr}\sqrt{\hat{\rho}^{1/2}\hat{\sigma}\hat{\rho}^{1/2}}. \tag{1.33}$$

It can be proven that the fidelity is symmetric under interchange of the its arguments [NC00]. Often, the target state is a pure state, in that case the fidelity reads,

$$F(\hat{\rho}, |\psi\rangle) = \sqrt{\langle\psi|\hat{\rho}|\psi\rangle}. \tag{1.34}$$

The time evolution of the open system's density matrix can be found from the Master equation of the Lindblad type [BP02],

$$\frac{d}{dt}\hat{\rho}(t) = -\frac{i}{\hbar}[\hat{H}_s, \hat{\rho}(t)] - \sum_i \frac{\gamma_i}{2}\left(\hat{L}_i^\dagger \hat{L}_i \hat{\rho}(t) + \hat{\rho}(t)\hat{L}_i^\dagger \hat{L}_i - 2\hat{L}_i \hat{\rho}(t)\hat{L}_i^\dagger \right), \tag{1.35}$$

where in the right hand side $\hat{H}_s$ is the Hamiltonian of the system and therefore the first term gives the unitary evolution. The second term gives the non-unitary contribution to the evolution. The operators $\hat{L}_i$ are called Lindblad operators but they will not be used in this work.

1.1.3 Projective and POVM measurements

Let a Hermitian operator $\hat{N}$ be a physical quantity that solves the eigenvalues-eigenvectors problem,

$$\hat{N}|\psi_n\rangle = n|\psi_n\rangle. \tag{1.36}$$

Let us define the operators $|\psi_n\rangle\langle\psi_n|$ that we call projective measurements. Also, let a system $\hat{\rho}$ be expressed in the basis that the eigenvectors of $\hat{N}$ provide,

$$\hat{\rho} = \sum_{n,m} c_{nm}|\psi_n\rangle\langle\psi_m|. \tag{1.37}$$

Then it is easy to see that the probability that the system is found to be in the state $|\psi_k\rangle$ is given by,

$$\mathrm{tr}(\hat{\rho}|\psi_k\rangle\langle\psi_k|) = |c_{kk}|^2. \tag{1.38}$$

Immediately after the measurement the state is found to be $|\psi_k\rangle$.

In several quantum information tasks, as for example in quantum state discrimination, one is interested in the probabilities of several outcomes and not in the state after the measurement. In that case one can define more general measurement operators, let $\hat{E}_m$, that are not orthogonal rank-one projectors but nevertheless positive definite. We call the operators $\hat{E}_m$ Positive Operator-Valued Measure (abbreviated POVM) element. A set of POVM elements $\{\hat{E}_m\}$ that satisfies $\sum_m \hat{E}_m = \mathbb{I}$, is simply called POVM and is sufficient to determine the probabilities p_m of different measurements outcomes [NC00],

$$p_m = \mathrm{tr}(\hat{\rho}\hat{E}_m). \tag{1.39}$$

1.2 Entanglement and entropy

1.2.1 The idea of entanglement

It is straightforward to see that when two or more states interact under some Hamiltonian, then the total outcome state may not be written as a tensor product. In such a case we say that the state is entangled [PV07]. For example if we consider that the transformation,

$$\hat{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \tag{1.40}$$

acts on the pure state $|i\rangle = |10\rangle$ then the output state $|f\rangle$ reads,

$$|f\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \tag{1.41}$$

and that is an entangled state. In fact the state in (1.41) is a maximally entangled state. In a $d-$dimensional Hilbert space, maximally entangled states are local unitarily equivalent to the state,

$$|\Psi\rangle = \frac{1}{\sqrt{d}}(|00\rangle + |11\rangle + \ldots + |d-1d-1\rangle). \tag{1.42}$$

It can be proven that the entanglement cannot increase under LOCC transformations and it remains unchanged under local unitary operations. The latter sentence rises the question of *how much* entanglement one state possess. This question is the starting point of a vast field of research. In this thesis we will take under consideration pure entangled states for which, in general, the starting point to understand the quantification of entanglement is to make the simple observation that when we take the partial trace of some entangled state we end up with a mixed state. So the question is to find the measures of mixedness of some density operator. As a result of Schmidt decomposition (1.7), the mixed states obtained when tracing out one or the other subsystem have the same set of eigenvalues, hence the same measures of mixedness. These measures should increase when the entanglement is increasing, and that is how their name entanglement monotones is justified.

1.2.2 von Neuman entropy and mutual information

Entanglement is a type of correlation that occurs in quantum systems. Correlation means that a subsystem possess information on another subsystem of the whole system. Therefore by tracing out one of the subsystems we loose a part of the available information. Note that this makes apparent that information is understood and studied via its loss.

A normal first thing to do is to define a quantity inspired by the classical entropy, namely the von Neumann entropy (or simply entropy from now on),

$$S(\hat{\rho}) = -\mathrm{tr}(\hat{\rho}\ln\hat{\rho}). \tag{1.43}$$

If the density operator $\hat{\rho}$ has eigenvalues λ_i then the entropy reads,

$$S(\hat{\rho}) = -\sum_i \lambda_i \ln \lambda_i. \tag{1.44}$$

From the practical point of view, in order to find the entropy of some state we have to find the eigenvalues of the matrix and then apply definition (1.44).

The entropy is positive and concave to its arguments [NC00],

$$S\left(\sum_i p_i \hat{\rho}_i\right) \geq \sum_i p_i S(\hat{\rho}_i) \tag{1.45}$$

where $\sum_i p_i = 1$.

The entropy possess also a number of other interesting properties, namely for a bipartite system described by the density operator $\hat{\rho}_{AB}$ it holds that $S(\hat{\rho}_A) = S(\hat{\rho}_B)$ as long as $\hat{\rho}_{AB}$ is pure state. Also the entropy is subadditive [NC00],

$$S(\hat{\rho}_{AB}) \quad \leq \quad S(\hat{\rho}_A) + S(\hat{\rho}_B) \tag{1.46}$$

and satisfies the triangle inequality,

$$S(\hat{\rho}_{AB}) \quad \geq \quad |S(\hat{\rho}_A) - S(\hat{\rho}_B)|. \tag{1.47}$$

Moreover it satisfies strong subadditivity [Weh78], namely for a tripartite system with density matrix $\hat{\rho}_{ABC}$ it holds,

$$S(\hat{\rho}_{ABC}) + S(\hat{\rho}_B) \quad \leq \quad S(\hat{\rho}_{AB}) + S(\hat{\rho}_{BC}) \tag{1.48}$$

Another interesting quantity, based on the entropy, is the mutual information [AC97a, AC97b, Cer98],

$$I(A:B) = S(\hat{\rho}_A) + S(\hat{\rho}_B) - S(\hat{\rho}_{AB}) \tag{1.49}$$

that measures the common information that the two subsystems share.

1.2.3 Purity and Tsallis entropies

Mixed states always satisfy $\mathrm{tr}(\hat{\rho}^n) \leq 1$ for $n \geq 1$ [NC00]. Note that the latter fact introduces an interesting quantity, i.e. the Schmidt number, that is the number of the non-zero Schmidt numbers as defined in equation (1.7).

For $n = 2$ we have the quantity $\mu_2(\hat{\rho}) = \mathrm{tr}(\hat{\rho}^2) \leq 1$ that is the so-called purity [NC00]. In this thesis we will refer to all quantities $\mathrm{tr}(\hat{\rho}^n)$ as purities. Based on the purities we can define a family of entanglement monotones, the Tsallis entropies of order α [Tsa88],

$$S^T_\alpha(\hat{\rho}) = \frac{1}{\alpha - 1}\left(1 - (\|\bar{\lambda}\|_\alpha)^\alpha\right) \tag{1.50}$$

for $\alpha \in \mathbb{R}$ and $\alpha \neq 1$. By $\|\bar{\lambda}\|_\alpha$ we denote the α-norm of the probability vector that the eigenvalues of $\hat{\rho}$ put together, so we can equivalently write,

$$S_\alpha^T(\hat{\rho}) = \frac{1}{\alpha - 1}\left(1 - \sum_i \lambda_i^\alpha\right). \tag{1.51}$$

In many cases we will use the notation $S_\alpha^T(\bar{\lambda})$ instead of $S_\alpha^T(\hat{\rho})$. The Tsallis entropies are positive and concave in their arguments [Vid00]. Note that in the limit $\alpha \to 1$ the von-Neumann entropy is recovered [Vid00] and as we will see in Chapter 5, that connection is the initial point for the replica method.

1.2.4 Rényi entropies

One can define an infinite number of entanglement monotones. Here we will introduce one more, namely the Rényi entropies [Ren60],

$$\begin{aligned} S_\alpha(\hat{\rho}) &= \frac{\alpha}{1-\alpha} \ln \|\bar{\lambda}\|_\alpha = \\ &= \frac{1}{1-\alpha} \ln \sum_i \lambda_i^\alpha \end{aligned} \tag{1.52}$$

for $\alpha \geq 0$ and $\alpha \neq 1$. Like for the Tsallis entropies, in many cases we will use the notation $S_\alpha(\bar{\lambda})$ instead of $S_\alpha(\hat{\rho})$.

In the limit $\alpha \to 1$ the von-Neumann entropy is recovered. For the Rényi entropies it holds that $S_0 \geq S_1 \geq S_\infty$, as Jensen's inequality implies. Also, the Rényi entropies are positive and concave in their arguments [Vid00].

We will revisit these entanglement monotones in Chapter 4 where their neat connection with majorization theory will provide some interesting and counter-intuitive results and in Chapter 5 where the replica method will be introduced to quantum optics.

Chapter 2

Quantum optics

2.1 Electromagnetic field

2.1.1 Quantization of the electromagnetic field

The classical theory of electromagnetism is fully described by a seemingly simple set of only five equations; the Maxwell equations and the Lorentz force. The Maxwell equations in vacuum and without sources are [Jac98],

$$\nabla \cdot \mathbf{B} = 0 \tag{2.1}$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial}{\partial t}\mathbf{B} \tag{2.2}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{2.3}$$

$$\nabla \times \mathbf{B} = \epsilon_0 \frac{\partial}{\partial t}\mathbf{E} \tag{2.4}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, μ_0 is the magnetic constant while ϵ_0 is the electric permeability of the vacuum. The Lorentz force describes the force that a moving charge q is feeling while it is moving with a velocity $\mathbf{u}$ within an electromagnetic field [Jac98],

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}). \tag{2.5}$$

By introducing the vector potential $\mathbf{A}$ of the magnetic field and by using the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, we have the following expressions,

$$\mu_0 \mathbf{B} = \nabla \times \mathbf{A} \tag{2.6}$$

$$\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A}. \tag{2.7}$$

Equations (2.6) and (2.7) and the last Maxwell equation (2.4) give rise to the wave equation without sources,

$$\nabla^2 \mathbf{A}(\mathbf{r}, t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A}(\mathbf{r}, t) \tag{2.8}$$

where $(\mathbf{r}, t)$ are the spacetime coordinates and $c = 1/\sqrt{\mu_0 \epsilon_0}$ is the speed of light in vacuum.

By taking the Fourier transform of $\mathbf{A}(\mathbf{r}, t)$, that is decomposing it into its orthonormal modes, and using equations (2.7) and (2.8) we can write,

$$\mathbf{E}(\mathbf{r}, t) = \sum_{j=1}^{2} \int d^3k \sqrt{\frac{\hbar \omega_k}{2\epsilon_0}} \varepsilon_{kj} \left(\alpha_{kj} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} + \alpha_{kj}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} \right) \tag{2.9}$$

where $\mathbf{k}$ is the propagation vector, ε_{kj} is the polarization vector with $j = 1, 2$ the possible polarizations, ω_k is the angular frequency, α_k is the complex amplitude of the k mode, and $\hbar$ is the reduced Planck constant.

Now we have all we need to preform the quantization of the electromagnetic field. This is achieved by upgrading the complex amplitude α_k and its complex conjugate to operators. This gives rise to the annihilation and creation operators [KL10],

$$\alpha_{kj} \rightarrow \hat{a}_{kj} \tag{2.10}$$

$$\alpha_{kj}^* \rightarrow \hat{a}_{kj}^\dagger . \tag{2.11}$$

Since photons are bosons we impose the following commutation relations,

$$\left[\hat{a}_{kj}, \hat{a}_{k'j'}^\dagger\right] = \delta_{kk'} \delta_{jj'} \tag{2.12}$$

$$\left[\hat{a}_{kj}, \hat{a}_{k'j'}\right] = 0 \tag{2.13}$$

$$\left[\hat{a}_{kj}^\dagger, \hat{a}_{k'j'}^\dagger\right] = 0. \tag{2.14}$$

The normal-ordered Hamiltonian of the quantized free electromagnetic field is,

$$\hat{H} = \sum_{j=1}^{2} \sum_{k} \hbar \omega_{kj} \left(\hat{N}_{kj} + \frac{1}{2} \right) = \sum_{k} \hat{h}_k \tag{2.15}$$

where

$$\hat{N}_{kj} = \hat{a}_{kj}^\dagger \hat{a}_{kj} . \tag{2.16}$$

is the number operator. The Hamiltonian of the free quantized electromagnetic field is the summation of an infinite number of harmonic oscillator's Hamiltonians, one for each mode.

One can define the following operators,

$$\hat{Q}_k = \sqrt{\frac{\hbar}{2\omega_k}}\left(\hat{a}_k + \hat{a}_k^\dagger\right) \tag{2.17}$$

$$\hat{P}_k = -i\sqrt{\frac{\hbar\omega_k}{2}}\left(\hat{a}_k - \hat{a}_k^\dagger\right) \tag{2.18}$$

that are the generalized position and momentum respectively of the quantum field. Note that we have dropped the polarization index j for simplicity. The Hamiltonian (2.15) yields the form,

$$\hat{h}_k = \frac{1}{2}\left(\hat{P}_k^2 + \omega_k^2\hat{Q}_k^2\right). \tag{2.19}$$

The operators $\hat{Q}_k$ and $\hat{P}_k$ are called quadrature operators and are the observables corresponding to the electric and the magnetic amplitude of the electromagnetic field. The quadrature operators satisfy the commutations relations,

$$\left[\hat{Q}_k, \hat{P}_{k'}\right] = i\hbar\delta_{kk'} \tag{2.20}$$

and from the commutation relation (2.20) one can derive the Heisenberg uncertainty relation,

$$\Delta\hat{Q}_k\Delta\hat{P}_k \geq \frac{\hbar}{2} \tag{2.21}$$

that is an upper bound of the product of the standard deviations of the generalized position and momentum.

For simplicity we drop the k index and we redefine the quadrature operators as follows,

$$\hat{q} = \sqrt{\omega}\hat{Q} = \sqrt{\frac{\hbar}{2}}\left(\hat{a} + \hat{a}^\dagger\right) \tag{2.22}$$

$$\hat{p} = \frac{1}{\sqrt{\omega}\hat{P}} = -i\sqrt{\frac{\hbar}{2}}\left(\hat{a} - \hat{a}^\dagger\right). \tag{2.23}$$

The quadrature operators $\hat{q}$ and $\hat{p}$ are Hermitian and therefore they satisfy the eigensystem problem,

$$\hat{q}|q\rangle = q|q\rangle \tag{2.24}$$

$$\hat{p}|p\rangle = p|p\rangle. \tag{2.25}$$

with real eigenvalues q and p and orthogonal eigenvectors

$$\langle q|q'\rangle = \delta(q-q') \tag{2.26}$$
$$\langle p|p'\rangle = \delta(p-p') \tag{2.27}$$

that satisfy completeness relations,

$$\int dq|q\rangle\langle q| = \mathbb{I} \tag{2.28}$$
$$\int dp|p\rangle\langle p| = \mathbb{I}. \tag{2.29}$$

The two bases $\{|q\rangle\}$ and $\{|p\rangle\}$ are transformed to one-another via the Fourier transform,

$$|q\rangle = \frac{1}{\sqrt{\pi}}\int dp e^{iqp}|p\rangle \tag{2.30}$$
$$|p\rangle = \frac{1}{\sqrt{\pi}}\int dq e^{-iqp}|q\rangle. \tag{2.31}$$

2.1.2 States of the electromagnetic field

2.1.2.1 Fock states

The number operator (2.16) satisfies the following eigenvalue-eigenvector problem,

$$\hat{N}_k|n_k\rangle = n_k|n_k\rangle \tag{2.32}$$

and therefore the Hamiltonian (2.15) has the same eigenvectors $|n_k\rangle$ as $\hat{N}_k$, while its eigenvalues read $\hbar\omega_k(n_k+1/2)$. Since $\hat{N}_k$ (and $\hat{H}$) are hermitian operators the vectors $|n_k\rangle$ form a complete and orthogonal basis $\{|n_k\rangle\}$, namely

$$\langle n_k|m_k\rangle = \delta_{nm} \tag{2.33}$$
$$\sum_{n=0}^{\infty}|n_k\rangle\langle n_k| = 1. \tag{2.34}$$

We refer to $|n_k\rangle$ as number states or Fock states with photon number n_k for the mode k. Since $\{|n_k\rangle\}$ is a complete and orthogonal basis we can express any state $|\psi\rangle$ in this basis,

$$|\psi\rangle = \sum_n c_n|n\rangle \tag{2.35}$$

where $c_n \in \mathbb{C}$ and

$$\sum_n |c_n|^2 = 1. \tag{2.36}$$

The annihilation operator $\hat{a}_k$ subtracts a photon while the creation operator $\hat{a}_k^\dagger$ adds a photon when they act on $|n_k\rangle$. This is described by the following relations,

$$\hat{a}_k|n_k\rangle = \sqrt{n_k}|n_k - 1\rangle \tag{2.37}$$

$$\hat{a}_k^\dagger|n_k\rangle = \sqrt{n_k + 1}|n_k + 1\rangle \tag{2.38}$$

and when the annihilation operator is acting upon the vacuum state $|0\rangle$ we get $\hat{a}_k|0\rangle = 0$. Clearly $\hat{a}_k$ and $\hat{a}_k^\dagger$ are non-hermitian operators, nevertheless they play an important role in quantum optics.

2.1.2.2 Coherent states and displacement operator

The coherent state $|\alpha\rangle$ is the eigenstate of the annihilation operator,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{2.39}$$

where α is a complex number. The coherent state corresponds to quantum physical system where the distribution of photons $P(n) = |\langle n|\alpha\rangle|^2$ is easily proved to be the Poisson distribution. Coherent states are a minimum uncertainty states, i.e. they achieve the lower bound imposed by Heisenberg's uncertainty relation [Leo10],

$$\Delta\hat{q}\Delta\hat{p} = \frac{\hbar}{2} \tag{2.40}$$

and moreover,

$$\Delta\hat{q} = \Delta\hat{p}. \tag{2.41}$$

The unitary displacement operator is defined as,

$$\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} \tag{2.42}$$

and if we act with the displacement operator on the vacuum state we take the coherent state,

$$\hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2}e^{\alpha a^\dagger}|0\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle = |\alpha\rangle. \tag{2.43}$$

The displacement operator has the following properties as well,

$$\hat{D}^{\dagger}(\alpha) = \hat{D}^{-1}(\alpha) = \hat{D}(-\alpha). \tag{2.44}$$

It is not difficult to find how the field operators transform under the action of the displacement operator [KL10],

$$\hat{D}^{\dagger}(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha \tag{2.45}$$

$$\hat{D}^{\dagger}(\alpha)\hat{a}^{\dagger}\hat{D}(\alpha) = \hat{a} + \alpha^{*} \tag{2.46}$$

$$\hat{D}^{\dagger}(\alpha)\hat{q}\hat{D}(\alpha) = \hat{q} + \sqrt{2\hbar}\Re(\alpha) \tag{2.47}$$

$$\hat{D}^{\dagger}(\alpha)\hat{p}\hat{D}(\alpha) = \hat{p} + \sqrt{2\hbar}\Im(\alpha) \tag{2.48}$$

while action of the displacement operator on the position eigenstate yields,

$$\hat{D}(\alpha)|q\rangle = e^{i\sqrt{2}q\Im(\alpha)/\sqrt{\hbar}}|q + \sqrt{2\hbar}\Re(\alpha)\rangle. \tag{2.49}$$

From the discussion so far it is apparent that the coherent state is the vacuum state displaced on phase space.

As we saw already the expansion of the coherent state on the Fock basis is given by,

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{2.50}$$

From equation (2.50) we can easily find that the probability of measuring n photons in the coherent state $|\alpha\rangle$ is given by the Poisson distribution,

$$P(n) = |\langle n|\alpha\rangle|^2 = e^{-\langle n\rangle}\frac{\langle n\rangle^n}{n!} \tag{2.51}$$

where $\langle n\rangle = |\alpha|^2$ is the mean and the variance of the photon number distribution.

Let us close the discussion on the coherent states by saying that they form a basis, something that is useful for many calculations. It requires care given the fact that the basis they form is not orthogonal since the inner product of two different coherent states is not zero,

$$\langle\beta|\alpha\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2-2\beta^{*}\alpha)} \neq \delta(\alpha-\beta) \tag{2.52}$$

moreover the basis is over-complete,

$$\int d^2\alpha|\alpha\rangle\langle\alpha| = \pi\mathbb{I} \tag{2.53}$$

where the double integral means that we have to integrate with respect to the real and imaginary part of α. Note that no subset of an over-complete basis forms a complete basis.

2.1.2.3 Squeezed states and squeezing operator

Coherent states are not the only ones that possess the minimum amount of uncertainty. There is also the class of squeezed states that, just like coherent states, achieve the bound imposed by Heisenberg's uncertainty relation but their quadratures' variances are not equal. A squeezed state can be understood as a state that occupies the same volume in phase space as a coherent state but is squeezed in some direction.

We will define the single-mode squeezed state by first introducing the squeezing operator,

$$S(\xi) = e^{-\frac{\xi}{2}\hat{a}^{\dagger 2}+\frac{\xi^*}{2}\hat{a}^2} \tag{2.54}$$

where $\xi = |\xi|e^{i\phi}$, $|\xi|$ is the squeezing parameter and ϕ is the squeezing angle. A single-mode squeezed state $|\alpha,\xi\rangle$ is produced when we act on the vacuum state $|0\rangle$ with the squeezing operator and then by displacing it using the operator defined in (2.42),

$$|\alpha,\xi\rangle = D(\alpha)S(\xi)|0\rangle. \tag{2.55}$$

Note that the operators $D(\alpha)$ and $S(\xi)$ do not commute and in general it holds that,

$$|\alpha,\xi\rangle = S(\xi)D(\beta)|0\rangle \tag{2.56}$$

where $\beta = \alpha\cosh|\xi| + \alpha^* e^{i\phi}\sinh|\xi|$.

Just like in the case of the displacement operator, for the single-mode squeezing operator it holds that $S(\xi)^\dagger = S^{-1}(\xi) = S(-\xi)$ and one can derive the following transformations [Ors07],

$$S(\xi)^\dagger \hat{a} S(\xi) = \hat{a}\cosh|\xi| - \hat{a}^\dagger e^{i\phi}\sinh|\xi| \tag{2.57}$$

$$S(\xi)^\dagger \hat{a}^\dagger S(\xi) = \hat{a}^\dagger\cosh|\xi| - \hat{a} e^{-i\phi}\sinh|\xi| \tag{2.58}$$

from where one may derive the transformations for $\hat{q}$ and $\hat{p}$ as well.

Single-mode squeezed states can be expanded in the Fock basis as follows [GA90],

$$
\begin{aligned}
|\alpha,\xi\rangle &= \frac{1}{\sqrt{\cosh r}} \exp\left\{ -\frac{|\alpha|^2 + \alpha^{*2} e^{i\phi} \tanh r}{2} \right\} \\
&\times \sum_{n=0}^{\infty} H_n \left(\frac{\alpha + \alpha^* e^{i\phi} \tanh r}{(2e^{i\phi} \tanh r)^{1/2}} \right) \left(\frac{e^{i\phi} \tanh r}{2} \right)^{n/2} \frac{|n\rangle}{\sqrt{n!}}
\end{aligned} \tag{2.59}
$$

where $H_n(x)$ is the Hermite polynomial of order n. From the expansion (2.59) one can find the distribution $P_{\mathrm{sq}}(n) = |\langle n|\alpha,\xi\rangle|^2$ of the photon number for single-mode squeezed. The mean photon number for single-mode squeezed state is found to be $\langle \hat{n} \rangle = |\alpha|^2 + \sinh^2 |\xi|$.

2.1.2.4 Two-mode squeezed states and two-mode squeezing operator

The single-mode squeezing can be generalized into the two-mode squeezing operator,

$$
S(\xi) = e^{-\xi \hat{a}_1^\dagger \hat{a}_2^\dagger + \xi^* \hat{a}_1 \hat{a}_2}. \tag{2.60}
$$

Where $\xi = |\xi| \mathrm{e}^{i\phi}$ is as defined before in the case of the single-mode squeezer. As it is apparent from its name, the two-mode squeezer acts on two-mode state. The action of the two-mode squeezer on the two-mode vacuum state $|00\rangle$ produces the two-mode vacuum squeezed state which may be expressed on the Fock basis,

$$
|00,\xi\rangle = \frac{1}{\cosh|\xi|} \sum_{n=0} \frac{(-\xi)^n}{|\xi|^n} \tanh^n |\xi| |n,n\rangle. \tag{2.61}
$$

Acting on the previous state with two displacement operators, one for each mode, a general two-mode squeezed state is produced. Note that in (2.61) it is apparent that the two-mode squeezing operator produces pairs of photons.

The modes are transformed as follows [Ors07],

$$
S(\xi)^\dagger \hat{a} S(\xi) = \hat{a}_1 \cosh|\xi| - \hat{a}_2^\dagger e^{i\phi} \sinh|\xi| \tag{2.62}
$$

$$
S(\xi)^\dagger \hat{a}^\dagger S(\xi) = \hat{a}_1^\dagger \cosh|\xi| - \hat{a}_2 e^{i\phi} \sinh|\xi|. \tag{2.63}
$$

2.1.2.5 Statistical mixtures

So far we have introduced the states of the electromagnetic field that plays an important role in quantum optics. All of these states were pure. In many situations though we deal with mixed states; states that are statistical mixtures of pure states. A statistical mixture of pure states can be represented on any complete or overcomplete basis, as for example the Fock basis,

$$\hat{\rho} = \sum_{n,m} \rho_{nm} |n\rangle\langle m| \tag{2.64}$$

where $\sum_n \rho_{nn} = 1$. An important example of mixed state is the thermal radiation $\hat{\rho}_{\text{th}}$. For thermal equilibrium at inverse temperature β_e we have $\rho_{nm} = P_n \delta_{n,m}$ with,

$$P_n = \left[1 - e^{-\hbar\omega\beta_e}\right] e^{-\hbar n\omega\beta_e} \tag{2.65}$$

where P_n is the probability that one mode of the field is excited with n photons. It is not difficult to find that,

$$\hat{\rho}_{\text{th}} = \sum_n \frac{\langle\hat{n}\rangle^n}{(1 + \langle\hat{n}\rangle)^{n+1}} |n\rangle\langle n| \tag{2.66}$$

where $\langle\hat{n}\rangle$ is the mean photon number of the thermal field,

$$\langle\hat{n}\rangle = \frac{1}{e^{\hbar\omega\beta_e} - 1}. \tag{2.67}$$

The thermal state will come up naturally in section 2.1.3 where the two-mode squeezing operator will serve as an amplifier and it will be discussed again in section 2.2.

2.1.3 Quantum optical transformations

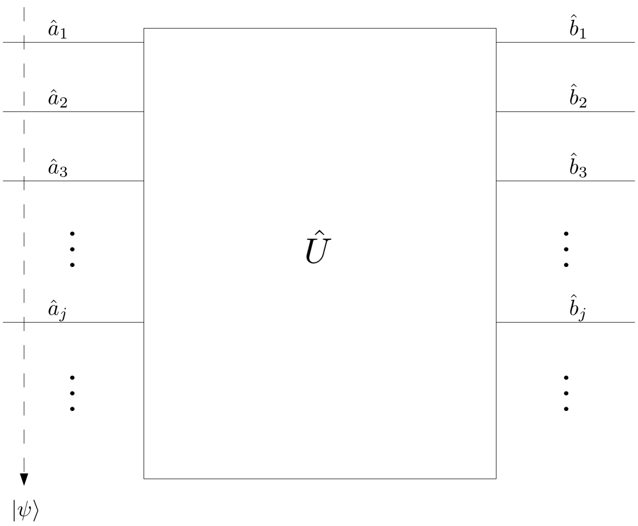

Figure 2.1: A general unitary transformation $\hat{U}$ is acting in some initial state $|\psi\rangle$. In order to find how to transform $|\psi\rangle$ we have to find how the modes $\hat{a}_1, \ldots, \hat{a}_j, \ldots$ are transformed into $\hat{b}_1, \ldots, \hat{b}_j, \ldots$.

In this thesis we will consider quantum optical transformations and we will explore several of their aspects. Therefore it is necessary to explain how a quantum optical element works from a mathematical point of view. Quantum optical transformation may be divided into two groups; the passive and the active ones. Passive are those that conserve the photon number of the input while active are those that do not. The quantum optical transformations are unitary transformations and there ought to be a Hermitian Hamiltonian that generates them. Here, we will give the transformations of the modes of some input field under the action of some unitary operation $\hat{U}$ that is generated by some Hamiltonian through $\hat{U} = e^{-\frac{i}{\hbar}\hat{H}}$.

2.1.3.1 Phase shift

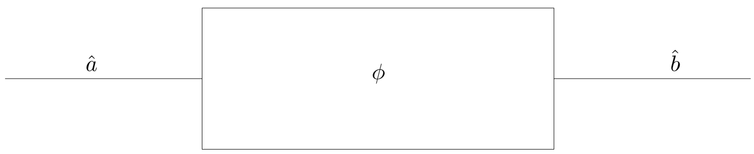

Figure 2.2: Phase shifter.

The Hamiltonian that corresponds to the evolution of the free field mode through free space without interaction with a medium reads,

$$\hat{H}_\phi = \hbar\phi\hat{a}^\dagger\hat{a} \tag{2.68}$$

this Hamiltonian may be written in terms of the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$ and that way it becomes apparent that the generator of the phase shifting is the operator $\hat{n}$. The corresponding unitary transformation reads,

$$\hat{U}_\phi = \exp\left(-\frac{i}{\hbar}\hat{H}_\phi\right) = \exp\left(-i\phi\hat{n}\right). \tag{2.69}$$

The single-mode field is transformed as,

$$\hat{b} = \hat{U}_\phi \hat{a} \hat{U}_\phi^\dagger = \hat{a} e^{-i\phi}. \tag{2.70}$$

The phase shifter is the most simple passive optical transformation.

2.1.3.2 Beam splitter

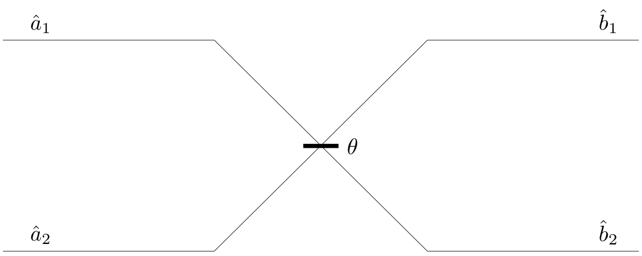

Figure 2.3: Beam splitter.

The beam splitter is a fundamental passive optical element. It can be realized for example with a simple glass plate with thin coating and no additional (pump) energy. Roughly speaking, the beam splitter absorbs an incoming photon and creates a new one in the output modes. This is described by the Hamiltonian,

$$\hat{H}_\theta = \hbar\theta e^{i\varphi} \hat{a}_1^\dagger \hat{a}_2 + \hbar\theta e^{-i\varphi} \hat{a}_1 \hat{a}_2^\dagger. \tag{2.71}$$

The modes' transformation reads [KL10],

$$\hat{b}_1 = \hat{U} \hat{a}_1 \hat{U}^\dagger = \cos\theta \hat{a}_1 - ie^{i\varphi} \sin\theta \hat{a}_2 \tag{2.72}$$

$$\hat{b}_2 = \hat{U} \hat{a}_2 \hat{U}^\dagger = -ie^{-i\varphi} \sin\theta \hat{a}_1 + \cos\theta \hat{a}_2 \tag{2.73}$$

or in matrix form,

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -ie^{i\varphi} \sin\theta \\ -ie^{-i\varphi} \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}. \tag{2.74}$$

By choosing appropriate phase reference, we usually set $\varphi = 0$ or $\varphi = \pi/2$. Note that the fact that the beam splitter is a passive element is reflected in the fact that transformation (2.74) does not mix $\hat{a}$ with $\hat{a}^\dagger$, so that the total photon number $\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2$ is conserved.

2.1.3.3 One-mode squeezer

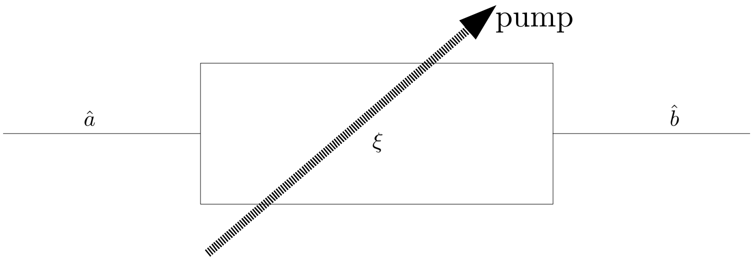

Figure 2.4: One-mode squeezer.

One basic non photon-number preserving optical transformation is the one-mode squeezer. This optical element (and the two-mode squeezer that we consider next) adds or removes energy to the initial field by exploiting the non-linear features of dielectric media. This transformation, that mixes annihilation and creation operators, is generated by the following Hamiltonian,

$$\hat{H}_{\xi,\varphi} = \hbar \xi e^{i\varphi} \hat{a}^2 + \hbar \xi e^{-i\varphi} \hat{a}^{\dagger 2}. \tag{2.75}$$

where as we have already seen $\xi = |\xi| e^{i\phi}$, with $|\xi|$ the squeezing parameter and ϕ the squeezing angle. In matrix form the transformation reads,

$$\begin{pmatrix} \hat{b} \\ \hat{b}^\dagger \end{pmatrix} = \begin{pmatrix} \cosh 2\xi & -ie^{-i\varphi} \sinh 2\xi \\ ie^{i\varphi} \sinh 2\xi & \cosh 2\xi \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}. \tag{2.76}$$

If we take for example the vacuum state $|0\rangle$ as an initial state, we obtain the squeezed state as defined in equation (2.55). We also note that the mean photon number $\hat{n} = \sinh^2 |\xi|$.

2.1.3.4 Two-mode squeezer

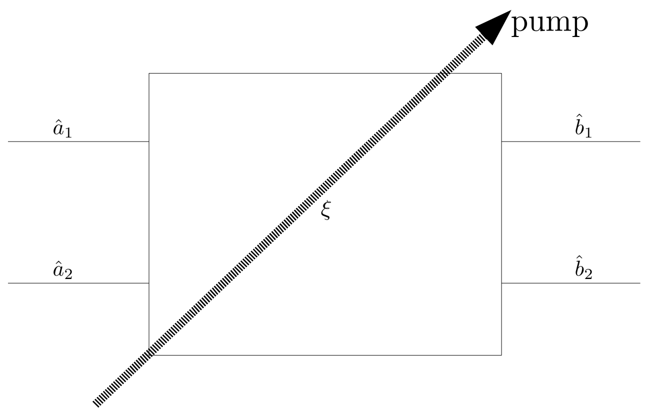

Figure 2.5: Two-mode squeezer.

The two mode squeezer is a transformation generated by the Hamiltonian,

$$\hat{H}_{\xi,\varphi} = \hbar\xi e^{i\varphi}\hat{a}_1\hat{a}_2 + \hbar\xi e^{-i\varphi}\hat{a}_1^\dagger\hat{a}_2^\dagger \tag{2.77}$$

and the corresponding matrix form of the modes' transformation is found to be,

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_1^\dagger \\ \hat{b}_2 \\ \hat{b}_2^\dagger \end{pmatrix} = \begin{pmatrix} \cosh\xi & 0 & 0 & -ie^{i\varphi}\sinh\xi \\ 0 & \cosh\xi & ie^{-i\varphi}\sinh\xi & 0 \\ 0 & ie^{i\varphi}\sinh\xi & \cosh\xi & 0 \\ -ie^{-i\varphi}\sinh\xi & 0 & 0 & \cosh\xi \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_1^\dagger \\ \hat{a}_2 \\ \hat{a}_2^\dagger \end{pmatrix}. \tag{2.78}$$

If we take for example as initial state the two-mode state $|\alpha 0\rangle$ and we apply the two-mode squeezing operator, then the reduced output, i.e. the state in one of the output modes, is the thermal state,

$$\hat{\rho}_{th} = (1-|\tau|^2)\sum_n |\tau|^{2n}|n\rangle\langle n| \tag{2.79}$$

where $|\tau| = \tanh|\xi|$. If (2.79) is compared with the density matrix (2.66) we see immediately that $\langle\hat{n}\rangle = \sinh^2|\xi|$.

2.1.3.5 Bloch-Messiah reduction

Let us first introduce briefly the notion of normal modes. As it is understood so far a mode operator is transformed following a unitary matrix that describes the transformation that is generated by some Hamiltonian $\hat{H}$. This Hamiltonian can always be diagonalized by some unitary matrix and be written in the form,

$$\hat{H} = \frac{\hbar}{2}\sum_i \hat{d}_i^\dagger \hat{d}_i \tag{2.80}$$

the modes $\hat{d}_i$ are the normal modes of the field.

In general all Gaussian quantum optical transformations are described by the Bogoliubov transformations [KL10], namely,

$$\hat{b}_i = \sum_j A_{ij}\hat{a}_j + B_{ij}\hat{a}_j^\dagger \tag{2.81}$$

$$\hat{b}_i^\dagger = \sum_j B_{ij}^*\hat{a}_j + A_{ij}^*\hat{a}_j^\dagger. \tag{2.82}$$

Since the operators $\hat{b}_i$ and $\hat{b}_i^\dagger$ must be annihilation and creation operators they should obey the correct commutation relations,

$$\left[\hat{b}_i, \hat{b}_j^\dagger\right] = \delta_{ij} \tag{2.83}$$

$$\left[\hat{b}_i, \hat{b}_j\right] = \left[\hat{b}_i^\dagger, \hat{b}_j^\dagger\right] = 0. \tag{2.84}$$

The commutations relations (2.83) and (2.84) impose some restrictions on the matrices A and B appearing in equations (2.81) and (2.82),

$$AB^T = \left(AB^T\right)^T \tag{2.85}$$

$$AA^\dagger = BB^\dagger + \mathbb{I}. \tag{2.86}$$

The inverse transformations to equations (2.81) and (2.82) read,

$$\hat{a}_i = \sum_j A_{ij}^*\hat{b}_j - B_{ij}^*\hat{b}_j^\dagger \tag{2.87}$$

$$\hat{a}_i^\dagger = \sum_j (-B_{ij})\hat{b}_j + A_{ij}\hat{b}_j^\dagger \tag{2.88}$$

while the corresponding restrictions for A and B read,

$$A^\dagger B = \left(A^\dagger B\right)^T \tag{2.89}$$

$$A^\dagger A = \left(B^\dagger B\right)^T + \mathbb{I}. \tag{2.90}$$

In general the matrices A and B are simultaneously diagonalized according to the singular value decomposition problem,

$$A = \mathcal{U} A_D \mathcal{V}^\dagger \quad (2.91)$$

$$B = \mathcal{U} B_D \mathcal{W}^\dagger. \quad (2.92)$$

The restrictions (2.89) and (2.90) lead to the relation $\mathcal{V}^* = \mathcal{W}$. The singular value decomposition problem now reads,

$$A = \mathcal{U} A_D \mathcal{V}^\dagger \quad (2.93)$$

$$B = \mathcal{U} B_D \mathcal{V}^T. \quad (2.94)$$

If we organize the modes in vectors $\bar{a} = (\hat{a}_1, \ldots, \hat{a}_i, \ldots)$ and $\bar{b} = \left(\hat{b}_1, \ldots, \hat{b}_i, \ldots\right)$, the Bogoliubov transformation reads,

$$\begin{pmatrix} \bar{b} \\ \bar{b}^\dagger \end{pmatrix} = \begin{pmatrix} \mathcal{U} & 0 \\ 0 & \mathcal{U}^* \end{pmatrix} \begin{pmatrix} A_D & B_D \\ B_D^* & A_D^* \end{pmatrix} \begin{pmatrix} \mathcal{V}^\dagger & 0 \\ 0 & \mathcal{V}^* \end{pmatrix} \begin{pmatrix} \bar{a} \\ \bar{a}^\dagger \end{pmatrix}. \quad (2.95)$$

At this point there is an important observation to be made. The first and the last block diagonal matrices in the right hand side of equation (2.95) do not mix annihilation and creation operators, therefore they represent passive optical elements, i.e. beam splitters and phase shifters. The matrix in the middle mixes annihilation and creation operators so it correspond to active optical elements and since the matrices A_D and B_D are diagonal then these optical elements are single-mode squeezers. The diagonal elements of A_D are proportional to $\cosh|\xi|$ while those of B_D are proportional to $\sinh|\xi|$. This is what is known as the Bloch-Messiah reduction. Any quantum optical transformation may be decomposed as the the successive action of passive optical interferometers, single-mode squeezers and passive optical interferometers again. From this point of view the most fundamental objects in quantum optics are beam splitters, phase shifters and single-mode squeezers [Bra05].

By substituting equations (2.81) and (2.82) into (2.80), considering the operators $\hat{b}^\dagger$ and $\hat{b}$ we get the most general quadratic Hamiltonian,

$$\hat{H} = \frac{\hbar}{2} \sum_{ij} \left(\hat{a}_i F_{ij} \hat{a}_j + 2\hat{a}_i^\dagger G_{ij} \hat{a}_j + \hat{a}_i^\dagger F_{ij}^* \hat{a}_j^\dagger \right) \quad (2.96)$$

where

$$F_{ij} = \sum_k A_{ki} B^*_{kj} \quad (2.97)$$

$$G_{ij} = \sum_k A_{ki} A^*_{kj}. \quad (2.98)$$

The Hamiltonian in equation (2.96) describes the dynamics when the modes are mixed under the action of Bogoliubov transformations.

2.1.4 Deterministic and probabilistic maps in quantum optics

Putting together the fundamental elements described so far, one can build composite quantum optical devices for fundamental or more advanced use. Note that we divide the composite quantum optical elements into two sets; the deterministic and the probabilistic ones. As their names imply, the deterministic devices work always in the sense that they produce always the same result. On the other hand the probabilistic devices produce the desired outcome with some probability of success. The probabilistic devices are designed in order to produce better output states than the deterministic ones, i.e. with fidelity closer to one when compared with the ideal output. The important feature that the probabilistic processing must possess is that it should be heralded; there is a photon-counting event heralding the fact that the desired result is obtained.

The probabilistic protocols can be described as follows. Let some initial state $\hat{\rho}_i$ and some map $\mathcal{T}[.]$ that deterministically transforms $\hat{\rho}_i$ into some final state $\mathcal{T}[\hat{\rho}_i] = \hat{\rho}_f$. Also let,

$$\hat{\rho}_f = P_{succ}\hat{\rho}_{succ} + (1 - P_{succ})\hat{\rho}_{fail} \quad (2.99)$$

where $\hat{\rho}_{succ}$ the desirable outcome of the procedure, while $\hat{\rho}_{fail}$ is a non-desirable state. Now we perform post-selection, i.e. measurements to some idler modes of the final state, so that we know when we would collect the desirable state at the output. In this manner the final state that occurs with probability P_{succ} reads,

$$\hat{\rho}_f = P_{succ}\hat{\rho}_{succ} \quad (2.100)$$

which is clearly non-normalized. The output state that is collected should be of course a physical state and therefore normalized. The normalization of the final state gives the probability of success,

$$\mathrm{tr}\,(\hat{\rho}_f) = P_{succ}\mathrm{tr}\,(\hat{\rho}_{succ}) \Leftrightarrow \mathrm{tr}\,(\hat{\rho}_f) = P_{succ} \quad (2.101)$$

where we have used the fact that the state $\hat{\rho}_{succ}$ is normalized to unity, i.e. $\text{tr}\left(\hat{\rho}_{succ}\right) = 1$. It is apparent that the ingredient that gives rise to the probabilistic nature comes from the post-selection. If we incorporate the post-selection step into the deterministic map $\mathcal{T}[.]$, we will get a map $\tilde{\mathcal{T}}[.]$ that will decrease the trace of the input state. This is another way to see the non-deterministic procedures, namely as trace-decreasing completely positive maps (in contrast to the trace-preserving completely positive maps that the deterministic procedures always are). The complete positivity means that the output is a valid quantum state, but the trace-decreasing property reflects that the final state needs to be normalized and therefore is non-deterministic procedure. The probabilistic or non-deterministic processes will be discussed again in Chapter 3.

2.1.4.1 Realization of the displacement operator

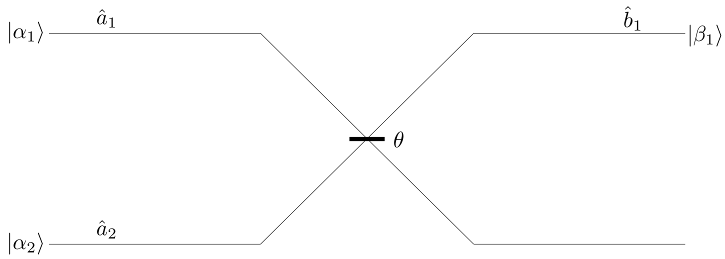

Figure 2.6: Realization of the displacement operator.

The displacement operator that was defined in equation (2.42) is implemented by a beam splitter. In the input there is the coherent state to be displaced $|\alpha_1\rangle$ and a bright coherent state $|\alpha_2\rangle$ with $|\alpha_2| \gg 1$. The upper mode transformation yields,

$$\hat{b}_1 = \cos\theta \hat{a}_1 + \sin\theta \alpha_2 \tag{2.102}$$

where we have replaced the operator $\hat{a}_2$ by its mean value α_2.

By choosing $\alpha_2 = \alpha / \sin\theta$, equation (2.102) becomes,

$$\hat{b}_1 = \cos\theta \hat{a}_1 + \alpha. \tag{2.103}$$

Therefore the coherent amplitude can be found to be transformed as,

$$\beta_1 = \cos\theta \alpha_1 + \alpha \tag{2.104}$$

which is the desired displacement operation if $\cos\theta \to 1$. In this limit the amplitude of the bright coherent state goes to infinity so in practical realizations inevitably, there is a trade-off between the transmissivity and the intensity of the bright coherent input state.

2.1.4.2 Deterministic amplification and attenuation

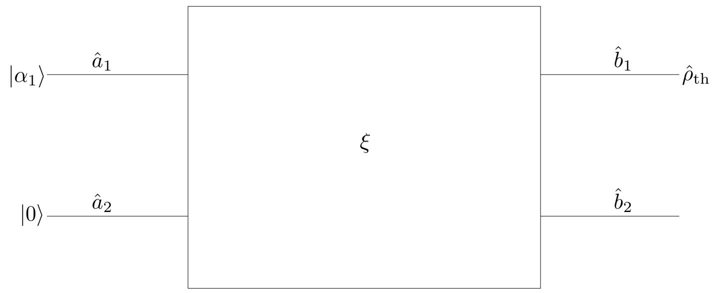

Figure 2.7: Realization of the quantum-limited amplifier.

When we talk about amplification or attenuation we may mean two things. One is that the mean field of the signal is increased in case of amplification or decreased in the case of attenuation. The second thing is that the mean photon number is increased when we amplify the input state or decreased if we attenuate it. Those two definitions coincide both for Gaussian states and non-Gaussian states under deterministic amplification. We will see in Chapter 3 that this is not anymore the case for probabilistic amplification and attenuation.

The quantum-limited amplifier is actually a two-mode squeezer where in one input port (we take this idler mode to be the lower one) the vacuum state is injected. From relation (2.78), that gives the modes' transformation, it is apparent that entanglement is produced and therefore when we trace out the idler mode the amplified state will be a mixed state. In other words, in quantum mechanical systems noise is induced inevitably [Cav82]. For example when a coherent state is the input state, then the full output state would be a two-mode squeezed state. By tracing out the degrees of freedom of the idler mode we obtain a thermal state (2.66), i.e a state with greater mean field amplitude $\langle\hat{a}\rangle$ but the second order moments are affected as well, meaning that the uncertainty of the position and momentum are greater than those of a coherent state. Since the mean field amplitude $\langle\hat{a}\rangle$ and the mean photon number $\langle\hat{n}\rangle$ are increased in comparison with the coherent state we say that the two-mode squeezer serves as an optical amplifier.

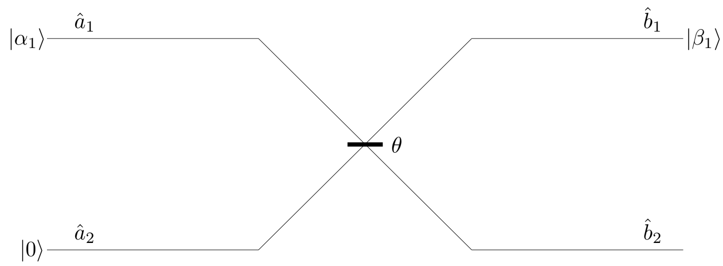

Figure 2.8: Realization of the quantum-limited attenuator.

To attenuate a Gaussian quantum signal deterministically, means to decrease the input state mean field amplitude or to decrease its mean photon number. To achieve this we could, for example, inject a coherent state in a beam splitter while the idler state is vacuum. From the transformation (2.74), if the state to be attenuated is $|\alpha_1\rangle$, then it is found that the attenuated state is $|\beta_1\rangle = |\cos\theta\alpha_1\rangle$, that is a state with reduced mean field amplitude and less photons. If the state to be attenuated is not a coherent or thermal state then entanglement will be generated and the final state would be attenuated but it would be a mixed state.

2.1.4.3 Quantum scissors

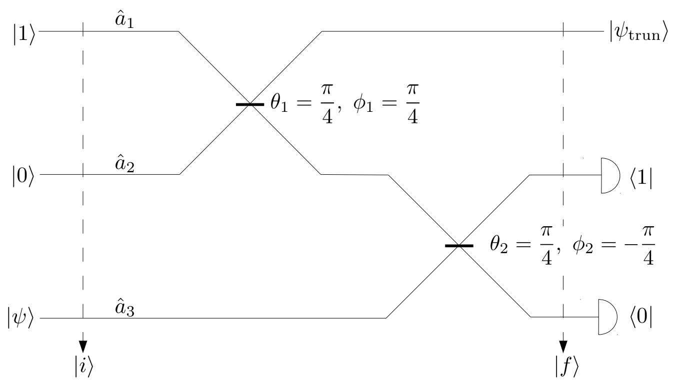

Figure 2.9: Realization of the quantum scissors.

The first probabilistic protocol that we will discuss are the quantum scissors. As the name may implies, quantum scissors truncate a state down to a superposition of $|0\rangle$ and $|1\rangle$. Let the initial, normalized state $|i\rangle$ be,

$$|\psi\rangle = \sum_k c_k |k\rangle \tag{2.105}$$

with $\sum_k |c_k|^2 = 1$. The one photon state $|1\rangle$ interacts with the vacuum state $|0\rangle$ with a balanced beam splitter. The state produced interacts with the state to be truncated $|\psi\rangle$, through another balanced beam splitter. Note that the phases of the beam splitters are not the same, as is noted in figure 2.9. Using formulas (2.74), it is straightforward to do the calculations and to find how the initial state $|i\rangle = |10\psi\rangle$ is transformed to the final state $|f\rangle$. Prior to post selection the final state may be written,

$$|f\rangle = \sum_{k=0}^{\infty} \frac{c_k}{2^{\frac{k}{2}}\sqrt{k!}} \sum_{l=0}^{k} \binom{k}{l} \left(\frac{1}{\sqrt{2}} \hat{a}_1^{\dagger} \hat{a}_2^{\dagger l} \hat{a}_3^{\dagger k-l} + \frac{1}{2} \hat{a}_2^{\dagger l+1} \hat{a}_3^{\dagger k-l} - \frac{1}{2} \hat{a}_2^{\dagger l} \hat{a}_3^{\dagger k-l+1} \right) |000\rangle \quad (2.106)$$

where the binomial expansion has been used.

The post-selection on $|1\rangle$ on the second mode and on $|0\rangle$ on the third puts restrictions if we act with the creation operators of the expression (2.106); only terms with one photon on the second mode and vacuum on the third mode will survive. The resulting, non-normalized state $|\tilde{\psi}_{\text{trun}}\rangle$ that will remain in the first mode is,

$$\begin{aligned} |\tilde{\psi}_{\text{trun}}\rangle &= \sum_{k=0}^{\infty} \frac{c_k}{2^{\frac{k}{2}}\sqrt{k!}} \sum_{l=0}^{k} \binom{k}{l} \left(\frac{1}{\sqrt{2}} \delta_{l,1}\delta_{k,l}|1\rangle + \frac{1}{2}\delta_{l,0}\delta_{k,l}|0\rangle - \frac{1}{2}\delta_{l,1}\delta_{k,l-1}|0\rangle \right) \\ &= \frac{1}{2}c_0|0\rangle + \frac{1}{2}c_1|1\rangle. \end{aligned} \quad (2.107)$$

The state $|\tilde{\psi}_{\text{trun}}\rangle$ in equation (2.107) is indeed a truncated version of the initial state $|\psi\rangle$. As we argued before, the fact that it is not normalized is a manifestation of the fact the procedure is probabilistic. The absolute square of the normalization factor gives the probability of success. The output of the procedure is,

$$|\psi_{\text{trun}}\rangle = c_0|0\rangle + c_1|1\rangle \quad (2.108)$$

while the probability of occurring for the above state is,

$$P_{succ} = \frac{1}{4}|c_0|^2 + \frac{1}{4}|c_1|^2. \quad (2.109)$$

The probability of success in (2.109) is not optimal. Generally speaking, if one involves unitary operations in he setup of figure 2.9 the probability of success can be increased, nevertheless it can never be equal to one. Note that a setup that would enhance the probability of success would be more complicated from the perspective of the actual realization of such a device.

2.1.4.4 Probabilistic amplification and attenuation

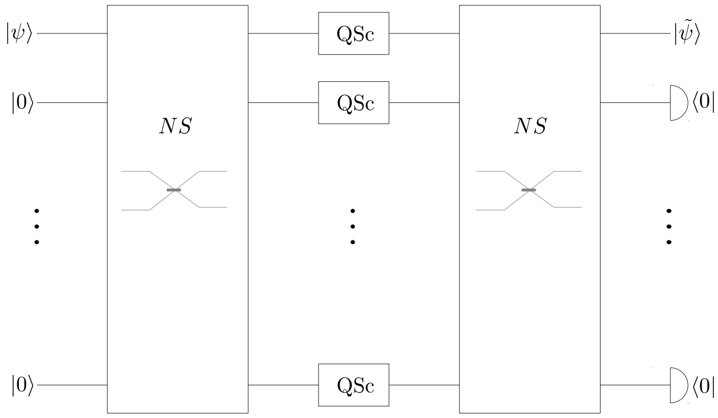

Figure 2.10: A setup for noiseless heralded amplification. The NS stands for N-splitter, meaning that the input beam is evenly divided into N parts with a grid of beam splitters. QSc stands for quantum scissors with θ_1 being chosen according to the gain and $\theta_2 = \pi/4$. Post-selection on the vacuum state takes place at the $N-1$ output modes.

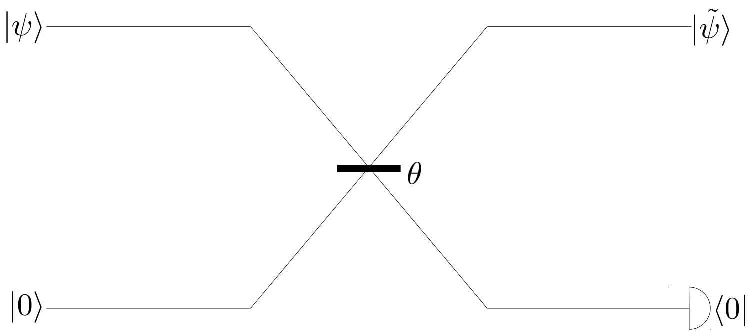

Figure 2.11: A setup for noiseless heralded attenuation. It consists of a single beam splitter where post-selection on vacuum state takes place at the lower output port.

The probabilistic amplifier may be described by the operator $\eta^{N/2} g^{\hat{n}}$, where η, N and $g > 1$ are parameters defined in the proposed realization proposed in [RL09] and depicted in figure 2.10. Other realizations have been proposed [XRL+10, FBB+10, UMW+10, ZFB11, OBS+12] as well.

If a state is expressed on Fock basis,

$$|\psi\rangle = \sum_n c_n |n\rangle \tag{2.110}$$

after the action of the probabilistic amplifier the weights that correspond to higher Fock states will be increased,

$$|\psi\rangle \propto \sum_n c_n g^n |n\rangle \tag{2.111}$$

where the proportional sign should be interpreted as that a normalization is required for the final state, as we saw this provides the success probability.

In the level of coherent states it means that

$$|\alpha\rangle \to |g\alpha\rangle \tag{2.112}$$

and the corresponding probability reads,

$$P_{succ} = \eta^N \exp\left(-(1-g^2)|\alpha|^2\right). \tag{2.113}$$

Note that the probabilistic amplifier, also called heralded noiseless linear amplifier (abbreviated HNLA), is an unbounded operator.

The probabilistic attenuator has the inverse action of the probabilistic amplifier. It has the same mathematical description as the noiseless probabilistic amplifier for $g = \nu < 1$. The setup is simply a beam splitter with post selection on vacuum in the one output port and therefore is a bounded operator.

2.2 Phase space description

From now on, and for the rest of the thesis we fix $\hbar = 2$, except for Section 5.3 where $\hbar = 1$. We will define below the three characteristic functions and the corresponding phase space representations or quasi-probability distributions. The characteristic functions and the representations carry all the information of the quantum system and therefore are useful for calculations.

2.2.1 Symmetric-ordered characteristic function and the Wigner representation

The symmetric characteristic function is defined as the mean value of the displacement operator of equation (2.42) which is in the symmetric order. Namely,

$$\chi_S(\beta,\beta^*) = \mathrm{tr}\left(\hat{\rho}\exp\left(\beta\hat{a}^\dagger - \beta^*\hat{a}\right)\right). \tag{2.114}$$

The Wigner representation is defined as the Fourier transform of the symmetric characteristic function.

$$W(\alpha,\alpha^*) = \frac{1}{\pi^2}\int d^2\beta \exp\left(-\beta\alpha^* + \beta^*\alpha\right)\chi_S(\beta,\beta^*). \tag{2.115}$$

It is straightforward to find that the Wigner function can assume the form,

$$W(q,p) = \frac{1}{4\pi}\int dx e^{\frac{i}{2}px}\Big\langle q - \frac{x}{2}\Big|\hat{\rho}\Big|q + \frac{x}{2}\Big\rangle. \tag{2.116}$$

The Wigner function is of course normalized,

$$\int dpdqW(q,p) = 1. \tag{2.117}$$

In quantum mechanical systems, since position and momentum cannot be measured at the same time for a single system, the Wigner representation is not a proper distribution, as for example it can take negative values. Often it is referred as quasi-distribution. Nevertheless, it carries all information about the state and therefore is useful. The marginals of the Wigner representation though give the correct probability distributions for the position and momentum,

$$\int dpW(q,p) = P(q) \tag{2.118}$$

$$\int dqW(q,p) = P(p). \tag{2.119}$$

When we calculate mean values involving creation and annihilation operators using the Wigner representation, the operator should be in symmetric form. For example the symmetric form of the operator $\hat{a}^{\dagger 2}\hat{a}$ is $1/3(\hat{a}^\dagger\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}\hat{a}^\dagger + \hat{a}\hat{a}^\dagger\hat{a}^\dagger)$.

From equations (2.119) it becomes apparent that the Wigner distribution describes the distribution we would take from the so-called homodyne detection [Leo10].

2.2.2 Antinormal ordered characteristic function and the Husimi Q representation

The antisymmetric characteristic function is defined as the mean value of the antisymmetric displacement operator,

$$\chi_A(\beta) = \mathrm{tr}\left(\hat{\rho}\exp(-\beta^*\hat{a})\exp(\beta\hat{a}^\dagger)\right). \tag{2.120}$$

The Husimi Q representation is defined as the Fourier transform of the antisymmetric characteristic function,

$$Q(\alpha,\alpha^*) = \int d^2\beta \exp\left(-\beta\alpha^* + \beta^*\alpha\right)\chi_A(\beta). \tag{2.121}$$

An equivalent way to define the Q representation reads,

$$Q(\alpha,\alpha^*) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle. \tag{2.122}$$

The Q representation is again normalized,

$$\int d^2aQ(\alpha,\alpha^*) = 1 \tag{2.123}$$

and it is always positive and bounded, i.e $0 \leq Q(\alpha,\alpha^*) \leq 1/\pi$.

When we calculate mean values involving creation and annihilation operators using the Q representation, the operator should be in antinormal form. As an example let us calculate the mean photon number $\langle\hat{n}\rangle$, the coherent amplitude $\langle\hat{a}\rangle$, the mean squared position $\langle\hat{q}^2\rangle$ and the mean squared momentum $\langle\hat{p}^2\rangle$ of a single-mode squeezed state $|\beta,\xi\rangle$, with $\xi = |\xi|\exp(i\phi)$. The single-mode squeezed state's Q representation is easily found from definition (2.122),

$$\begin{aligned} Q_{\beta,\xi}(\alpha,\alpha^*) \quad = \quad & \frac{1}{\pi\cosh|\xi|}\exp\big(-(1+\tanh|\xi|\cos\phi)(\Re\alpha-\Re\beta)^2 \\ & -(1-\tanh|\xi|\cos\phi)(\Im\alpha-\Im\beta)^2 \\ & -2(\Re\alpha-\Re\beta)(\Im\alpha-\Im\beta)\tanh|\xi|\sin\phi\big). \end{aligned} \tag{2.124}$$

In order to find the mean photon number we have first to bring it in antinormal ordering,

$$\langle\hat{a}^\dagger\hat{a}\rangle = \langle\hat{a}\hat{a}^\dagger - 1\rangle = \langle\hat{a}\hat{a}^\dagger\rangle - 1. \tag{2.125}$$

where we have used the commutation relation $\left[\hat{a},\hat{a}^{\dagger}\right]=1$. So the mean photon number reads,

$$\langle\hat{a}\hat{a}^{\dagger}\rangle-1=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}d\Re\alpha d\Im\alpha Q_{\beta,\xi}(\alpha^{*},\alpha)(\alpha\alpha^{*}-1)=|\beta|^{2}+\sinh^{2}|\xi|. \tag{2.126}$$

Continuing the single-mode squeezed state example, extra care is required when we calculate the mean value $\langle\hat{q}^2\rangle$ and $\langle\hat{p}^2\rangle$. We have to bring the operators $\hat{q}^2$ and $\hat{p}^2$ in terms of $\hat{a}$ and $\hat{a}^{\dagger}$. Since $\hbar=2$ we have,

$$\hat{q} = \hat{a}+\hat{a}^{\dagger} \tag{2.127}$$
$$\hat{p} = -i(\hat{a}+\hat{a}^{\dagger}) \tag{2.128}$$

and therefore,

$$\hat{q}^2 = \hat{a}^2+\hat{a}^{\dagger 2}+2\hat{a}\hat{a}^{\dagger}-1 \tag{2.129}$$
$$\hat{p}^2 = -\hat{a}^2-\hat{a}^{\dagger 2}+2\hat{a}\hat{a}^{\dagger}-1 \tag{2.130}$$

or

$$\hat{q}^2 = q_A^2-1 \tag{2.131}$$
$$\hat{p}^2 = p_A^2-1 \tag{2.132}$$

where $q_A^2=\hat{a}^2+\hat{a}^{\dagger 2}+2\hat{a}\hat{a}^{\dagger}$ and $p_A^2=-\hat{a}^2-\hat{a}^{\dagger 2}+2\hat{a}\hat{a}^{\dagger}$. As we have seen, when we calculate some mean value using the Q representation the result corresponds to the antinormally-ordered operator, so in the case of $\hat{q}^2$ and $\hat{p}^2$ the calculation would actually return the values for $\hat{q}_A^2$ and $\hat{p}_A^2$. Therefore in order to obtain the correct values we should subtract 1 as equations (2.129) and (2.130) imply. That is,

$$\langle\hat{q}^2\rangle = \int d^2\alpha Q(\alpha,\alpha^*)(q^2-1) \tag{2.133}$$
$$\langle\hat{p}^2\rangle = \int d^2\alpha Q(\alpha,\alpha^*)(p^2-1). \tag{2.134}$$

Note that since $\hbar=2$ we have $q=2\Re\alpha$ and $p=2\Im\alpha$. We get,

$$\langle\hat{q}^2\rangle = q_\beta^2+\cosh 2|\xi|\left(1-\tanh 2|\xi|\cos\phi\right) \tag{2.135}$$
$$\langle\hat{p}^2\rangle = p_\beta^2+\cosh 2|\xi|\left(1+\tanh 2|\xi|\cos\phi\right). \tag{2.136}$$

The operators $\hat{q}$ and $\hat{p}$ are trivially in antinormal-ordering (and in normal and symmetric as well) and we easily get,

$$\langle \hat{q} \rangle = q_\beta \tag{2.137}$$
$$\langle \hat{p} \rangle = p_\beta. \tag{2.138}$$

From equations (2.135), (2.136), (2.137) and (2.137) we readily find,

$$\begin{aligned}\Delta\hat{q}_\phi^2 &= \cosh 2|\xi| \,(1 - \tanh 2|\xi| \cos\phi) \\ &= e^{-2|\xi|}\cos^2\frac{\phi}{2} + e^{2|\xi|}\sin^2\frac{\phi}{2}\end{aligned} \tag{2.139}$$
$$\begin{aligned}\Delta\hat{p}_\phi^2 &= \cosh 2|\xi| \,(1 + \tanh 2|\xi| \cos\phi) \\ &= e^{-2|\xi|}\sin^2\frac{\phi}{2} + e^{2|\xi|}\cos^2\frac{\phi}{2}.\end{aligned} \tag{2.140}$$

If $\xi = 0$ it means that there is no squeezing and therefore we have a coherent state $|\beta\rangle$, equations (2.139) and (2.140) read,

$$\Delta\hat{q}^2 = 1 \tag{2.141}$$
$$\Delta\hat{p}^2 = 1. \tag{2.142}$$

Form equations (2.141)and (2.142) it is apparent that if we represent a coherent state on q versus p diagram, that is a phase space, then we would get a circle centered at (q_β, p_β) with diameter $\sqrt{\Delta\hat{q}^2} = \sqrt{\Delta\hat{p}^2} = 1$. If we go back to the single-mode squeezed state, we notice that for $\phi = 0$ equation (2.139) is minimized while equation (2.140) is maximized,

$$\Delta\hat{q}_0^2 = e^{-2\xi} \tag{2.143}$$
$$\Delta\hat{p}_0^2 = e^{2\xi}. \tag{2.144}$$

It is straightforward to find that for a single-mode squeezed state $|\beta, \xi\rangle$, the direction of squeezing is $\phi/2$. From equations (2.143) and (2.144) it is apparent that the state $|\beta, \xi\rangle$ may be represented on the $q - p$ plain as a squeezed coherent state in the $\phi/2$ direction and anti-squeezed in the $\pi/2 - 2\phi$ direction, i.e. it is a ellipse centered at (q_β, p_β) with minor axis $\sqrt{\Delta\hat{q}_0^2} = e^{-\xi}$ and major axis $\sqrt{\Delta\hat{p}_0^2} = e^{\xi}$.

The plots of the Q representation (2.124) are given in figures 2.13 and 2.14

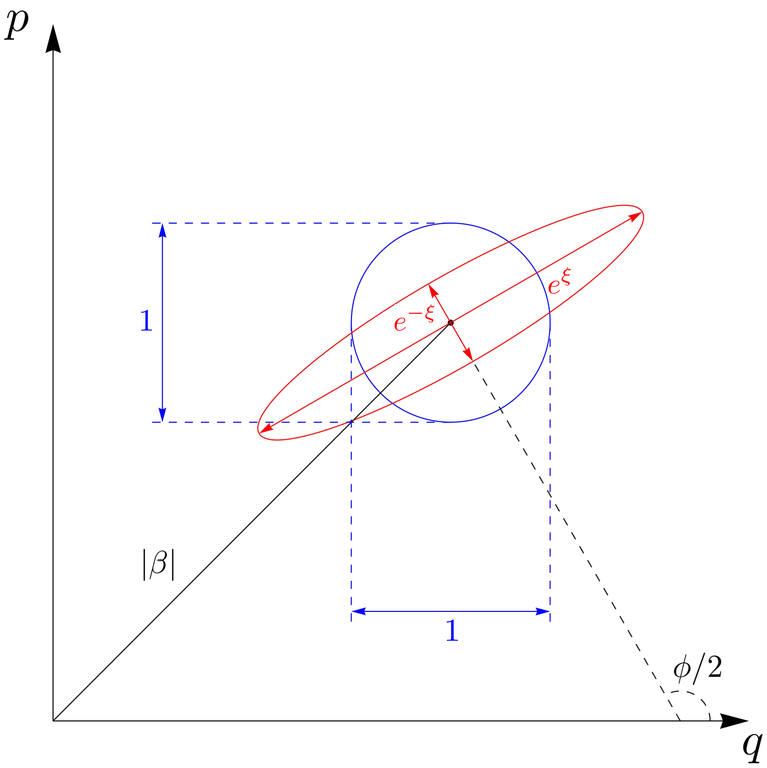

Figure 2.12: Coherent and squeezed state on the $q-p$ plain.

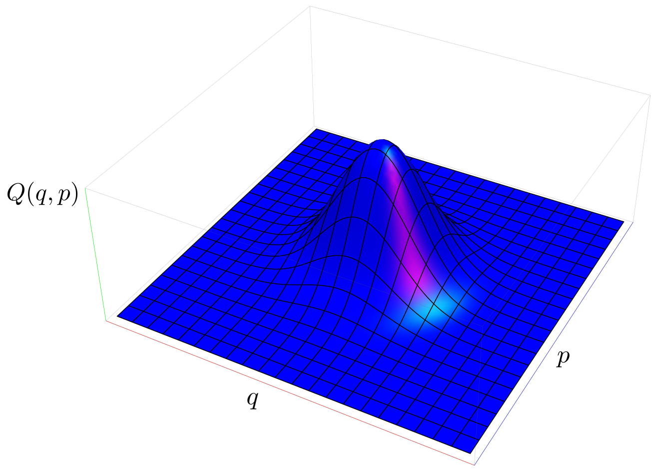

Figure 2.13: Q representation of a coherent state, i.e. a single-mode squeezed state with $|\xi| = 0$.

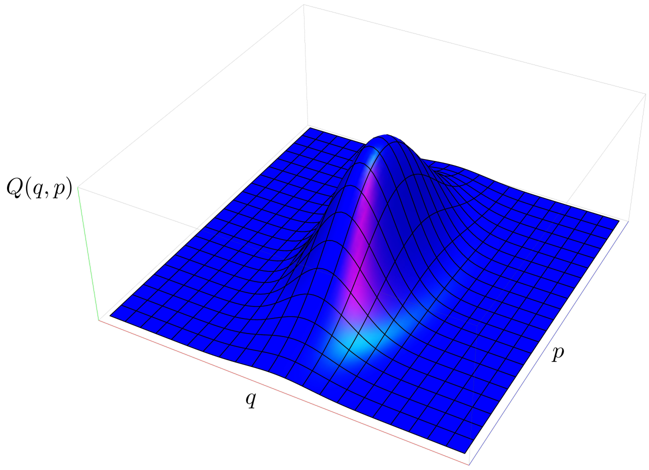

Figure 2.14: Q representation of a single-mode squeezed state with $|\xi| = 0.8$ and $\phi = 0$, i.e squeezed in the q direction.

Since in the definition of the Q representation (2.122) $|\alpha\rangle$ represents a coherent state, it becomes apparent that the Q representation is the outcome of the measurement of position and momentum at the same time, i.e. it corresponds to the so-called heterodyne measurement [Leo10].

2.2.3 Normal ordered characteristic function and the Glauber P representation

The last possible characteristic function is the normal-ordered one,

$$\chi_N(\beta) = \mathrm{tr}\left(\hat{\rho}\exp(\beta\hat{a}^\dagger)\exp\left(-\beta^*\hat{a}\right)\right). \tag{2.145}$$

The Fourier transform of the normal-ordered characteristic function defines the Glauber P representation,

$$P(\alpha,\alpha^*) = \int d^2\beta\exp\left(-\beta\alpha^* + \beta^*\alpha\right)\chi_N(\beta). \tag{2.146}$$

The connection between the density matrix $\hat{\rho}$ and the P representation is,

$$\hat{\rho} = \int d^2\alpha P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|. \tag{2.147}$$

The P representation is normalized,

$$\int d^2\alpha P(\alpha,\alpha^*) = 1. \tag{2.148}$$

The Wigner and the Q representations can be found without the use of the characteristic functions. In contrast, it is more handy to find the P representation by first finding the normal-ordered characteristic function. As an example we will consider the thermal state of the expression (2.79). In the coherent states' basis the thermal state can be written,

$$\begin{aligned}\hat{\rho}_{th} &= \frac{1-|\tau|^2}{\pi^2}\sum_n |\tau|^{2n}\int d^2\alpha \int d^2\alpha' |\alpha\rangle\langle\alpha|n\rangle\langle n|\alpha'\rangle\langle\alpha'| \\ &= \frac{1-|\tau|^2}{\pi^2}\int d^2\alpha \int d^2\alpha' \exp\left(-\frac{|\alpha|^2}{2}-\frac{|\alpha'|^2}{2}+|\tau|^2\alpha^*\alpha'\right)\times \\ &\quad \times|\alpha\rangle\langle\alpha'|. \end{aligned} \tag{2.149}$$

From equations (2.145) and (2.149) it is straightforward to find the normal-ordered characteristic function for the thermal state ,

$$\chi_N(\beta) = \exp\left(-\frac{|\tau|^2}{1-|\tau|^2}|\beta|^2\right). \tag{2.150}$$

The P representation is found from equations (2.146) and (2.150),

$$P_{th}(\alpha,\alpha^*) = \frac{1}{\pi}\frac{1-|\tau|^2}{|\tau|^2}\exp\left(-|\alpha|^2\frac{1-|\tau|^2}{|\tau|^2}\right) \tag{2.151}$$

where $|\tau| = \tanh|\xi|$.

To calculate the mean value of a combination of annihilation and creation operators using the P representations we should first bring this combination to normal order, meaning that all annihilation operators should be placed on the right while the creation operator on the left. If we consider for example the thermal state (2.151), the mean value $\langle\hat{a}^\dagger\hat{a}\rangle$ reads,

$$\langle\hat{a}^\dagger\hat{a}\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\Re\alpha d\Im\alpha P_{th}(\alpha,\alpha^*)\alpha\alpha^* = \sinh^2|\xi|. \tag{2.152}$$

2.3 Symplectic form and transformations of Gaussian states

In this thesis we mainly use the Q representation, therefore in this section we will focus on how the Q representation is transformed. We will discuss the transformation of the Wigner function as well though, since it is important in many applications in quantum optics.

Let a density matrix $\hat{\rho}$ have a Wigner representation $W_{\hat{\rho}}(q,p)$. If $\hat{\rho}$ is transformed according to the Krauss decomposition,

$$\rho' = \sum_k \hat{F}_k \rho \hat{F}_k^\dagger \tag{2.153}$$

then from definition (2.116) it is straightforward to see that in order to find $W_{\hat{\rho}'}(q',p')$ one has to transform (q,p) with the inverse transformation prescribed from F. The same holds for Q. For example consider the displacement transformation $\hat{\rho} \to \hat{D}(\alpha)\hat{\rho}\hat{D}(\alpha)^\dagger$,

$$\begin{aligned} W(q,p) &= \frac{1}{4\pi}\int dx e^{\frac{i}{2}px}\left\langle q - \frac{x}{2}\middle|\hat{D}(\alpha)\hat{\rho}\hat{D}(\alpha)^\dagger\middle|q+\frac{x}{2}\right\rangle \\ &= W(q-2\Re(\alpha), p-2\Im(\alpha)). \end{aligned} \tag{2.154}$$

Let us now focus on Gaussian states. In order to proceed we will first consider a system of n modes that each mode has quadrature operators $\hat{q}_i$ and $\hat{p}_i$. The total Hilbert space of the n-mode system is given by,

$$\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i. \tag{2.155}$$

We can group the $2n$ quadratures in two -or more ways, for example,

$$\hat{\bar{r}} = (\hat{q}_1, \dots, \hat{q}_n, \hat{p}_1, \dots, \hat{p}_n)^T, \tag{2.156}$$

$$\hat{\bar{R}} = (\hat{q}_1, \hat{p}_1, \dots, \hat{q}_n, \hat{p}_n)^T. \tag{2.157}$$

Note that the commutation relation now reads,

$$\left[\hat{\bar{r}}_k, \hat{\bar{r}}_{k+n}\right] = i\mathbf{\Omega}_{k,k+n} \tag{2.158}$$

or

$$\left[\hat{\bar{R}}_k, \hat{\bar{R}}_j\right] = i\tilde{\mathbf{\Omega}}_{k,j} \tag{2.159}$$

where

$$\mathbf{\Omega} = \begin{pmatrix} \mathbb{O}_{n\times n} & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & \mathbb{O}_{n\times n} \end{pmatrix} \tag{2.160}$$

and

$$\tilde{\mathbf{\Omega}} = \bigoplus_{i=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{2.161}$$

A multivariate Gaussian Wigner representation reads,

$$W(\bar{r}) = \frac{1}{\pi^n \sqrt{\det \boldsymbol{\gamma}}} \exp\left(-(\bar{r} - \bar{d})^T \boldsymbol{\gamma}^{-1} (\bar{r} - \bar{d})\right) \tag{2.162}$$

where $\bar{d}$ are the $2n$ first moments,

$$\bar{d} = \langle \hat{\bar{r}} \rangle = \text{tr}(\hat{\rho}\hat{\bar{r}}) \tag{2.163}$$

while the $2n \times 2n$ matrix $\boldsymbol{\gamma}$ is the covariance matrix,

$$\boldsymbol{\gamma}_{kj} = \langle \{\hat{\bar{r}}_k - \bar{d}_k, \hat{\bar{r}}_j - \bar{d}_j\} \rangle = \text{tr}\left(\hat{\rho}\{\hat{\bar{r}}_k - \bar{d}_k, \hat{\bar{r}}_j - \bar{d}_j\}\right) . \tag{2.164}$$

By $\{\hat{A}, \hat{B}\}$ we denote the anti-commutator, $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$.

In order to correspond to a physical state, the covariance matrix should obey the inequality,

$$\boldsymbol{\gamma} + i\mathbf{\Omega} \geq 0 \tag{2.165}$$

that originates from Heisenberg's uncertainty principle. Note that the $\geq$ sign in inequality (2.165) should be interpreted as a positive-semidefiniteness.

Also, a multivariate Gaussian Q representation has the form,

$$Q(\bar{r}) = \frac{1}{\pi^n \sqrt{\det \mathbf{\Gamma}^{-1}}} \exp\left(-(\bar{r} - \bar{d})^T \mathbf{\Gamma} (\bar{r} - \bar{d})\right) \tag{2.166}$$

where $\mathbf{\Gamma} = (\boldsymbol{\gamma} + 2\mathbb{I})^{-1}$.

If the n-mode system is in vacuum state, the its covariance matrix reads,

$$\boldsymbol{\gamma}_{\text{vac}} = 2 \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{pmatrix} . \tag{2.167}$$

As it is apparent by now, from the vacuum state we can construct any other Gaussian state by the use of displacement and squeezing operators. The covariance matrix of a coherent state is again $\boldsymbol{\gamma}_{\text{vac}}$ since when the vacuum state is displaced, in order to obtain a coherent state, only the first moments are affected.

The covariance matrix of a squeezed state with $\phi = 0$ is,

$$\boldsymbol{\gamma}_{\text{sq}} = 2\begin{pmatrix} e^{-2|\xi|} & 0 \\ 0 & e^{2|\xi|} \end{pmatrix}. \tag{2.168}$$

while if $\phi \neq 0$ the covariance matrix reads,

$$\boldsymbol{\gamma}_{\text{sq}} = 2\begin{pmatrix} e^{-2|\xi|}\cos^2\frac{\phi}{2} + e^{2|\xi|}\sin^2\frac{\phi}{2} & -\sin\frac{\phi}{2}\cos\frac{\phi}{2}(e^{2|\xi|} - e^{-2|\xi|}) \\ -\sin\frac{\phi}{2}\cos\frac{\phi}{2}(e^{2|\xi|} - e^{-2|\xi|}) & e^{-2|\xi|}\cos^2\frac{\phi}{2} + e^{2|\xi|}\sin^2\frac{\phi}{2} \end{pmatrix}.$$

As we have seen before, the state obtained as the reduced output of a two-mode squeezer when the input state is $|\alpha 0\rangle$, where $|\alpha\rangle$ is a coherent state, has density matrix given in equation (2.79). It is straightforward to find that the covariance matrix corresponding to the thermal state reads,

$$\boldsymbol{\gamma}_{\text{th}} = 2\begin{pmatrix} \cosh 2|\xi| & 0 \\ 0 & \cosh 2|\xi| \end{pmatrix}. \tag{2.169}$$

The covariance matrix for squeezed thermal vacuum is found to be,

$$\boldsymbol{\gamma}_{\text{sq,th}} = \cosh 2|\xi| \boldsymbol{\gamma}_{sq}. \tag{2.170}$$

Finally, the covariance matrix for the two-mode squeezed state reads,

$$\boldsymbol{\gamma}_{\text{tms}} = 2\begin{pmatrix} \cosh 2|\xi| & \sinh 2|\xi| & 0 & 0 \\ \sinh 2|\xi| & \cosh 2|\xi| & 0 & 0 \\ 0 & 0 & \cosh 2|\xi| & -\sinh 2|\xi| \\ 0 & 0 & -\sinh 2|\xi| & \cosh 2|\xi| \end{pmatrix}. \tag{2.171}$$

An important group of transformations $\mathbf{M}$ is the symplectic group $\mathbf{M} \in Sp(2n, \mathbb{R})$. They represent Gaussian unitary transformations on the density matrix, i.e. $\hat{U}_G \hat{\rho}_G \hat{U}_G^\dagger$. They transform the first and the second moments as,

$$\bar{d}' = \mathbf{M}\bar{d} \tag{2.172}$$

$$\boldsymbol{\gamma}' = \mathbf{M}\boldsymbol{\gamma}\mathbf{M}^T \tag{2.173}$$

and they leave the $\boldsymbol{\Omega}$ matrix invariant,

$$\mathbf{M}\boldsymbol{\Omega}\mathbf{M}^T = \boldsymbol{\Omega}, \tag{2.174}$$

or in other words the commutation relations are invariant under the symplectic transformations.

All symplectic transformations satisfy the relation $\det \mathbf{M} = 1$, moreover the passive transformations satisfy $\mathbf{M}\mathbf{M}^T = \mathbb{I}$, so in that case M is a real rotation matrix. The Williamson's theorem [SCS99] states that all Gaussian states with covariance matrix $\boldsymbol{\gamma}$, are diagonalized by symplectic matrices $\mathbf{M}$,

$$\mathbf{M}\boldsymbol{\gamma}\mathbf{M}^T = \mathrm{diag}(\nu_1, \ldots, \nu_n; \nu_1, \ldots, \nu_n) \tag{2.175}$$

where ν_i are the symplectic eigenvalues that practically can be found as eigenvalues of the matrix $\sqrt{(i\boldsymbol{\Omega\gamma})^\dagger (i\boldsymbol{\Omega\gamma})}$.

The von Neumann entropy of a Gaussian n-mode state is found to be,

$$S(\hat{\rho}_G) = \sum_{i=1}^{n} g\left(\nu_i - \frac{1}{2}\right) \tag{2.176}$$

where,

$$g(x) = \begin{cases} (x+1)\log_2(x+1) - x\log_2 x, & x > 0 \\ 0, & x = 0 \end{cases}. \tag{2.177}$$

When the case at hand involves non-Gaussian states, there is not such handy formulas for the calculation of the entropy, therefore other methods are required for the calculation of the von Neumann entropy. We will address this problem in Chapter 5.

The phase space transformations that we will meet in this thesis are the rotation $\mathbf{R}(\theta)$, the beam splitter $\mathbf{B}(T)$ and the one-mode squeezer $\mathbf{S}(\xi)$.

$$\mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{2.178}$$

$$\mathbf{B}(T) = \begin{pmatrix} \sqrt{T} & \sqrt{1-T} & 0 & 0 \\ -\sqrt{1-T} & \sqrt{T} & 0 & 0 \\ 0 & 0 & \sqrt{T} & \sqrt{1-T} \\ 0 & 0 & -\sqrt{1-T} & \sqrt{T} \end{pmatrix}, \tag{2.179}$$

$$\mathbf{S}(\xi) = \begin{pmatrix} e^{-|\xi|} & 0 \\ 0 & e^{|\xi|} \end{pmatrix} \tag{2.180}$$

where $T = \cos^2\theta$, therefore the beam splitter's phase space transformation corresponds to the application of a rotation from a mode to the other mode.

2.4 Second quantization of the bosonic field

2.4.1 Preliminaries

From classical electrodynamics one performs the second quantization and obtains quantum optics. We will now briefly discuss what happens when one proceeds with the so-called second quantization in a more general context using the Lagrangian formalism and functional analysis in order to be able to analyze interesting physics in Section 5.3. In second quantization one upgrades the eigenfunctions to operators. In that way the quantum mechanical position operator $\hat{\bar{x}}$ corresponds to the field operator $\hat{\phi}(\bar{x})$ that creates a boson in position $\bar{x}$. The quantum mechanical position eigenstate $|\bar{x}\rangle$ corresponds to the field state $|\phi\rangle$. So that it holds,

$$\hat{\phi}(\bar{x})|\phi\rangle = \phi(\bar{x})|\phi\rangle. \tag{2.181}$$

In field theory the space that the eigenvectors $\{|\phi\rangle\}$ form, is referred as Fock space. The basis $\{|\phi\rangle\}$ is complete and orthogonal,

$$\hat{\mathbb{I}} = \int \mathcal{D}\phi(\bar{x})|\phi\rangle\langle\phi| \tag{2.182}$$

$$\langle\phi_a|\phi_b\rangle = \prod_{\bar{x}} \delta(\phi_a(\bar{x}) - \phi_b(\bar{x})) \equiv \delta[\phi_a - \phi_b] \tag{2.183}$$

where the integration $\mathcal{D}\phi(\bar{x})$ means that one integrates over all possible formations of the field.

The action $S[\phi]$ is a functional of the field,

$$S[\phi] = \int_{t_i}^{t_f} dt \int d^3x \mathcal{L}[\phi] \tag{2.184}$$

where $\mathcal{L}[\phi]$ is the Lagrangian density. From the least action principle it is straightforward to find the equations of motion of the field under consideration,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{2.185}$$

where $\partial_\mu = \partial/\partial x^\mu$, with $x^0 = t$. The field variable that is conjugate to ϕ, is the momentum of the field,

$$\pi(t,\bar{x}) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi(t,\bar{x}))}. \tag{2.186}$$

The Hamiltonian of the field is given by the Legendre transform of the Lagrangian,

$$\hat{H} = \int d^3x(\pi(t,\bar{x})\partial_0\phi(t,\bar{x}) - \mathcal{L}) \tag{2.187}$$

and can be written as,

$$\hat{H} = \int d^3x \mathcal{H}(\hat{\phi},\hat{\pi}). \tag{2.188}$$

The position and momentum field operators satisfy the equal-time commutation relations,

$$[\hat{\phi}(t,\bar{x}), \hat{\phi}(t,\bar{x}')] = i\delta^3(\bar{x} - \bar{x}'). \tag{2.189}$$

For the momentum field operator it holds that,

$$\hat{\pi}(\bar{x})|\pi\rangle = \pi(\bar{x})|\pi\rangle \tag{2.190}$$

and it satisfies completeness and orthogonality relations,

$$\hat{\mathbb{I}} = \int \frac{\mathcal{D}(\pi(\bar{x}))}{2\pi}|\pi\rangle\langle\pi| \tag{2.191}$$

$$\langle\pi_a|\pi_b\rangle = \delta[\pi_a - \pi_b]. \tag{2.192}$$

The overlap of the field eigenstate with the momentum eigenstate reads,

$$\langle\phi|\pi\rangle = \exp\left(i\int d^3x\pi(\bar{x})\phi(\bar{x})\right). \tag{2.193}$$

2.4.2 Transition amplitude

An expression for calculating the transition amplitude $\langle\phi_f|e^{-i(t_f-t_i)\hat{H}}|\phi_i\rangle$, i.e. the amplitude from an initial state $|\phi_i\rangle$ at $t = t_i$ to some final state $|\phi_f\rangle$ at $t = t_f$, can be found if we begin by dividing the time interval (t_i, t_f) into N equal steps $\Delta t = (t_f - t_i)/(N+1)$

and the use of equations (2.182) and (2.183),

$$\begin{aligned}\langle\phi_f|e^{-i(t_f-t_i)\hat{H}}|\phi_i\rangle &= \lim_{N\to\infty}\int\prod_{j=0}^{N+1}d\phi_j\int\prod_{j=1}^{N}\frac{d\pi_j}{2\pi}\delta(\phi_{N+1}-\phi_j)\delta(\phi_0-\phi_i)\\ &\times\prod_{j=1}^{N}\langle\phi_j|e^{-i\Delta t\hat{H}_j}|\pi_j\rangle\prod_{j=1}^{N}\langle\pi_j|\phi_{j-1}\rangle. \end{aligned}\tag{2.194}$$

Since $\Delta t\to 0$, we expand the exponential in equation (2.194) and keep terms up to the first order,

$$\langle\phi_j|e^{-i\Delta t\hat{H}_j}|\pi_j\rangle \approx \langle\phi_j|(1-i\Delta t\hat{H}_j)|\pi_j\rangle = (1-i\Delta tH_j)\langle\phi_j|\pi_j\rangle \tag{2.195}$$

with

$$H_j = \int d^3x\mathcal{H}(\pi_j(\bar{x}),\phi_j(\bar{x})). \tag{2.196}$$

From equation (2.193) we readily take,

$$\begin{aligned}\langle\phi_f|e^{-i(t_f-t_i)\hat{H}}|\phi_i\rangle &= \lim_{N\to\infty}\int\prod_{j=0}^{N+1}d\phi_j\int\prod_{j=1}^{N}\frac{d\pi_j}{2\pi}\delta(\phi_{N+1}-\phi_j)\delta(\phi_0-\phi_i)\\ &\times\exp\left(-i\Delta t\sum_{j=1}^{N}\int d^3x\frac{\mathcal{H}(\pi_j,\phi_j)-\pi_j(\phi_j-\phi_{j-1})}{\Delta t}\right)\end{aligned}\tag{2.197}$$

and by taking the continuum limit in equation (2.197) we obtain,

$$\begin{aligned}\langle\phi_f|e^{-i(t_f-t_i)\hat{H}}|\phi_i\rangle &= \int\mathcal{D}\pi\int_{\phi(t_i,\bar{x})=\phi_i(\bar{x})}^{\phi(t_f,\bar{x})=\phi_f(\bar{x})}\mathcal{D}\phi\exp\left(i\int_{t_i}^{t_f}dt\int d^3x\right.\\ &\times\left.\left(\pi(t,\bar{x})\frac{\partial\phi(t,\bar{x})}{\partial t}-\mathcal{H}(\pi(t,\bar{x}),\phi(t,\bar{x}))\right)\right)\end{aligned}\tag{2.198}$$

where $\mathcal{D}\pi$ and $\mathcal{D}\phi$ are functional integrals and the argument of the integrations involves functions and not operators.

2.4.3 Thermal density matrix and partition function

The thermal density matrix for the canonical ensemble has the well-known form,

$$\hat{\rho} = \frac{e^{-\beta\hat{H}}}{Z} \tag{2.199}$$

where $\hat{H}$ is the Hamiltonian, Z is the partition function and $\beta = 1/kT$, with T the temperature and k the Boltzmann's constant. For a quantum field system we have,

$$\rho[\phi_f, \phi_i] = \frac{1}{Z}\langle\phi_f|e^{-\beta\hat{H}}|\phi_i\rangle. \tag{2.200}$$

The numerator in equation (2.200) looks like the transition amplitude (2.198) but with imaginary time $\tau = it$. That means that we can see $\langle\phi_f|e^{-\beta\hat{H}}|\phi_i\rangle$ as the transition amplitude from the field $|\phi_i\rangle$ at time $\tau = 0$ to the field $|\phi_f$ at time $\tau = \beta$. We can write,

$$\begin{aligned}\langle\phi_f|e^{-\beta\hat{H}}|\phi_i\rangle &= \int\mathcal{D}\pi \int\limits_{\phi(0,\bar{x})=\phi_i(\bar{x})}^{\phi(\beta,\bar{x})=\phi_f(\bar{x})} \mathcal{D}\phi \exp\Bigg(\int\limits_0^\beta dt\int d^3x \\ &\times \left(i\pi(\tau,\bar{x})\frac{\partial\phi(\tau,\bar{x})}{\partial\tau} - \mathcal{H}(\pi(\tau,\bar{x}),\phi(\tau,\bar{x}))\right)\Bigg).\end{aligned} \tag{2.201}$$

The partition function is defined,

$$Z(\beta) = \mathrm{tr}e^{-\beta\hat{H}} = \sum_\alpha\int d\phi_a\langle\phi_\alpha|e^{-\beta\hat{H}}|\phi_\alpha\rangle \tag{2.202}$$

where the summation is over all possible states. In the imaginary time formalism the partition function assumes the form,

$$\begin{aligned}Z(\beta) &= \int\mathcal{D}\pi \int\limits_{\text{periodic}} \mathcal{D}\phi \exp\Bigg(\int\limits_0^\beta dt\int d^3x \\ &\times \left(i\pi(\tau,\bar{x})\frac{\partial\phi(\tau,\bar{x})}{\partial\tau} - \mathcal{H}(\pi(\tau,\bar{x}),\phi(\tau,\bar{x}))\right)\Bigg).\end{aligned} \tag{2.203}$$

where the integration over ϕ has the constraint $\phi(0,\bar{x}) = \phi(\beta,\bar{x})$.

From equations (2.199) and (2.201) we readily obtain,

$$\begin{aligned}\rho[\phi_f,\phi_i] &= \frac{1}{Z(\beta)}\int\mathcal{D}\pi \int\limits_{\phi(0,\bar{x})=\phi_i(\bar{x})}^{\phi(\beta,\bar{x})=\phi_f(\bar{x})} \mathcal{D}\phi \exp\Bigg(\int\limits_0^\beta dt\int d^3x \\ &\times \left(i\pi(\tau,\bar{x})\frac{\partial\phi(\tau,\bar{x})}{\partial\tau} - \mathcal{H}(\pi(\tau,\bar{x}),\phi(\tau,\bar{x}))\right)\Bigg).\end{aligned} \tag{2.204}$$

The expressions (2.204) and (2.203) can be easily generalized for an arbitrary number of fields. Note that if the system under consideration admits a conserved charge, then we must make the replacement,

$$\mathcal{H}(\pi,\phi) \to \mathcal{H}(\pi,\phi) - \mu\mathcal{N}(\pi,\phi) \tag{2.205}$$

where π, ϕ is the conserved charge density.

A more elegant expression for equations (2.204) and (2.203) is,

$$\rho[\phi_f, \phi_i] \quad = \quad \frac{1}{Z(\beta)} \int \mathcal{D}\pi \int\limits_{\phi(0,\bar{x})=\phi_i(\bar{x})}^{\phi(\beta,\bar{x})=\phi_f(\bar{x})} \mathcal{D}\phi \exp\left(\int\limits_0^\beta dt \int d^3x \mathcal{L}_E[\phi] \right) \tag{2.206}$$

and

$$Z(\beta) = \int \mathcal{D}\pi \int\limits_{\text{periodic}} \mathcal{D}\phi \exp\left(\int\limits_0^\beta dt \int d^3x \mathcal{L}_E[\phi] \right) \tag{2.207}$$

where $\mathcal{L}_E[\phi]$ is the Euclidean Lagrangian Density. For example the free Euclidean Klein-Gordon Lagrangian density reads,

$$\mathcal{L}_E[\phi] = \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2. \tag{2.208}$$

Chapter 3

Noiseless amplification and attenuation of quantum light

3.1 Noiseless amplification and attenuation of Gaussian states

3.1.1 Heralded noiseless amplification and attenuation

Typically, what we mean by amplification in quantum optics is to amplify the intensity of a quantum signal, i.e. a quantum state. That means that if the mean photon number of the initial state is n, the mean photon number of the amplified state is $n_g > n$, or equivalently (if state is Gaussian) the mean field α of the initial state is smaller than the mean field α_g of the resulting state, i.e. $|\alpha_g| > |\alpha|$. Such a transformation is provided by the two-mode squeezer $\hat{S}(r)$ that we have already met. The state to be amplified, let it be a pure state $|\Psi\rangle$, is coupled with some environment state $|e\rangle$ which usually is taken to be the vacuum state $|0\rangle$ via the operator $\hat{S}(r)$. The resulting, mixed output state, let $\hat{\rho}_{|\Psi\rangle}$ will possess more noise than the initial state $|\Psi\rangle$. If for example the initial state was a coherent state, the amplified state would be a thermal state. This is deeply rooted in quantum mechanics as a consequence of the unitary evolution and it is connected to the result of tracing out one of the output modes; noise is the price we must pay in any deterministic quantum state phase-insensitive amplification process. This can be seen in the ideal (quantum-noise limited) optical amplifier, which is described by the evolution [Cav82]

$$\hat{a}_{\text{out}} = g\,\hat{a}_{\text{in}} + \sqrt{g^2-1}\,\hat{b}^{\dagger}_{\text{vac}} \tag{3.1}$$

where $\hat{a}_{\text{in}}$ and $\hat{a}_{\text{out}}$ denote the input and ouput bosonic mode operators, $\hat{b}_{\text{vac}}$ is the bosonic operator associated with an ancilla mode initially in the vacuum state, and $g > 1$ is the

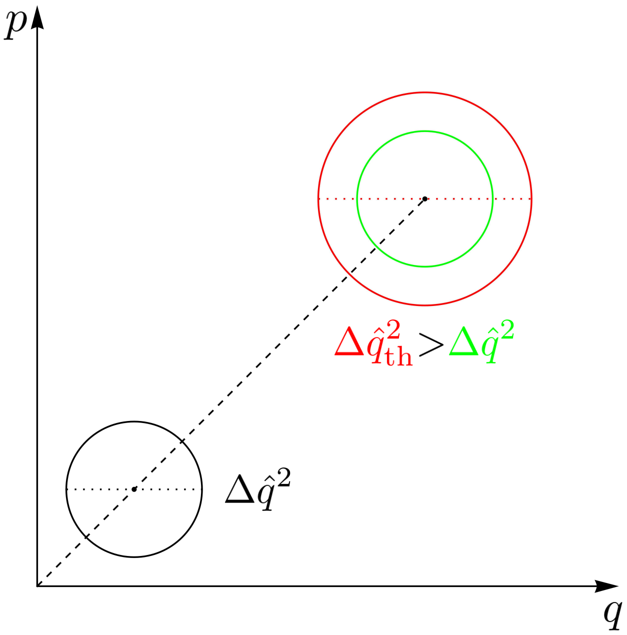

Figure 3.1: If an initial coherent state (black circle) is amplified deterministically it evolves to a thermal state (red circle), if the same initial coherent state is probabilistically amplified it remains a coherent state (green circle).

amplitude gain. The term in $\hat{b}^{\dagger}_{\text{vac}}$ necessarily adds some noise, which originates from the vacuum fluctuations of mode $\hat{b}_{\text{vac}}$ and can be associated with spontaneous emission. Remarkably, if one drops the constraint that the amplifier is deterministic, it becomes possible to define a *noiseless* amplification process, which probabilistically amplifies any coherent state $|\alpha\rangle$ with no added noise, that is

$$|\alpha\rangle \rightarrow |g\alpha\rangle \tag{3.2}$$

In other words, one can trade a noisy trace-preserving process for a noiseless but trace-decreasing one. Such a scheme was proposed by Ralph and Lund [RL09], based on an optical quantum scissor setup. It is called an *heralded* noiseless linear amplifier (HNLA), in the sense that the success of the noiseless amplification can be heralded by some detection event (we know when the noiseless amplification has succeeded). Strictly speaking, the HNLA operator is unbounded, so it is actually impossible to implement a *perfect* noiseless amplifier, albeit with zero success probability. However, the perfect HNLA can be approximated as closely as we wish by truncating the input Fock space to an increasingly large photon number N. More precisely, the action of an approximate HNLA on a Fock state $|n\rangle$ in the truncated space $\{|0\rangle, |1\rangle, \ldots, |N\rangle\}$ can be mathematically described by some filtration operator $\hat{F}$, which works as

$$\hat{F}|n\rangle = \eta^{N/2} g^n |n\rangle \tag{3.3}$$

where N and η are two parameters defining the optical HNLA setup of Ref. [RL09]. Note that $0 < \eta < 1/2$ implies that $g = \sqrt{(1-\eta)/\eta} > 1$. In the limit $N \to \infty$, applying the filtration operator $\hat{F}$ on a coherent state $|\alpha\rangle$ gives

$$\hat{F}|\alpha\rangle \simeq \eta^{N/2} e^{(g^2-1)|\alpha|^2/2} |g\alpha\rangle \tag{3.4}$$

which is proportional to the desired noiselessly amplified coherent state $|g\alpha\rangle$. One can see that $\hat{F}$ tends to a perfect HNLA as $N \to \infty$, while its success probability

$$P_{\mathrm{succ}|\alpha} \simeq \eta^N e^{(g^2-1)|\alpha|^2} \tag{3.5}$$

tends to zero. Note that the success probability is state-dependent and diverges for large input amplitudes (large α). This is precisely related to the fact that $\hat{F} \propto g^{\hat{n}}$ is an unbounded operator in the infinite-dimensional Fock space (this is why we need to have the prefactor η^N, which vanishes for a perfect HNLA as $N \to \infty$). As expected, only an approximate HNLA with non-zero success probability can be realized physically if we keep N finite. Various possible implementations of the HNLA have been found and experimentally demonstrated [XRL+10, FBB+10, UMW+10, ZFB11, OBS+12], but they all share this property that the higher is the fidelity between the actual output state and target state, the lower is the success probability.

The HNLA may serve as a tool for quantum entanglement distillation or for breeding Schrödinger cat states ($|\alpha\rangle + |-\alpha\rangle$) [RL09]. More recently, it has also been shown useful to carry out continuous-variable quantum error correction on a lossy line [Ral11], or, in conjunction with noiseless attenuation, as a tool to convert a lossy line into a lossless line [MSM+12]. In this Chapter, we will investigate its ability to serve as an heralded phase-insensitive single-mode squeezer. We will mainly be interested in the perfect HNLA, so we will disregard the above truncation effect and simply use $\hat{F} \propto g^{\hat{n}}$ as a filtration operator, remembering that the proportionality constant is related to the normalization of the actual output state and would vanish in the limit of a perfect HNLA.

The same formalism applies when we deal with noiseless attenuation, only of course in that case $g < 1$. The difference is found in the realization setup. In the case of noiseless amplification the probabilistic heralded amplification may be realized with a scheme based on the unbalanced quantum scissors, while as we have already seen in Section 2.1.4, in the case of noiseless attenuation the realization is simply a beam splitter with post-selection on vacuum state in one of the outputs.

3.1.2 General properties of the noiseless amplifier and attenuator

The noiseless amplifier probabilistically enhances the amplitude of a coherent state as

$$|\alpha\rangle \rightarrow |g\alpha\rangle \tag{3.6}$$

where $g > 1$ is the amplitude gain. It can be described by a quantum filter F (a trace-decreasing CP map with a single Kraus operator F) such that

$$\hat{\rho} \rightarrow \hat{F}\hat{\rho}\hat{F}^{\dagger} \tag{3.7}$$

where the filter $F = cg^{\hat{n}}$ is diagonal in the Fock state basis $|n\rangle$ and c is a real constant. The trace non-increasing condition $\hat{F}^{\dagger}\hat{F} \leq \mathbb{I}$ implies that $|c|^2 g^{2n} \leq 1$, $\forall n$, which is possible only if $c = 0$; hence, the success probability $\mathrm{Tr}(\hat{F}\hat{\rho}\hat{F}^{\dagger})$ of this *ideal* noiseless amplifier vanishes. Mathematically, this is because the operator $g^{\hat{n}}$ is unbounded for $g > 1$. However, a *non-ideal* version of the noiseless amplifier can be defined by truncating the Fock state basis at $|N\rangle$. Then, the trace non-increasing condition is fulfilled provided $|c|^2 g^{2N} = 1$; hence, the success probability scales as g^{-2N} and can be made strictly larger than zero as long as N is finite. In other words, a noiseless amplifier can be implemented with non-zero success probability only within a finite-dimensional subspace of the Fock space. We will ignore this subtlety in the rest of this Chapter, and consider the ideal noiseless amplifier that is simply associated with the quantum filter $g^{\hat{n}}$.

Noiseless attenuation corresponds to the same quantum filter, but taking $g = \nu < 1$. In contrast with noiseless amplification, it corresponds to a bounded operator $\nu^{\hat{n}}$ for $\nu < 1$, so it can be implemented exactly with a success probability that is strictly larger than zero. Indeed, the quantum filter $\nu^{\hat{n}}$ can be realized, for instance, by processing the input state through a beam splitter of amplitude reflectance ν whose auxiliary input port is prepared in the vacuum state $|0\rangle$, and then conditioning on projecting the state of the auxiliary output port onto the vacuum state $|0\rangle$, as shown in Fig. 3.2 [MSM+12] .

It is easy to see that for an input state $|\psi\rangle = \sum_n c_n|n\rangle$, with $\sum_n |c_n|^2 = 1$, the final state will read

$$|\tilde{\psi}\rangle \propto \sum_n \nu^n\, c_n|n\rangle. \tag{3.8}$$

Intuitively, we understand that the heralded filtering operation preferentially keeps low-n Fock states since ν^n exponentially decays with n, so in this sense the state is attenuated. Conversely, if we formally consider amplitude reflectance larger than 1, we will get an output state which can be interpreted as a noiselessly amplified state, where large-n Fock

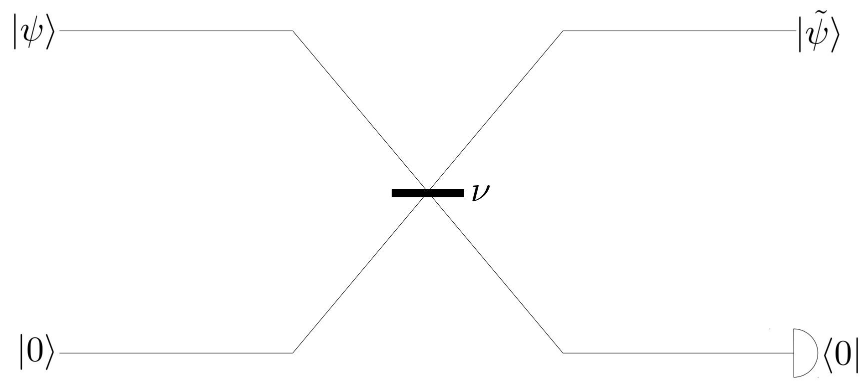

Figure 3.2: Noiseless attenuator. In a beam splitter with amplitude reflectance ν the lower input mode is set to vacuum and we post-select on vacuum in the lower output mode.

states are preferentially post-selected. This formal equivalence allows us to analyze the effect of both conditional operations simultaneously.

Let us clarify the intuition behind saying that g^n amplifies the state, or ν^n attenuates the state. It so happens that this intuition holds true as far as the mean photon number $\langle\hat{n}\rangle$ is concerned, but may be contradicted if we probe the mean field amplitude $\langle\hat{a}\rangle$ of certain non-Gaussian states as we will discuss later. As a first step, we will prove here that $\langle\hat{n}\rangle$ is necessarily increased (decreased) under the action of noiseless amplifier g^n (attenuator ν^n). For simplicity, we consider single-mode states, but the argument can be extended to multimode states. An arbitrary input state $\hat{\rho}$ can be expressed in Fock basis as

$$\hat{\rho} = \sum_{n,m=0}^{\infty} \rho_{mn}|n\rangle\langle m|. \tag{3.9}$$

where $\rho \geq 0$ and $\sum_{n=0}^{\infty} \rho_{nn} = 1$. The amplified (attenuated) state is

$$\tilde{\hat{\rho}} = \frac{\sum_{n,m=0}^{\infty} g^{n+m} \rho_{mn}|n\rangle\langle m|}{\sum_{n=0}^{\infty} g^{2n} \rho_{nn}} \tag{3.10}$$

and its mean photon number is given by

$$\langle\tilde{n}\rangle = \frac{\sum_{n=0}^{\infty} n g^{2n} \rho_{nn}}{\sum_{n=0}^{\infty} g^{2n} \rho_{nn}} \tag{3.11}$$

We shall assume that this state is physical, i.e., the sum in the denominator exists and has a finite value (this is not necessarily true for some input states and $g > 1$). The derivative of equation (3.11) with respect to g is

$$\frac{d\langle\tilde{n}\rangle}{dg} = \frac{1}{N^2}\sum_{m,n=0}^{\infty} n(n-m)e_{nm}, \tag{3.12}$$

where

$$N = \sum_{n=0}^{\infty} g^{2n}\rho_{nn} \tag{3.13}$$

and

$$e_{nm} = e_{mn} = 2g^{2(n+m)-1}\rho_{nn}\rho_{mm} \geq 0. \tag{3.14}$$

Equation (3.12) can be rewritten as

$$\frac{d\langle\tilde{n}\rangle}{dg} = \frac{1}{N^2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}(n-m)^2 e_{nm} \tag{3.15}$$

from which we conclude that

$$\frac{d\langle\tilde{n}\rangle}{dg} \geq 0. \tag{3.16}$$

Thus, the mean photon number of any physical state increases monotonically with g. Equation (3.16) is valid for all $g > 0$, no matter if one considers attenuation ($g < 1$) or amplification ($g > 1$), for $g = 1$ the mean photon number remains constant.

If the input state is a Fock state, which is an eigenstate of the operator $g^{\hat{n}}$, then the mean photon number remains constant under noiseless amplification or attenuation. Let us now discuss how noiseless amplification (attenuation) transforms Gaussian states, paying a particular attention to the properties of the mean field. Let us begin by recalling that the operator $g^{\hat{n}}$ transforms a coherent state $|\alpha\rangle$ as

$$g^{\hat{n}}|\alpha\rangle = e^{(g^2-1)|\alpha|^2/2}|g\alpha\rangle. \tag{3.17}$$

where α is a complex number. This transformation suggests to decompose any input state into the overcomplete basis of coherent states. Then, one has to evolve every component coherent state according to the transformation (3.17). A natural idea may be to use the Glauber P representation of the input state, that is

$$\hat{\rho} = \int d^2\alpha\, P(\alpha)|\alpha\rangle\langle\alpha|. \tag{3.18}$$

If we apply the transformation of equation (3.17) to both sides of equation (3.18), we obtain the P representation of the transformed state $\tilde{\hat{\rho}} \propto g^{\hat{n}}\hat{\rho}g^{\hat{n}}$, namely [Bla13]

$$\tilde{P}(\alpha) \propto e^{(1-1/g^2)|\alpha|^2} P\Big(\frac{\alpha}{g}\Big). \tag{3.19}$$

where the symbol $\propto$ indicates that $\tilde{P}$ needs to be normalized. In the case at hand, we find it more elegant to consider instead the Husimi Q-function. For an arbitrary quantum state $\hat{\rho}$, the Q-function is defined as

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle, \tag{3.20}$$

For a Gaussian state with covariance matrix γ and vector of mean values of quadrature operators $d = (\langle\hat{x}\rangle, \langle\hat{p}\rangle)^T$, it can be expressed as

$$Q(\alpha) = \frac{1}{\pi\sqrt{\det\Gamma^{-1}}}\exp\left(-(\bar{r}-\bar{d})^T\Gamma(\bar{r}-\bar{d})\right). \tag{3.21}$$

Here $\Gamma = (\gamma + 2I)^{-1}$, I denotes the identity matrix, and $r = (\alpha_R, \alpha_I)^T$, where α_R and α_I denote the real and imaginary parts of α. We use the convention where $\hbar = 2$.

We recall that $Q(\alpha)$ can be viewed as the probability density for the complex outcome α of a heterodyne measurement performed on state $\hat{\rho}$, which consists in projecting onto the coherent-state basis. It is then possible to back-propagate each coherent state through the noiseless amplifier or attenuator, as done in Ref. [FC12]. The Q-function $\tilde{Q}(\alpha)$ of the transformed state $\tilde{\hat{\rho}} \propto g^{\hat{n}}\hat{\rho}g^{\hat{n}}$ can then be written as

$$\tilde{Q}(\alpha) \propto \frac{1}{\pi}\langle\alpha|g^{\hat{n}}\hat{\rho}g^{\hat{n}}|\alpha\rangle = e^{(g^2-1)|\alpha|^2}Q(g\alpha), \tag{3.22}$$

where we have used equation (3.17) and the symbol $\propto$ indicates that $\tilde{Q}$ needs to be normalized. Note that if $Q(\alpha)$ is expressed as in equation (3.21), its Gaussian form is preserved by transformation (3.22), so that Gaussian input states are mapped onto Gaussian output states. In particular, the corresponding transformations on γ and d can be determined by looking at the exponent in $\tilde{Q}(\alpha)$. After some algebra, we find

$$\tilde{\Gamma} = g^2\Gamma - \frac{g^2-1}{4}I \tag{3.23}$$

which implies that the covariance matrix transforms as

$$\tilde{\gamma} = \left[g^2(\gamma+2I)^{-1} - \frac{g^2-1}{4}I\right]^{-1} - 2I. \tag{3.24}$$

Similarly, the vector of mean values transforms as

$$\tilde{d} = 2g\left[(g^2+1)I - \frac{g^2-1}{2}\gamma)\right]^{-1} d. \tag{3.25}$$

Here, tilde denotes the parameters of the Gaussian state after noiseless amplification or attenuation. Note that equation (3.25) agrees with a formula for the displacement of a noiselessly amplified Gaussian state derived in Ref. [WLR13].

Since the operator $g^{\hat{n}}$ commutes with a unitary phase shift $e^{i\phi\hat{n}}$, we can without loss of generality assume that the covariance matrix is diagonal,

$$\gamma = \begin{pmatrix} 2V_x & 0 \\ 0 & 2V_p \end{pmatrix}, \tag{3.26}$$

where V_x and V_p denote the variances of amplitude and phase quadratures, respectively, which obey the Heisenberg uncertainty relation $V_xV_p \geq 1$. If we insert the diagonal covariance matrix (3.26) into equation (3.25), we get

$$\tilde{d}_j = \frac{2g}{(1+g^2)+V_j(1-g^2)}d_j, \tag{3.27}$$

where $j = x, p$. The effective amplification gain is thus different for the amplitude and phase quadratures, and it depends on the variance V_j of the quadrature j, namely

$$G_{\text{eff},j} \equiv \frac{\tilde{d}_j}{d_j} = \frac{2g}{(1+g^2)+V_j(1-g^2)}. \tag{3.28}$$

This can be rewritten as

$$\frac{G_{\text{eff},j}-g}{G_{\text{eff},j}} = \frac{g^2-1}{2}(V_j-1) \tag{3.29}$$

so that for the noiseless amplifier ($g > 1$) we have

$$G_{\text{eff}}^{V_j<1} < g < G_{\text{eff}}^{V_j>1} \tag{3.30}$$

while for the noiseless attenuator ($g < 1$) we have

$$G_{\text{eff}}^{V_j>1} < g < G_{\text{eff}}^{V_j<1} \tag{3.31}$$

In other words, in both cases, the effective gain is sublinear for the squeezed quadrature ($V < 1$) and superlinear for the antisqueezed quadrature ($V > 1$). Of course, in between these cases, we have simply a linear effective gain $G_{\text{eff}}^{V_j=1/2} = g$. At this point, we note that the squeezed quadrature with the lowest possible variance must be considered in order to find the minimum effective gain that the noiseless amplifier may exhibit, as well as the maximum effective gain that the noiseless attenuator may exhibit.

Let us prove that noiseless amplification always increases the amplitude of Gaussian states. Since the HNLA is an unbounded operator it may generate unphysical states from certain input Gaussian states. The starting point of the proof is that in practice we will always observe physical states and a state is physical if and only if the output covariance matrix is positive definite, which is equivalent to the matrix inequalities $I/2 > g^2\Gamma + (1-g^2)I/4 > 0$. Both these inequalities are equivalent to

$$\max_j(V_j) < \frac{g^2+1}{g^2-1}. \tag{3.32}$$

and the denominator of equation (3.28) vanishes if the variance V_j reaches this upper bound, making $G_{\text{eff},j}$ diverge. Taking this constraint into account, the squeezed quadrature variance of the input state is lower bounded by

$$\min_j(V_j) > \frac{g^2-1}{g^2+1} \tag{3.33}$$

which, when plugged into equation (3.28), gives

$$G_{\text{eff},j} > \frac{1+g^2}{2g} > 1 \tag{3.34}$$

Thus, when $g > 1$, one finds that $G_{\text{eff},j} > 1$ for all input states leading to physical output states.

Similarly, by considering the case $g < 1$, one can prove that the noiseless attenuation always reduces the amplitude of Gaussian states and $G_{\text{eff},j} < 1$. In this latter case, there is no physical constraint on the admissible input states because noiseless attenuation is a physically allowed operation that can be implemented with finite success probability on any input state. Thus, the variance of the squeezed quadrature is simply lower bounded by $\min_j(V_j) > 0$. When plugging this into equation (3.28), we obtain

$$G_{\text{eff},j} < \frac{2g}{1+g^2} < 1 \tag{3.35}$$

for all input states.

3.1.3 Squeezed states: the noiseless amplifier as a universal squeezer

In the following, we use the notation

$$|\alpha,\xi\rangle = D(\alpha)S(\xi)|0\rangle \tag{3.36}$$

for denoting a coherent squeezed state, where $|0\rangle$ is the vacuum state, $D(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ is the displacement operator, and $S(\xi) = \exp((\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger 2})/2)$ is the squeezing operator with $\xi = re^{i\phi}$, where $r > 0$ is the squeezing strength and ϕ is the squeezing angle ($\phi = 0$ refers to squeezing of the x quadrature, while $\phi = \pi$ stands for squeezing of the p quadrature).

For simplicity, we first consider the action of $\hat{F}$ on a vacuum squeezed state, i.e., $\alpha = 0$. The expansion of such a state in the Fock basis reads as a superposition of even Fock states [GA90],

$$|0,\xi\rangle = \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \sqrt{\binom{2n}{n}} \left(-\frac{e^{i\phi}\tanh r}{2}\right)^n |2n\rangle \tag{3.37}$$

so that applying the $\hat{F}$ operator gives rise to an additional weight g^{2n} in each term of this superposition, which results into another vacuum squeezed state

$$\hat{F}|0,\xi\rangle \propto \sqrt{\frac{\cosh r'}{\cosh r}}\, |0,\xi'\rangle \tag{3.38}$$

The resulting squeezing parameter $\xi' = r'e^{i\phi}$ is related to the original one by the relation

$$\tanh r' = g^2 \tanh r \tag{3.39}$$

while the phase ϕ is unchanged. (Note that the condition $g^2 \tanh r < 1$ must be satisfied for this expression to make sense.) Thus, if the operation is successful, the resulting state is squeezed along the same angle ϕ as the original state, but with a stronger parameter $r' > r$ regardless of ϕ, as implied by equation (3.39). This is yet another very peculiar property of this heralded transformation, namely *phase-insensitive squeezing*. The signal-to-noise ratio is thus conserved in this transformation, just as for a (deterministic) phase-sensitive amplifier [OPK93, LAR+93, GLP98], while the transformation is actually phase-insensitive; this is obvious since $\hat{F}$ only depends on $\hat{n}$. Note that the success probability

$$P_{\mathrm{succ}|0,\xi} \propto \frac{\cosh r'}{\cosh r} \tag{3.40}$$

is independent of the squeezing angle ϕ of the original state. (It is of course always lower than one, given the meaning of the proportionality sign as explained above.)

Consider now the action of $\hat{F}$ on a coherent squeezed state, whose expansion in the Fock basis reads [GA90]

$$|\alpha,\xi\rangle = \frac{1}{\sqrt{\cosh r}} \exp\left\{-\frac{|\alpha|^2 + \alpha^{*2}e^{i\phi}\tanh r}{2}\right\}$$
$$\times \sum_{n=0}^{\infty} H_n\left(\frac{\alpha + \alpha^* e^{i\phi}\tanh r}{(2e^{i\phi}\tanh r)^{1/2}}\right) \left(\frac{e^{i\phi}\tanh r}{2}\right)^{n/2} \frac{|n\rangle}{\sqrt{n!}} \tag{3.41}$$

where $H_n(x)$ is the Hermite polynomial of order n, defined as

$$H_n(x) = n! \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m}{m!\,(n-2m)!} (2x)^{n-2m} \tag{3.42}$$

Acting with the $\hat{F}$ operator on this state gives an additional weight g^n in each term of this superposition, which can be written as

$$\begin{aligned} \hat{F}|\alpha,\xi\rangle \propto & \frac{1}{\sqrt{\cosh r}} \exp\left\{ -\frac{|\alpha|^2 + \alpha^{*2} e^{i\phi} \tanh r}{2} \right\} \\ & \times \sum_{n=0}^{\infty} H_n\left(\frac{\alpha + \alpha^* e^{i\phi} \tanh r}{(2e^{i\phi} \tanh r)^{1/2}} \right) \frac{1}{n!} \left(\left(\frac{e^{i\phi} \tanh r}{2} \right)^{1/2} g\hat{a}^\dagger \right)^n |0\rangle \end{aligned} \tag{3.43}$$

With the help of the generating function of Hermite polynomials [AS65]

$$\sum_{n=0}^{\infty} \frac{t^n H_n(x)}{n!} = e^{2xt-t^2} \tag{3.44}$$

we can simplify equation (3.43) as

$$\begin{aligned} \hat{F}|\alpha,\xi\rangle \propto & \frac{1}{\sqrt{\cosh r}} \exp\left\{ -\frac{|\alpha|^2 + \alpha^{*2} e^{i\phi} \tanh r}{2} \right\} \\ & \times \exp\left\{ (\alpha + \alpha^* e^{i\phi} \tanh r) g\hat{a}^\dagger - \frac{e^{i\phi} \tanh r}{2} g^2 \hat{a}^{\dagger 2} \right\} |0\rangle \end{aligned} \tag{3.45}$$

Now, let us see how the operator in the right-hand side of equation (3.45) acts on $|0\rangle$. Using the fact that

$$|0, re^{i\phi}\rangle = S(re^{i\phi})|0\rangle = e^{-\nu/2} e^{-\tau \hat{a}^{\dagger 2}/2} |0\rangle \tag{3.46}$$

where $\tau = e^{i\phi} \tanh r$ and $\nu = \ln(\cosh r)$, we can write

$$\exp\left\{ -\frac{e^{i\phi} \tanh r}{2} \hat{a}^{\dagger 2} \right\} |0\rangle = \sqrt{\cosh r}\, |0, re^{i\phi}\rangle \tag{3.47}$$

If we define the parameter r' according to equation (3.39), we can use equation(3.47) with r replaced by r' in order to reexpress equation(3.45) as

$$\begin{aligned} \hat{F}|\alpha,\xi\rangle \propto & \sqrt{\frac{\cosh r'}{\cosh r}} \exp\left\{ -\frac{|\alpha|^2 + \alpha^{*2} e^{i\phi} \tanh r}{2} \right\} \\ & \times \exp\left\{ (\alpha + \alpha^* e^{i\phi} \tanh r) g\hat{a}^\dagger \right\} |0, \xi'\rangle \end{aligned} \tag{3.48}$$

where we note the appearance of a vacuum squeezed state $|0, \xi'\rangle$ of parameter $\xi' = r' e^{i\phi}$. As it is linear in the bosonic mode operator, the exponential operator acting on this state effects a displacement of the state. To calculate this displacement, we start by rewriting the vacuum squeezed state as

$$|0,\xi'\rangle = \frac{1}{\sqrt{\cosh r'}} \sum_{n=0}^{\infty} \frac{H_n(0)}{\sqrt{n!}} \left(\frac{e^{i\phi}\tanh r'}{2}\right)^{n/2} |n\rangle \tag{3.49}$$

where

$$H_n(0) = \begin{cases} \frac{(-1)^{n/2}\, n!}{(n/2)!} & \text{for even } n \\ 0 & \text{for odd } n \end{cases} \tag{3.50}$$

If we define $\gamma = g(\alpha + \alpha^* e^{i\phi}\tanh r)$, we may calculate the action of the exponential $e^{\gamma\hat{a}^\dagger}$ on the state $|0,\xi'\rangle$ by using the expansion

$$e^{\gamma\hat{a}^\dagger}|n\rangle = \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} \sqrt{\frac{(n+k)!}{n!}} |n+k\rangle \tag{3.51}$$

The expression of $e^{\gamma\hat{a}^\dagger}|0,\xi'\rangle$ is thus a double summation over n and k, which we may express by relabeling the variables as

$$e^{\gamma\hat{a}^\dagger}|0,\xi'\rangle = \frac{1}{\sqrt{\cosh r'}} \sum_{n=0}^{\infty} \frac{c_n}{\sqrt{n!}} |n\rangle \tag{3.52}$$

with

$$c_n = \sum_{k=0}^{n} \binom{n}{k} \gamma^{n-k}\, H_k(0) \left(\frac{e^{i\phi}\tanh r'}{2}\right)^{k/2} \tag{3.53}$$

Using equation (3.50) for $H_k(0)$, which only contributes to the sum for even k, we can rewrite this as

$$c_n = n! \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m}{m!\,(n-2m)!} \gamma^{n-2m} \left(\frac{e^{i\phi}\tanh r'}{2}\right)^{m} \tag{3.54}$$

Using the explicit expression for the Hermite polynomial, equation (3.42), we get

$$c_n = H_n\left(\frac{\gamma}{(2e^{i\phi}\tanh r')^{1/2}}\right) \left(\frac{e^{i\phi}\tanh r'}{2}\right)^{n/2} \tag{3.55}$$

Replacing γ by its definition and inserting Eqs. (3.52) and (3.55) into equation (3.48), we obtain

$$\begin{aligned} \hat{F}|\alpha,\xi\rangle \propto & \frac{1}{\sqrt{\cosh r}} \exp\left\{-\frac{|\alpha|^2 + \alpha^{*2} e^{i\phi}\tanh r}{2}\right\} \\ & \times \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} H_n\left(\frac{g(\alpha + \alpha^* e^{i\phi}\tanh r)}{(2e^{i\phi}\tanh r')^{1/2}}\right) \\ & \times \left(\frac{e^{i\phi}\tanh r'}{2}\right)^{n/2} |n\rangle \end{aligned} \tag{3.56}$$

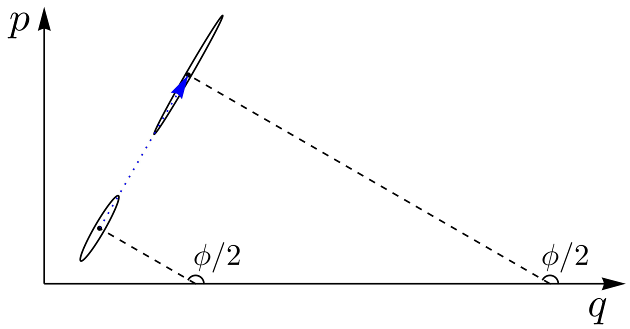

Figure 3.3: When noiseless amplification is applied to an initially squeezed state, the squeezing is enhanced while the direction remains unchanged as equation (3.39) implies. Also, the initial state is displaced according to formula (3.57).

Noting the similarity with equation (3.41), this can be reexpressed as another coherent squeezed state $|\alpha', \xi'\rangle$. By defining the new displacement amplitude α' such that

$$\alpha' + \alpha'^* e^{i\phi} \tanh r' = g(\alpha + \alpha^* e^{i\phi} \tanh r) \tag{3.57}$$

we obtain finally

$$\hat{F}|\alpha, \xi\rangle \propto \sqrt{\frac{\cosh r'}{\cosh r}} \exp\left\{\frac{\alpha'^*(\alpha' + \alpha'^* e^{i\phi} \tanh r')}{2}\right\} \times \exp\left\{-\frac{\alpha^*(\alpha + \alpha^* e^{i\phi} \tanh r)}{2}\right\} |\alpha', \xi'\rangle \tag{3.58}$$

In summary, we see that the coherent squeezed state $|\alpha, \xi\rangle$ has been transformed by $\hat{F}$ into another coherent squeezed state $|\alpha', \xi'\rangle$, where the transformation of the squeezing amplitude $\xi = re^{i\phi} \to \xi' = r'e^{i\phi}$ is governed by equation (3.39), while the transformation of the coherent amplitude $\alpha \to \alpha'$ is defined via equation (3.57). The latter can be rewritten as a transformation between the quadrature components before amplification $(\langle x\rangle, \langle p\rangle)$ and those after amplification $(\langle \tilde{x}\rangle, \langle \tilde{p}\rangle)$. Such a transformation is generally complicated, but if we consider an x-squeezed state ($\phi = 0$), it takes the simple form

$$\langle \tilde{x}\rangle = \frac{1 + \tanh r}{1 + \tanh r'} g\langle x\rangle \qquad \langle \tilde{p}\rangle = \frac{1 - \tanh r}{1 - \tanh r'} g\langle p\rangle \tag{3.59}$$

If the initial squeezing vanishes ($r = 0$), we recover the transformations which characterize the action of the HNLA on a coherent state, $\langle \tilde{x}\rangle = g\langle x\rangle$ and $\langle \tilde{p}\rangle = g\langle p\rangle$. However, for an x-squeezed state, we see that the prefactor of gx is smaller than one while the prefactor of gp is larger than one (remember $r' > r$). Of course, a similar behavior prevails for squeezed states along any quadrature since the transformation is phase insensitive. Thus, we conclude that the amplification gain becomes sublinear in g for the squeezed quadrature and superlinear in g for the antisqueezed quadrature.

Finally, the success probability can be expressed as

$$P_{\mathrm{succ}|\alpha,\xi} \propto \frac{\cosh r'}{\cosh r} \exp\Big\{\mathrm{Re}\Big[\alpha'^*(\alpha' + \alpha'^* e^{i\phi}\tanh r')\Big]\Big\} \\ \times \exp\Big\{-\mathrm{Re}\Big[\alpha^*(\alpha + \alpha^* e^{i\phi}\tanh r)\Big]\Big\}. \tag{3.60}$$

We observe that it is state-dependent, just as for coherent states ($r = 0$), in which case we get

$$P_{\mathrm{succ}|\alpha,0} \propto \exp\{|\alpha'|^2 - |\alpha|^2\} \tag{3.61}$$

in agreement with equation (3.5).

We can derive the same results by considering the phase space formalism. Namely, we can easily check that Eqs. (3.24) and (3.25) are consistent with the formulas obtained in Section 3.1.2 for the noiseless amplification of a squeezed state of light, whose covariance matrix is given by

$$\gamma = \begin{pmatrix} 2\mathrm{e}^{-2r} & 0 \\ 0 & 2\mathrm{e}^{2r} \end{pmatrix} \tag{3.62}$$

with $|\xi| = s$ being the squeezing parameter. We have

$$\Gamma = (\gamma + 2I)^{-1} = \begin{pmatrix} \frac{1+\tanh r}{4} & 0 \\ 0 & \frac{1-\tanh r}{4} \end{pmatrix} \tag{3.63}$$

implying

$$\tilde{\Gamma} = \begin{pmatrix} \frac{1+g^2\tanh r}{4} & 0 \\ 0 & \frac{1-g^2\tanh r}{4} \end{pmatrix} \tag{3.64}$$

so we conclude that the covariance matrix of the amplified state is that of another squeezed state

$$\tilde{\gamma} = \begin{pmatrix} 2\mathrm{e}^{-2r'} & 0 \\ 0 & 2\mathrm{e}^{2r'} \end{pmatrix} \tag{3.65}$$

with stronger squeezing (the output squeezing parameter s' satisfies $\tanh s' = g^2 \tanh s$). The transformation of the vector of mean values, equation (3.25), gives

$$\begin{pmatrix} \langle\tilde{x}\rangle \\ \langle\tilde{p}\rangle \end{pmatrix} = g \begin{pmatrix} \frac{1+\tanh r}{1+\tanh r'} & 0 \\ 0 & \frac{1-\tanh r}{1-\tanh r'} \end{pmatrix} \begin{pmatrix} \langle x\rangle \\ \langle p\rangle \end{pmatrix} \tag{3.66}$$

in perfect agreement with equations (3.59) ([GKC12]).

As we have already seen $\hat{F}$ is an unbounded operator which can only be approximately implemented by truncating the Fock space at a photon number N in order to get a

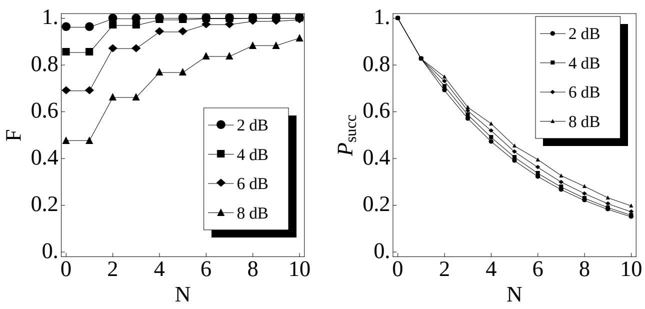

Figure 3.4: Fidelity F of the approximate phase-insensitive squeezer (with respect to the ideal one) as a function of the truncation size N for $g = 1.1$. We consider a vacuum squeezed state at the input with several values of the squeezing r (from 2 dB to 8 dB). The success probability P_{succ} is also shown. For $N = 2$ and a squeezing of 4 dB, we have $F = 0.9694$ and $P_{\mathrm{suc}} = 0.7099$

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non-zero success probability. We have investigated this truncation effect for a vacuum squeezed state of various squeezing parameters r. In figure 3.4, we exhibit the fidelity $F = |\langle 0, \xi' | 0, \xi' \rangle_{\mathrm{tr}}|^2$ between the ideal output squeezed state $|0, \xi'\rangle$ and the (renormalized) truncated output state $|0, \xi'\rangle_{\mathrm{tr}}$ resulting from applying the truncated operator $\hat{F}_{\mathrm{tr}} = g^{\hat{n}}/g^N$ onto the truncated input squeezed state $|0, \xi\rangle_{\mathrm{tr}}$. The fidelity can be expressed as

$$F = \frac{1}{\cosh r'} \sum_{n=0}^{N} \frac{1}{n!} \left(\frac{\tanh r'}{2} \right)^n H_n(0)^2 \equiv f_N(r') \tag{3.67}$$

and it is easy to check that $\lim_{N\to\infty} f_N(r') = 1$, $\forall r'$, so that the truncated process becomes perfect in the limit of an infinite large space. The success probability reads

$$P_{\mathrm{succ}} = \frac{f_N(r')}{g^{2N} f_N(r)} \tag{3.68}$$

which tends to zero in the limit of a large N, as expected. In figure 3.4, we plot F and P_{succ} as a function of the truncation size N. We take a value of the gain $g = 1.1$ and consider several values of the squeezing r assuming $\phi = 0$ (remember $\xi = re^{\mathrm{i}\phi}$). We observe that for small values of N, the fidelity is close to one if the squeezing is not too large, while the success probability remains acceptable. For example, if N is as low as 2 photons and the squeezing r corresponds to 4 dB, we get $F = 0.9694$ with $P_{\mathrm{suc}} = 0.7099$.

3.1.4 A paradox resolved

Any local operation on an entangled quantum state whose members are spacelike separated is well known to preserve causality in the sense that acting on one member cannot lead to consequences that would be instantaneously measurable on another member.

Thus, although the nonlocality inherent to quantum theory seemingly leads to a so-called "action at a distance", it does not open a way to instantaneous signaling. This feature was dubbed by Shimony the "peaceful coexistence" between quantum mechanics and special relativity [Shi78]. As explained below, it seems that the ability to squeeze a quantum state independently of its phase could contradict this peaceful coexistence and lead to instantaneous signaling between entangled parties on an heralded basis. While we, of course, do not expect this possibility to hold, it is intriguing enough to deserve a serious analysis. In this Section we examine the mechanism behind this peaceful co-existence between the phase-insensitive squeezer and special relativity. Let us start by analyzing the action of the HNLA on one mode of an EPR pair, or more precisely a two-mode vacuum squeezed state of parameter s,

$$|\mathrm{EPR}_s\rangle = (\cosh s)^{-1} \sum_{n=0}^{\infty} (\tanh s)^n \, |n\rangle|n\rangle \tag{3.69}$$

Remember that tracing out one of the modes, results in a thermal state of mean photon number $\nu = (\sinh s)^2$,

$$\hat{\rho}_s^{\mathrm{th}} = (\cosh s)^{-2} \sum_{n=0}^{\infty} (\tanh s)^{2n} \, |n\rangle\langle n| \tag{3.70}$$

with a covariance matrix

$$\gamma_s^{\mathrm{th}} = \begin{pmatrix} 2\cosh 2s & 0 \\ 0 & 2\cosh 2s \end{pmatrix} \tag{3.71}$$

Assume that Alice and Bob share such an EPR state and that Bob applies the HNLA, as depicted in figure 3.5. As discussed in [RL09], applying the operator $\mathbb{I} \times \hat{F}$ gives an additional weight g^n in each term of the superposition (3.69). Thus, when successful, this yields a stronger entangled EPR state, namely

$$|\mathrm{EPR}_{s'}\rangle = (\cosh s')^{-1} \sum_{n=0}^{\infty} (\tanh s')^n \, |n\rangle|n\rangle \tag{3.72}$$

with squeezing parameter $s' > s$ satisfying

$$\tanh s' = g \tanh s \tag{3.73}$$

In a second time, Alice may measure the photon number in her mode (i.e., she applies a projective measurement in the $\{|n\rangle\}$ basis as shown in figure 3.5), so that she prepares a mixture of photon number states (with a geometric distribution) on Bob's side, which

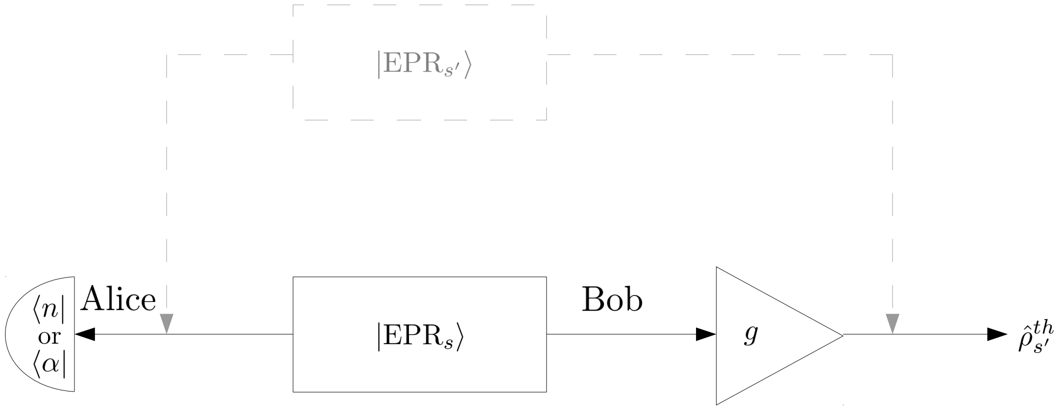

Figure 3.5: Simple illustration of the peaceful coexistence of the HNLA with special relativity. Alice and Bob share an entangled state $|\mathrm{EPR}_s\rangle$ and we compare two situations: (i) Bob amplifies his mode with the HNLA of gain g, which creates a stronger entangled state $|\mathrm{EPR}_{s'}\rangle$, and then Alice measures her mode, thereby preparing some mixture at Bob's side; (ii) Alice measures her mode first, thereby preparing some mixture at Bob's side, and Bob later amplifies each component pure state of this mixture. It is verified that these two situations yield the same average state $\hat{\rho}_{s'}^{\mathrm{th}}$ at Bob's side as a consequence of Bayes rule (both photon number or heterodyne measurement are considered).

is the thermal state

$$\hat{\rho}_{s'}^{\mathrm{th}} = (\cosh s')^{-2} \sum_{n=0}^{\infty} (\tanh s')^{2n} \, |n\rangle\langle n| \tag{3.74}$$

with a mean photon number $\nu' = (\sinh s')^2$. Of course, this preparation "at a distance" just corresponds to some possible ensemble realizing Bob's state, which may also have been obtained simply by performing a partial trace of $|\mathrm{EPR}_{s'}\rangle$ over Alice's mode.

Now, instead of assuming that Bob's amplification was done before Alice's measurement, we may also consider the opposite situation, that is, Alice first measures her part of the state $|\mathrm{EPR}_s\rangle$, which prepares a photon number state $|n\rangle$ at Bob's side with probability

$$p_n = \frac{(\tanh s)^{2n}}{(\cosh s)^2} \tag{3.75}$$

Then, applying the HLNA on $|n\rangle$ results in the same state $|n\rangle$ with a success probability $P_{\mathrm{succ}|n} \propto g^{2n}$. To get the probability of the $|n\rangle$ component in the resulting state, we need to use Bayes rule for conditional probabilities, namely,

$$p_{n|\mathrm{succ}} = \frac{P_{\mathrm{succ}|n} \, p_n}{P_{\mathrm{succ}}} \propto g^{2n} \frac{(\tanh s)^{2n}}{(\cosh s)^2} \propto \frac{(\tanh s')^{2n}}{(\cosh s')^2} \tag{3.76}$$

where P_{succ} is the average success probability. Thus, as expected, the resulting mixture of photon number states exactly coincides with the expression of Bob's state given by equation (3.74), so that measuring on Alice's side before or after amplifying on Bob's side does not make any difference. Even though the photon number states $|n\rangle$ are unaffected by the HNLA, the mean photon number of Bob's state is enhanced ($\nu \to \nu'$) by the HNLA due to the fact that higher photon-number states have a higher probability to be sucessfully transformed (into themselves), which introduces just the right bias. This is the simplest illustration of the "peaceful coexistence" that we can find.

Another simple case occurs if Alice performs an heterodyne measurement on her mode of the EPR state, that is, she performs a POVM measurement based on projectors onto coherent states $|\alpha\rangle$ (as also shown in Fig. 2). Remember that the thermal state $\hat{\rho}_s^{\text{th}}$ can also be written as a Gaussian mixture of coherent states, namely

$$\hat{\rho}_s^{\text{th}} = \frac{1}{\pi\nu} \int \mathrm{d}^2\alpha \; \mathrm{e}^{-|\alpha|^2/\nu} \; |\alpha\rangle\langle\alpha| \tag{3.77}$$

where the mean photon number ν is related to the squeezing parameter s via $\nu = (\sinh s)^2$. Starting from the entangled state $|\text{EPR}_s\rangle$, if Bob amplifies his mode before Alice's measurement, we need to consider the effect of heterodyne measurement on Alice's mode of the entangled state $|\text{EPR}_{s'}\rangle$. The resulting state that is prepared "at a distance" (on Bob's side) is obviously a Gaussian mixture of coherent states, which reads as

$$\hat{\rho}_{s'}^{\text{th}} = \frac{1}{\pi\nu'} \int \mathrm{d}^2\alpha \; \mathrm{e}^{-|\alpha|^2/\nu'} \; |\alpha\rangle\langle\alpha| \tag{3.78}$$

with a mean photon number $\nu' = (\sinh s')^2$. This is also simply the state obtained by tracing $|\text{EPR}_{s'}\rangle$ over Alice's mode.

Alternatively, if Alice performs her heterodyne measurement before Bob's amplification, she first prepares the coherent states $|\alpha\rangle$ on Bob's side with the probability distribution

$$p_\alpha = \frac{\mathrm{e}^{-|\alpha|^2/\nu}}{\pi\nu} \tag{3.79}$$

Each coherent state $|\alpha\rangle$ is transformed by the HNLA into an amplified coherent state $|g\alpha\rangle$ with probability $P_{\text{succ}|\alpha} \propto \mathrm{e}^{(g^2-1)|\alpha|^2}$, so that Bob's state conditional on the success of the HNLA can be written as

$$\begin{aligned} \hat{\rho}_{.|\text{succ}} &= \int \mathrm{d}^2\alpha \; \frac{P_{\text{succ}|\alpha}\, p_\alpha}{P_{\text{succ}}} \; |g\alpha\rangle\langle g\alpha| \\ &\propto \frac{1}{\pi\nu} \int \mathrm{d}^2\alpha \; \mathrm{e}^{(g^2-1)|\alpha|^2} \; \mathrm{e}^{-|\alpha|^2/\nu} \; |g\alpha\rangle\langle g\alpha| \\ &\propto \frac{1}{\pi\nu} \int \mathrm{d}^2\alpha \; \mathrm{e}^{g^2|\alpha|^2} \; \mathrm{e}^{-|\alpha|^2/(\tanh s)^2} \; |g\alpha\rangle\langle g\alpha| \end{aligned} \tag{3.80}$$

where we have used the identity $(\tanh s)^2 = \nu/(1+\nu)$. By making the change of variable $g\alpha \to \alpha$ in equation (3.80), we recover the thermal state of equation (3.78) with a mean

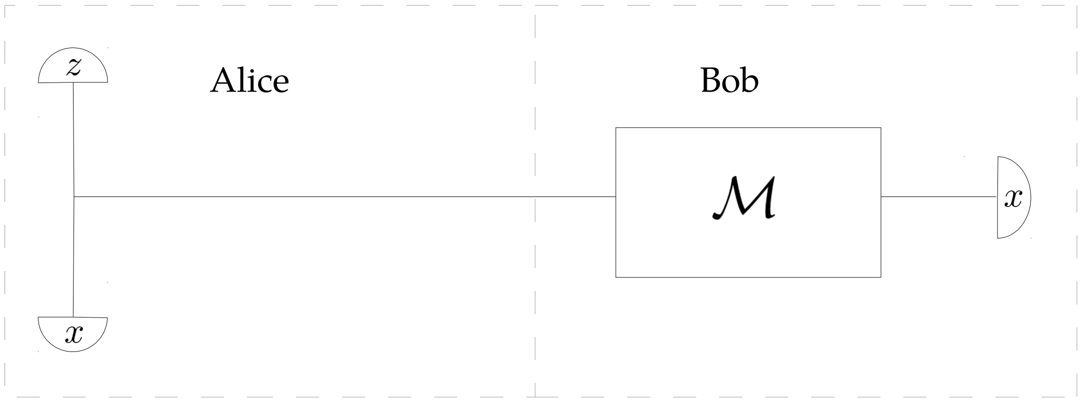

Figure 3.6: With the use of the deterministic cloning machine Bob can know which measurement Alice did. Therefore a deterministic cloner cannot exist.

photon number ν'. Thus, the mean photon number enhancement $\nu \to \nu'$ is due here to the combined effect of the amplification gain g with the bias induced by Bayes rule.

These two examples are of course not very surprising since we knew from the beginning that whatever Alice does on her part of the entangled state (measuring it or not), Bob is left with the same reduced state and the fact that he noiselessly amplifies it before or after Alice's measurement is irrelevant (this notion is even meaningless for a spacelike interval between the two events). We will consider a more subtle scenario exploiting the link between causality, cloning, and amplification, which seemingly provides a genuine way to signaling.

It has been known since a famous paper by Dieks [Die82] that the quantum no-cloning principle can be proven using a thought experiment which connects it to the impossibility of instantaneous signaling. Assuming that Alice and Bob share an entangled state, it appears that Alice could instantaneously communicate to Bob if the latter had a perfect quantum cloning transformation at his disposal. Alice would simply perform a specific measurement on her component of the state depending on the information she wishes to communicate, while Bob would acquire the information by perfectly cloning his component of the state before measuring it. Since signaling is impossible, perfect quantum cloning is precluded. Note that imperfect quantum cloning is nevertheless possible (see, e.g., [CF05] for a review) and, interestingly enough, the minimum cloning noise that is needed to comply with causality in Dieks' thought experiment exactly coincides with the noise of the best imperfect cloning transformation that is allowed by quantum mechanics [Gis98, NC03].

Let us see the Dieks' scheme in some detail. In a setup like in figure 3.6 Alice and Bob have one part of compound state, for example two entangled electrons, with total spin

0,

$$|\Psi\rangle = \frac{|+-\rangle + |-+\rangle}{\sqrt{2}} \tag{3.81}$$

where $|+\rangle$ and $|-\rangle$ are the eigenvectors of $\hat{S}_x$, with $|0\rangle = (|+\rangle + |-\rangle)/\sqrt{2}$ and $|1\rangle = (|+\rangle - |-\rangle)/\sqrt{2}$ while $|0\rangle$ and $|1\rangle$ are the eigenvectors of the $\hat{S}_z$.

Bob possesses a multiplication or cloning machine. That means that if an electron with a certain spin direction enters to the machine, in the output there will be N electrons in the exact same state as the input. Initially Alice decides to measure either the x spin direction or the z spin direction. If she measures x spin direction then cloning machine on Bob's side will produce N copies of $|+\rangle$ or $|-\rangle$ so since Bob measures always the x spin direction then he will find all the spins in the same state ($|+\rangle$ or $|-\rangle$). If Alice decides to measure the z spin direction then when Bob measures the x direction spin, half of the electrons will be found to be in the state $|0\rangle$ and the other half in the state $|1\rangle$. In that way Bob knows what measurement Alice did and therefore information could be send without respecting causality. With this reasoning one can exclude the possibility that there a cloning machine is feasible. Note that Dieks paper proceeds with more formal proof of the no-cloning theorem (see also [WZ82]) based on the linearity of quantum mechanics.

Note that the cloning machine in the setup described above is deterministic. One may think if there is the possibility to violate causality by means of probabilistic protocols. Here, we start from the possible realization of the continuous-variable Gaussian cloning transformation [CIR00] in terms of a phase-insensitive amplifier of amplitude gain $\sqrt{2}$ followed by a balanced beam splitter, whose output ports yield the two clones [CI01, BCI$^+$01]. In this realization, the imperfection of the clones originates from the added noise of the amplifier, and the optimal (imperfect) cloner precisely corresponds to the ideal quantum-limited amplifier of equation (3.1). If we replace this amplifier by the HNLA with the same gain, we obtain a probabilistic heralded perfect cloner, which yields two perfect clones when it succeeds. More precisely, the fidelity of the clones can be made arbitrary close to one at the cost of a decreasing success probability. Following Dieks' reasoning, it seems that such a probabilistic perfect cloner could enable Alice to communicate instantaneously to Bob, though on an heralded basis only. It must be stressed that this possibility for heralded instantaneous signaling is in reckless contradiction with causality. Even if the success probability is very low and the fidelity is slightly below one, Bob should not *at all* be able to acquire *any* information on Alice's bit while knowing that he has succeeded. We will show how this loophole is avoided, thanks to a subtle interplay between quantum mechanics and Bayes rule for conditional probabilities.

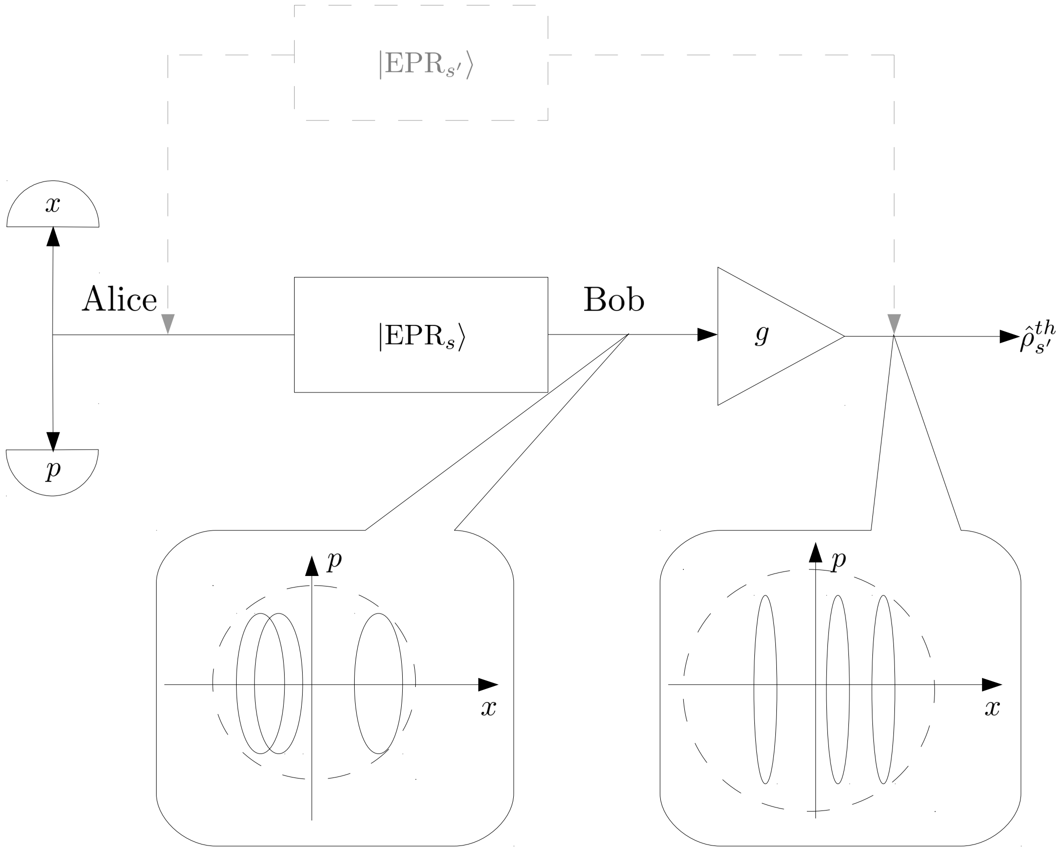

Figure 3.7: Continuous-variable analog to Dieks' scheme exploiting the heralded phase-insensitive squeezer, which seemingly opens the possibility for instantaneous signaling on an heralded basis. Alice and Bob share an entangled state $|\mathrm{EPR}_s\rangle$, and Alice measures the x or p quadrature in order to send the bit 0 or 1, respectively. This results into two possible (indistinguishable) mixtures of squeezed states at Bob's side. The perfect cloning (or amplification) of each component squeezed state of these two mixtures realized with the heralded phase-insensitive squeezer of gain g seems to provide Bob with a way to discriminate the phase-encoded information (x or p), which would lead to heralded signaling. In fact, the state dependence of this heralded transformation "conspires" with Bayes rule and forbids this to happen. Bob's mixtures coincide with the same state $\hat{\rho}^{\mathrm{th}}_{s'}$ regardless of Alice's bit.

We introduce a continuous-variable analog to Dieks' scheme (see Fig. 3). In such a scheme, Alice and Bob share an entangled state $|\mathrm{EPR}_s\rangle$, and Alice decides to perform a homodyne measurement of the x or p quadrature on her mode. The choice between x and p corresponds to the bit of information she wishes to instantaneously communicate to Bob. If she measures the x (p) quadrature, this prepares an ensemble of x-squeezed (p-squeezed) states on Bob's side. Then, Bob clones these squeezed states by using the HNLA and attempts to gain some knowledge on the phase-encoded information bit. To understand why signaling is impossible, it is necessary to investigate the noiseless amplification of these two indistinguishable ensembles of x- or p-squeezed states.

Assume that Alice wants to communicate a bit 0, so she measures the x quadrature of her mode. As sketched in figure 3.7, this prepares on Bob's side an ensemble of x-squeezed states of mean vector $(x, 0)$ and covariance matrix

$$\gamma_r^{\mathrm{sq}} = \begin{pmatrix} 2\mathrm{e}^{-2r} & 0 \\ 0 & 2\mathrm{e}^{2r} \end{pmatrix} \tag{3.82}$$

with x being drawn from a Gaussian distribution of mean 0 and variance $\sigma^2 = \mathrm{e}^{2r} - \mathrm{e}^{-2r}$. Here r is the single-mode squeezing parameter of each component state of this mixture, and, given equation (3.71), it must be related to the two-mode squeezing parameter s of the EPR state by $\mathrm{e}^{2r} = \cosh 2s$. This expresses that the p-variance of each component x-squeezed state is equal to the variance of the thermal state $\hat{\rho}_s^{\mathrm{th}}$ obtained simply by tracing over Alice's mode. Using the identities

$$\mathrm{e}^{2r} = \frac{1+\tanh r}{1-\tanh r} \qquad \cosh 2s = \frac{1+(\tanh s)^2}{1-(\tanh s)^2} \tag{3.83}$$

this relation between r and s can also be equivalently reexpressed as

$$\tanh r = (\tanh s)^2 \tag{3.84}$$

Since the HNLA transforms the single-mode squeezing parameter r of each component state according to equation (3.39) while it transforms the two-mode squeezing parameter s of the EPR state according to equation (3.73), we get simply

$$\tanh r' = (\tanh s')^2 \tag{3.85}$$

which implies that the p-variance of each component x-squeezed state after amplification remains precisely equal to that of the thermal state $\hat{\rho}_{s'}^{\mathrm{th}}$ obtained by tracing the amplified state $|\mathrm{EPR}_{s'}\rangle$ over Alice's mode (see Fig. 3).

The next step is to verify that the x-variance of Bob's state after amplification coincides with the p-variance. More generally, we need to verify that Bob's resulting state is independent of Alice's choice to measure one particular quadrature (here x). The ensemble of x-squeezed states before amplification can be written as a mixture

$$\hat{\rho} = \int \mathrm{d}x \; p_x \, \hat{\rho}_r^{\mathrm{sq}}(x) \tag{3.86}$$

where $\hat{\rho}_r^{\mathrm{sq}}(x)$ stands for an x-squeezed state of parameter r that is centered on $(x, 0)$. The Gaussian distribution

$$p_x = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big\{ -\frac{x^2}{2\sigma^2} \Big\} \tag{3.87}$$

has a variance that can be reexpressed as

$$\sigma^2 = \frac{4 \tanh r}{1 - (\tanh r)^2} \tag{3.88}$$

Each squeezed state $\hat{\rho}_r^{\mathrm{sq}}(x)$ is transformed by the HNLA into another squeezed state $\hat{\rho}_{r'}^{\mathrm{sq}}(x')$ where r' is related to r via equation (3.39) and x' is related to x via equation (3.59). Using equation (3.60), the success probability can be written as

$$\begin{aligned} P_{\mathrm{succ}|x} \propto \frac{\cosh r'}{\cosh r} \exp\Big\{ \frac{x'^2(1+\tanh r')}{4} \Big\} \\ \times \exp\Big\{ -\frac{x^2(1+\tanh r)}{4} \Big\} \end{aligned} \tag{3.89}$$

Putting all this together, Bob's resulting state conditionally on the success of the HNLA can be written as

$$\begin{aligned} \hat{\rho}_{.|\mathrm{succ}} &= \int \mathrm{d}x \; \frac{P_{\mathrm{succ}|x} \, p_x}{P_{\mathrm{succ}}} \, \hat{\rho}_{r'}^{\mathrm{sq}}(x') \\ &\propto \int \mathrm{d}x \; \exp\Big\{ \frac{x'^2(1+\tanh r') - x^2(1+\tanh r)}{4} \Big\} \\ &\qquad \times \exp\Big\{ -\frac{1-(\tanh r)^2}{8 \tanh r} \, x^2 \Big\} \, \hat{\rho}_{r'}^{\mathrm{sq}}(x') \end{aligned} \tag{3.90}$$

We wish to prove that $\hat{\rho}_{.|\mathrm{succ}}$ coincides with the thermal state $\hat{\rho}_{s'}^{\mathrm{th}}$, which can also be written as a mixture of x-squeezed states

$$\hat{\rho}_{s'}^{\mathrm{th}} = \int \mathrm{d}x' \; p_{x'} \, \hat{\rho}_{r'}^{\mathrm{sq}}(x') \tag{3.91}$$

where

$$p_{x'} = \frac{1}{\sqrt{2\pi\sigma'^2}} \exp\Big\{ -\frac{x'^2}{2\sigma'^2} \Big\} \tag{3.92}$$

is a Gaussian distribution of variance

$$\sigma'^2 = \frac{4\tanh r'}{1-(\tanh r')^2} \tag{3.93}$$

in analogy with equation (3.88). This boils down to checking that, for all x, we have

$$\begin{aligned}&\left(\frac{1-(\tanh r)^2}{8\tanh r}+\frac{1+\tanh r}{4}\right)x^2\\ &\quad=\left(\frac{1-(\tanh r')^2}{8\tanh r'}+\frac{1+\tanh r'}{4}\right)x'^2\end{aligned} \tag{3.94}$$

which simplifies to

$$\frac{(1+\tanh r)^2}{\tanh r}\,x^2 = \frac{(1+\tanh r')^2}{\tanh r'}\,x'^2 \tag{3.95}$$

and holds as a consequence of Eqs. (3.39) and (3.59).

Therefore, we have verified that if Alice wants to send a bit 0 and Bob's thermal state is thereby decomposed into a Gaussian mixture of x-squeezed states $\hat{\rho}_r^{\mathrm{sq}}(x)$, the resulting state becomes a Gaussian mixture of amplified x-squeezed states $\hat{\rho}_{r'}^{\mathrm{sq}}(x')$ with a stronger squeezing but which nevertheless remains phase-invariant, as illustrated in figure 3.7. The same reasoning is of course true regardless of the initially squeezed quadrature, so that the resulting state would be exactly the same if Alice wanted to send a bit 1 and Bob's thermal state was decomposed into a Gaussian mixture of p-squeezed states. Hence, no information whatsoever is available to Bob and causality is preserved, even on a heralded basis.

3.2 Noiseless amplification and attenuation of non-Gaussian states

3.2.1 Mean photon number and mean-field amplitude

The heralded noiseless amplifier (attenuator) is a transformation that increases (decreases) the complex amplitude α of a coherent state $|\alpha\rangle$ without adding noise, that is $|\alpha\rangle \to |g\alpha\rangle$. We have proven, that the same behavior holds true for the mean amplitude of any (possibly mixed) Gaussian state. One could therefore naively expect that this remains true for all states. Surprisingly, we will show that the mean amplitude of a non-Gaussian state can actually be attenuated by noiseless amplification, or amplified by noiseless attenuation. We first illustrate this counterintuitive effect on two simple and instructive examples of states that can be expressed as superpositions of a finite number of Fock states. In a third example, a non-Gaussian mixed state will also be shown to exhibit this effect. In the next subsection, we will design a scheme for experimentally verifying the mean field amplification by noiseless attenuation that is robust against most experimental imperfections.

As a first example, let us consider the superposition of vacuum and single-photon state,

$$|\Psi_1\rangle = c_0|0\rangle + c_1|1\rangle, \tag{3.96}$$

where without loss of any generality we assume that c_0 and c_1 are real and $c_0^2 + c_1^2 = 1$. The coherent amplitude of this state then reads

$$A_1 \equiv \langle\Psi_1|\hat{a}|\Psi_1\rangle = c_1 c_0 = c_1\sqrt{1-c_1^2}, \tag{3.97}$$

where $\hat{a}$ denotes the annihilation operator. After noiseless amplification with gain $g > 1$, the state becomes

$$|\tilde{\Psi}_1\rangle = g^{\hat{n}}|\Psi_1\rangle = c_0|0\rangle + gc_1|1\rangle, \tag{3.98}$$

and its amplitude changes to

$$\tilde{A}_1 = \frac{g\sqrt{1-c_1^2}c_1}{1+(g^2-1)c_1^2}. \tag{3.99}$$

The effective amplification gain is given by $\tilde{A}_1/A_1$, and we get

$$G_{\text{eff}}^{(1)} = \frac{g}{1+(g^2-1)c_1^2}. \tag{3.100}$$

If the probability of single-photon state satisfies $c_1^2 > 1/(g+1)$, then $G_{\text{eff}} < 1$ hence the noiseless amplification attenuates the complex amplitude of the state. This effect can be understood by noting that the mean amplitude of the superpositions (3.96) is maximized when $c_0 = c_1 = 1/\sqrt{2}$. If the amplification gain becomes large enough, then it enhances the imbalance between the amplitudes of the vacuum and single-photon contributions, which results in effective reduction of the mean field. In the limit of very large amplification gain, the noiselessly amplified state approaches a Fock state $|1\rangle$, whose mean field vanishes.

Similar conclusions hold also for the noiseless attenuation. The effective amplitude gain is given again by equation (3.100) but with $g = \nu < 1$,

$$G_{\text{eff}}^{(1)} = \frac{\nu}{1+(\nu^2-1)c_1^2}. \tag{3.101}$$

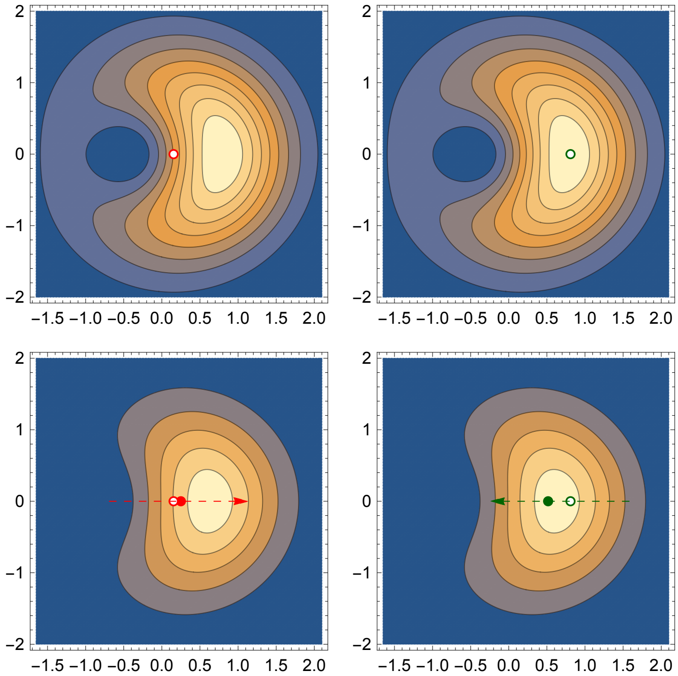

Figure 3.8: Contour plots for the Q representations for the state $|\Psi_1\rangle = c_0|0\rangle + c_1|1\rangle$ before attenuation (upper row) and after attenuation (lower row). The coefficients have the values $c_0 = \sqrt{0.1}$ and $c_1 = \sqrt{0.9}$ while the attenuation is $\nu = 0.5$. The dots represent the mean values $\langle\hat{\alpha}\rangle^2$ (left column) and $\langle\hat{\alpha}^\dagger\hat{\alpha}\rangle$ (right column). Hollow points represent mean values before attenuation while full points correspond to mean values after attenuation. The mean value $\langle\hat{\alpha}^\dagger\hat{\alpha}\rangle$ behaves as expected; it is decreased due to attenuation. The mean value $\langle\hat{\alpha}\rangle^2$ exhibits the effect that it is increased via attenuation.

If $c_1^2 > 1/(1+\nu)$ then $G_{\text{eff}}^{(1)} > 1$ because starting from a state where the single-photon component is dominant, the noiseless attenuation drives it closer to the balanced superposition $(|0\rangle + |1\rangle)/\sqrt{2}$.

As a second example, let us consider superposition of the three lowest Fock states,

$$|\Psi_2\rangle = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle, \tag{3.102}$$

where for the sake of simplicity we again assume real c_j, and $c_0^2 + c_1^2 + c_2^2 = 1$. The amplitude reads

$$A_2 = \langle\Psi_2|\hat{a}|\Psi_2\rangle = c_1(c_0 + \sqrt{2}c_2). \tag{3.103}$$

Since the formula contains two terms, constructive or destructive quantum interference can occur. After noiseless amplification, the complex amplitude becomes

$$\tilde{A}_2 = \frac{gc_1(c_0 + \sqrt{2}g^2c_2)}{c_0^2 + g^2c_1^2 + g^4c_2^2} \tag{3.104}$$

and the effective amplification gain can be expressed as

$$G_{\text{eff}}^{(2)} = \frac{g}{c_0^2 + g^2 c_1^2 + g^4 c_2^2} \frac{c_0 + \sqrt{2} g^2 c_2}{c_0 + \sqrt{2} c_2}. \tag{3.105}$$

By suitably choosing c_0 and c_2, the factor $c_0 + \sqrt{2} g^2 c_2$ in the numerator can be made arbitrarily small and we may even achieve zero gain. This can be interpreted as the arising of a destructive interference between the vacuum and two-photon components in the noiselessly amplified state, hence decreasing its mean field. Similarly, in case of noiseless attenuation, we can choose the parameters such that the factor $c_0 + \sqrt{2} c_2$ will be very small and the gain will be very large. Here, the destructive interference that makes the mean field of the initial state very small is disturbed as a result of noiseless attenuation, hence increasing the mean field. Interestingly, this mechanism of interference disturbance is robust against imperfections in the process of noiseless attenuation, so it is a good candidate for an experimental demonstration.

We note that the same type of counterintuitive effects may also be exhibited by non-Gaussian mixtures of Gaussian states. Indeed, as a third example, consider the binary mixture of two coherent states $|\alpha\rangle$ and $|\beta\rangle$,

$$\hat{\rho}\hat{\rho}_3 = p|\alpha\rangle\langle\alpha| + (1-p)|\beta\rangle\langle\beta|, \tag{3.106}$$

where $p \in [0, 1]$. The amplitude of this state reads

$$A_3 = \text{Tr}(\hat{\rho}_3 \hat{a}) = p\,\alpha + (1-p)\beta. \tag{3.107}$$

After noiseless amplification, each coherent state $|\alpha\rangle$ is mapped onto $|g\alpha\rangle$ with weight factor $e^{(g^2-1)|\alpha|^2}$. Hence, the resulting state is also a mixture of two coherent states with amplified amplitudes and modified weight,

$$\tilde{\hat{\rho}}_3 = p'|g\alpha\rangle\langle g\alpha| + (1-p')|g\beta\rangle\langle g\beta|, \tag{3.108}$$

where

$$p' = \frac{p\, e^{(g^2-1)|\alpha|^2}}{p\, e^{(g^2-1)|\alpha|^2} + (1-p)\, e^{(g^2-1)|\beta|^2}}. \tag{3.109}$$

Its amplitude is given by

$$\tilde{A}_3 = g[p'\alpha + (1-p')\beta] \tag{3.110}$$

so that the effective amplification gain reads

$$G_{\text{eff}}^{(3)} = g\, \frac{p'\alpha + (1-p')\beta}{p\,\alpha + (1-p)\beta}. \tag{3.111}$$

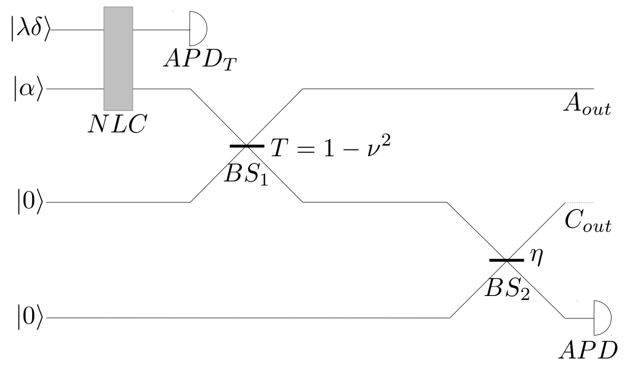

Figure 3.9: Proposed experimental setup. Coherent states are injected into signal and idler ports of a nonlinear crystal where parametric down-conversion with a low gain λ occurs. Conditional photon addition is heralded by a click of a single-photon detector APD_T. BS_1 is a beam splitter with amplitude reflectance ν and noiseless attenuation is heralded by a no-click of the single-photon detector APD. Imperfect detection with efficiency $\eta < 1$, is modeled by coupling to an auxiliary mode C prepared in a vacuum state, where η is the transmittance of BS_2.

This gain can be complex, and we can have $|G_{\text{eff}}^{(3)}| < 1$ for $g > 1$. To see this, take the example of two coherent states with real amplitudes $\alpha = 1$ and $\beta = -0.9$ that are mixed with $p = 1/3$. If we process this mixture in a noiseless amplifier of gain $g = 2$, we get an effective gain $G_{\text{eff}}^{(3)} = 0.063$ smaller than unity. Thus, we observe a mean field reduction by noiseless amplification of a non-Gaussian mixture of coherent states. Conversely, if we set $g = \nu < 1$ in equation (3.111), we get a formula for the effective gain of the noiseless attenuation of state (3.106), and it is easy to find examples where it is larger than 1. Thus, noiseless attenuation may enhance the mean field amplitude of a non-Gaussian mixture of coherent states.

3.2.2 Proposal for an experimental setup

In this subsection, we propose and analyze an optical setup that enables to experimentally demonstrate the counterintuitive effect of mean field enhancement by noiseless attenuation. The suggested scheme is illustrated in Fig. 3.9. The non-Gaussian state is generated from an input coherent state by conditional photon addition. A coherent state $|\alpha\rangle$ is injected into the signal input port of a nonlinear crystal where a non-degenerate parametric down-conversion with a low parametric gain $\lambda \ll 1$ takes place. A click of the trigger single-photon detector APD_T heralds the generation of a photon pair in the crystal and the addition of a photon to the signal beam. The noiseless attenuation $\nu^{\hat{n}}$ is implemented by sending the signal beam through a beam splitter BS_1 with reflectance $R = \nu^2$ and transmittance $T = 1 - \nu^2$. The auxiliary input port of BS_1 is prepared in vacuum state, and the auxiliary output port is measured with single-photon detector APD. Assuming ideal detector with unit detection efficiency, the noiseless attenuation is heralded by a no-click of the detector. In practice, the detection efficiency will be rather low, so conditioning on no-clicks will result in a combination of noiseless attenuation and usual losses. In what follows, we will first assume an ideal APD and then we will provide a more realistic description which will account for imperfect state preparation and inefficient single-photon detection.

In order to increase the flexibility of the setup we suggest to also inject a weak auxiliary coherent state $|\lambda\delta\rangle$ to the idler input port of the nonlinear crystal. The detector APD_T can then be triggered either by the idler photon generated in the crystal or by a photon from the auxiliary input coherent beam. If these two photons are indistinguishable, then one obtains a coherent superposition of the photon addition and identity operations, and the resulting conditionally prepared state reads,

$$|\Psi\rangle = \frac{1}{\sqrt{N}}(\hat{a}^\dagger + \delta)|\alpha\rangle. \tag{3.112}$$

Here $N = 1 + |\alpha^* + \delta|^2$ is a normalization factor and the parameters α and δ can be independently set to any desired value by tuning the amplitudes of the coherent beams injected into the signal and idler ports of the nonlinear crystal, respectively. Note that in the limit $\alpha = 0$ the state becomes the superposition of vacuum and single-photon states as studied in the previously. For the sake of simplicity, we shall assume that both α and δ are real. In this case, the complex amplitude of $|\Psi\rangle$ is real as well,

$$A = \alpha + \frac{\alpha + \delta}{1 + (\alpha + \delta)^2}. \tag{3.113}$$

After noiseless attenuation, the state transforms into

$$|\tilde{\Psi}\rangle \propto \nu^{\hat{n}}(\hat{a}^\dagger + \delta)|\alpha\rangle \propto (\nu\hat{a}^\dagger + \delta)|\nu\alpha\rangle. \tag{3.114}$$

where we have used the identity $\nu^{\hat{n}}\hat{a}^\dagger = \hat{a}^\dagger\nu^{\hat{n}+1}$. We see that the structure of the state remains unaltered but its parameters change according to $\alpha \to \nu\alpha$ and $\delta \to \delta/\nu$. Therefore, the amplitude of the noiselessly attenuated state (3.114) can be expressed as

$$\tilde{A} = \nu\alpha + \frac{\nu\alpha + \delta/\nu}{1 + (\nu\alpha + \delta/\nu)^2}. \tag{3.115}$$

The effective gain $G_{\text{eff}} = \tilde{A}/A$ is plotted in figures 3.10 and 3.11 as a function of ν and δ respectively. We can see that the effective amplitude gain can be both positive and

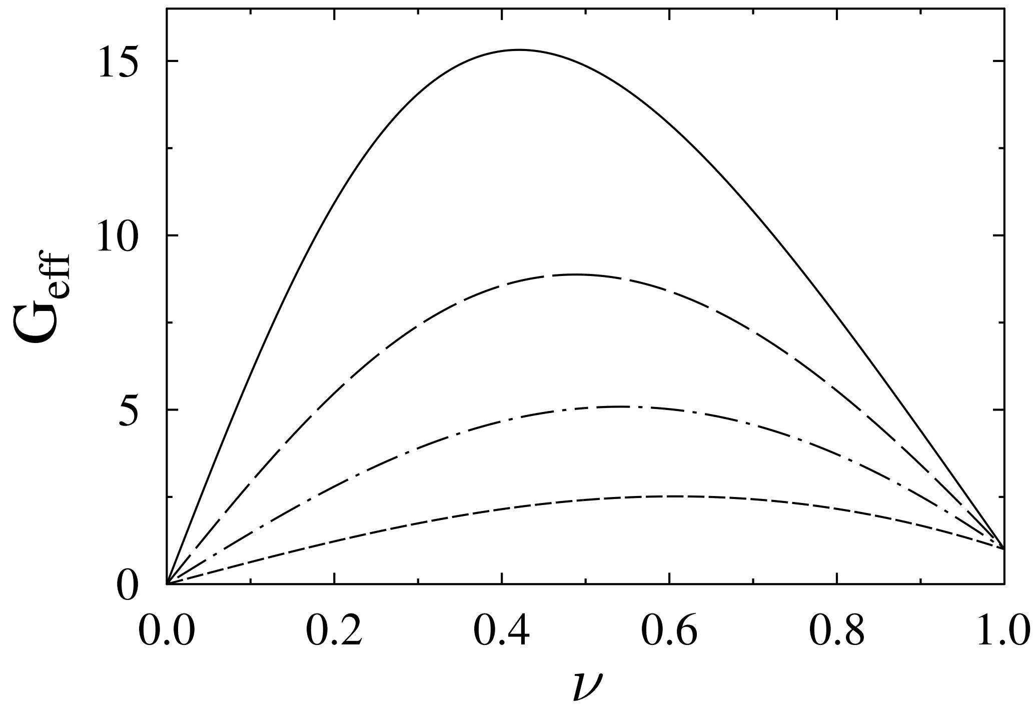

Figure 3.10: Noiseless attenuation of a non-Gaussian state given in equation(3.112). The amplitude gain G_{eff} is plotted as a function of the attenuation factor ν for four different values of detection efficiency $\eta = 1$ (solid line), $\eta = 0.75$ (long dashed line), $\eta = 0.5$ (dot dashed line) and $\eta = 0.25$ (short dashed line). The other parameters read $\alpha = 0.25$, $\delta = -0.55$, and $p = 1$.

negative and for suitable parameter values the gain can be much larger than 1. The large gain occurs in the neighborhood of a point where $A = 0$ (figure 3.11). In the proposed experiment, one could seek an optimal working point δ_{opt} such that $G_{\text{eff}} > 1$ while the input and output amplitudes are large enough so the amplification effect would be observable and not buried in noise.

Let us now include the effect of inefficient single-photon detection into our model. As illustrated in Fig. 3.9, this can be done by including another auxiliary mode C, which is coupled to mode B by the beam splitter BS_2 with transmittance η equal to the detection efficiency of the APD. While mode B is projected onto vacuum state, mode C is traced over. The output state before measurement on mode B reads,

$$\hat{U}(\hat{a}^\dagger + \delta)|\alpha\rangle_A|0\rangle_B|0\rangle_C. \tag{3.116}$$

Here $\hat{U}$ is a unitary operation describing the mode coupling effected by the two beam splitters BS_1 and BS_2,

$$U = \begin{pmatrix} \nu & \sqrt{\eta T} & \sqrt{(1-\eta)T} \\ -\sqrt{T} & \nu\sqrt{\eta} & \nu\sqrt{1-\eta} \\ 0 & -\sqrt{1-\eta} & \sqrt{\eta} \end{pmatrix} \tag{3.117}$$

where $T = 1 - \nu^2$. Hence,

$$\hat{U}\hat{a}^\dagger\hat{U}^\dagger = \nu\hat{a}^\dagger + \sqrt{T}(\sqrt{\eta}\hat{b}^\dagger + \sqrt{1-\eta}\hat{c}^\dagger), \tag{3.118}$$

where $\hat{b}^\dagger$ and $\hat{c}^\dagger$ denote creation operators of modes B and C, respectively, so that the output state before measurement on mode B can be rewritten as

$$[\nu\hat{a}^\dagger + \sqrt{T\eta}\,\hat{b}^\dagger + \sqrt{T(1-\eta)}\,\hat{c}^\dagger + \delta]\,\hat{U}|\alpha\rangle_A|0\rangle_B|0\rangle_C. \tag{3.119}$$

We also use the fact that a passive linear optical network transforms input coherent states onto output coherent states, so that

$$\hat{U}|\alpha\rangle_A|0\rangle_B|0\rangle_C = |\nu\alpha\rangle_A|\sqrt{T\eta}\alpha\rangle_B|\sqrt{T(1-\eta)}\alpha\rangle_C. \tag{3.120}$$

After projection of mode B onto vacuum, the unnormalized conditional state reads

$$|\tilde{\Psi}_\eta\rangle = (\nu a^\dagger + \sqrt{T(1-\eta)}\,c^\dagger + \delta)\;|\nu\alpha\rangle_A|\sqrt{T(1-\eta)}\alpha\rangle_C. \tag{3.121}$$

The amplitude of the output signal mode A is then given by

$$\tilde{A}_\eta = \nu\alpha + \frac{\nu\alpha(1-\eta T) + \nu\delta}{[\alpha(1-\eta T)+\delta]^2 + 1 - \eta T}. \tag{3.122}$$

A second effect that we take into account is the imperfect mode overlap in conditional photon addition. With some probability, the photon may be added to a different mode and in this case the input state remains the coherent state $|\alpha\rangle$. Thus, a realistic input state can be modeled as a mixture of the state (3.112) and the coherent state $|\alpha\rangle$,

$$\hat{\rho} = p|\Psi\rangle\langle\Psi| + (1-p)|\alpha\rangle\langle\alpha|, \tag{3.123}$$

where $p \in [0,1]$. After noiseless attenuation, the (un-normalized) state of the output signal mode reads

$$\hat{\rho}_{\text{out}} = \frac{p}{N}\text{Tr}_C[|\tilde{\Psi}_\eta\rangle\langle\tilde{\Psi}_\eta|] + (1-p)|\nu\alpha\rangle\langle\nu\alpha|. \tag{3.124}$$

The amplitude of this output state can be expressed as

$$\tilde{A}_{\eta,p} = p'\tilde{A}_{\eta} + (1-p')\nu\alpha, \tag{3.125}$$

where

$$p' = \frac{p}{p + (1-p)\frac{1+(\alpha+\delta)^2}{[\alpha(1-\eta T)+\delta]^2+1-\eta T}}. \tag{3.126}$$

As shown in figures 3.11 and 3.12, the effect of amplitude enhancement by noiseless attenuation persists even for $p = 0.75$ and low detection efficiency $\eta = 0.25$, although it becomes less pronounced with decreasing η. When fixing the parameters δ, α, and p, there exists a detection efficiency threshold η_{th} such that if $\eta \leq \eta_{\mathrm{th}}$ then $|G_{\mathrm{eff}}| < 1$ for any $0 < \nu < 1$, so the noiseless attenuation does not any more increase the amplitude of the considered input state. The dependence of G_{eff} on ν and η shown in Figs. 3.10

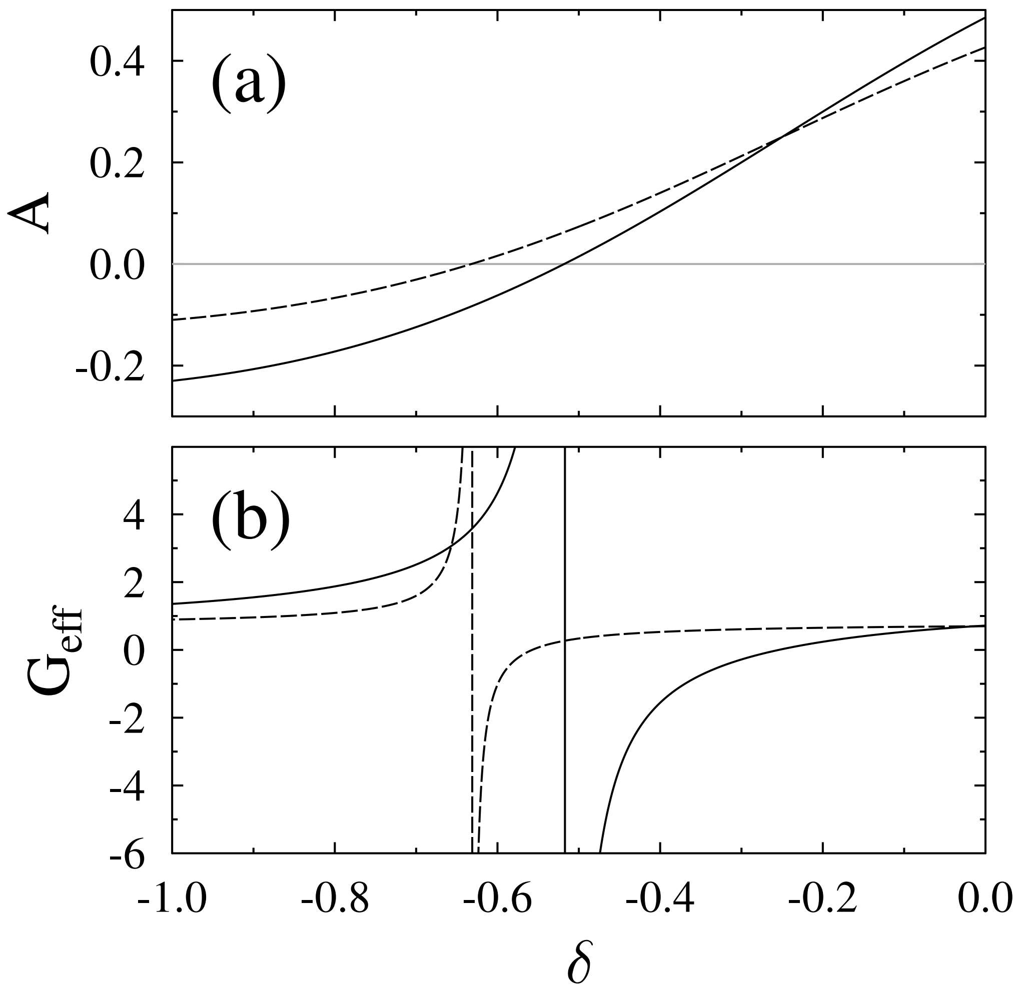

Figure 3.11: Dependence of the input amplitude A (a) and amplitude gain G_{eff} (b) on the displacement parameter δ. The other parameters read $\alpha = 0.25$, $\nu = 1/\sqrt{2}$, and $\eta = 1$, $p = 1$ (solid line) and $\eta = 0.25$, $p = 0.75$ (dashed line).

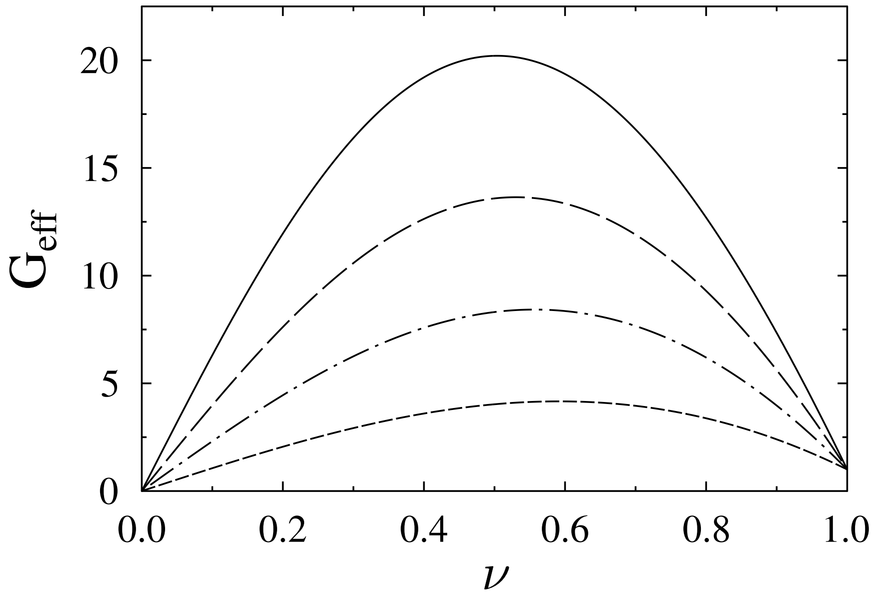

Figure 3.12: The same as Fig. 3.10 but $p = 0.75$ and $\delta = -0.65$.

and 3.12 suggests that the efficiency threshold can be derived from the condition

$$\left.\frac{dG_{\text{eff}}}{d\nu}\right|_{\nu=1} = 0, \tag{3.127}$$

which yields

$$\eta_{\text{th}} = \frac{\left[1+(\alpha+\delta)^2\right]\left[\alpha+\alpha(\alpha+\delta)^2+p(\alpha+\delta)\right]}{2p^2(\alpha+\delta)(1+2\alpha^2+2\alpha\delta)-2p\alpha[1+(\alpha+\delta)^2]}. \tag{3.128}$$

Explicitly, for $\alpha = 0.25$, $\delta = -0.55$, and $p = 1$ (Fig. 3.10), we obtain $\eta_{\text{th}} = 0.0284$, while for $\alpha = 0.25$, $\delta = -0.65$, and $p = 0.75$ (Fig. 3.12), we obtain $\eta_{\text{th}} = 0.0146$. We have also confirmed these results by numerical calculations.

Note that with a suitable choice of α and δ, the amplitude amplification by noiseless attenuation can be observed for any $\eta > 0$ and $p > 0$, in which case $\eta_{\text{th}} = 0$. This can be proved by noting that for a real α satisfying $|\alpha| < p/2$, there exists real δ such that the amplitude of the input state (3.123) vanishes. After (imperfect) noiseless attenuation, the state will possess nonzero amplitude, hence the gain will be infinite. Therefore, by continuity, there exists a region of parameters for which the initial amplitude is nonzero and $|G_{\text{eff}}| > 1$. Even severe experimental inefficiencies and imperfections can thus be compensated and we can achieve high effective gains by a careful tuning of α and δ.

In an experimental realization of the proposed setup, values for the vacuum conditioning efficiency η and for the purity parameter p like those used in figures 3.11 and 3.12 are realistic and probably even too conservative. An efficient vacuum conditioning can be obtained by loosening the spectral and spatial filtering in front of the APD detector so to bring losses in the heralding channel to a minimum. The increased rate of unwanted background counts can be limited by using pulsed laser sources for gating the time interval when the absence of APD clicks should be detected. Note, however, that higher levels of background counts only decrease the no-click heralding rate, without compromising the quality of the generated states. Finally, a regime of small coherent state amplitude α and high reflectance ν should be preferentially used in an experiment in order to avoid saturation effects in the APD detector.

3.3 Conclusions

In this Chapter, we have found that the HNLA acts as a phase-insensitive or universal single-mode squeezer in the sense that any squeezed state is transformed into another squeezed state with a stronger squeezing of the quadrature that was initially squeezed. This is not to be confused with other notions of universal squeezing such as that some operations squeezes the initial state always in the same manner. The universal squeezing in the context of this Chapter is superimposed with a nonlinear amplification of the mean field of the input state, as the gain is proportional to g corrected with an additional factor that underamplifies the initially squeezed quadrature and overamplifies the initially antisqueezed quadrature. Note that if one considers probabilistic noiseless attenuation, i.e. $g = \nu < 1$ the inverse effect occurs, namely the the initial squeezed states becomes less squeezed while the phase remains unchanged. This comes from the fact that the mathematical description of the probabilistic noiseless amplification and probabilistic noiseless attenuation have the same mathematical description; to obtain results concerning attenuation from results concerning amplification one has to simply substitute $g \rightarrow \nu$, where $g > 1,\ g > 0$ and $\nu < 1,\ \nu > 0$.

Such an ability to squeeze a quantum state independently of its phase, though it is predicted within quantum mechanics, seems to contradict the celebrated "peaceful coexistence" with special relativity as it might lead to an heralded version of instantaneous signaling between entangled parties. We have examined the reasons why this intriguing possibility fails by inserting this phase-insensitive squeezer in a continuous-variable analog of Dieks' scheme, which is a thought experiment aimed at ruling out the possibility of perfect cloning (or amplification) based on the link with signaling. Our analysis shows that if Alice encodes information in the phase of her measured quadrature of an entangled two-mode state, Bob obtains different mixtures of squeezed states by applying the phase-insensitive squeezer on his mode, but all these mixtures realize a same thermal state. Hence, signaling is indeed impossible.

Although it was expected, such a "coincidence" is remarkable if one recognizes that the HNLA effects an enhancement of the squeezing strength according to equation (3.39) together with a nonlinear amplification of the mean field according to equation (3.59). Although the formulas do not suggest it at first sight, a simplification occurs due to the conjunction of the filtration operator $\hat{F}$ characterizing the HNLA with Bayes rule for conditional probabilities, which guarantees that causality is preserved.

Since the class of squeezed states is broadly used in quantum information protocols, we expect that this universal squeezer may offer new perspectives in quantum optics, going beyond those that have been investigated today such as quantum entanglement distillation, quantum error correction, or loss compensation. Our analysis may yield useful tools in particular when turning to squeezed-state protocols, quantum noise reduction, or phase estimation [RHG+92].

We have investigated the behavior of quantum states of light under the action of heralded noiseless attenuation and amplification. By considering certain non-Gaussian states, we have found out that noiselessly amplifying the state may be accompanied by a decrease of its mean field amplitude $\langle \hat{a} \rangle$. Conversely, noiselessly attenuating the state may come with an increased coherent amplitude. We have proven that such counterintuitive effects cannot occur for Gaussian states, so these are specific to non-Gaussian states. Our work thus reveals that the intuition based on ordinary deterministic phase-insensitive amplifiers and lossy channels is not directly applicable to noiseless amplifiers and attenuators, and some unexpected interconnection between the mean field amplitude and mean photon number may occur when noiselessly amplifying or attenuating non-Gaussian states.

We have proposed an experimental scheme that is feasible with current technology and should enable the observation of amplitude enhancement by noiseless attenuation with a coherently-displaced single-photon added coherent state under realistic experimental conditions. An alternative way to observe this effect may be based on the virtual noiseless amplifier or attenuator [FC12, CWA+14], where the amplification or attenuation effect is emulated by post-processing the experimental data.

Chapter 4

Majorization theory in quantum optics

4.1 Elements of majorization theory

4.1.1 Accumulation of probability vectors

What does it mean that a vector is more *disordered* or more *mixed* than another one? This question has many answers that depend on the system under consideration and it appears in many cases-with no apparent connection between them. In most cases and particularly in physics, the above question is answered by the means provided by the theory of majorization [Mui03, HLP78, MO79, Arn87]. The theory of majorization is an extremely vast area of mathematics and it had been studied from the early twentieth century up to date. It finds application in many areas of physics, chemistry, economical and social sciences. As it will be understood later on when a vector majorizes another one is the strongest statement than one can prove in the context of which one has *more spread out* components. To begin to get a feeling let us consider the following simple example attributed to Lorentz.

Consider a community of n members. Each individual i has a wage w_i. The question we want to answer now is if this community's wages are balanced. Let us denote as $\mathbf{w}^{\downarrow}$ a vector with the wages of each individual as components. The down-pointing arrow denotes that the components are in a non-increasing order, i.e. from richest to poorest. We define a quantity called *accumulation* as $S_k = \sum_{i=1}^{k} w_i^{\downarrow}$ and we plot the proportion of population k/n versus the proportion of income S_k/S_n for $k = 0, \ldots, n$. It is apparent that if the income for every person is the same then the plot will be a straight line.

A slight depart from equality of incomes will create a convex line above the absolute equality line as depicted in figure 4.1.

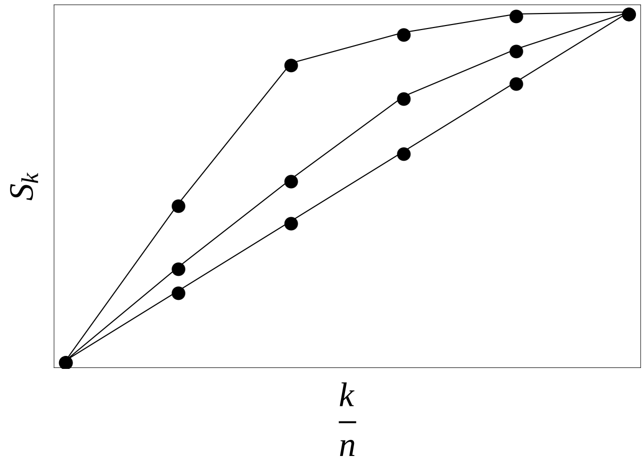

Figure 4.1: The closer to the straight the more equal the money is distributed among individuals.

So, the intuition we obtain from this paradigm is that two communities with income vectors $\mathbf{w}^{\downarrow}$ and $\mathbf{s}^{\downarrow}$ may be compared in terms of the summations of their accumulations. Vector $\mathbf{w}^{\downarrow}$ is more even than $\mathbf{s}^{\downarrow}$ if,

$$\sum_{i=1}^{k} w_i^{\downarrow} \leq \sum_{i=1}^{k} s_i^{\downarrow}, \; k = 1, \ldots, n \tag{4.1}$$

obviously it holds that,

$$\sum_{i=1}^{n} w_i^{\downarrow} = \sum_{i=1}^{n} s_i^{\downarrow} = T, \tag{4.2}$$

where T is the total wealth. Equation (4.1) will be transformed to the first formal definition of majorization. Namely, consider two d-dimensional real positive vectors $\mathbf{p}$ and $\mathbf{q}$. We say that $\mathbf{p}$ is majorized by $\mathbf{q}$, symbolized by $\mathbf{p} \prec \mathbf{q}$, if and only if

$$\sum_{i=1}^{k} p_i^{\downarrow} \leq \sum_{i=1}^{k} q_i^{\downarrow} \tag{4.3}$$

for $k = 1, \ldots, d-1$ and

$$\sum_{i=1}^{d} p_i^{\downarrow} = \sum_{i=1}^{d} q_i^{\downarrow}, \tag{4.4}$$

where the down-pointing arrow on $\mathbf{p}$ and $\mathbf{q}$ indicates that the components are sorted in non-increasing order. Equation (4.4) is automatically satisfied if $\mathbf{p}$ and $\mathbf{q}$ are vectors normalized to unity, e.g., if they are probability distributions. Majorization only provides a partial order, in the sense that if $\mathbf{p}$ is *not* majorized by $\mathbf{q}$ (symbolized by $\mathbf{p} \not\prec \mathbf{q}$) then this does not imply that $\mathbf{p} \succ \mathbf{q}$. When both $\mathbf{p} \not\prec \mathbf{q}$ and $\mathbf{q} \not\prec \mathbf{p}$ hold, we say that the two vectors are *incomparable*. Moreover as we will see in the next subsection, majorization is something stricter than partial ordering, it provides preordering.

4.1.2 Doubly stochastic matrices

The definition of majorization through Eqs. (4.3) and (4.4) is very handy for calculation purposes, but there is clearer definition in what sense $\mathbf{p}$ is more disordered than $\mathbf{q}$. A more intuitive, but equivalent, definition is to say that $\mathbf{p}$ is majorized by $\mathbf{q}$ if and only if there exists a set of $d-$dimensional permutation matrices $\mathbf{\Pi}_n$ and a probability distribution $\{t_n\}$ such that

$$\mathbf{p} = \sum_n t_n \, \mathbf{\Pi}_n \mathbf{q}. \tag{4.5}$$

The above definition says that $\mathbf{p}$ is majorized by $\mathbf{q}$ if and only if we can obtain $\mathbf{p}$ by randomly permuting the components of vector $\mathbf{q}$ and afterwards taking the average over all permutations.

The notion of majorization is closely related to the notion of doubly stochastic matrices. A real $d \times d$ matrix $\mathbf{D} = [D_{ij}]$ is doubly stochastic if all its entries are non-negative, and each row and each column sums up to 1. The following theorem gives the relation between majorization and doubly stochastic matrices.

Theorem 1: $\mathbf{p} \prec \mathbf{q}$ if and only if there exists a doubly stochastic matrix $\mathbf{D}$ such that $\mathbf{p} = \mathbf{D}\mathbf{q}$.

The set of doubly stochastic matrices of a given dimension is convex. All extremal points of this convex set are the permutation matrices $\mathbf{\Pi}_n$, so any doubly stochastic matrix can be expressed as a convex combination of permutation matrices. This is expressed in the following theorem.

Theorem 2: The $d \times d$ doubly stochastic matrices $\mathbf{D}$ form a convex set whose extremal points are all the $d \times d$ permutation matrices $\mathbf{\Pi}_n$.

The convex set of $d \times d$ doubly stochastic matrices is called Birkhoff's polytope. It admits $d!$ vertices (i.e., the number of $d \times d$ permutation matrices) and its dimension is

$(d-1)^2$. Note that if we want to express a point (a doubly stochastic matrix) belonging to this $(d-1)^2$-dimensional polytope as a convex combination of the extremal points, Caratheodory's theorem implies that we would need $(d-1)^2+1$ extremal points at most.

One naturally expects that majorization theory should be connected with various measures of "disorder", such as the various entropies. Indeed, since $\mathbf{p} \prec \mathbf{q}$ means that $\mathbf{p}$ is more disordered than $\mathbf{q}$, any measure of disorder $S : \mathbb{R}^d \to \mathbb{R}$ should satisfy

$$S(\mathbf{p}) \geq S(\mathbf{q}) \tag{4.6}$$

for all $\mathbf{p}$ and $\mathbf{q}$ such that $\mathbf{p} \prec \mathbf{q}$. A function S obeying this property is called Schur-concave. Consider, for example, the Shannon entropy

$$S_1(\mathbf{p}) = -\sum_{i=1}^{d} p_i \ln p_i \tag{4.7}$$

or the Rényi entropy

$$S_\alpha(\mathbf{p}) = \frac{1}{1-\alpha} \ln \Big(\sum_{i=1}^{d} {p_i}^{\alpha} \Big) \tag{4.8}$$

of order $\alpha \geq 0$, $\alpha \neq 1$. (In the limit $\alpha \to 1$, the Rényi entropy converges to the Shannon entropy.) These functions can be seen to be Schur-concave as a consequence of the following theorem [HLP78].

Theorem 3: $\mathbf{p} \prec \mathbf{q}$ if and only if $\sum_{i=1}^{d} h(p_i) \geq \sum_{i=1}^{d} h(q_i)$ for all concave functions h.

Here let us note that any function that is concave in the usual sense and symmetric is Schur-concave as well.

The usefulness of majorization in quantum information theory appears first if we wish to compare two density matrices instead of probability distributions. Consider the density matrices ρ and σ of a $d-$level quantum system, and their respective vectors of eigenvalues $\boldsymbol{\lambda}(\rho)$ and $\boldsymbol{\lambda}(\sigma)$. We have the following theorem.

Theorem 4: $\boldsymbol{\lambda}(\rho) \prec \boldsymbol{\lambda}(\sigma)$ if and only if state ρ can be obtained from state σ by applying a random mixture of unitaries.
The proof goes simply by noting that there is a unitary transformation that aligns the eigenbasis of σ with that of ρ, and that each permutation of the eigenstates can be realized by a unitary transformation.

Let us close be saying that majorization provides is a preordering theory. That means that there is no distinction between two vectors such that one of them is transformed to the other by applying a permutation matrix. This is easily understood by considering the relation

$$\mathbf{p} = \mathbf{\Pi}\mathbf{q}, \tag{4.9}$$

where $\mathbf{\Pi}$ is a permutation matrix. In order to check majorization relations we order the vectors as equations (4.3) and (4.4) imply, so the action of the permutation matrix is suppressed and the vectors $\mathbf{p}$ and $\mathbf{q}$ become equivalent in terms of majorization.

4.1.3 Catalysis and LOCC transformations

The connection with quantum information theory becomes even more evident in the context of comparing pure bipartite entangled states. Indeed, majorization theory gives the means to determine whether one pure bipartite state is convertible into another one using LOCC (local operation and classical communication). Consider two $d-$level systems A and B, which can be thought of as belonging to Alice and Bob, respectively. Any bipartite pure states of these systems can be written in the Schmidt form

$$|\Psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i}|i\rangle_A|i\rangle_B, \tag{4.10}$$

where $\{|i\rangle_A\}$ and $\{|i\rangle_B\}$ are suitable orthonormal bases of the systems A and B, respectively. The reduced density matrix of system A is $\rho_A^\Psi \equiv \mathrm{tr}_B|\Psi\rangle\langle\Psi| = \sum_{i=1}^d \lambda_i|i\rangle\langle i|_A$, and similarly for B (the two reduced density matrices have the same eigenvalues λ_i). We have the following theorem [Nie99, NV01].

Theorem 5: State $|\Psi\rangle$ can be converted deterministically into state $|\Phi\rangle$ by means of LOCC if and only if $\boldsymbol{\lambda}_\Psi \prec \boldsymbol{\lambda}_\Phi$, where $\boldsymbol{\lambda}_\Psi$ is the vector of eigenvalues of $\rho_A^\Psi \equiv \mathrm{tr}_B|\Psi\rangle\langle\Psi|$ and similarly for $\boldsymbol{\lambda}_\Phi$.

For conciseness, we will write simply $\Psi \prec \Phi$ instead of $\boldsymbol{\lambda}_\Psi \prec \boldsymbol{\lambda}_\Phi$ in what follows. A consequence of this theorem is that $\Psi \prec \Phi$ if and only if $\mu(\Psi) \geq \mu(\Phi)$ for all measures of entanglement μ. A measure of entanglement, or entanglement monotone, is a non-negative function of the state which does not increase on average under LOCC and vanishes on separable states [Vid00]. A common example is the entropy of entanglement, $E(\Psi) \equiv S_1(\boldsymbol{\lambda}_\Psi)$. Since, according to Theorem 5, converting Ψ into Φ is possible when $\Psi \prec \Phi$, this means that $\mu(\Psi) \geq \mu(\Phi)$ must hold for all entanglement monotones. Conversely, the maximum probability of success of converting Ψ into Φ by means of an LOCC protocol satisfies $P(\Psi \to \Phi) \leq \min_\mu \frac{\mu(\Psi)}{\mu(\Phi)}$, where the minimum is taken over all entanglement monotones [Vid00]. Since a strategy exists where $P(\Psi \to \Phi) = 1$, this implies that $\mu(\Psi) \geq \mu(\Phi)$, $\forall \mu$.

According to Theorem 5, if $\boldsymbol{\lambda}_\Psi$ and $\boldsymbol{\lambda}_\Phi$ are incomparable, then there does not exist a strategy to convert one state into the other by LOCC with probability 1. Remarkably, it has been shown that one may nevertheless be able to accomplish such a transformation deterministically with the use of an auxiliary entangled state, an effect called catalysis [JP99]. If two states $|\Psi\rangle$ and $|\Phi\rangle$ have incomparable $\boldsymbol{\lambda}$ vectors, then, under certain conditions, there exists an entangled state $|C\rangle$ that the two parties can share, called a catalyst state, such that

$$|\Psi\rangle \otimes |C\rangle \underset{\text{LOCC}}{\longrightarrow} |\Phi\rangle \otimes |C\rangle \tag{4.11}$$

is possible. The term "catalysis" is justified because the entangled state $|C\rangle$ is recovered after the LOCC transformation. Note that if converting Ψ into Φ by catalysis is possible, then all *additive* measures of entanglement must satisfy $\mu(\Psi) \geq \mu(\Phi)$. In particular, we must have $S_\alpha(\boldsymbol{\lambda}_\Psi) \geq S_\alpha(\boldsymbol{\lambda}_\Phi)$ for all $\alpha \geq 0$.

4.1.4 Weak majorization

By relaxing the equation (4.4) in the definition of majorization we obtain the so-called weak majorization. We say that $\mathbf{p}$ is weakly majorized by $\mathbf{q}$, denored as $\mathbf{p} \prec_w \mathbf{q}$ if and only if

$$\sum_{i=1}^{k} p_i^{\downarrow} \leq \sum_{i=1}^{k} q_i^{\downarrow} \tag{4.12}$$

for $k = 1, \ldots, d$. As previously, there is an equivalent definition involving matrices, namely $\mathbf{p}$ is weakly majorized by $\mathbf{q}$ if and only if

$$\mathbf{p} = \mathbf{D}_w \mathbf{q}, \tag{4.13}$$

where $\mathbf{D}_w$ is a substochastic matrix, that is a non-negative matrix that has all his entries smaller than the elements of some doubly stochastic.

Also, similarly to case of majorization, $\mathbf{p} \prec \mathbf{q}$ if and only if $\sum_{i=1}^{d} h(p_i) \geq \sum_{i=1}^{d} h(q_i)$ for all *increasing* concave functions h.

At this point let us underline the role of the column stochasticity (the column's elements sum to unity) compared with the row stochasticity (the row's elements sum to unity).

A column stochastic matrix (some authors refer to it simply as stochastic) preserves the norm of the probability vector that acts upon, in other words it transforms probability vectors to probability vectors. The row stochasticity guarantees that the invariant $d-$dimesional vector, i.e. the maximally mixed vector $(\frac{1}{d}, \ldots, \frac{1}{d})$, is transformed to itself. There is an interesting known result concerning the case of infinite probability vectors. That is that weak majorization is enough to prove majorization [BZ07]. Since both vectors are normalized as probability vectors, the matrix that transforms the one vector to the other ought to be at least column stochastic, i.e a matrix matrix with positive entries, that its columns sum to unity while the rows sum to $|r| < 1$. In other words in the case of normalized infinite vectors, the row stochasticity can be dropped and replaced by row substochasticity.

4.2 Application of majorization theory to Gaussian bosonic transformations

4.2.1 Entanglement generation in a beam splitter

Now we are ready to proceed with the main target of this Chapter, that is the unfolding of the entropic characteristics of one fundamental quantum optical transformation, namely the beam splitter that has already been discussed in Chapter 2. More precisely we prove that a beam splitter, one of the most common optical components, fulfills several classes of majorization relations, which govern the amount of quantum entanglement that it can generate. First, we show that the state resulting from k photons impinging on a beam splitter majorizes the corresponding state with any larger photon number $k' > k$, implying that the entanglement monotonically grows with k. Then, we examine parametric infinitesimal majorization relations as a function of the beam-splitter transmittance, and find that there exists a parameter region where majorization is again fulfilled, implying a monotonic increase of entanglement by moving towards a balanced beam splitter. We also identify regions with a majorization default, where the output states become incomparable. In this latter situation, we find examples where catalysis may nevertheless be used in order to recover majorization. The catalyst states can be as simple as a path-entangled single-photon state or a two-mode vacuum squeezed state. Then we review the majorization properties of another fundamental quantum optical transformation, namely the quantum optical amplifier.

As we have discussed already, in quantum optics, one of the most common transformations consists in the linear coupling between two-modes of the electromagnetic field, as effected, for instance, by a beam splitter in bulk optics or an optical coupler in fiber optics [WM94]. We have seen that mathematically it corresponds to a rotation in phase space, namely

$$\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} \to \begin{pmatrix} \hat{a}' \\ \hat{b}' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}, \tag{4.14}$$

where $\hat{a}$ and $\hat{b}$ are the annihilation operators of the two-modes that are coupled, while the angle $\theta \in [0, \pi/2]$ is a coupling parameter related to the transmittance $\tau = \cos^2\theta$ of the beam splitter. The beam splitter is called balanced when $\tau = 1/2$ or $\theta = \pi/4$. The transformation (4.14) belongs to the set of Gaussian unitaries as it corresponds, in the Heisenberg picture, to a linear canonical transformation in the annihilation (creation) operators $\hat{a}^{(\dagger)}$ and $\hat{b}^{(\dagger)}$, or equivalently to a quadratic Hamiltonian, namely $H = i(\hat{a}^\dagger\hat{b} - \hat{a}\hat{b}^\dagger)$, see [WPG+12] for a review on Gaussian transformations.

The beam splitter is very conveniently modeled within the so-called symplectic formalism by focusing on the action of the rotation (4.14) on the first- and second-order moments of the quadrature operators. This enables treating complex optical circuits made of beam splitters and other optical devices in a very concise way, which is sufficient for many purposes, e.g. when the goal is to predict distributions in phase space such as Wigner functions. However, if we want to make predictions about entropies or entanglement, we need to move back to state space and work with density operators. Such calculations are often nontrivial despite the simplicity of the transformation in phase-space representation. For example, consider a single-photon state $|1\rangle$ impinging on a balanced beam splitter, the other input mode being in the vacuum state $|0\rangle$. The two-mode output state is obtained simply by inverting transformation (4.14) and writing the input-mode annihilation operators $\hat{a}$ and $\hat{b}$ as functions of the output-mode annihilation operators $\hat{a}'$ and $\hat{b}'$. The input state being $\hat{a}^\dagger|0\rangle$, we can write the output state as

$$2^{-1/2}(\hat{a}'^\dagger + \hat{b}'^\dagger)|0\rangle = (|1\rangle|0\rangle + |0\rangle|1\rangle)/\sqrt{2}, \tag{4.15}$$

which is well known to be a path-entangled state of one photon, characterized by an entanglement entropy of 1 bit as measured by the reduced von Neumann entropy of any output mode. However, whenever we consider higher photon-number states at the input and arbitrary transmittances, it becomes much harder to find closed formulas for the entanglement entropy.

As another example illustrating the difficulty of treating a beam splitter in state space, let us consider an arbitrary input state in mode $\hat{a}$ that is coupled to a thermal field in mode $\hat{b}$. The transformation $\hat{a} \to \hat{a}'$ can be viewed as a thermal bosonic channel, which is a special case of a Gaussian phase-insensitive bosonic channel [GGL+04a]. In order to derive the channel capacity, a crucial element is to determine the input state in mode

$\hat{a}$ that results in the minimum-entropy output state in mode $\hat{a}'$. Although it is very tempting to assume that this extremal input state is the vacuum state $|0\rangle$ [HG12], this has only recently been proven [GGC+14] after many years of elaboration. It is linked to the Holevo-Werner conjecture, which states that Gaussian mixtures of Gaussian states achieve the capacity of such Gaussian channels [HW01].

We exploit majorization theory in order to study the entanglement generated by an optical beam splitter and a two-mode squeezer. Majorization provides a preorder relation between bipartite pure quantum states and gives a necessary and sufficient condition for the existence of a deterministic LOCC (local operations and classical communication) transformation from one state to another [Nie99, NV01]. Here, we will show that a beam splitter obeys two classes of majorization relations, which bear some similarity to those characterizing another optical component, namely a two-mode squeezer [GNL+12]. Specifically, we will prove that when feeding the input mode of a beam splitter with k photons while the other input mode is in the vacuum state, the resulting two-mode output state majorizes the state corresponding to any larger photon number $k' > k$. This implies that any bipartite entanglement measure on the output modes increases with k in a monotonic fashion.

Then, we examine majorization relations with respect to the coupling parameter θ (or, equivalently, the transmittance τ) of the beam splitter. For a fixed photon-number input state $|k\rangle$ in one port and vacuum in the other port, we probe the existence of majorization relations between the output states corresponding to different θ's, which we call parametric majorization. We find that in a region of finite width, the output state with parameter θ majorizes the output state with $\theta' > \theta$, which implies that entanglement can only increase with the coupling between the two modes in this region. Interestingly, we also disprove parametric majorization in other regions of the parameter θ, which means that the corresponding output states are then incomparable. In some cases, however, we can prove that these incomparable output states can be catalyzed [JP99], that is, if we supplement both states with an appropriate catalyst state, then the new states become comparable (one is majorized by the other). Remarkably, the catalyst state can be as simple as a path-entangled single-photon state or a two-mode vacuum squeezed state. In order to study a first class of majorization relations characteristic of a beam splitter we use the definition of majorization involving doubly stochastic matrices. Let $|\Psi^{(k)}(\theta)\rangle$ be the output state of a beam splitter if the input state is $|k,0\rangle$, as shown in figure 4.2. Denoting by $\mathcal{U}(\theta)$ the unitary transformation resulting from the beam splitter, we have

$$|\Psi^{(k)}(\theta)\rangle = \mathcal{U}(\theta)\,|k,0\rangle = \sum_{n=0}^{k} \sqrt{P_n^{(k)}(\theta)}\,|n,k-n\rangle, \tag{4.16}$$

where

$$P_n^{(k)}(\theta) = \binom{k}{n} \cos^{2n}\theta \, \sin^{2(k-n)}\theta. \tag{4.17}$$

The reduced density matrix of the first output mode is

$$\rho^{(k)}(\theta) = \sum_{n=0}^{k} P_n^{(k)}(\theta)\, |n\rangle\langle n|, \tag{4.18}$$

where $P_n^{(k)}(\theta)$ can be interpreted as the probability that n photons are transmitted out of the k incident photons if the transmittance of the beam splitter is $\tau = \cos^2\theta$.

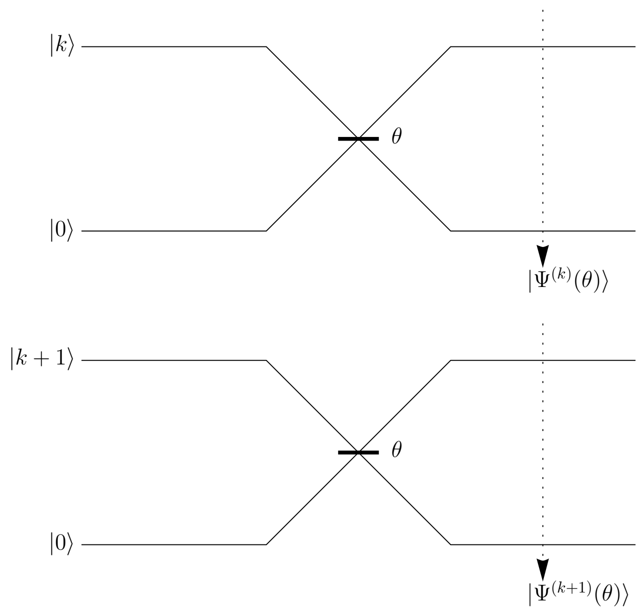

Figure 4.2: Majorization relations with respect to the input photon number. In the upper scheme, the input state is $|k, 0\rangle$, while in the lower scheme it is $|k+1, 0\rangle$. The output state $|\Psi^{(k)}(\theta)\rangle$ majorizes $|\Psi^{(k+1)}(\theta)\rangle$, and therefore the generated entanglement $\mu(\Psi^{(k)}(\theta))$ monotonically grows with k for all θ.

We wish to prove a majorization relation between $P_n^{(k)}(\theta)$ and $P_n^{(k+1)}(\theta)$, that is, we want to prove that there exists a doubly stochastic matrix $\mathbf{D}$ such that

$$\mathbf{P}^{(k+1)}(\theta) = \mathbf{D}^{(k+1)}\mathbf{P}^{(k)}(\theta), \tag{4.19}$$

where $\mathbf{P}^{(k)}(\theta)$ is a vector having the eigenvalues $P_n^{(k)}(\theta)$ of $\rho^{(k)}(\theta)$ as components. Using Pascal's identity for the binomial coefficients, we obtain the recurrence equation

$$\begin{aligned} P_n^{(k+1)}(\theta) &= \binom{k+1}{n} \cos^{2n}\theta \, \sin^{2(k+1-n)}\theta \\ &= \left(\binom{k}{n-1} + \binom{k}{n}\right) \cos^{2n}\theta \, \sin^{2(k+1-n)}\theta \\ &= P_{n-1}^{(k)}(\theta)\, \cos^2\theta + P_n^{(k)}(\theta)\, \sin^2\theta. \end{aligned} \tag{4.20}$$

This simply expresses the fact that the probability of transmitting n photons out of $k+1$ incident photons is the sum of two mutually exclusive possibilities: either $n-1$ photons (out of k) are transmitted and the $(k+1)$-th photon goes through, or n photons (out of k) are transmitted and the $(k+1)$-th photon is reflected. We can expand equation (4.20) as

$$\begin{aligned} P_0^{(k+1)}(\theta) &= 0 + \sin^2\theta\, P_0^{(k)}(\theta), \\ P_1^{(k+1)}(\theta) &= \cos^2\theta\, P_0^{(k)}(\theta) + \sin^2\theta\, P_1^{(k)}(\theta), \\ P_2^{(k+1)}(\theta) &= \cos^2\theta\, P_1^{(k)}(\theta) + \sin^2\theta\, P_2^{(k)}(\theta), \\ &\vdots \\ P_{k+1}^{(k+1)}(\theta) &= \cos^2\theta\, P_k^{(k)}(\theta) + 0. \end{aligned} \tag{4.21}$$

This can be put in the form of equation (4.19) by defining

$$\mathbf{P}^{(k+1)}(\theta) = \begin{pmatrix} P_0^{(k+1)}(\theta) \\ P_1^{(k+1)}(\theta) \\ P_2^{(k+1)}(\theta) \\ \vdots \\ P_k^{(k+1)}(\theta) \\ P_{k+1}^{(k+1)}(\theta) \end{pmatrix}, \; \mathbf{P}^{(k)}(\theta) = \begin{pmatrix} P_0^{(k)}(\theta) \\ P_1^{(k)}(\theta) \\ P_2^{(k)}(\theta) \\ \vdots \\ P_k^{(k)}(\theta) \\ 0 \end{pmatrix}, \tag{4.22}$$

where a zero entry has been inserted in the vector $\mathbf{P}^{(k)}$ in order to make $\mathbf{P}^{(k)}$ and $\mathbf{P}^{(k+1)}$ of equal dimension. The doubly stochastic matrix is

$$\mathbf{D}^{(k+1)} = \begin{pmatrix} \sin^2\theta & 0 & 0 & \cdots & \cos^2\theta \\ \cos^2\theta & \sin^2\theta & 0 & \cdots & 0 \\ 0 & \cos^2\theta & \sin^2\theta & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sin^2\theta \end{pmatrix}, \tag{4.23}$$

where the right-most entry of the fist row has been chosen so as to fulfill the doubly-stochastic conditions (it plays no role since the last entry of $\mathbf{P}^{(k)}$ is zero).

Thus, we have proven the majorization relation

$$\Psi^{(k+1)}(\theta) \prec \Psi^{(k)}(\theta), \qquad \forall\theta, \tag{4.24}$$

which implies that when increasing the number of the incident photons, the 2-mode output state can only be more entangled and the corresponding 1-mode reduced states are getting more disordered. This also implies that the two parties (Alice and Bob) can convert state $|\Psi^{(k+1)}(\theta)\rangle$ into state $|\Psi^{(k)}(\theta)\rangle$ by using a deterministic LOCC transformation. To achieve such a transformation, Alice can, for example, perform a two-outcome POVM (Positive Operator Valued Measure) measurement with the following Kraus operators,,

$$\mathcal{F}_1^{(k)} = \sum_{n=0}^{k} \sqrt{\frac{k+1-n}{k+1}} |n\rangle\langle n| \tag{4.25}$$

and

$$\mathcal{F}_2^{(k)} = \sum_{n=0}^{k} \sqrt{\frac{n+1}{k+1}} |n\rangle\langle n+1|\,, \tag{4.26}$$

satisfying $\mathcal{F}_1^\dagger \mathcal{F}_1 + \mathcal{F}_2^\dagger \mathcal{F}_2 = \mathbb{I}$ for all k. Then, she must communicate her outcome to Bob, who has to apply proper local unitaries. If outcome 1 occurs, then Bob should apply the unitary

$$\mathcal{U}_1^{(k)} = \sum_{n=0}^{k} |n\rangle\langle n+1| + |k+1\rangle\langle 0| \tag{4.27}$$

corresponding to a cyclic shift in the space spanned by $\{|0\rangle, \cdots |k+1\rangle\}$. The second term on the right-hand side of equation (4.27) ensures unitarity (it plays no role since Bob's reduced state is supported by $\{|1\rangle, \cdots |k+1\rangle\}$ when outcome 1 occurs). If outcome 2 occurs, then Bob does nothing, that is, he applies the unitary $\mathcal{U}_2^{(k)} = \mathbb{I}$. It is easy to check that the transformation $|\Psi^{(k+1)}(\theta)\rangle \to |\Psi^{(k)}(\theta)\rangle$ takes place for both outcomes, so the LOCC transformation is indeed deterministic.

We now examine the scenario that is summarized in figures 4.3 and 4.4. The input state is fixed to $|k, 0\rangle$, but we change the angle θ parameterizing the transmittance of the beam splitter by an infinitesimal amount ε. Note that we take $\theta \geq 0$, $\varepsilon > 0$, and $\theta + \varepsilon \leq \frac{\pi}{4}$. (For angles greater than $\frac{\pi}{4}$, the transmittance and reflectance just interchange their roles.) An equivalent way to see this scenario is depicted in figure 4.4. Our goal is thus to probe whether the intermediate state $|\Psi^{(k)}(\theta)\rangle$ majorizes or not the final state

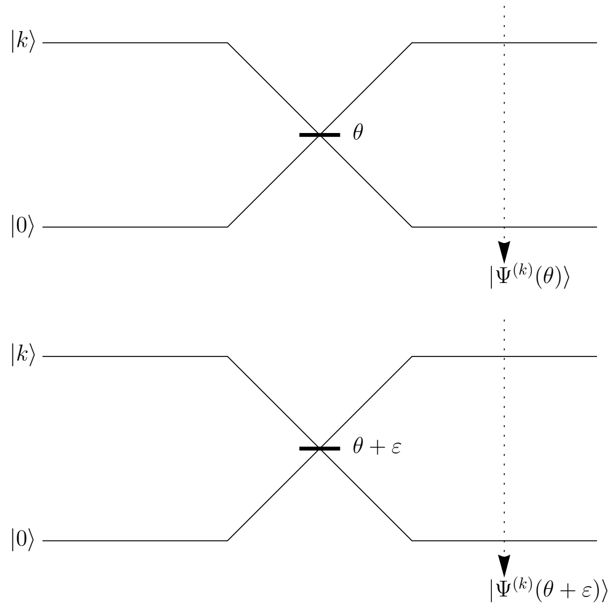

Figure 4.3: Majorization relations with respect to the coupling parameter θ (or transmittance $\tau = \cos^2\theta$). In both schemes the input state is $|k, 0\rangle$ but the angles differ by ε. For a specific parameter region, the corresponding output states $|\Psi^{(k)}(\theta)\rangle$ and $|\Psi^{(k)}(\theta+\varepsilon)\rangle$ are proven to satisfy a majorization relation.

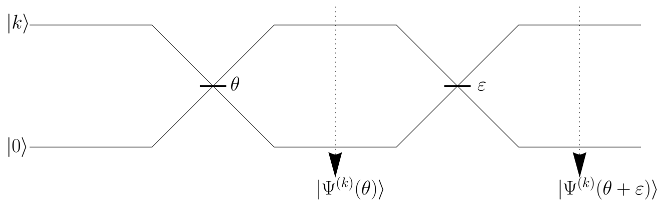

Figure 4.4: Same situation as in figure 4.3, viewed as a sequence of two beam splitters with angles θ and ε. For a specific parameter region, $|\Psi^{(k)}(\theta)\rangle$ majorizes $|\Psi^{(k)}(\theta+\varepsilon)\rangle$ for all k.

$|\Psi^{(k)}(\theta+\varepsilon)\rangle$. To this end, we find it easier to use the first definition of majorization, involving the accumulations of the ordered vectors of eigenvalues of the reduced density matrix. We will refer to these vectors as OSC (ordered Schmidt coefficients).

Let $\mathbf{P}^{\downarrow}(\theta)$ be an OSC vector, whose components are the elements of the binomial distribution (4.17). From now on, we drop the index k as it is fixed. This OSC vector will not keep the same ordering as the parameter θ varies, so we will adopt the notation $\mathbf{P}^{\downarrow r}(\theta)$, where $r = 1, 2, \ldots$ labels the regions of parameter θ in which the ordering of the OSC vector remains the same. More precisely, we have a change of ordering every time two eigenvalues $P_n(\theta)$ and $P_m(\theta)$ are equal, which occurs at

$$\theta = \arctan\left(\frac{(k-n)!n!}{(k-m)!m!}\right)^{\frac{1}{2(n-m)}}. \tag{4.28}$$

Indeed, it can be shown that if $P_n(\theta) = P_m(\theta)$ for $m \neq n$, then $\frac{dP_n(\theta)}{d\theta} \neq \frac{dP_m(\theta)}{d\theta}$, i.e., the two eigenvalues cross. The cross-over points between regions are the different solutions $\theta_1 < \theta_2 < ...$ of the above equations. We define the region $r = 1$ as $\theta \in [0, \theta_1)$, the region $r = 2$ as $\theta \in [\theta_1, \theta_2)$, etc.

Our goal now is to check whether the infinitesimal majorization relation

$$\mathbf{P}^{\downarrow r}(\theta + \varepsilon) \prec \mathbf{P}^{\downarrow r}(\theta) \tag{4.29}$$

holds or not within region r, taking the limit of an infinitesimal angle ε. Using the definition of equation (4.3), we have to prove

$$\begin{aligned} &\sum_{n=0}^{j} P_n^{\downarrow r}(\theta + \varepsilon) \leq \sum_{n=0}^{j} P_n^{\downarrow r}(\theta), \qquad j = 0, \dots k-1 \\ \Leftrightarrow \quad &\sum_{n=0}^{j} \frac{P_n^{\downarrow r}(\theta)}{d\theta} \leq 0, \qquad j = 0, \dots k-1. \end{aligned} \tag{4.30}$$

By defining the vector of accumulation derivatives

$$a_j^{\downarrow r}(\theta) = \sum_{n=0}^{j} \frac{dP_n^{\downarrow r}(\theta)}{d\theta}, \tag{4.31}$$

the infinitesimal majorization relations can be written simply as

$$a_j^{\downarrow r}(\theta) \leq 0, \qquad j = 0, \dots k-1. \tag{4.32}$$

We do not need to consider the last accumulation derivative $a_k^{\downarrow r}(\theta) = 0$ since equation (4.4) is always satisfied (the OSC vectors are normalized).

The violation of at least one relation in equations. (4.32) is sufficient to disproof majorization in region r. A priori, if the above majorization relations do not hold, there may nevertheless be a majorization in the opposite direction if all relations are satisfied with $\geq$ instead of $\leq$. However, the $(k-1)$-th accumulation is the same in all regions no matter what the ordering is, and its derivative

$$a_{k-1}(\theta) = -2k \sin^{2k-1}\theta \, \cos\theta \tag{4.33}$$

respects equation (4.32) with a strict inequality (except in the trivial cases $k = 0$ or $\theta = 0$). Hence, majorization is never possible in the opposite direction.

It is easy to see that in the region $r = 1$, the components of the OSC vector are

$$P_n^{\downarrow 1}(\theta) = \binom{k}{n} \sin^{2n}\theta \cos^{2(k-n)}\theta. \tag{4.34}$$

This region extends until the second largest eigenvalue becomes equal to the largest eigenvalue and the two switch places. Beyond this cross-over point, we enter the second region $r = 2$. Now, let us prove that the infinitesimal majorization relation $\mathbf{P}^{\downarrow 1}(\theta+\varepsilon) \prec \mathbf{P}^{\downarrow 1}(\theta)$ always holds in region $r = 1$. We have

$$\frac{dP_n^{\downarrow 1}(\theta)}{d\theta} = \left[\frac{2n}{\tan\theta} - 2(k-n)\tan\theta\right] P_n^{\downarrow 1}(\theta) \tag{4.35}$$

and

$$P_n^{\downarrow 1}(\theta) = P_0^{\downarrow 1}(\theta) \binom{k}{n} \tan^{2n}\theta, \tag{4.36}$$

from which we can express the vector of accumulation derivatives as

$$a_j^{\downarrow 1}(\theta) = P_0^{\downarrow 1}(\theta) \sum_{n=0}^{j} \left[2n - 2(k-n)\tan^2\theta\right] \binom{k}{n} \tan^{2n-1}\theta. \tag{4.37}$$

The summation in equation (4.37) can be expressed in a closed form as

$$a_j^{\downarrow 1}(\theta) = -P_0^{\downarrow 1}(\theta)\, 2(k-j) \binom{k}{j} \tan^{2j+1}\theta, \tag{4.38}$$

which is non-positive for $j = 0, \ldots k-1$. Thus, infinitesimal majorization always holds within region $r = 1$, which means that all states are comparable within this region,

$$\Psi^{(k)}(\theta+\varepsilon) \prec \Psi^{(k)}(\theta), \qquad \forall k \geq 0, \tag{4.39}$$

and all measures of entanglement increase with θ.

At the cross-over point between regions $r = 1$ and $r = 2$, the first two components of the OSC vector switch places and the derivative of the first accumulation becomes $\frac{dP_1(\theta)}{d\theta}$, which is positive at this point. Therefore, majorization is violated from the left boundary of region $r = 2$ until the point where this derivative ceases to be positive (and possibly beyond that point). In general, in every region that begins with a positive derivative of the first accumulation, which is equal to $\frac{dP_n(\theta)}{d\theta}$ for some n, majorization is violated at least until the point where this derivative ceases to be positive. From equations (4.35) and (4.36), we find that the derivative $\frac{dP_n(\theta)}{d\theta}$ ceases to be positive up to the value $\theta_n^+ \leq \arctan\left(\frac{n}{k-n}\right)^{\frac{1}{2n-1}}$.

In order to illustrate how parametric majorization behaves, let us exhibit three examples, corresponding respectively to a single-photon, two-photon, and three-photon state impinging on the beam splitter.

Let us proceed with some examples:

Example 1:
We first consider the case of a single photon ($k = 1$). The OSC vector in region $r = 1$ is

$$\mathbf{P}^{\downarrow 1}(\theta) = \begin{pmatrix} \cos^2\theta \\ \sin^2\theta \end{pmatrix}. \tag{4.40}$$

In order to find all possible regions we have to find all solutions of $\cos^2\theta = \sin^2\theta$ in $[0, \frac{\pi}{4})$.There is no solution in this region (equality is reached at the boundary point $\theta = \frac{\pi}{4}$), which means that there is a single region $r = 1$ and, as we proved earlier, parametric majorization holds everywhere.

Example 2:
We now move to the case of two photons ($k = 2$). The OSC vector in region $r = 1$ is

$$\mathbf{P}^{\downarrow 1}(\theta) = \begin{pmatrix} \cos^4\theta \\ 2\cos^2\theta\sin^2\theta \\ \sin^4\theta \end{pmatrix}, \tag{4.41}$$

where this ordering holds for $[0, \arctan\frac{1}{\sqrt{2}})$. There is a second region $r = 2$ corresponding to $[\arctan\frac{1}{\sqrt{2}}, \frac{\pi}{4})$, where the OSC vector is

$$\mathbf{P}^{\downarrow 2}(\theta) = \begin{pmatrix} 2\cos^2\theta\sin^2\theta \\ \cos^4\theta \\ \sin^4\theta \end{pmatrix}. \tag{4.42}$$

As proven in full generality, majorization holds in region $r = 1$. However, in region $r = 2$, the accumulation derivatives are given by

$$\begin{aligned} a_0^{\downarrow 2}(\theta) &= \sin 4\theta \\ a_1^{\downarrow 2}(\theta) &= -4\cos\theta\sin^3\theta. \end{aligned} \tag{4.43}$$

The accumulation derivative $a_0(\theta)$ is positive in $[0, \frac{\pi}{4})$, so majorization does not hold in the entire region $r = 2$. This means that there ought to be measures of disorder that decrease instead of increase as a function of θ in region $r = 2$. Indeed, we observe in figure 4.5 that although the Shannon entropy increases, all other Rényi entropies of order

$\alpha > 1$ exhibit a decreasing behavior somewhere within the region $r = 2$. In particular, the Rényi entropy of order $\alpha \to \infty$, which is the min-entropy and is directly related to the leading probability of the OSC vector, starts decreasing immediately when we enter the second region at $\theta = \arctan \frac{1}{\sqrt{2}}$.

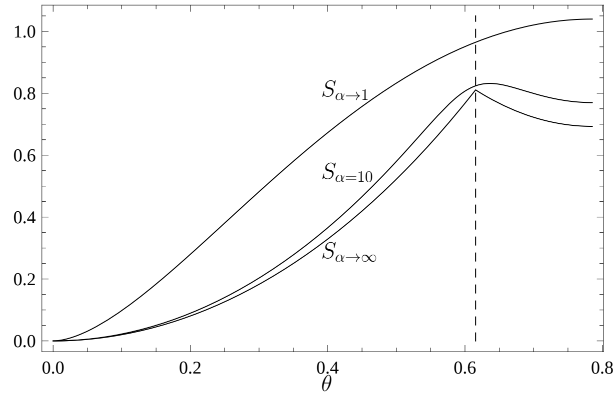

Figure 4.5: Entanglement Rényi entropies resulting from a 2-photon state impinging on a beam splitter as a function of the coupling parameter θ (related to the transmittance $\tau = \cos^2\theta$). The vertical dashed line denotes the boundary between parameter regions $r = 1$ and $r = 2$. The von Neumann entropy ($\alpha \to 1$) keeps increasing in region $r = 2$, while higher-order Rényi entropies have a different behavior and start decreasing somewhere in region $r = 2$. The min-entropy ($\alpha \to \infty$) exhibits a non-differentiable point right at the crossover point and decreases throughout the entire region $r = 2$, reflecting the default of majorization.

Example 3:

As a last example, we consider the case of three photons ($k = 3$). We have two cross-over angles and the OSC vectors in the corresponding three regions read,

$$\begin{aligned}
\mathbf{P}^{\downarrow 1}(\theta) &= \begin{pmatrix} \cos^6\theta \\ 3\cos^4\theta\sin^2\theta \\ 3\cos^2\theta\sin^4\theta \\ \sin^6\theta \end{pmatrix} \\
\mathbf{P}^{\downarrow 2}(\theta) &= \begin{pmatrix} 3\cos^4\theta\sin^2\theta \\ \cos^6\theta \\ 3\cos^2\theta\sin^4\theta \\ \sin^6\theta \end{pmatrix} \\
\mathbf{P}^{\downarrow 3}(\theta) &= \begin{pmatrix} 3\cos^4\theta\sin^2\theta \\ 3\cos^2\theta\sin^4\theta \\ \cos^6\theta \\ \sin^6\theta \end{pmatrix}.
\end{aligned} \tag{4.44}$$

In region $r = 1$, it is easy to confirm that majorization holds, as it should. In regions $r = 2$ and $r = 3$, the accumulation derivatives are given, respectively, by

$$\begin{aligned} a_0^{\downarrow 2}(\theta) &= 3\cos^3\theta(-1+3\cos 2\theta)\sin\theta \\ a_1^{\downarrow 2}(\theta) &= -\frac{3}{2}\sin^3 2\theta \\ a_2^{\downarrow 2}(\theta) &= -6\cos\theta\sin^5\theta \end{aligned} \tag{4.45}$$

and

$$\begin{aligned} a_0^{\downarrow 3}(\theta) &= 3\cos^3\theta(-1+3\cos 2\theta)\sin\theta \\ a_1^{\downarrow 3}(\theta) &= \frac{3}{2}\sin 4\theta \\ a_2^{\downarrow 3}(\theta) &= -6\cos\theta\sin^5\theta. \end{aligned} \tag{4.46}$$

The quantity $a_0^{\downarrow 2}(\theta)$ is positive in the interval $[0, \arctan\frac{1}{\sqrt{2}})$, which means that we are sure that there is no majorization in the interval $[\arctan\frac{1}{\sqrt{3}}, \arctan\frac{1}{\sqrt{2}})$, i.e., from the left boundary of region $r = 2$ up to where $a_0^{\downarrow 2}(\theta)$ remains positive. In region $r = 3$, $a_2^{\downarrow 3}(\theta)$ is always positive, while the other accumulation derivatives are negative within this region. Hence, for $r = 3$, the states are always incomparable. This last violation of majorization is, however, not visible with the Rényi entropies. In figure 4.6, we display the evolution of entropies across the three regions.

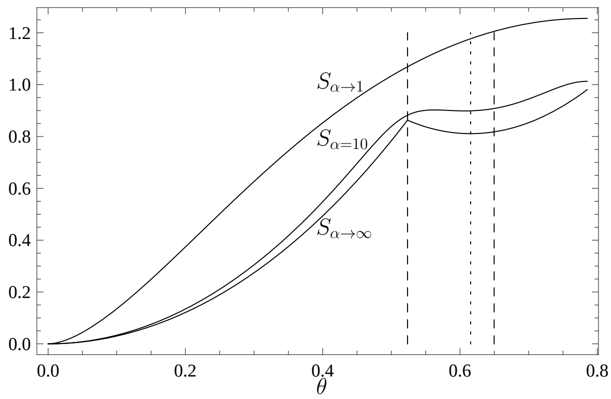

Figure 4.6: Entanglement Rényi entropies resulting from a 3-photon state impinging on a beam splitter as a function of θ. The two vertical dashed lines at θ_1 and θ_2 separate the three regions $r = 1, 2, 3$, while the dotted line corresponds to the local minimum of the min-entropy. Majorization is violated in the region $r = 2$ from the left boundary of this region at θ_1 up to the local minimum of the min-entropy. The majorization violation throughout the entire region $r = 3$ is not manifested by the behavior of the Rényi entropies.

We have shown that one can always expect a default of majorization when the leading

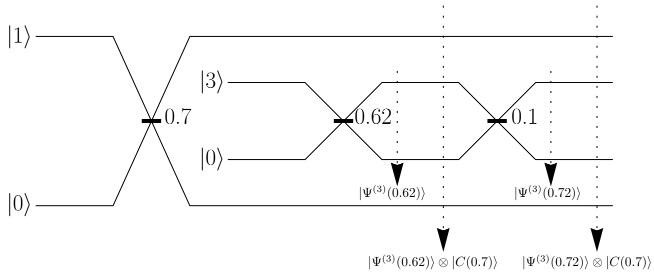

Figure 4.7: Schematic of the catalyzed conversion between the incomparable states resulting from a 3-photon state impinging on a beam splitter with angles $\theta = 0.62$ and $\theta + \omega = 0.72$. The catalyst is the entangled state obtained from a beam splitter with angle $\theta = 0.7$ and one single-photon input state.

term in the probability vector changes, and this majorization default prevails at least up to the point where the first accumulation derivative ceases to be positive, or, equivalently, until we reach the local minimum of the min-entropy within this region. Beyond the case where the first two components of the probability vector switch places, it appears difficult to provide general rules for predicting the existence or absence of majorization for an arbitrary k, and one has to treat the problem on a case-by-case basis. The situation also becomes more complicated if we take a non-infinitesimal angle ε such that the pair of angles θ and $\theta + \varepsilon$ belong to different regions.

It is natural to ask whether the incomparable states that occur when we change the parameter θ can nevertheless be made comparable through catalysis. We will show that this is indeed possible in certain cases, and will provide an example for this. Note that not all incomparable states can be catalyzed: some necessary conditions have to be respected [JP99, DK01, ZG00, SDT05]. To solve the problem of whether catalysis is possible or not in generality is difficult due to the fact that one has to reorder the vector resulting from the tensor product of the state to be catalyzed and the catalyst state.

Consider two angles θ and $\theta + \omega$ that give OSC vectors $\mathbf{P}^{\downarrow r}(\theta)$ and $\mathbf{P}^{\downarrow r'}(\theta + \omega)$ which are incomparable. These angles may be within different ordering regions, which is the case in the following example where $r = 2$ and $r' = 3$. We take $k = 3$, $\theta = 0.62$, and $\omega = 0.10$, which gives

$$\mathbf{P}^{\downarrow 2}(0.62) = \begin{pmatrix} 0.44439 \\ 0.290641 \\ 0.226491 \\ 0.0384782 \end{pmatrix} \tag{4.47}$$

and

$$\mathbf{P}^{\downarrow 3}(0.72) = \begin{pmatrix} 0.416698 \\ 0.320544 \\ 0.180565 \\ 0.0821927 \end{pmatrix} \tag{4.48}$$

such that $\mathbf{P}^{\downarrow 3}(0.72) \not\prec \mathbf{P}^{\downarrow 2}(0.62)$. The path-entangled single-photon state $\cos\theta|1\rangle|0\rangle + \sin\theta|0\rangle|1\rangle$ resulting from a single photon impinging on a beam splitter with angle $\theta = 0.7$ is sufficient to serve as a catalyst for these two states. It corresponds to the binary probability vector

$$\mathbf{C}(0.7) = \begin{pmatrix} 0.584984 \\ 0.415016 \end{pmatrix}, \tag{4.49}$$

and one can easily verify from equations (4.47), (4.48), and (4.49), that $\mathbf{P}^{\downarrow 3}(0.72) \otimes \mathbf{C}(0.7) \prec \mathbf{P}^{\downarrow 2}(0.62) \otimes \mathbf{C}(0.7)$, implying that the catalyzed conversion is possible. This is summarized in figure 4.7. We could also use as a catalyst a two-mode squeezed vacuum state $\propto \sum_{n=0}^{\infty} \tanh^n r\,|n\rangle|n\rangle$ with squeezing parameter $r = 1.38$. Several other numerical examples can be found and some of them, like the ones provided above, are experimentally accessible.

Note that, due to the additivity of the Rényi entropies, one should look for catalyzable incomparable states only in regions where all of the Rényi entropies increase (see also Refs. [AN08, DFL+05]). However, in the case at hand, we have got numerical evidence that the sole behavior of the min-entropy seems to give a necessary and sufficient condition for the existence of catalysis. The latter property will be examined in a forthcoming work.

4.2.2 Entanglement generation in a two-mode squeezer

For the sake of completeness we will briefly discuss the majorization relations that hold in a two-mode squeezer for Fock states input. The main target of the work [GNL+12] where such a scenario is explored is to prove the so-called minimum entropy conjecture mentioned before, therefore to prove that maximum capacity of Gaussian channel is achieved by Gaussian input states. This was proven later [GGC+14] but we will not discuss such a result since our purpose is to explore the entropic characteristics of quantum optical operation mainly for non classical states.

We consider a two-mode squeezer that was introduced in chapter 2,

$$\hat{U} = \mathrm{e}^{-\xi \hat{a}_A^\dagger \hat{a}_B^\dagger + \xi^* \hat{a}_A \hat{a}_B} \tag{4.50}$$

where as before, A is the system under consideration while B is the mode that we trace out in the output. It can be shown that the final pure state of a two-mode squeezer, when the input state is Fock and vacuum product state $|m0\rangle$,

$$|\Psi_\lambda^{(m)}\rangle = \sum_{k=0}^{\infty} \sqrt{p_k^{(m)}(\lambda)}|k+m, k\rangle \tag{4.51}$$

with,

$$p_k^{(m)}(\lambda) = (1-\lambda^2)^{m+1}\lambda^{2k}\binom{k+m}{k} \tag{4.52}$$

and we denote $\lambda = \tanh|\xi|$.

We want to compare in terms of majorization the two situations corresponding to m and $m+1$ photons. It easy to see that the probabilities in (4.52) are the eigenvalues of the reduced density matrix in the output. Also they form an infinite-dimensional vector $\mathbf{p}^{(m)}$, so if weak majorization is proven among $\mathbf{p}^{(m+1)}$ and $\mathbf{p}^{(m)}$ is enough for majorization relations to hold as well. Indeed one can find,

$$\mathbf{p}^{(m+1)} = \mathbf{D}_c \mathbf{p}^{(m)} \tag{4.53}$$

where D_c is a column stochastic matrix of the form,

$$\mathbf{D}_c = (1-\lambda^2)\begin{pmatrix} 1 & 0 & 0 & \dots \\ \lambda^2 & 1 & 0 & \dots \\ \lambda^4 & \lambda^2 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.54}$$

The elements of the matrix in (4.54) can be shown that are given by,

$$D_{c_{ij}} = (1-\lambda^2)\lambda^{2(i-j)}\Theta(i-j) \tag{4.55}$$

where $\Theta(x) = 1$ for $x \geq 0$ and $\Theta(x) = 0$ otherwise. It is easy to see that D_c is column stochastic therefore $\mathbf{p}^{(m+1)} \prec \mathbf{p}^{(m)}$, just like in the case where the interaction between the modes was a beam splitter.

We now move to the second class of majorization. The input state is fixed to $|m0\rangle$ and we compare the two reduced output states, one that corresponds to λ and the other that corresponds to $\lambda' < \lambda$. It can be shown that the following relation holds,

$$\mathbf{p}^{(m)}(\lambda) = \mathbf{F}_c^{(m)}(\lambda, \lambda')\mathbf{p}^{(m)}(\lambda') \tag{4.56}$$

where the elements of the column stochastic matrix $\mathbf{F}_c(\lambda, \lambda')$ are,

$$F_{c_{ij}}^{(m)}(\lambda, \lambda') = \binom{j+m}{m}^{-1} \left(\frac{1-\lambda^2}{1-\lambda'^2}\right)(L_{i-j}^{(m,j)}\lambda^2 - L_{i-j-1}^{(m,j+1)}\lambda'^2)\lambda^{1(i-j-1)}\Theta(i-j) \tag{4.57}$$

with

$$L_j^{(m,i)} = i\binom{i+m}{m}\binom{j+m}{m}\lambda'^{-2i}B(\lambda'^2; i, 1+m) \tag{4.58}$$

and $B(z;a,b) = \int_0^z dx x^{a-1}(1-x)^{b-1}$ is the incomplete beta function.

Equation (4.56) and the column stochasticity of matrix $\mathbf{F}_c(\lambda, \lambda')$ provide that the majorization relations $\mathbf{p}^{(m)}(\lambda) \prec \mathbf{p}^{(m)}(\lambda')$, $\forall \lambda' < \lambda$ are valid. This is in contrast to the result provided for the parametric default of majorization relation in a beam splitter. Since there are valid majorization relations one can find LOCC transformations from the more random state to the less random one [GNL+12] but we not will consider them here.

4.3 Conclusions

We have found several classes of majorization relations characterizing a beam splitter, or more generally the linear coupling between a pair of bosonic modes. More formally, we have proven that the passive Bogoliubov transformation of equation (4.14) fulfills some majorization relations, which enable comparing the output states corresponding to various input photon numbers k as well as various coupling parameters θ (or transmittances $\tau = \cos^2\theta$). We have reviewed the majorization relations that prevail with an active Bogoliubov transformation (a two-mode squeezer or parametric amplifier) [GNL+12]. Interestingly and counter-intuitively, the behavior of the passive and active Bogoliubov transformations are different in terms of partial ordering of the reduced outputs.

We have shown that for any value of the transmittance parameter θ, the output states resulting from injecting Fock states $|k\rangle$ in one port of the beam splitter and vacuum $|0\rangle$ in the other port obey a chain of majorization relations $\Psi^{(k+1)}(\theta) \prec \Psi^{(k)}(\theta)$, for all $k \geq 0$. As a consequence, the output states can only be more entangled when increasing the number of incident photons, and we have found an explicit deterministic LOCC transformation that maps $\Psi^{(k+1)}(\theta)$ onto $\Psi^{(k)}(\theta)$.

In contrast, we have found that the situation is more complicated when varying the parameter θ and keeping k constant. In that case, we have shown that there exists a first region in the space of parameter θ where a parametric infinitesimal majorization relation holds, by taking the limit of an infinitesimal angle $\varepsilon > 0$ for any $k \geq 0$, namely $\Psi^{(k)}(\theta + \varepsilon) \prec \Psi^{(k)}(\theta)$. This implies a monotonic increase of the entanglement of the output states when decreasing the transmittance and moving towards a balanced beam splitter. However, beyond some value of the parameter θ, we have shown the existence of a default of majorization, which occurs because the ordering of the OSC vectors changes in such a way that the leading probability is replaced by another one. This majorization default holds from the left boundary of this new ordering region at least up to the local minimum of the min-entropy. Moreover, by examining specific examples, we have shown that one may find more violations of majorization for non-infinitesimal angles ε within the same ordering region or between different ordering regions.

Finally, we have provided an example of two incomparable states, resulting from different values of θ, whose conversion can nevertheless be catalyzed with the help of an experimentally accessible state, such as a single-photon path-entangled state or a two-mode squeezed vacuum state. Catalysis schemes like the one in figure 4.7 may potentially be used for authentication protocols based on entanglement-assisted LOCC [Bar99, JS00]. Further investigations should also include a more general solution to the catalysis process in the parameter-varying case, the analysis of majorization relations in more complicated optical circuits in the spirit of [LM02] or the application in the context of the non-classicality of quantum states [ACR05]. More ambitiously, one may address phase transitions and critical phenomena in a field-theoretical approach [ZBF+06] under the prism of parametric majorization, where the parameter could be the temperature of a thermal field (Section 5.3 and [GKK13]).

Chapter 5

von Neumann entropy revisited

5.1 The replica method

5.1.1 Calculating the von Neumann entropy using the moments

Calculating the von Neumann entropy $S(\hat{\rho}) = -\mathrm{tr}(\hat{\rho}\ln\hat{\rho})$ of a bosonic mode that is found in state $\hat{\rho}$ is often an intractable task because it requires finding the infinite vector of eigenvalues of $\hat{\rho}$. This can sometimes be circumvented by using the replica method, which relies on the identity $\log Z = \lim_{n\to 0^+}(Z^n - 1)/n$. Using $x\log x = \lim_{n\to 1^+}(x^n - x)/(n-1) = \frac{\partial}{\partial n}(x^n)\big|_{n=1^+}$, we may reexpress the von Neumann entropy as

$$S(\hat{\rho}) = -\frac{\partial\,\mathrm{tr}(\hat{\rho}^n)}{\partial n}\Big|_{n=1^+} \tag{5.1}$$

The trick is to find an analytical expression of $\mathrm{tr}(\hat{\rho}^n)$ as a function of $n \in \mathbb{N}^*$ and then computing the derivative at $n = 1$, avoiding the need to diagonalize $\hat{\rho}$. This method also makes apparent the connection between the von Neumann entropy and other widely used measures of disorder, such as Tsallis and Rényi entropies. It has been used with great success in the context of spin glasses and quantum field theory [MPV87, CW94, HLW94, CC04, CC05, RT06, BB08, GKK13], being often justified based on the analyticity of $\mathrm{tr}(\hat{\rho}^n)$ and its derivative with respect to n in the neighborhood of $n = 1$ [CC04, CC05].

To be more precise, the replica method gives the mean value $\langle\ln\hat{A}\rangle$ of some Hermitian operator $\hat{A}$ by the use of the moments $\langle\hat{A}^n\rangle$. Therefore, in order for the replica method to be applicable, two conditions should be satisfied. First, one must ensure that the moments $\langle\hat{A}^n\rangle$ carry all the information for the corresponding probability distribution, and therefore it makes sense to use the moments to find any other mean value. When considering $\hat{A} = \hat{\rho}$, where $\hat{\rho}$ is some density matrix, this is always true. Something that is shown using the Hausdorff moment problem. Second, once one has obtained the $\mathrm{tr}(\hat{\rho}^n) = f(n)$ one assumes that the function $f(n),\ n \in \mathbb{N}^*$ can be expanded to reals, i.e. $f(x),\ x \in \mathbb{R}$. Many authors refer to this step as analytic continuation, but this is not correct since it is not mathematically correct to do analytic continuation from the positive integers to reals. There are examples where the so-called *replica symmetry breaking* occurs, i.e $f(n) \neq f(x)$. These examples do not concern density matrices though.

Now let us argue that the knowledge of $\mathrm{tr}(\hat{\rho}^n),\ \forall\ n \in \mathbb{N}^*$, is equal to the knowledge of the eigenvalues of the density matrix $\hat{\rho}$. In what follows, we define as moments the quantity $m_n = \mathrm{tr}(\hat{\rho}^n)$. Hausdorff's moment problem states that if X is a random variable in the interval $[0, 1]$, then the integer-order moments

$$m_n \equiv \mathbb{E}(X^{n-1}) \tag{5.2}$$

uniquely determine the distribution $P(X)$ if and only if

$$(-1)^k(\Delta^k m)_n \geq 0,\ \forall\ k \in \mathbb{N}^* \tag{5.3}$$

where

$$(\Delta^k m)_n = \sum_{j=0}^{k} (-1)^j \binom{k}{i} m_{n+k-i}. \tag{5.4}$$

In the case at hand, we will consider as random variable the eigenvalues λ_i of the (hermitian) density operator, that is $\Lambda \in [0, 1]$ with

$$P(\Lambda = \lambda_i) = \lambda_i \tag{5.5}$$

The moments,

$$m_n = \mathbb{E}(\hat{\rho}^{n-1}) = \mathrm{tr}(\hat{\rho}^{n-1}\hat{\rho}) = \mathrm{tr}(\hat{\rho}^n) \tag{5.6}$$

are easily found to satisfy,

$$(-1)^k(\Delta^k m)_n = \sum_j \lambda_j^n (1 - p_j)^k > 0. \tag{5.7}$$

Therefore the knowledge of the moments uniquely determines the density operator in the basis of its eigenvectors, or equivalently the eigenvalues of the density operator. Thus, we can find in principle any expectation value from the knowledge of $\mathrm{tr}(\hat{\rho}^n)$ for $n \in \mathbb{N}^*$. What we need is a recipe to derive the von Neumann entropy from the moments. An obvious choice comes from observing that if we treat n as a real variable we have,

$$-\left.\frac{\partial \mathrm{tr}(\hat{\rho}^n)}{\partial n}\right|_{n=1} = -\mathrm{tr}(\hat{\rho}\ln\hat{\rho}) = S. \tag{5.8}$$

Here we should make two important remarks. First, the trick played in equation (5.8) may not be unique; there could be other recipes to determine the von Neumann entropy. What is important is that Hausdorff's moment problem guarantees that these other recipes would give the same result as equation (5.8). Second, extra care should be taken regarding the following fact. During the calculation of $\mathrm{tr}(\hat{\rho}^n)$, we consider n to be a natural number in $\mathbb{N}^*$. We only consider n to be real in order to apply the step in equation (5.8), but this does not imply that $\mathrm{tr}(\hat{\rho}^x)$, with $x \in \mathbb{R}$, is found by simply substituting the natural variable n with the real variable x. What Hausdorff's moment problem guarantees is that, in principle, $\mathrm{tr}(\hat{\rho}^x)$ could be uniquely determined from $\mathrm{tr}(\hat{\rho}^n)$ with some proper recipe. One recipe is given by equation (5.1).

One may object that the derivative in equation (5.8) it is not a correct thing to do in the sense that by taking $\mathrm{tr}(\hat{\rho}^n)$ and doing the substitution $n \;\to\; x$ may not give the correct $\mathrm{tr}(\hat{\rho}^x)$, as if one computes $\mathrm{tr}(\hat{\rho}^x)$ considering $x \;\in\; \mathbb{R}$ (or more generally $\mathrm{tr}(\hat{\rho}^\alpha)$ for $\alpha \in \mathbb{C}$) from the start. To put it in a formal way, let us consider the quantities,

$$\mathrm{tr}\hat{\rho}^n = f_0(n) \longrightarrow f_0(\alpha),\; n \in \mathbb{N},\; \alpha \in \mathbb{C} \tag{5.9}$$

and

$$\mathrm{tr}\hat{\rho}^\alpha = f(\alpha). \tag{5.10}$$

The function $f(\alpha)$ should be analytic and since it represents moments it should be bounded,

$$|f(\alpha)| < 1. \tag{5.11}$$

Equations $f_0(\alpha)$ and $f(\alpha)$ should be equal when their arguments are integers (in our case we care about only positive integers). So we can write,

$$\begin{aligned} f(\alpha) &= f_0(\alpha) + g(\alpha) \\ g(\alpha) &= h(\alpha)\sin\pi\alpha \\ g(n) &= 0 \end{aligned} \tag{5.12}$$

where $h(\alpha)$ is any analytic function. From equation (5.11) we have,

$$|f_0(\alpha) + g(\alpha)| < 1. \tag{5.13}$$

The problematic term in the previous equation is $g(\alpha)$ since it hides a hyperbolic sinus. That means that the term

$$|g(\alpha)| = |\sin \pi\alpha||h(\alpha)| \tag{5.14}$$

should be bounded but the hyperbolic sinus diverges like $|\sin \pi\alpha| \sim \mathcal{O}(e^{\pi|y|})$. The function $h(\alpha)$ must compensate this divergence meaning that $|h(\alpha)| \sim \mathcal{O}(e^{-|\tau||y|})$ and therefore,

$$|g(\alpha)| \sim \mathcal{O}(e^{(\pi-|\tau|)|y|}). \tag{5.15}$$

From Carlson's theorem [Car14], equations (5.11), (5.12) and (5.15) imply that $g(\alpha) = 0,\ \alpha \in \mathbb{C}$. That means that by computing $\mathrm{tr}\hat{\rho}^n,\ n \in \mathbb{N}$, and simply substituting $n \to \alpha,\ \alpha \in \mathbb{C}$, it is the same as if one computes $\mathrm{tr}\hat{\rho}^\alpha,\ \alpha \in \mathbb{C}$, from the start.

5.1.2 Examples from classical probability theory

First, consider the Gaussian distribution over some real variable x with mean value μ and standard deviation σ,

$$P^{(G)}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{5.16}$$

The entropy of this distribution can be found by calculating the moments, which are

$$m_n^{(G)} = \frac{1}{(2\pi)^{\frac{n-1}{2}}\sqrt{n}\sigma^{n-1}} \tag{5.17}$$

and then by finding the derivative of equation (5.17) with respect to n at $n=1$,

$$-\left.\frac{dm_n^{(G)}}{dn}\right|_{n=1} = \frac{1}{2}\ln(2e\pi\sigma^2) \tag{5.18}$$

which is indeed the well-known entropy of the Gaussian distribution.

Now, let us consider the Poisson distribution,

$$P^{(P)}(k) = \frac{\lambda^k}{k!}e^{-\lambda},\ \lambda > 0,\ k \in \mathbb{N}. \tag{5.19}$$

The moments read,

$$m_n^{(P)} = \sum_{k=0}^{\infty} \frac{\lambda^{nk}}{k!^n} e^{-n\lambda} \tag{5.20}$$

and from the latter we find,

$$-\left.\frac{dm_n^{(P)}}{dn}\right|_{n=1} = \lambda(1 - \ln\lambda) + e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \ln k! \tag{5.21}$$

which is indeed the entropy of the Poisson distribution. This result reminds us the simple fact that a non-summation expression for the entropy is not always available, even for a simple classical distribution. As we see in this paper, a similar situation prevails when considering the output entropy of a parametric amplifier that is fed with an arbitrary Fock state $|m\rangle$ with $m > 1$.

5.2 Application of replica method to quantum optics

In quantum optics and continuous-variable quantum information theory, the so-called Gaussian transformations are ubiquitous [WPG+12]. For instance, the coupling between two modes of the electromagnetic field as effected by a beam splitter in bulk optics or an optical coupler in fiber optics is modeled by the (passive) quadratic Hamiltonian $H = i\hat{a}_1\hat{a}_2^\dagger - i\hat{a}_1^\dagger\hat{b}_2$, where $\hat{a}_1$ and $\hat{a}_2$ are the bosonic annihilation operators of the modes. This operation can be shown to preserve the Gaussian character of the quantum state, or more precisely the quadratic exponential form of its characteristic function. The corresponding transformation in phase space is the rotation $\hat{a}_1 \to \cos\theta\,\hat{a}_1 + \sin\theta\,\hat{a}_2$ and $\hat{a}_2 \to \cos\theta\,\hat{a}_2 - \sin\theta\,\hat{a}_1$, where $\cos^2\theta$ is the transmittance. Another generic Gaussian coupling between two modes of the electromagnetic field results from parametric down-conversion in a nonlinear medium, which is modeled by the (active) quadratic Hamiltonian $H = i\hat{a}_1\hat{a}_2 - i\hat{a}_1^\dagger\hat{a}_2^\dagger$. It effects the Bogoliubov transformation $\hat{a}_1 \to \cosh r\,\hat{a}_1 + \sinh r\,\hat{a}_2^\dagger$ and $\hat{a}_2 \to \cosh r\,\hat{a}_2 + \sinh r\,\hat{a}_1^\dagger$, where $\cosh^2 r$ is the parametric amplification gain. It corresponds to two-mode squeezing or parametric amplification in the context of quantum optics, but also describes a much wider range of physical situations, such as the Unruh radiation in an accelerating frame [Ful73, Dav75, Unr76] or the Hawking radiation emitted by a black hole [Haw74, Haw75].

While they are common in quantum optics and related fields, these Gaussian transformations are nevertheless poorly understood in terms of quantum entropy generation. For example, when amplifying the state of the electromagnetic field using parametric down-conversion, the output state suffers from some intrinsic quantum noise, which is an increasing function of the amplification gain. It is a central problem to characterize this noise and be able to determine the von Neumann entropy of the output state, this being indispensable for instance to compute the capacities of Gaussian bosonic channels [HW01, GGL+04a, GNL+12]. Unfortunately, the output entropy is generally not accessible for an arbitrary input state because it is difficult – usually impossible – to diagonalize the corresponding density operator in an infinite-dimensional Fock space. With the exception of Gaussian states, e.g., the vacuum state (resulting after amplification in a thermal state whose entropy is given by a well-known formula), few analytical results are available as of today. This problem is also linked to several entropic conjectures on Gaussian optimality in the context of bosonic channels. Notably, determining the capacity of a multiple-access or broadcast Gaussian bosonic channel is pending on being able to prove such entropic conjectures, see e.g [Guh08, YS05, GSE07].

In this chapter, we demonstrate that the *replica method* can be successfully exploited in order to overcome this problem and find the exact analytical expression of the output entropy of Gaussian transformations applied on non-trivial input states. The replica method is well known to be a very useful tool in statistical physics, especially with disordered systems [MPV87], and in quantum field theory [CW94, HLW94]. Here, we first apply it to the field of quantum optics and show that it enables accessing the entropy generated by a quantum optical amplifier, paving the way towards a quantum entropic characterization of all Gaussian transformations generated by quadratic Hamiltonians. To illustrate the power of this approach, we calculate the output entropy when amplifying a binary superposition of the vacuum and an arbitrary Fock state, which yields a surprisingly simple analytical expression.

5.2.1 Example: The thermal state

Let us consider the thermal state $\hat{\rho}_0 = (1-|\tau|^2)\sum_{k=0}^{\infty}|\tau|^{2k}\,|k\rangle\langle k|$ with a mean photon number $N = |\tau|^2/(1-|\tau|^2)$, where $\tau \equiv \tanh\xi$ and ξ is the squeezing parameter (see below). Since $\hat{\rho}_0$ is in a diagonal form, it is of course straightforward to calculate its entropy, giving the well-known formula $S(\hat{\rho}_0) = g(N) \equiv (N+1)\log(N+1) - N\log N$. However, we may also start with its non-diagonal representation in the coherent-state basis $\{|\alpha\rangle\}$, where α is a complex number, namely

$$\hat{\rho}_0 = \frac{1}{\pi}\frac{1-|\tau|^2}{|\tau|^2}\int d^2\alpha\, e^{-\frac{1-|\tau|^2}{|\tau|^2}|\alpha|^2}|\alpha\rangle\langle\alpha| \tag{5.22}$$

By making the change of variable $\alpha \to |\tau|\ \alpha$ and by using $\langle\alpha|\beta\rangle = e^{-(|\alpha|^2+|\beta|^2-2\alpha^*\beta)/2}$, we can write

$$\mathrm{tr}(\hat{\rho}_0^n) = \frac{(1-|\tau|^2)^n}{\pi^n}\int d^2\alpha_1 \ldots \int d^2\alpha_n\, e^{-\bar{\alpha}^\dagger M \bar{\alpha}} \tag{5.23}$$

where $\bar{\alpha} = (\alpha_1, \ldots \alpha_n)^T$ is a column vector and M is the $n \times n$ circulant matrix

$$M = \begin{pmatrix} 1 & -|\tau|^2 & 0 & \ldots & 0 \\ 0 & 1 & -|\tau|^2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -|\tau|^2 & 0 & 0 & \ldots & 1 \end{pmatrix}. \tag{5.24}$$

Equation (5.23) is a simple Gaussian integral, which, using the determinant $\det M = 1 - |\tau|^{2n}$, can be expressed as

$$\mathrm{tr}(\hat{\rho}_0^n) = \frac{(1-|\tau|^2)^n}{1-|\tau|^{2n}} \tag{5.25}$$

Then, we readily find that

$$-\frac{\partial}{\partial n}\mathrm{tr}(\hat{\rho}_0^n)\Big|_{n=1} = \ln\frac{1}{1-|\tau|^2} + \frac{|\tau|^2}{1-|\tau|^2}\ln\frac{1}{|\tau|^2} \tag{5.26}$$

which coincides with the above expression $S(\hat{\rho}_0) = g(N)$ for the entropy of a thermal state, as expected.

5.2.2 Entropic characterization of the two mode squeezer: Amplifying a Fock state

Consider now the problem of expressing the entropy S_m generated by amplifying an arbitrary Fock state $|m\rangle$. Thus, we consider a two-mode squeezer of parameter $\xi = |\xi|\, e^{i\phi}$, applying the unitary transformation

$$\hat{U} = e^{-\xi\hat{a}^\dagger\hat{b}^\dagger + \xi^*\hat{a}\hat{b}} \tag{5.27}$$

on the initial state $|m\rangle_a|0\rangle_b$ (subscript a refers to the signal mode, while b refers to the idler mode). It can be shown that the vector of eigenvalues of the reduced output state $\hat{\rho}_m$ of the signal mode is given by [GNL+12, NGS+12]

$$p_k^{(m)} = (1-|\tau|^2)^{m+1}\binom{k+m}{k}|\tau|^{2k}, \quad k \in \mathbb{N} \tag{5.28}$$

where $|\tau| = \tanh|\xi|$, from which we find

$$\mathrm{tr}(\hat{\rho}_m^n) = (1-|\tau|^2)^{n(m+1)} \sum_{k=0}^{\infty} \binom{k+m}{k}^n |\tau|^{2nk}. \tag{5.29}$$

Equation (5.29) can be re-expressed in a closed form as

$$\mathrm{tr}(\hat{\rho}_m^n) = \frac{(1-|\tau|^2)^{n(m+1)}}{|\tau|^{2n}} \mathrm{Li}_{-n}^{(m)}(|\tau|^{2n}). \tag{5.30}$$

where

$$\mathrm{Li}_{-n}^{(m)}(\zeta) \equiv \sum_{k=0}^{\infty} \binom{k+m}{k}^n \zeta^{k+1} \tag{5.31}$$

and $\mathrm{Li}_{-n}^{(1)}(\zeta)$ denotes the polylogarithm of order $-n$ [OLB+10]. Applying equation (5.1) to equation (5.30) and taking into account that $\mathrm{Li}_{-1}^{(m)}(\zeta) = \zeta/(1-\zeta)^{m+1}$, we obtain the entropy

$$\begin{aligned} S_m = -\frac{\partial}{\partial n}\mathrm{tr}(\hat{\rho}_m^n)\Big|_{n=1} &= \ln\frac{|\tau|^2}{(1-|\tau|^2)^{m+1}} \\ &- \tfrac{(1-|\tau|^2)^{m+1}}{|\tau|^2}\tfrac{\partial}{\partial n}\mathrm{Li}_{-n}^{(m)}(|\tau|^{2n})\Big|_{n=1}. \end{aligned} \tag{5.32}$$

From equation (5.28), the corresponding output state can be written as

$$\hat{\rho}_m = \sum_{k=0}^{\infty}(1-|\tau|^2)^{m+1}\binom{k+m}{k}|\tau|^{2k}\,|k\rangle\langle k| \tag{5.33}$$

so that we obtain a closed expression (5.30) where $\mathrm{Li}_{-n}^{(m)}(\zeta)$ was defined in (5.31) The function $\mathrm{Li}_{-n}^{(1)}(\zeta)$ is known as the polylogarithm of order $-n$. It is connected with the Eulerian polynomials in the following way,

$$\mathrm{Li}_{-n}^{(1)}(\zeta) = \frac{1}{(1-\zeta)^{n+1}} \sum_{k=0}^{n-1} E(n,k)\,\zeta^{k+1} \tag{5.34}$$

where

$$E(n,k) \equiv \sum_{j=0}^{k} \binom{n+1}{j} (-1)^j (k-j+1)^n \tag{5.35}$$

are the Eulerian numbers.

Equation (5.32) can be written more explicitly as

$$S_m = (m+1)\,S_0 + \ln m! - \left(1-|\tau|^2\right)^{m+1} \times \sum_{k=0}^{\infty} \binom{k+m}{k} |\tau|^{2k} \ln \frac{(k+m)!}{k!}. \tag{5.36}$$

Closed expressions of S_m are hard to extract for $m > 1$ since the function $\mathrm{Li}_{-n}^{(m)}(\zeta)$ cannot be written in a non-summation form for $m > 1$. This is not so unexpected as it is similar to the case of a Poisson distribution. Note that S_m can also be calculated analytically using the standard definition of the entropy $-\mathrm{tr}(\hat{\rho}\ln\hat{\rho})$, but the replica method provides an alternative way to achieve this calculation which is straightforward and remains applicable even when the formula $-\mathrm{tr}(\hat{\rho}\ln\hat{\rho})$ cannot be exploited.

One can easily verify numerically from equation (5.36) that $S_{m+1} > S_m$. We will not provide an analytical proof on this because this result is already known [GNL+12]. Let us underline that replica was introduced to calculate the von Neumann entropy and indeed in the last paradigm replica method was successful.

5.2.3 Entropic characterization of the two mode squeezer: Amplifying a superposition of Fock states

We will now show that the same procedure makes it possible to express the entropy analytically in situations where no diagonal form is available for the output state, so the replica method becomes essential. Consider the amplification of a binary superposition of the type

$$|\psi\rangle = \frac{|0\rangle + z|m\rangle}{\sqrt{1+z^2}} \tag{5.37}$$

where we take $z \in \mathbb{R}$ without loss of generality. By using the Baker-Campbell-Hausdorff relation, the unitary transformation (5.27) can be rewritten in the form

$$\hat{U} = \mathrm{e}^{-\nu}\mathrm{e}^{-\tau\hat{a}^\dagger\hat{b}^\dagger}\mathrm{e}^{-\nu(\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b})}\mathrm{e}^{\tau^*\hat{a}\hat{b}} \tag{5.38}$$

where $\nu = \ln\cosh|\xi|$ and $\tau = \frac{\xi}{|\xi|}\tanh|\xi|$, so that the joint output state $|\Psi\rangle$ of the two modes can be expressed in the double coherent-state basis $|\alpha\rangle_a|\beta\rangle_b$, namely

$$\begin{aligned}\langle\alpha,\beta|\Psi\rangle &= \frac{1}{\sqrt{1+z^2}}\Big(\langle\alpha,\beta|\hat{U}|0,0\rangle + z\,\langle\alpha,\beta|\hat{U}|m,0\rangle\Big)\\ &= \frac{1}{\sqrt{1+z^2}}\left((1-|\tau|^2)^{\frac{1}{2}} + z\frac{(1-|\tau|^2)^{\frac{m+1}{2}}}{\sqrt{m!}}\right)\\ &\quad\times e^{-(|\alpha|^2+|\beta|^2)/2-\tau\alpha^*\beta^*+\tau^*\alpha\beta}. \end{aligned} \tag{5.39}$$

From equation (5.39), we can easily write the reduced output state $\hat{\rho}$ obtained by tracing $|\Psi\rangle$ over the idler mode and paying attention to the non-orthogonality of coherent states. Using the notation $\bar{\alpha} = (\alpha_1, \ldots \alpha_n)^T$, we get

$$\mathrm{tr}(\hat{\rho}^n) = \frac{(1-|\tau|^2)^n}{\pi^n(1+z^2)^n}\prod_{j=1}^{n}\int d^2\alpha_j\left|1+c\alpha_j^m\right|^2 e^{-\bar{\alpha}^\dagger M\bar{\alpha}} \tag{5.40}$$

where the matrix M is defined as in equation (5.24) and

$$c = \frac{z}{\sqrt{m!}}(1-|\tau|^2)^{m/2}. \tag{5.41}$$

In order to bring this back to a Gaussian integral, we use the so-called "sources" trick [Zei09], exploiting the identity $x^m e^{-x^2} = \frac{\partial^m}{\partial\lambda^m}e^{-x^2+\lambda x}|_{\lambda=0}$. Then, equation (5.40) becomes

$$\begin{aligned}\mathrm{tr}(\hat{\rho}^n) &= \frac{(1-|\tau|^2)^n}{\pi^n(1+z^2)^n}\,\Pi_{\partial\lambda}(n)\prod_{j=1}^{n}\int d^2\alpha_j\\ &\quad\times\exp\left(-\bar{\alpha}^\dagger M\bar{\alpha}+\bar{\alpha}^\dagger\bar{\lambda}+\bar{\lambda}^\dagger\bar{\alpha}\right)\Big|_{\bar{\lambda}=\bar{0}}\end{aligned} \tag{5.42}$$

where $\Pi_{\partial\lambda}(n) \equiv \prod_{j=1}^{n}\left|1+c\,\partial^m/\partial\lambda_j^m\right|^2$ is a differential operator in the variables $\bar{\lambda} = (\lambda_1, \ldots \lambda_n)^T$. Note here that λ_j and λ_j^* are treated as independent variables, instead of their real and imaginary parts. The derivatives with respect to all λ's have been pushed in front of the integrals in equation (5.42), so that we get a Gaussian integral that is immediately computable, resulting in

$$\mathrm{tr}(\hat{\rho}^n) = \frac{\mathrm{tr}(\hat{\rho}_0^n)}{(1+z^2)^n}\,\Pi_{\partial\lambda}(n)\exp\left(\bar{\lambda}^\dagger N\bar{\lambda}\right)\Big|_{\bar{\lambda}=\bar{0}}, \tag{5.43}$$

where $\mathrm{tr}(\hat{\rho}_0^n)$ is given by equation (5.25) and corresponds to a vacuum input state ($z = 0$). In equation (5.43), we have defined the circulant matrix $N = (1 - |\tau|^{2n})^m M^{-1}$, with

$$M^{-1} = \frac{1}{1-|\tau|^{2n}} \begin{pmatrix} 1 & |\tau|^2 & \dots & |\tau|^{2(n-1)} \\ |\tau|^{2(n-1)} & 1 & \dots & |\tau|^{2(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ |\tau|^2 & |\tau|^4 & \dots & 1 \end{pmatrix} \tag{5.44}$$

while the operator $\Pi_{\partial\lambda}(n)$ can be expanded as

$$\Pi_{\partial\lambda}(n) = \sum_{k=0}^{n} c^{2k}\, \Pi_{2k}(n) \tag{5.45}$$

where each $\Pi_{2k}(n)$ contains $\binom{n}{k}^2$ terms that return a non-zero result when acting on $\exp\left(\bar{\lambda}^\dagger N \bar{\lambda}\right)$ and taking the value at $\bar{\lambda} = \bar{0}$, see Appendix A for details. The term with $k = 0$ in equation (5.45) is simply $\Pi_0(n) = 1$, so that taking $z = 0$ trivially results into $\mathrm{tr}(\hat{\rho}^n) = \mathrm{tr}(\hat{\rho}_0^n)$. The term with $k = n$ gives, when acting on the exponential of equation (5.43),

$$\Pi_{2n} \exp\left(\bar{\lambda}^\dagger N \bar{\lambda}\right)\Big|_{\bar{\lambda}=\bar{0}} = m!^n \frac{1-|\tau|^{2n}}{|\tau|^{2n}}\, \mathrm{Li}_{-n}^{(m)}(|\tau|^{2n}) \tag{5.46}$$

where $\mathrm{Li}_{-n}^{(m)}(|\tau|^{2n})$ is defined in equation (5.31). Thus, we recognize that this term is connected with the case of an input Fock state $|m\rangle$, something that can also be seen by taking the limit $z \to \infty$ in equation (5.43). If we put all pieces together, we obtain the expression

$$\begin{aligned} \mathrm{tr}(\hat{\rho}^n) = \frac{\mathrm{tr}\hat{\rho}_0^n}{(1+z^2)^n} \Bigg\{ & \left[1 + z^2 \left(\frac{1-|\tau|^2}{1-|\tau|^{2n}} \right)^m \right]^n \\ & -z^{2n} \left(\frac{1-|\tau|^2}{1-|\tau|^{2n}} \right)^{mn} + F^{(m)}(n) + z^{2n} \frac{\mathrm{tr}\hat{\rho}_m^n}{\mathrm{tr}\hat{\rho}_0^n} \Bigg\} \end{aligned} \tag{5.47}$$

where $\hat{\rho}_m$ is the reduced output state resulting from the amplification of $|m\rangle$, and $F^{(m)}(n)$ is defined in Appendix A. Now, applying equation (5.1) to equation (5.47), we get

$$S(z) = \frac{1}{1+z^2}\, S_0 + \frac{z^2}{1+z^2}\, S_m - \frac{\partial}{\partial n} F^{(m)}(n)\Big|_{n=1}. \tag{5.48}$$

Finally, we prove in Appendix A that the last term of the right-hand side of equation (5.48) vanishes, so that equation (5.48) simplifies into the expression,

$$S(z) = \frac{1}{1+z^2}\, S_0 + \frac{z^2}{1+z^2}\, S_m \tag{5.49}$$

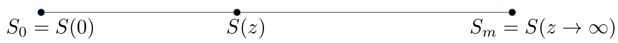

Figure 5.1: The von Neumann entropy $S(z)$, function of the superposition parameter z, is pictured by a point belonging to a one-dimensional convex polytope. The two extremal points of the polytope are the entropies S_0 and S_m corresponding to the two extreme cases, i.e., the input states $|0\rangle$ and $|m\rangle$.

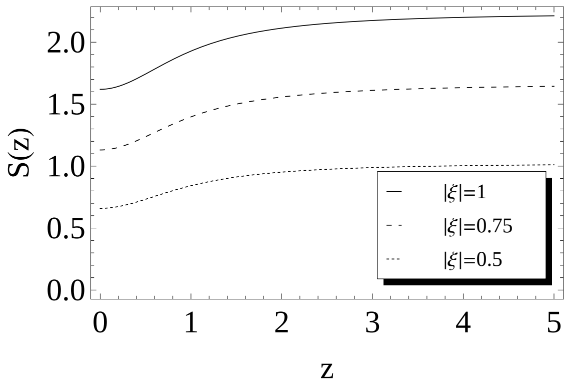

Figure 5.2: Plot of the von Neumann entropy $S(z)$ as a function of the superposition parameter z for $m = 1$ and several values of the squeezing parameter ξ. Since $S_1 > S_0$, equation (5.49) implies that the curve $S(z)$ is always above S_0.

Intriguingly, the output entropy is thus a simple convex combination of S_0 and S_m with the exact same weights as if we had lost coherence between the components $|0\rangle$ and $|m\rangle$ of the input superposition. This is schematically pictured in figure 5.1. We have also numerically verified this behavior, which, to our knowledge, had never been observed before. It is illustrated in figure 5.2, where we show that the entropy is a monotonically increasing function of the superposition parameter z.

5.3 Application of replica method to bosonic field theory

The use of the replica method can be expanded to bosonic field theory. In particular the replica method can be used to calculate the geometric entropy, in the framework of a quantum system. One of the first calculations [BKL+86] was performed in the eighties for the case of a scalar field propagating in a black hole background. Some years later, a similar problem, in the framework of a quantum field theory, was addressed by several authors [Sre93, KS94, CW94, CC04, CC06, CCD08, CC09]. Geometric entropy, generally speaking, is a measure of the information loss after cutting out a spatial region of the system. It caught attention because of its characteristic behavior; for a system in its ground state, it grows like the boundary surface of the excluded subregion, a property that the black hole entropy exhibits as well. In fact, in the context of quantum field theory, pioneering work on the geometric entropy was driven in part by the suggested connection to the Bekenstein-Hawking black hole entropy [Bek73].

From the very beginning, geometric entropy has been tightly related with the presence of spatial entanglement in a quantum system. Entanglement is a fundamental ingredient of quantum mechanics leading to strong correlations between subsystems and from the early days of quantum mechanics up until now, it has been playing an increasingly important role in understanding and controlling quantum systems. The interest in it has been renewed [PV98, AEP+02, ON02, OMF+02, VLR+03] after the developments of the quantum information science in which it is viewed as a resource in quantum information processing. Geometric entropy has been considered as a measure of spatial entanglement when the system under consideration is in a pure quantum state with a density matrix of the form $\hat{\rho} = |\Psi\rangle\langle\Psi|$. By defining an *in* and an *out* spatial region and tracing out the *in* degrees of freedom, one obtains the reduced density matrix for the *out* region, $\hat{\rho}_{\rm out} = {\rm tr}_{in}\hat{\rho}$. The geometric entropy is then defined as the von Neumann entropy: $S_{\rm out} = -{\rm tr}\hat{\rho}_{\rm out} \ln \hat{\rho}_{\rm out}$.

When the system is in a thermal state, the geometric entropy can be defined, following the von Neumann definition, in an analogous way. However, in this case, it does not have the same properties as the entanglement entropy in a pure state system, and it is no longer a good estimator of entanglement since it mixes correlations of different types [CW94, CC04, WVH+08, Woo01, HAK+07], from genuine quantum to thermal correlations. Since it measures the thermal information loss, geometric entropy becomes an extensive quantity at the limit of an infinite system, and loses the *area law* behavior that characterizes a pure state system. As an alternative probe for the amount of correlations between different parts of a system in the case of thermal states, the notion of the so-called mutual information [AC97a, HV01, WVH+08, GPW05, DY09] has been proposed, which, roughly speaking, eliminates the contribution of the extensive part of the thermal entropy from the geometric entropy and can be considered as an upper limit for the entanglement entropy.

In any case, geometric entropy has been considered as a convenient construction, playing the role of an order parameter, for the investigation of finite temperature conformal quantum field systems [CC04] and in the context of the AdS/CFT correspondence, aiming at the physics of strongly coupled Quark-Gluon Plasma, the weakly coupled deconfined phase of Yang-Mills theories or the phase structure of large N QCD at a finite density [FNT08, FO10].

In the rest of this Chapter, we derive, for a simple spatial partition, the geometric entropy in a free bosonic quantum field theory at finite temperature, starting from the canonical or the grand canonical partition functional of the system. Having found it, we subtract its extensive part, that is, the part related to the amount of information that is lost due to the mixed nature of the system. The result quantifies the spatial correlations between the different parts of the system, it exhibits the known area law behavior and it can be related to the mutual information for the specific partition.

The underlying reason for the present study is connected to the Bose-Einstein condensation that characterizes the system, which has been in the center of theoretical and experimental investigations during the last fifteen years after the production of the condensate in the laboratory [AEM+95, DMA+95].

Bose-Einstein condensation has the characteristics of a phase transition albeit, theoretically at least, it can take place in an ideal system [LD65, Kap81, HW81, HW82, PS03, PS08, GNZ09]. It is then natural to search for the interconnection between this phase transition and the spatial correlations in the Bose system. Our findings indicate that, indeed, the Bose-Einstein condensation influences the behavior of the mutual information; we find that its derivative with respect to the temperature, $\partial I_m/\partial \mathrm{T}$, has a finite discontinuity at the critical temperature both at the non-relativistic and the relativistic limit. Thus, we show how this phase transition leaves its fingerprint on a quantum informational quantity like mutual information.

Note that in this Section we fix $\hbar = 1$.

5.3.1 Geometric entropy at finite temperature

Our starting point is the thermal density matrix $\hat{\rho} = \mathrm{e}^{-\beta \hat{H}}/Z$ of a quantum field system and its Fock space representation,

$$\rho[\Phi', \Phi] = \frac{1}{Z}\langle \Phi' | \mathrm{e}^{-\beta \hat{H}} | \Phi \rangle, \tag{5.50}$$

where Φ denotes a single scalar field or a collection of fields. The matrix element (5.50) can be written as a functional integral,

$$\rho[\Phi', \Phi] = \frac{1}{Z(\beta)} \int\limits_{\Phi(0,\vec{x})=\Phi(\vec{x})}^{\Phi(\beta,\vec{x})=\Phi'(\vec{x})} \mathcal{D}\Phi(\tau, \vec{x})\, \mathrm{e}^{-\int_0^\beta \mathrm{d}\tau \int \mathrm{d^D x} \mathcal{L}[\Phi]} \tag{5.51}$$

where in (5.51) the action contains the usual free Klein-Gordon Lagrangian, see equation (2.208).

It is worth noting that at the zero temperature limit, $\beta \to \infty$, (5.51) is just an expression for the ground state density matrix [CW94]. It is then natural to expect that our result will reproduce, at this limit, the known [KS94, CW94, CC04] entanglement entropy. To derive the geometric entropy we follow the usual line of reasoning and we divide the D dimensional space on which our system is defined, into two regions $A: \ (x_1 > 0, \vec{x}_\perp)$ and $B: \ (x_1 < 0, \vec{x}_\perp)$. Tracing out the "in" region, and gluing along the axis $x_1 > 0$, n copies of the resulting reduced density matrix, we find [KS94, CW94, CC04],

$$\mathrm{tr}(\hat{\rho}_R)^n = \frac{1}{Z^n(\beta)} \int\limits_{M_n} \mathcal{D}\Phi \mathrm{e}^{-S[\Phi]} \equiv \frac{Z_n(\beta)}{Z^n(\beta)}. \tag{5.52}$$

In Z_n the fields are defined on a $D+1$ dimensional space $M_n = R_{D-1} \times C_n$. The subspace R_{D-1} is an Euclidean space with metric $ds^2 = dx_2^2 + \cdots + dx_D^2$ while C_n is a two dimensional Riemann space consisting of n sheets glued together along the positive x_1 axis. This n folded structure turns eventually [CW94] the (τ, x_1) plane into a flat cone with an angle deficit $\delta = 2\pi(1-n)$ at the origin. Having found $\mathrm{tr}(\hat{\rho}_R)^n$ the geometric entropy is defined through the relation,

$$\begin{aligned} -\lim_{n\to 1} \frac{\mathrm{tr}(\hat{\rho}_R)^n - 1}{n-1} &= -\mathrm{tr}(\hat{\rho}_R \ln \hat{\rho}_R) \equiv S_g \\ &= -\Big(\frac{\partial}{\partial n} - 1\Big) \ln Z_n \Big|_{n=1}. \end{aligned} \tag{5.53}$$

For a free bosonic theory, the partition function can be deduced by following standard steps [CW94],

$$\begin{aligned} \ln Z_n(\beta) &= \ln \Big[\det_{M_n}(-\partial_E^2 + m^2) \Big]^{-\frac{1}{2}} = -\frac{1}{2} \mathrm{tr}_{M_n} \Big[\ln(-\partial_E^2 + m^2) \Big] \\ &= \frac{1}{2} \int\limits_{0^+}^{\infty} \frac{dT}{T} \mathrm{e}^{-Tm^2} \mathrm{tr}_{M_n} \mathrm{e}^{-T(-\partial_E^2)} \\ &= \frac{1}{2} \frac{1}{(4\pi)^{(D-1)/2}} V_{D-1} \int\limits_{0^+}^{\infty} \frac{dT}{T^{(D+1)/2}} \mathrm{e}^{-Tm^2} \mathrm{tr}_{C_n} \mathrm{e}^{-T(-\partial_E^2)} \end{aligned} \tag{5.54}$$

where $\partial_E^2 = \partial_\tau^2 + \partial_{\vec{x}}^2 + m^2$. Due to the locality of the action, the partition function in (5.54) is not expected to depend explicitly on the details of the Riemann surface. Thus, in order to calculate the non-trivial trace appearing in (5.54), we start with the finite temperature propagator of a free particle in cartesian coordinates,

$$A_{\beta_n}(\vec{x}', \vec{x}) = \langle \vec{x}' | \mathrm{e}^{-T(-\partial_E^2)} | \vec{x} \rangle_{\beta_n}. \tag{5.55}$$

In (5.55) the $|\vec{x}\rangle$ denotes the the eigenstates of the position operator and the subscript

$\beta_n = \beta n$ indicates the periodic boundary conditions imposed on the thermal Green's function. As it is obvious, they are dictated by the n folded structure of the Riemann space C_n. The next step is to transfer the result onto a two dimensional cone with angle deficit $2\pi(1-n)$,

$$ds^2 = d\rho^2 + \rho^2 n^2 d\theta^2, \; 0 \leq \theta \leq 2\pi. \tag{5.56}$$

One can easily find that the free thermal propagator in (5.55) assumes the form [Sim95],

$$\begin{aligned} A_{\beta_n}(\vec{x}', \vec{x}) &= \frac{1}{4\pi T} \sum_{\nu=-\infty}^{\infty} \mathrm{e}^{-\frac{1}{4T}\left[(x_1'-x_1)^2+(x_0'-x_0-\nu\beta_n)^2\right]} \\ &= \frac{1}{4\pi T} \sum_{\nu=-\infty}^{\infty} \mathrm{e}^{-\frac{1}{4T}(\vec{x}'-\vec{x})^2+\frac{\nu\beta_n}{2T}(x_0'-x_0)-\frac{(\nu\beta_n)^2}{4T}}. \end{aligned} \tag{5.57}$$

The above expression can be written in the conical metric (5.56) by making the replacements $x_0 = \rho\sin(n\theta)$, $x_1 = \rho\cos(n\theta)$, and using the expansion [DJ88],

$$\mathrm{e}^{iz\cos(n\theta)} = \sum_{m=-\infty}^{\infty} c_m J_{\frac{|m|}{n}}(z)\mathrm{e}^{im\theta}, \; c_m = i^{\frac{|m|}{n}}, \tag{5.58}$$

where $J_{\frac{|m|}{n}}$ are Bessel functions of the first kind. Thus, the thermal propagator on the surface (5.56) reads,

$$\begin{aligned} A_{\beta_n}(\rho', \theta'; \rho, \theta; n) &= \frac{1}{4\pi it} \sum_{\nu,m,m_1,m_2} \mathrm{e}^{-\frac{(\nu\beta)^2}{4it}} \mathrm{e}^{-\frac{\rho'^2+\rho^2}{4it}} \times \\ &\times \mathrm{e}^{im(\theta'-\theta)} \mathrm{e}^{im_1\theta'+im_2\theta} \mathrm{e}^{-\frac{i\pi(m_1+m_2)}{2n}} J_{\frac{|m|}{n}}\left(\frac{\rho'\rho}{2t}\right) \times \\ &\times J_{\frac{|m_1|}{n}}\left(\frac{\nu\beta_n}{2t}\rho'\right) J_{\frac{|m_2|}{n}}\left(\frac{\nu\beta_n}{2t}\rho\right) \times \\ &\times i^{-\frac{|m|}{n}} i^{\frac{|m_1|}{n}} (-i)^{\frac{|m_2|}{n}}. \end{aligned} \tag{5.59}$$

In the last expression the rotation $T \to it$ has been adopted in order to secure convergence of all our intermediate steps.

Tracing out (5.59) we find,

$$\begin{aligned} A_{\beta_n} &= \mathrm{tr}_{C_n} \mathrm{e}^{-T(-\partial_E^2)} \\ &= \frac{1}{2it} \sum_{\nu,m,m_1} \mathrm{e}^{-\frac{(\nu\beta)^2}{4it}} i^{-\frac{|m|}{n}} \times \\ &\times \int_0^\infty d\rho\rho \mathrm{e}^{-\frac{\rho^2}{2it}} J_{\frac{|m|}{n}}\left(\frac{\rho^2}{2t}\right) J^2_{\frac{|m_1|}{n}}\left(\frac{\nu\beta_n}{2t}\rho\right). \end{aligned} \tag{5.60}$$

At this point we stress the fact that for $n \neq 1$, the trace over the conical metric (5.56), that is the integration over ρ, must be performed before the summations over m or m_1. The relevant calculations can be facilitated by using the fact that, apart for $Z_{n=1}$ which is easily calculated, we are only interested in the derivative of (5.60) with respect to n,

$$\begin{aligned}(\partial_n A_{\beta_n})_{n=1} &= \frac{1}{2it}\sum_{\nu} \mathrm{e}^{-\frac{(\nu\beta)^2}{4it}}\partial_n\left[\sum_m \int_0^\infty d\rho\rho \mathrm{e}^{-\frac{\rho^2}{2it}} i^{-\frac{|m|}{n}} J_{\frac{|m|}{n}}\left(\frac{\rho^2}{2t}\right)\right]_{n=1} + \\ &+\frac{1}{2it}\partial_n\left[\sum_{\nu,m} \mathrm{e}^{-\frac{(\nu\beta_n)^2}{4it}} \int_0^\infty d\rho\rho J_{\frac{|m|}{n}}\left(\frac{\nu\beta_n}{2t}\rho\right)\right]_{n=1}. \end{aligned} \tag{5.61}$$

In obtaining the last expression we have used the identities,

$$\sum_m i^{-m} J_m(z) = \mathrm{e}^{-iz}, \quad \sum_m J_m^2(z) = 1. \tag{5.62}$$

In Appendix B we prove that,

$$\begin{aligned} &\frac{1}{2it}\sum_m \int_0^\infty d\rho\rho \mathrm{e}^{-\frac{\rho^2}{2it}} i^{-\frac{|m|}{n}} J_{\frac{|m|}{n}}\left(\frac{\rho^2}{2t}\right) \\ \underset{it\to T}{=} \quad & n\frac{V_2}{4\pi T} + \frac{1}{12}\left(\frac{1}{n} - n\right) + \mathcal{O}\left(\frac{T}{V_2}\right) \end{aligned} \tag{5.63}$$

and

$$\frac{1}{2it}\sum_m \int_0^\infty d\rho\rho J^2_{\frac{|m|}{n}}\left(\frac{\nu\beta_n}{2t}\rho\right) \underset{it\to T}{=} \frac{V_2}{4\pi T} + \mathcal{O}\left(\frac{T}{V_2}\right). \tag{5.64}$$

In the above equations we introduced an upper cutoff R in the ρ-integrals and we have written as $V_2 = \pi R^2$ the volume of the two dimensional subspace. Substituting the first term in the rhs of (5.63) into (5.61) and feeding with the result (5.61) we find (see Appendix B) that it leads to the logarithm of the partition function,

$$\frac{1}{2}\frac{V_{D-1}V_2}{(4\pi)^{\frac{D+1}{2}}}\int_{0^+}^\infty \frac{dT}{T^{\frac{D+3}{2}}}\mathrm{e}^{-Tm^2}\sum_\nu \mathrm{e}^{-\frac{(\nu\beta)^2}{4T}} = \ln Z_1(\beta). \tag{5.65}$$

Following the same steps for the first term in the rhs of (5.64) we can prove that it is connected to the thermal entropy of the system,

$$\begin{aligned}&\frac{1}{2}\frac{V_{D-1}V_2}{(4\pi)^{\frac{D+1}{2}}}\partial_n\left[\int_{0^+}^{\infty}\frac{dT}{T^{\frac{D+3}{2}}}\mathrm{e}^{-Tm^2}\sum_{\nu}\mathrm{e}^{-\frac{(\nu\beta)^2}{4T}}\right]\\ =\;&\frac{1}{2}V_D\int\frac{d^Dp}{(2\pi)^D}\left(\ln(1-\mathrm{e}^{-\beta\omega})-\frac{\beta\omega}{\mathrm{e}^{\beta\omega}-1}\right)\end{aligned}\tag{5.66}$$

where $\omega^2 = p^2 + m^2$. The contribution to (5.61) of the second term in the rhs of (5.63) assumes the form,

$$\begin{aligned}&\frac{1}{12}\frac{1}{(4\pi)^{\frac{D-1}{2}}}V_{D-1}\int_{0}^{\infty}\frac{dT}{T^{\frac{D+1}{2}}}\mathrm{e}^{-Tm^2}\sum_{\nu}\mathrm{e}^{-\frac{(\nu\beta)^2}{4T}}\\ =\;&\frac{\pi}{6}V_{D-1}\int\frac{d^Dp}{(2\pi)^D\omega}\frac{1}{\tanh(\frac{\omega\beta}{2})}.\end{aligned}\tag{5.67}$$

Collecting everything together and using (5.54) we get the geometric entropy,

$$\begin{aligned}S_g\;=\;&\frac{\pi}{6}V_{D-1}\int\frac{d^Dp}{(2\pi)^D\omega}\frac{1}{\tanh(\frac{\omega\beta}{2})}+\\ &+\frac{1}{2}V_D\int\frac{d^Dp}{(2\pi)^D}\left(\ln(1-\mathrm{e}^{-\beta\omega})-\frac{\beta\omega}{\mathrm{e}^{\beta\omega}-1}\right).\end{aligned}\tag{5.68}$$

At the limit $p\to\infty$, $\tanh\left(\frac{\sqrt{p^2+m^2}\beta}{2}\right)\to 1$ and consequently the fist integral in (5.68) diverges. The same divergence appears in the case of zero temperature,

$$\begin{aligned}S_g(\beta=\infty)\;=\;&\frac{\pi}{6}V_{D-1}\int\frac{d^Dp}{(2\pi)^D}\frac{1}{\sqrt{p^2+m^2}}\\ \to\;&\frac{1}{12}\frac{V_{D-1}}{(4\pi)^{\frac{D-1}{2}}}m^{D-1}\Gamma\Big(-\frac{D-1}{2},\frac{m^2}{\Lambda^2}\Big).\end{aligned}\tag{5.69}$$

After this observation we are led to write,

$$\begin{aligned}S_g(\beta)\;=\;&S_g(\beta=\infty)+\frac{\pi}{3}V_{D-1}\int\frac{d^Dp}{(2\pi)^D\omega}\frac{1}{\mathrm{e}^{\beta\omega}-1}+\\ &+\frac{1}{2}S_{\text{thermal}}.\end{aligned}\tag{5.70}$$

Some comments are in order at this point. The first term in the last expression represents the well known [KS94, CW94, CC04] entanglement entropy at zero temperature. This is a quantity that diverges in the absence of an ultraviolet cutoff, while it grows like the boundary surface of the excluded subregion. This fact clearly indicates the existence of very strong quantum correlations between fields defined at neighboring points, a direct consequence of a local quantum field theory. A quantitative explanation of such a behavior can be traced back to the uncertainty relations. Even at zero temperature, the notion of a sharp, well defined, boundary surface is more classical than quantum. The divergences appearing in $S_g(\mathrm{T} = 0)$ are connected to the fact that in (5.69) we integrate down to zero distance, driving to infinity the density of the reduced density matrix eigenvalues. The second term is finite and well-defined for $m^2 > 0$. It is also proportional to the boundary surface and it is an increasing function of the temperature. We can consider it as a measure of the number of degrees of freedom that have been excited on the boundary surface due to the non-zero temperature and, consequently, as a measure of the thermal correlations between the partitions. The last term is the thermal entropy of the subsystem, an obviously extensive quantity. Subtracting this term from the geometric entropy we are led to find the following quantity,

$$I_m(\beta) = S_g(\beta = \infty) + \frac{\pi}{3} V_{D-1} \int \frac{d^D p}{(2\pi)^D \omega} \frac{1}{\mathrm{e}^{\beta\omega} - 1}. \tag{5.71}$$

which turns out to be proportional to the mutual information. In general the mutual information is a measure of all correlations, thermal and quantum. We use the following definition,

$$I(A : B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}). \tag{5.72}$$

For the case in hand the entropy of the combined system AB is just the total thermal entropy, $S(\rho_{AB}) = S_{\text{thermal}}$. The entropies of each one of the two subsystems are equal due to the way we have divided our system. Moreover, each one of them contains a part which is one half of the total thermal entropy of the system. Thus, their extensive thermal contribution to the mutual information is equal to $S(\rho_{AB})$ and when subtracting the latter, all contributions due to the thermal entropy will be eliminated. So, what (5.71) represents is the mutual information of the system divided by 2, $I_m(\beta) \equiv I(A : B)/2$.

5.3.2 Mutual information and Bose-Einstein condensation

Almost all of the technical details needed for the current section have already been exposed in the previous one. We underline that we still consider the geometric entropy, when we cut space into two halves. The basic difference of the analysis that follows, lies on the fact that we are now interested in charged scalar (non- interacting) fields. The field theoretical description will be based on complex fields while the introduction of a chemical potential (as a Lagrange multiplier) will ensure the conservation of the charge. The obtained result is novel and connects the formation of a Bose-Einstein condensation and spatial correlations in system as that under consideration. Note that calculations are exact.

In this framework, the partition function of the system assumes the form,

$$Z(\beta) = \int\limits_{\beta-\text{periodic}} \mathcal{D}\phi\mathcal{D}\phi^* \exp\left\{ -\int\limits_0^\beta d\tau \int d^D x \mathcal{L}[\phi, \phi^*]\right\}. \tag{5.73}$$

The Lagrangian entering the last expression can be written [Kap81, HW81] as follows,

$$\mathcal{L}[\phi, \phi^*] = \phi^* \left[-(\partial_\tau - \mu)^2 - \partial_{\vec{x}}^2 + m^2 \right] \phi. \tag{5.74}$$

Following the same steps as in the previous section, we find,

$$\ln Z_n(\beta) = \frac{1}{(4\pi)^{\frac{D-1}{2}}} V_{D-1} \int\limits_0^\infty \frac{dT}{T^{\frac{D+1}{2}}} \mathrm{e}^{-Tm^2} \mathrm{tr}_{C_n} \mathrm{e}^{-T(-\partial_E^2 + 2\mu\partial_0 - \mu^2)}. \tag{5.75}$$

Once again we start from the free thermal propagator in Cartesian coordinates,

$$\begin{aligned} A_{\beta_n}(\vec{x}', \vec{x}) \quad &= \quad \frac{1}{4\pi T} \sum_{\nu=-\infty}^{\infty} \exp\left\{ -\frac{1}{4T}(\vec{x}' - \vec{x})^2 + \right. \\ &\left. + \left(\frac{\nu\beta_n}{2T} + \mu\right)(x_0' - x_0) - \frac{(\nu\beta_n)^2}{4T} - \mu\nu\beta_n \right\} \end{aligned} \tag{5.76}$$

to arrive at the traced quantity that is relevant for the final calculation in (5.75),

$$\begin{aligned} \mathrm{tr}_{C_n} \mathrm{e}^{-T(-\partial_E^2 + 2\mu\partial_0 - \mu^2)} \quad &\to \quad n \mathrm{tr}_{C_1} \mathrm{e}^{-T(-\partial_E^2 + 2\mu\partial_0 - \mu^2)} + \\ &+ \frac{1}{12}\left(\frac{1}{n} - n\right) \sum_\nu \mathrm{e}^{-\frac{(\nu\beta)^2}{4T} - \mu\nu\beta} + \\ &+ \mathcal{O}\left(\frac{T}{V_2}\right) \end{aligned} \tag{5.77}$$

where the arrow underlines the fact that we have followed the same steps as from (5.57) to (5.64) and we have kept only the non-extensive terms that are relevant for determining the mutual information.

In Appendix B we show that,

$$\begin{aligned} I_m \quad &= \quad \frac{\pi}{6} V_{D-1} \int \frac{d^D p}{(2\pi)^D \omega} \left\{ \frac{1}{\tanh\left[\frac{(\omega-\mu)\beta}{2}\right]} + \right. \\ &\left. + \frac{1}{\tanh\left[\frac{(\omega+\mu)\beta}{2}\right]} \right\}. \end{aligned} \tag{5.78}$$

Isolating the (diverging) zero temperature contribution we find,

$$\begin{aligned} I_m &= S_g(\beta=\infty) + \frac{\pi}{6} V_{D-1} \int \frac{d^D p}{(2\pi)^D \omega} \left\{ \frac{1}{\mathrm{e}^{(\omega-\mu)\beta}-1} + \right. \\ &\left. + \frac{1}{\mathrm{e}^{(\omega+\mu)\beta}-1} \right\}. \end{aligned} \tag{5.79}$$

For the system in hand the zero temperature geometric entanglement entropy reads,

$$\begin{aligned} S_g(\beta=\infty) &= \frac{\pi}{3} V_{D-1} \int \frac{d^D p}{(2\pi)^D \omega} \\ &\to \frac{1}{6} \frac{V_{D-1}}{(4\pi)^{\frac{D-1}{2}}} m^{D-1} \Gamma\Big(-\frac{D-1}{2}, \frac{m^2}{\Lambda^2}\Big). \end{aligned} \tag{5.80}$$

To reveal the physical content of our results we shall focus on the well-studied $D = 3$ case which hosts the Bose-Einstein condensation. As it is well-known [LD65, Kap81, HW81] the quantitative realization of the phenomenon is different at the two opposite limits, the non-relativistic $\rho \ll m^3$ and the ultra-relativistic one $\rho \gg m^3$, as these are defined by the total charge density of the system.

Beginning from the non-relativistic case, in which the charge density is very low and the anti-particle contribution can be omitted [HW81], we rewrite (5.79) in the form,

$$I_m^{NR}(\beta) = S_g(\beta=\infty) + \frac{\pi}{6} \frac{V_2}{m} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\mathrm{e}^{(\frac{p^2}{2m}-\mu_{NR})\beta}-1} \tag{5.81}$$

where we noted as $\mu_{NR}(\beta) = \mu - m \leq 0$ the non-relativistic chemical potential. The integral appearing in (5.81) is the total density of particles occupying excited states,

$$\rho^e = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\mathrm{e}^{(\frac{p^2}{2m}-\mu_{NR})\beta}-1} = \left(\frac{m}{2\pi\beta}\right)^{3/2} \sum_{n=1}^{\infty} \frac{z_{NR}^n}{n^{3/2}}, \tag{5.82}$$

where $z_{NR} = \mathrm{e}^{\beta\mu_{NR}} \leq 1$. For temperatures below a certain critical value T_C we have $\mu(T_C) = m$ and the above quantity is a constant,

$$\begin{aligned} \rho^e &= \left(\frac{m}{2\pi\beta}\right)^{3/2} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \\ &= \left(\frac{2\pi m}{\beta}\right)^{3/2} \zeta\Big(\frac{3}{2}\Big), \; 0 \leq \mathrm{T} \leq \mathrm{T}_C. \end{aligned} \tag{5.83}$$

At exactly the critical temperature the number (5.83) becomes the conserved total particle density of the system,

$$\rho^e = \rho = \left(\frac{2\pi m}{\beta_C}\right)^{3/2} \zeta\left(\frac{3}{2}\right). \tag{5.84}$$

As an immediate consequence we get for the mutual information,

$$I_m^{NR} = S_g(\mathrm{T}=0) + \frac{\pi}{6}\frac{V_2}{m}\rho\left(\frac{\mathrm{T}}{\mathrm{T}_C}\right)^{3/2}, \ \mathrm{T} < \mathrm{T}_C. \tag{5.85}$$

Above the critical temperature the system passes to the gas phase in which all of the particles occupy excited states. The mutual information reads now,

$$I_m^{NR} = S_g(\mathrm{T}=0) + \frac{\pi}{6}\frac{V_2}{m}\rho, \ \mathrm{T} > \mathrm{T}_C. \tag{5.86}$$

Thus, the Bose-Einstein condensation and the relevant phase transition are reflected in a discontinuity of the derivative with respect to the temperature of the mutual information,

$$\left.\frac{\partial I_m^{NR}}{\partial \mathrm{T}}\right|_{\mathrm{T}=\mathrm{T}_C^-} - \left.\frac{\partial I_m^{NR}}{\partial \mathrm{T}}\right|_{\mathrm{T}=\mathrm{T}_C^+} = \frac{\pi^2}{2}\zeta^{2/3}\left(\frac{3}{2}\right)V_2\rho^{1/3}. \tag{5.87}$$

When $\rho \gg m^3$ we are approaching the ultra-relativistic limit, the critical temperature rises at relativistic high values $\mathrm{T}_C = (3|\rho|/m)^{1/2} \gg m$ and the behavior of the system changes. Below the critical temperature, one easily finds that [LD65, Kap81],

$$\begin{aligned} I_m &= S_g(\mathrm{T}=0) + \frac{\pi}{6}V_2 \int \frac{d^3p}{(2\pi)^3\omega}\left(\frac{1}{\mathrm{e}^{(\omega-\mu)\beta}-1} + \frac{1}{\mathrm{e}^{(\omega+\mu)\beta}-1}\right) \\ &\approx S_g(\mathrm{T}=0) + \frac{\pi V_2}{12}\frac{|\rho|}{m}\left(\frac{\mathrm{T}}{\mathrm{T}_C}\right)^2, \ \mathrm{T} < \mathrm{T}_C. \end{aligned} \tag{5.88}$$

The integral that appears in (5.87) and (5.88) is not the charge density of the system and, consequently, is not a conserved quantity even for temperatures above the critical one. However, it is not hard to confirm [LD65] that at high temperatures $\mathrm{T} > \mathrm{T}_C$ it behaves as following,

$$\begin{aligned} &\int \frac{d^3p}{(2\pi^3)\omega}\left(\frac{1}{\mathrm{e}^{(\omega-\mu)\beta}-1} + \frac{1}{\mathrm{e}^{(\omega+\mu)\beta}-1}\right) \\ = &\ \frac{\mathrm{T}^2}{6} - \frac{\mathrm{T}}{2\pi}(m^2-\mu^2)^{1/2} - \frac{m^2}{4\pi^2}\ln\left(C\frac{m}{\mathrm{T}}\right) + \\ &+\frac{1}{4\pi^2}(m^2-\mu^2) + \mathcal{O}\left(\frac{m^2}{\mathrm{T}^2}\right) \end{aligned} \tag{5.89}$$

where $C = \mathrm{e}^{\gamma_E - 1}/4\pi$ and γ_E is the Euler-Macheroni constant.

As in the non-relativistic case, the derivative of the mutual information with respect to the temperature possesses a discontinuity that reflects the underlying phase transition,

$$\left.\frac{\partial I_m}{\partial \mathrm{T}}\right|_{\mathrm{T}=\mathrm{T}_C^-} - \left.\frac{\partial I_m}{\partial \mathrm{T}}\right|_{\mathrm{T}=\mathrm{T}_C^+} = -\frac{\pi\sqrt{3}}{9} V_2 \left(\frac{|\rho|}{m}\right)^{1/2}. \tag{5.90}$$

The last result completes our study for the influence of the Bose-Einstein condensation on the entropy of entanglement in an ideal Bose system at finite temperature and non-zero chemical potential.

5.4 Conclusions

The main propose of this chapter was to demonstrated the power of the replica method. This tool, used in several fields of physics, provides a new angle of attack to access quantum entropic measures for fundamental Gaussian transformations. The entropic characteristics of such transformations can be normally accessed as long as Gaussian states are considered, using the symplectic formalism, but otherwise the problem is generally unsolvable. For instance, it could be proven only recently that the state minimising the output entropy of an optical amplifier is simply the vacuum state [GGC+14]. Although the amplifier is a common Gaussian operation, the difficulty behind this proof was that no diagonal representation of $\hat{\rho}$ is available, as is most often the case when non-Gaussian states are considered. The replica method should hopefully enable going beyond this proof as it provides a trick to overcome this difficulty: $\mathrm{tr}(\hat{\rho}^n)$ is expressed for n replicas of state $\hat{\rho}$ by using Gaussian integrals, without accessing its eigenvalues.

We have illustrated this procedure with the amplification of a state of the form $|0\rangle + z|m\rangle$. This allowed us to unveil a remarkably simple behavior for the entropy of the amplified state, namely that it is a convex combination of the extremal points S_0 and S_m, as expressed by equation (5.49). It must be stressed that this analytical result is non-trivial as we do not expect similar expressions for the entropy resulting from other superpositions, such as $|1\rangle + z|2\rangle$ or $|0\rangle + z|1\rangle + z'|2\rangle$. Take for instance a coherent state, which is an infinite superposition of Fock states: the entropy of the amplified state is simply S_0, just as for the vacuum state.

Future works will further explore this avenue in order to achieve a better entropic characterization of fundamental quantum optical operations or even perhaps solve some pending conjectures on bosonic Gaussian channels. More generally, we anticipate that the replica method will become an invaluable tool in quantum optics and continuous-variable quantum information theory.

Moreover, to a more familiar to the replica method field, we have performed two types of calculations and we have arrived at results with a clear physical content. First, we calculated the geometric entropy in an ideal Bose system at finite temperature and we confirmed the expected result; it combines the genuine quantum correlations with the thermal fluctuations, and it becomes an extensive quantity for an infinite system. Due to the simplicity of the system under consideration, we were able to explicitly subtract from the geometric entropy its extensive component which coincides with the corresponding thermal entropy. In this way we found the so-called mutual information, which grows like the surface that bounds the space region in which a system lives.

The second novel calculation we performed refers to a Bose system at finite temperature and in an environment with finite charge density. We found that, at the critical temperature, the temperature derivative of the mutual information exhibits a finite discontinuity, and we explicitly calculated it. This result connects the condensation that appears in an ideal quantum Bose system with the spatial correlations between two regions of the system. This connection was shown by using a purely informational tool namely, the quantum mutual information by introducing to thermal field theory.

Summary and outlook

This thesis was concerned with theoretical quantum optics and its interconnection with continuous-variable quantum information theory. The main focus was on Gaussian bosonic transformations, which play a special role in optical continuous-variable quantum communication. More specifically, we have investigated how non-Gaussian bosonic states evolve under such transformations, in which case the sympletic formalism cannot be used. We have considered both deterministic transformations (such as optical squeezers and beam splitters) and probabilistic transformations (such as noiseless amplifiers and attenuators), and have studied the entanglement production under such transformations. This led us to consider the mean-field characteristics of the noiseless amplifier/attenuator when acting on several classes of states, giving rise to some counter-intuitive behavior, as well as the entropic characteristic of Bogoliubov transformations, yielding some intriguing new analytical results. The tools that we have developed to reach these goals were based on majorization theory and the replica method. To our knowledge, there were only very few earlier applications of the former technique and none of the later technique in the area of quantum optics. The contributions of this thesis can be structured into the following three topics (corresponding respectively to Chapters 3, 4, and 5):

Noiseless amplification and attenuation of quantum light

Amplification of quantum light is an important operation in quantum information processing. It is rooted in the problem of how to transmit a quantum signal over large distances so that the signal does not lose its quantum properties, i.e., the quantum state should be protected from decoherence. Therefore, having at hand a setup that could amplify a quantum signal without introducing noise is of great importance and its study is necessary. Such a setup has been proposed and for reasons rooted deeply in the unitary evolution of closed systems such a transformation is probabilistic, albeit heralded. Another practical application of this so-called heralded noiseless amplifier can be found in relation with quantum error correction. In this thesis, we have studied some of the characteristics of such a probabilistic heralded transformation, namely its action on squeezed states and the proof that such a device acts as a universal squeezer.

Beyond practical applications, the heralded noiseless amplifier and attenuator have raised instant signaling paradoxes that have been discussed and resolved in the thesis. Furthermore, the study of noiseless amplification and attenuation has been completed by considering their effect on non-Gaussian states of light and finding counterintuitive physical effects concerning the behavior of the coherent (or mean-field) amplitude of the signal state.

Majorization relations in quantum optics

The theory of majorization, or preorder theory, when used in quantum mechanics provides the means to analyze the behavior of entanglement transformations. This is done without specific entanglement monotones since using majorization theory for proving (or for disproving) majorization relations, i.e. which state is more disordered, provides the means for very strict conclusions about all entanglement monotones. We have used preorder theory to analyze the entanglement production of a most common Gaussian transformation, namely a beam splitter, in two different scenarios: when the number of input photons is increasing and when the transmittance is varying. The results were unexpected and interesting and may be proven useful in future works on quantum optical circuits. Moreover, the author believes that majorization theory could be used in many-body systems and in particular with Bose-Hubbard models and the various phase transitions that such systems exhibit.

Quantum entropy

The most studied, interesting and important entanglement monotone is the (reduced) von Neumann entropy for a pure state (or geometric entropy in quantum field theory). This is because it is connected with several quantum informational conjectures concerning the amount of information a specific quantum operation can transmit. The difficult part is generally how to compute this entropy. To this end, we made an attempt in this thesis to introduce the replica method to the community of quantum optics and quantum information theory. With this method, one does not have to compute the eigenvalues of the density matrix in order to find the entropy, which is particularly useful when dealing with infinite-dimensional state spaces as it is the case for bosonic states. The replica method is based on the fact that $\mathrm{tr}(\hat{\rho}^n),\ \forall\ n\ \in\ \mathbb{N}^*$ carries equivalent information as the eigenvalues of $\hat{\rho}$, and on the assumption that this quantity is well behaved in the region of $n = 1$. One has to find the derivative of $\mathrm{tr}(\hat{\rho}^n)$ with respect to n at $n = 1$, which gives the entropy with a minus sign.

In this thesis, we have discussed the justification of the replica method and considered the amplification (using a two-mode squeezer) of some non-trivial input states (e.g., superposition of the vacuum and a Fock state). The author believes that the replica method will serve as an excellent tool for attacking and proving several entropic conjectures in quantum optics.

As a more mathematically challenging problem but more familiar ground for the use of the replica method, a quantum informational signature was found for the phase transition from thermal gas to Bose-Einstein condensation in systems with finite temperature and chemical potential. Namely, this signature is the discontinuity of the derivative of the mutual information of two spatial regions of the field. This problem may at first sight seem detached to the rest of this work but it becomes clear that when one geometrically cuts out a region in a field, this can be seen as a fundamental quantum operation as well.

Appendix A

Calculations for Section 5.2.3

In the main text, it is shown that if the input state is a superposition $|\psi\rangle = (|0\rangle + z|m\rangle)/\sqrt{1+z^2}$, then the output state $\hat{\rho}$ is such that

$$\mathrm{tr}(\hat{\rho}^n) = \frac{\mathrm{tr}(\hat{\rho}_0^n)}{(1+z^2)^n}\,\Pi_{\partial\lambda}(n)\exp\left(\bar{\lambda}^\dagger N\bar{\lambda}\right)\Big|_{\bar{\lambda}=\bar{0}}, \tag{A.1}$$

where $\mathrm{tr}(\hat{\rho}_0^n)$ corresponds to a vacuum input state, i.e., $z=0$. Here, we define the matrix $N = (1-|\tau|^{2n})^m M^{-1}$, with

$$M^{-1} = \frac{1}{1-|\tau|^{2n}}\begin{pmatrix} 1 & |\tau|^2 & \dots & |\tau|^{2(n-1)} \\ |\tau|^{2(n-1)} & 1 & \dots & |\tau|^{2(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ |\tau|^2 & |\tau|^4 & \dots & 1 \end{pmatrix} \tag{A.2}$$

being a circular matrix. The differential operator $\Pi_{\partial\lambda}(n)$ has the form

$$\Pi_{\partial\lambda}(n) = \sum_{k=0}^{n} c^{2k}\,\Pi_{2k}(n) \tag{A.3}$$

where each $\Pi_{2k}(n)$ contains $\binom{n}{k}^2$ terms that give a non-zero result when acting on $\exp\left(\bar{\lambda}^\dagger N\bar{\lambda}\right)$ and taking the value at $\bar{\lambda} = \bar{0}$. These terms are all the derivatives of even order $(2,\ 4,\ \dots, 2n)$ with respect to λ such that the number of λ is equal to the number of λ^* for each derivative. For example, by keeping terms that return non-zero result in Eq. (A.1), we have

$$
\begin{aligned}
k = 0 \quad &: \quad \Pi_0(n) = 1 \\
k = 1 \quad &: \quad \Pi_2(n) = \frac{\partial^{2m}}{\partial\lambda_1\partial\lambda_1^*} + \dots \\
k = 2 \quad &: \quad \Pi_4(n) = \frac{\partial^{4m}}{\partial\lambda_1\partial\lambda_1^*\partial\lambda_2\partial\lambda_2^*} + \dots \\
&\vdots \\
k = n \quad &: \quad \Pi_{2n}(n) = \frac{\partial^{2mn}}{\partial\lambda_1\partial\lambda_1^*\dots\partial\lambda_n\partial\lambda_n^*}
\end{aligned}
\tag{A.4}
$$

It can be verified that the term with $k = n$ gives, when acting on the exponential of Eq. (A.1),

$$
\Pi_{2n} \exp\left(\bar{\lambda}^\dagger N \bar{\lambda}\right)\Big|_{\bar{\lambda}=\bar{0}} = m!^n \frac{1-|\tau|^{2n}}{|\tau|^{2n}} \, \mathrm{Li}_{-n}^{(m)}(|\tau|^{2n}) \tag{A.5}
$$

where $\mathrm{Li}_{-n}^{(m)}(|\tau|^{2n})$ is defined in the main text. In other words, this term is connected with the entropy S_m when the input state is the Fock state $|m\rangle$, something that can be seen by taking the limit $z \to \infty$ in Eq. (A.1). Also, it is not difficult to see that for each operator with $k = 0, 1 \dots, n$ in Eq. (A.4), there are exactly $\binom{n}{k}$ terms where the derivatives with respect to conjugate pairs of λ's appear. For example, such a term is $\partial^6/\partial\lambda_1\partial\lambda_1^*\partial\lambda_2\partial\lambda_2^*\partial\lambda_3\partial\lambda_3^*$. These terms will give a result with no dependence on $|\tau|$ in the numerator when it acts on the exponential of Eq. (A.1). If we extract all these terms and gather them together, substituting c with its definition

$$
c = \frac{z}{\sqrt{m!}}(1-|\tau|^2)^{m/2}, \tag{A.6}
$$

we can write

$$
\sum_{k=0}^{n} \binom{n}{k} z^{2k} \left(\frac{1-|\tau|^2}{1-|\tau|^{2n}}\right)^{mk} = \left[1 + z^2\left(\frac{1-|\tau|^2}{1-|\tau|^{2n}}\right)^m\right]^n. \tag{A.7}
$$

From all this, we obtain the expression

$$
\begin{aligned}
\mathrm{tr}(\hat{\rho}^n) = \frac{\mathrm{tr}\hat{\rho}_0^n}{(1+z^2)^n} \Bigg\{ & \left[1 + z^2\left(\frac{1-|\tau|^2}{1-|\tau|^{2n}}\right)^m\right]^n \\
& - z^{2n}\left(\frac{1-|\tau|^2}{1-|\tau|^{2n}}\right)^{mn} + F^{(m)}(n) + z^{2n}\frac{\mathrm{tr}\hat{\rho}_m^n}{\mathrm{tr}\hat{\rho}_0^n} \Bigg\}
\end{aligned}
\tag{A.8}
$$

where $\hat{\rho}_m$ is the output state resulting from an input state $|m\rangle$. In Eq. (A.8), we subtracted the term proportional to z^{2n} as we have used it twice; one in the first term of the form $[\ldots]^n$ and a second time for the very last term. In the same equation, $F^{(m)}(n)$ gathers all terms except for the first and last one of Eq. (A.1), but without returning any term with no dependence on $|\tau|$ in the numerator, something that we denote as $\tilde{\Pi}_{2k}(n)$ in the expression

$$F^{(m)}(n) = \sum_{k=1}^{n-1} \frac{z^{2k}}{m!^k} \left(\frac{1-|\tau|^2}{1-|\tau|^{2n}} \right)^{mk} \times \tilde{\Pi}_{2k}(n) \exp\left(\bar{\lambda}^\dagger N \bar{\lambda}\right). \tag{A.9}$$

In the main text, it is shown that by taking the derivative of Eq. (A.8) with respect to n and keeping the value at $n = 1$, we get

$$S(z) = \frac{1}{1+z^2} S_0 + \frac{z^2}{1+z^2} S_m - \frac{\partial}{\partial n} F^{(m)}(n)\Big|_{n=1}. \tag{A.10}$$

We shall now prove that the last term in the right hand side of Eq. (A.10) is equal to zero. It can be found that Eq. (A.9) assumes the form,

$$\begin{aligned} F^{(m)}(n) &= \sum_{k=0}^{n-1} \frac{z^{2k}}{m!^k} \left(\frac{1-|\tau|^2}{1-|\tau|^{2n}} \right)^{mk} \times \\ &\quad \times \sum_{l=0}^{(n-1)k} A_k(n,l) |\tau|^{2(l+m-1)} \end{aligned} \tag{A.11}$$

where $A_k(n,l)$ are unknown coefficients satisfying the constraints,

$$\begin{aligned} A_k(1,l) &= 0 \\ A_0(n,l) &= 0. \end{aligned} \tag{A.12}$$

Now, Eq. (A.11) may be written,

$$F^{(m)}(n) = \sum_{k=0}^{n-1} R^{(m)}(n,k). \tag{A.13}$$

Using the Euler-McLaurin summation formula [Apo69] we get,

$$\begin{aligned} F^{(m)}(n) &= \int_0^{n-1} dx R^{(m)}(n,x) + \\ &+\frac{1}{2}\Big[R^{(m)}(n,n-1) + R^{(m)}(n,0)\Big] + \\ &+\sum_{r=1}^{p} \frac{B_{2r}}{(2r)!}\Big[R^{(m)(2r-1)}(n,n-1) - \\ &-R^{(m)(2r-1)}(n,0)\Big] + \text{Rem} \end{aligned} \tag{A.14}$$

where B_{2r} are the Bernoulli numbers and we symbolize as $R^{(m)(\mu)}(n,a)$ the differentiation with respect to the second argument,

$$R^{(m)(\mu)}(n,a) = \frac{\partial^\mu}{\partial x^\mu} R^{(m)}(n,x)\Big|_{x=a}. \tag{A.15}$$

Differentiation with respect to the first argument will be denoted explicitly as $\partial_n R^{(m)}(n,x)$. The last term in Eq. (A.14) is the remainder and has the form,

$$\text{Rem} = -\int_0^{n-1} dx \frac{P_{2p}(x)}{(2p)!} R^{(m)(2p)}(n,x). \tag{A.16}$$

For $x > 0$ we get the periodic Bernoulli functions $P_n(x) = B_n(x - [x])$, where $[x]$ is the largest integer x, while $P_n(0) = B_n$.

If we perform the derivative of Eq. (A.14) with respect to n at $n = 1$, we see, taking Eq. (A.12) into account, that all terms vanish. This is because three kind of terms appear,

$$R^{(m)}(1,0) = 0 \tag{A.17}$$

$$\int_0^0 dx(\dots) = 0 \tag{A.18}$$

$$\partial_n R^{(m)}(n,0)\Big|_{n=1} = \partial_n 0 = 0 \tag{A.19}$$

$$R^{(m)(2r)}(1,0) = 0. \tag{A.20}$$

Thus, we conclude that

$$\frac{d}{dn} F^{(m)}(n)\Big|_{n=1} = 0. \tag{A.21}$$

as advertised.

Appendix B

Calculations for Section 5.3.2

In this Appendix we shall prove those of the formulas appearing in the text for which summations over m or ν must be performed. To begin with, let us discuss (5.63). The relevant integral diverges and calls for the introduction of a cutoff,

$$\begin{aligned} & \frac{1}{2it}\sum_m \int\limits_0^\infty d\rho\rho \mathrm{e}^{-\frac{\rho^2}{2it}} i^{-\frac{|m|}{n}} J_{\frac{|m|}{n}}\left(\frac{\rho^2}{2t}\right) \\ \underset{it\to T}{\rightarrow} \; & \frac{1}{2T}\sum_m \int_0^R d\rho\rho \mathrm{e}^{-\frac{\rho^2}{2T}} I_{\frac{|m|}{n}}\left(\frac{\rho^2}{2T}\right). \end{aligned} \tag{B.1}$$

To handle the last integral we make an intermediate step by introducing the following expression,

$$\begin{aligned} F_n(\alpha) &= \frac{1}{2}\sum_{m=-\infty}^{\infty} \int\limits_0^\infty dq \mathrm{e}^{-\alpha q} I_{\frac{|m|}{n}}(q) \\ &= \frac{1}{2\sqrt{\alpha^2-1}} \coth\left(\frac{\alpha+\sqrt{\alpha^2-1}}{2}\right) \end{aligned} \tag{B.2}$$

which is also a regularized version (for $\alpha \to 1^+$) of the integral entering (5.63). Taking the limit $\alpha = 1+\epsilon, \;\epsilon \to 0^+$ one easily finds that,

$$F_n(\alpha) = \frac{n}{2\epsilon} + \frac{1}{12}\left(\frac{1}{n} - n\right) + \mathcal{O}(\epsilon). \tag{B.3}$$

We immediately see that the diverged part of the integral appears for $n = 1$. In this case the integration in (B.1) is trivial and we are led to the conclusion,

$$\frac{1}{\epsilon} \to \frac{\pi R^2}{4\pi T} = \frac{V_2}{4\pi T}. \tag{B.4}$$

Combining this identification with the finite part appearing in (B.3) we get the confirmation of (5.63).

Our next concern is (5.65). Using the identities,

$$\begin{aligned}
\mathrm{e}^{-\frac{(\nu\beta)^2}{4T}} &= (4\pi T)^{\frac{D+1}{2}} \int \frac{d^{D+1}p}{(2\pi)^{D+1}} \mathrm{e}^{-Tp^2+ip_0\nu\beta}, \\
\sum_{\nu=-\infty}^{\infty} \mathrm{e}^{ip_0\nu\beta} &= \frac{2\pi}{\beta} \sum_{k=-\infty}^{\infty} \delta(p_0-\omega_k),\ \ \omega_k = \frac{2\pi k}{\beta}
\end{aligned} \tag{B.5}$$

we recast the integral appearing in (5.65) into the form,

$$\begin{aligned}
&\frac{1}{2}\frac{V_{D-1}V_2}{(4\pi)^{\frac{D+1}{2}}} \int_{0^+}^{\infty} \frac{dT}{T^{\frac{D+3}{2}}} \mathrm{e}^{-Tm^2} \sum_{\nu} \mathrm{e}^{-\frac{(\nu\beta)^2}{4T}} \\
&= \frac{V_D}{4} \int \frac{d^D p}{(2\pi)^D} \sum_k \int_{0^+}^{\infty} \frac{dT}{T} \mathrm{e}^{-T\left[(\beta\omega)^2+(2\pi k)^2\right]}.
\end{aligned} \tag{B.6}$$

To obtain the last result we wrote $V_2 = \int_0^\beta d\tau \int_{-\infty}^{\infty} dx_1 \to \beta L$, we rescaled $T \to T\beta^2$ and we used the abbreviation $\omega^2 = p^2 + m^2$.

Performing the integral over T and neglecting an irrelevant (infinite) constant we get,

$$\int_{0^+}^{\infty} \frac{dT}{T} \mathrm{e}^{-T\left[(\beta\omega)^2+(2\pi k)^2\right]} = -\ln\left((\beta\omega)^2 + (2\pi k)^2\right). \tag{B.7}$$

The summation over k is standard [Kap81],

$$\frac{1}{2}\sum_k \ln\left((\beta\omega)^2 + (2\pi k)^2\right) = \frac{1}{2}\beta\omega + \ln(1-\mathrm{e}^{-\beta\omega}). \tag{B.8}$$

The last result proves (5.65).

Following the same line of reasoning we can prove (5.67). Using, once again, the identities (B.5) we rewrite the relevant integral in the form,

$$\begin{aligned}
&\frac{1}{12}\frac{V_{D-1}}{(4\pi)^{\frac{D-1}{2}}} \int_0^{\infty} \frac{dT}{T^{\frac{D+1}{2}}} \mathrm{e}^{-Tm^2} \sum_{\nu} \mathrm{e}^{-\frac{(\nu\beta)^2}{4T}} \\
&= \frac{\pi}{3\beta} V_{D-1} \int \frac{d^D p}{(2\pi)^D} \sum_k \frac{1}{\omega^2+\omega_k^2}.
\end{aligned} \tag{B.9}$$

The summation is easily performed,

$$\sum_k \frac{1}{\omega^2+\omega_k^2} = \frac{\beta}{2\omega}\frac{1}{\tanh(\beta\omega)}. \tag{B.10}$$

Combining (B.9) and (B.10) we immediately obtain (5.67) of the text.

Our next concern is Eq. (5.64). The relevant integral diverges and the introduction of a cutoff is necessary. To this end let us discuss the integral,

$$\begin{aligned}&\sum_m \int_0^\infty d\rho\rho \mathrm{e}^{-\frac{\rho^2}{R^2}} J^2_{\frac{|m|}{n}}\left(\frac{\nu\beta_n}{2t}\rho\right) \\ = \;& \frac{R^2}{2}\mathrm{e}^{-\frac{(\nu\beta R)^2}{8t^2}} \sum_m I_{\frac{|m|}{n}}\left(\frac{(\nu\beta R)^2}{8t^2}\right) = \frac{R^2}{2}\end{aligned} \tag{B.11}$$

which can be considered (at the limit $R\to\infty$) as a regularized version of the integral appearing in (5.64). Note that the divergence in (B.11) is independent of n and, contrary to (B.3), there is no finite part for $n\neq 1$. This completes the proof of (5.64).

To prove (17) it is enough to follow the road we followed to arrive at (B.8). Beginning from the relation,

$$\begin{aligned}&\frac{1}{2}\frac{V_{D-1}V_2}{(4\pi)^{\frac{D+1}{2}}}\int_{0^+}^\infty \frac{dT}{T^{\frac{D+3}{2}}}\mathrm{e}^{-Tm^2}\sum_\nu \mathrm{e}^{-\frac{(\nu\beta_n)^2}{4T}} \\ = \;& \frac{V_D}{4n}\int\frac{d^Dp}{(2\pi)^D}\sum_k\int_{0^+}^\infty \frac{dT}{T}\mathrm{e}^{-T[(\beta_n\omega)^2+(2\pi k)^2]}\end{aligned} \tag{B.12}$$

we only have to perform a differentiation with respect to n to arrive at the result indicated in (5.66),

$$\begin{aligned}&\partial_n\left\{\frac{1}{n}\int\frac{d^Dp}{(2\pi)^D}\left[-\frac{1}{2}\beta\omega n-\ln\left(1-\mathrm{e}^{-\beta\omega n}\right)\right]\right\}\Bigg|_{n=1} \\ = \;& \int\frac{d^Dp}{(2\pi)^D}\left[\ln\left(1-\mathrm{e}^{-\beta\omega}\right)-\frac{\beta\omega}{\mathrm{e}^{\beta\omega}-1}\right].\end{aligned} \tag{B.13}$$

The last relation we have to prove is (5.78) of the text. We begin by using the Poisson summation formula to find that,

$$\begin{aligned}\sum_\nu \mathrm{e}^{-\frac{(\nu\beta)^2}{4T}-\mu\nu\beta} &= \sum_k\int_{-\infty}^\infty dx \mathrm{e}^{2\pi i k x}\mathrm{e}^{-\frac{(x\beta)^2}{4T}-\mu x\beta} \\ &= \frac{\sqrt{4\pi T}}{\beta}\sum_k \mathrm{e}^{-T(\omega_k+i\mu)^2}.\end{aligned} \tag{B.14}$$

With the help of this result we get the mutual information,

$$I_m = \frac{2\pi}{3\beta} V_{D-1} \int \frac{d^D p}{(2\pi)^D} \sum_k \frac{1}{(\omega_k + i\mu)^2 + \omega^2}. \tag{B.15}$$

The sum in the last expression can be easily performed if we rewrite it in the form,

$$\begin{aligned} \sum_k \frac{1}{(\omega_k + i\mu)^2 + \omega^2} \quad &= \quad \frac{1}{2}\frac{\omega - \mu}{\omega} \sum_k \frac{1}{\omega_k^2 + (\omega - \mu)^2} + \\ &+ \frac{1}{2}\frac{\omega + \mu}{\omega} \sum_k \frac{1}{\omega_k^2 + (\omega + \mu)^2}. \end{aligned} \tag{B.16}$$

Using for each term the formula (B.10) we get the result indicated in (5.78).

It would be useful to compare our result indicated in (5.68) with the corresponding result derived in the framework of a two dimensional conformal scalar field theory [CC04] with central charge $c = 1/2$. This can be done by identifying the ultraviolet cutoff Λ with the inverse lattice spacing $1/\alpha$ and the mass m with the inverse finite size of excluded interval $1/l$ (that is, the infrared cutoff). Given that our result is valid at the limit $L^2 m^2 \to \infty$, the comparison is meaningful only for $\beta/\alpha \to \infty$ or $l/\beta \to \infty$. Applying (5.68) for $D = 1$ we get the result,

$$S_g = \begin{cases} \frac{1}{6} \ln\left(\frac{\Lambda}{m}\right) \to \frac{1}{6} \ln\left(\frac{l}{\alpha}\right) & \text{if } \beta \to \infty \\ \frac{\pi}{6}\frac{1}{m\beta} \to \frac{\pi}{6}\frac{l}{\beta} & \text{if } \beta \to 0 \end{cases}. \tag{B.17}$$

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