nature photonics
SUPPLEMENTARY INFORMATION
DOI: 10.1038/NPHOTON.2014.216

Ultimate classical communication rates of quantum optical channels

V. Giovannetti[1], R. García-Patrón[2,3], N. J. Cerf[2], and A. S. Holevo[4,5]
[1]*NEST, Scuola Normale Superiore and Istituto Nanoscienze-CNR, I-56127 Pisa, Italy*
[2]*QuIC, Ecole Polytechnique de Bruxelles, CP 165, Université Libre de Bruxelles, 1050 Bruxelles, Belgium*
[3]*Max-Planck Institut für Quantenoptik, Hans-Kopfermann Str. 1, D-85748 Garching, Germany*
[4]*Steklov Mathematical Institute, RAS, Moscow*
[5]*National Research University Higher School of Economics (HSE), 101000 Moscow, Russia*
(Dated: July 24, 2014)

SUPPLEMENTAL INFORMATION

Four fundamental Gaussian channels

A compact way to represent bosonic Gaussian channels is obtained by expressing the density matrices ρ of the mode as $\rho = (1/\pi)\int d^2z \chi(z)D(-z)$ where the integral is performed over all z complex and where $D(z) = \exp[za^\dagger - z^*a]$ is the displacement operator of the system while $\chi(z) = \mathrm{Tr}[\rho D(z)]$ is the symmetrically ordered characteristic function of ρ [1]. Accordingly the maps $\mathcal{E}_\eta^N$, $\mathcal{A}_\kappa^N$, $\tilde{\mathcal{A}}_\kappa^N$ and $\mathcal{N}_n$ can be assigned to the input-output mappings $\chi(z) \to \chi'(\rho) = \mathrm{Tr}[\Phi(\rho)D(z)]$ where [2, 3]

$$\chi'(z) = \begin{cases} \chi(\sqrt{\eta}z)\, e^{-(1-\eta)(N+1/2)|z|^2}, & (\Phi = \mathcal{E}_\eta^N) \\ \chi(z)\, e^{-n|z|^2}, & (\Phi = \mathcal{N}_n) \\ \chi(\sqrt{\kappa}z)\, e^{-(\kappa-1)(N+1/2)|z|^2}, & (\Phi = \mathcal{A}_\kappa^N) \\ \chi(-\sqrt{\kappa-1}z^*)\, e^{-\kappa(N+1/2)|z|^2}, & (\Phi = \tilde{\mathcal{A}}_\kappa^N), \end{cases} \tag{1}$$

with $\eta \in [0, 1]$, $\kappa \in [1, \infty[$ and $N, n \in [0, \infty[$.

Gaussian Channel Decomposition

As shown in Ref. [4], a generic phase-covariant single-mode channel Φ_τ^y of loss/gain parameter $\tau \geq 0$ and added noise $y \geq |\tau - 1|$ can be expressed as the concatenation of a quantum-limited lossy channel followed by a quantum-limited amplifier $\Phi = \mathcal{A}_{\kappa_0} \circ \mathcal{E}_{\eta_0}$ (see Figure 1 of the main text) with $\tau = \eta_0\kappa_0$ and $y = \kappa_0(1-\eta_0) + (\kappa_0 - 1)$. Similarly for phase-contravariant single-mode channels, the decomposition reads $\Phi_\tau^y = \tilde{\mathcal{A}}_{\kappa_0} \circ \mathcal{E}_{\eta_0}$ where now $\tau = \eta_0(1 - \kappa_0)$ and $y = (\kappa_0 - 1)(1-\eta_0) + \kappa_0$. The explicit values of the parameter η_0 and κ_0 for the maps $\mathcal{E}_\eta^N$, $\mathcal{A}_\kappa^N$, $\tilde{\mathcal{A}}_\kappa^N$ and $\mathcal{N}_n$ are [4, 5]

- Thermal channel $\mathcal{E}_\eta^N$: $\eta_0 = \eta/(1 + (1 - \eta)N)$, $\kappa_0 = 1 + (1 - \eta)N$;
- Additive classical noise channel $\mathcal{N}_n$: $\eta_0 = 1/(n + 1)$, $\kappa_0 = n + 1$ (see also [2]);
- Amplifier channel $\mathcal{A}_\kappa^N$: $\eta_0 = k/[k + (k - 1)N]$, $\kappa_0 = k + (k - 1)N$.

Linking the minimal entropy to the classical capacity

The m-mode energy-constrained χ-capacity of a BGC Φ is expressed as

$$C_\chi(\Phi^{\otimes m}; E) = \sup_{\text{ENS}} \{S(\Phi^{\otimes m}[\sum_j p_j\rho_j]) - \sum_j p_j S(\Phi^{\otimes m}[\rho_j])\}, \tag{2}$$

with the maximization being performed over all possible ensemble ENS $= \{p_j, \rho_j\}$ whose average state $\rho = \sum_j p_j\rho_j$ belongs to the set $\mathcal{B}_E$ fulfilling the average energy constraint

$$\mathrm{Tr}[H^{(m)}\rho_{\text{ENS}}] \leq mE\ . \tag{3}$$

where $H^{(m)} = \sum_{j=1}^m a_j^\dagger a_j$ is the total photon number operator in the m modes.

An upper-bound for (2) can be easily obtained by replacing the first term entering the maximization on the rhs with the maximum output entropy attainable within the set $\mathcal{B}_E$, i.e.

$$S_{max}^{(E)}(\Phi^{\otimes m}) = \sup_{\rho\in\mathcal{B}_E} S(\Phi^{\otimes m}(\rho))\ . \tag{4}$$

For the maps (1) this quantity is additive in m and it is attained by $[\rho_G^{(E)}]^{\otimes m}$ with $\rho_G^{(E)}$ being a Gaussian state representing the Gibbs thermal state whose mean energy coincides with E,

$$S_{max}^{(E)}(\Phi^{\otimes m}) = m\, S_{max}^{(E)}(\Phi) = m\, S(\Phi(\rho_G^{(E)}))\ , \tag{5}$$

(see e.g. [6–9]). Accordingly, we can write

$$C_\chi(\Phi^{\otimes m}; E) \leq m\ S(\Phi(\rho_G^{(E)})) - \inf_{\text{ENS}} \sum_j p_j S(\Phi^{\otimes m}[\rho_j]). \tag{6}$$

In contrast with the conventional version of the min-entropy conjecture [2], we define here the min-entropy quantity

$$S_{\min}^{(<)}[\Phi^{\otimes m}] := \inf_{\rho\in\mathcal{B}^{(<)}} S(\Phi^{\otimes m}(\rho))\,, \tag{7}$$

associated with the map $\Phi^{\otimes m}$, where the minimization is now restricted over the set $\mathcal{B}^{(<)}$ of m-mode states ρ having

DOI: 10.1038/NPHOTON.2014.216

bounded mean input energy, i.e., $\mathrm{Tr}[H^{(m)}\rho] < \infty$ (the minimization can be further restricted to the pure states of $\mathcal{B}^{(<)}$ due to the concavity of S). This allows us to rewrite the upper bound as

$$C_\chi(\Phi^{\otimes m}; E) \;\le\; m\, S(\Phi(\rho_G^{(E)})) - S^{(<)}_{\min}[\Phi^{\otimes m}]\,, \quad (8)$$

where the last inequality follows by exploiting the fact that for all ρ_j entering in one of the allowed ensembles ENS $= \{p_j, \rho_j\}$, the term $S(\Phi^{\otimes m}[\rho_j])$ can be lower bounded by $S^{(<)}_{\min}[\Phi^{\otimes m}]$. This can be seen by noticing that in order to satisfy Eq. (3) all states ρ_j which are associated with a not null probability p_j must be in $\mathcal{B}^{(<)}$.

Finally, owing to the proof of the conjecture (2) of the main text derived in this paper, one gets the simple expression

$$C_\chi(\Phi^{\otimes m}; E) \;\le\; m\, \left[S(\Phi(\rho_G^{(E)})) - S(\Phi(|0\rangle\langle 0|))\right] \quad (9)$$

and hence

$$C(\Phi; E) \;\le\; S(\Phi(\rho_G^{(E)})) - S(\Phi(|0\rangle\langle 0|))\,. \quad (10)$$

It turns out that for the channels (1) these bounds are attainable by exploiting Gaussian encodings formed by Gaussian distribution of coherent states [10], yielding the expressions given in Eqs. (12), (14), (16), and (18).

Mininal output entropy and classical capacity of the four fundamental Gaussian channels

Exploiting the conjecture (2) of the main text, we simply have to write the explicit expression of $S(\Phi(|0\rangle\langle 0|))$ for each of the four fundamental classes of bosonic Gaussian channels. Together with the expression of $S(\Phi(\rho_G^{(E)}))$, we can use Eq. (10) to compute the classical capacity for each of the four fundamental channels (as explained, the upper bound is achieved with a Gaussian encoding). The minimal entropy and corresponding capacities are listed below.

- Thermal channel $\mathcal{E}_\eta^N$:

$$S^{(<)}_{\min}[\mathcal{E}_\eta^N] = g((1-\eta)N)\,, \quad (11)$$
$$C(\mathcal{E}_\eta^N; E) = g(\eta E + (1-\eta)N) - g((1-\eta)N)\,; \quad (12)$$

- Additive classical noise channel $\mathcal{N}_n$:

$$S^{(<)}_{\min}(\mathcal{N}_n) = g(n)\,, \quad (13)$$
$$C(\mathcal{N}_n; E) = g(E+n) - g(n)\,; \quad (14)$$

- Amplifier channel $\mathcal{A}_\kappa^N$:

$$S^{(<)}_{\min}[\mathcal{A}_\kappa^N] = g((\kappa-1)(N+1))\,, \quad (15)$$
$$C(\mathcal{A}_\kappa^N; E) = g(\kappa E + (\kappa-1)(N+1)) - g((\kappa-1)(N+1))\,; \quad (16)$$

- Contravariant amplifier channel $\tilde{\mathcal{A}}_\kappa^N$:

$$S^{(<)}_{\min}[\tilde{\mathcal{A}}_\kappa^N] = g(\kappa(N+1)-1)\,, \quad (17)$$
$$C(\tilde{\mathcal{A}}_\kappa^N; E) = g(\kappa N + (\kappa-1)(E+1)) - g(\kappa(N+1)-1)\,; \quad (18)$$

In all these expressions, the function $g(x) = (x+1)\log_2(x+1) - x\log_2 x$ refers to the von Neumann entropy of a Gibbs thermal Bosonic state with a mean photon number equal to x.

Classical capacity of an arbitrary single-mode Gaussian channel

Using the same approach, an identity analogous to Eq. (2) of the main text can be shown to hold also for arbitrary non-degenerate single-mode channels Ψ. Indeed, as discussed in Refs. [11, 12], with few notable exceptions of channels with degenerate noise or signal treated in Ref. [13] all the single-mode channels can be expressed as a proper concatenation of one of the maps Φ of Eq. (1) together with two squeezing and/or displacement unitary transformations $\mathcal{U}$ and $\mathcal{V}$ acting on the input and/or the output of the communication line, i.e. $\Psi = \mathcal{U}\circ\Phi\circ\mathcal{V}$ (the symbol "$\circ$" representing concatenation of super-operators). From Eq. (2) of the main text it then follows that also for these maps the min-entropy conjecture applies: this time however $S^{(<)}_{\min}[\Phi^{\otimes m}]$ is achieved over a Gaussian input obtained by properly squeezing the vacuum state to compensate the action of $\mathcal{V}$. As different from the case of the BGCs of Eq. (1) this however will guarantee the optimality of Gaussian inputs for $C(\Psi; E)$ and the additivity of $C_\chi(\Psi; E)$ only for large enough values of the mean energy E (see e.g. Refs. [5, 14]).

Entanglement of Formation

The two-mode Gaussian state $\rho(\kappa, N)$ resulting from applying a two-mode squeezing operation U_κ to a thermal state $\rho_G^{(N)}$, of covaraince matrix $(2N+1)\mathrm{diag}(1,1)$ tensor the vacuum state (covariance matrix $\mathrm{diag}(1,1)$) reads

$$\gamma = \begin{pmatrix} aI & c\sigma_z \\ c\sigma_z & bI \end{pmatrix}, \quad (19)$$

where $\sigma_z = \mathrm{diag}(1,-1)$, $a = 2(N+1)\kappa - 1$, $b = 2(N+1)\kappa - (2N+1)$ and $c = 2(N+1)\sqrt{\kappa(\kappa-1)}$.

The upper bound It is easy to see that the covariance matrix can be decomposed as

$$\gamma = \gamma_0 + M \quad (20)$$

where

$$\gamma_0 = \begin{pmatrix} (2\kappa-1)I & 2\sqrt{\kappa(\kappa-1)}\sigma_z \\ 2\sqrt{\kappa(\kappa-1)}\sigma_z & (2\kappa-1)I \end{pmatrix} \quad (21)$$

is the covariance matrix of a two-mode squeezed vacuum state $\rho(\kappa, 0)$, and

$$M = 2N \begin{pmatrix} \kappa I & \sqrt{\kappa(\kappa-1)}\sigma_z \\ \sqrt{\kappa(\kappa-1)}\sigma_z & \kappa - 1 I \end{pmatrix}. \tag{22}$$

The matrix M can easily be shown to be positive, as it can be diagonalized to $\mathrm{diag}(2\kappa - 1, 0)$. This implies that one can generate the Gaussian state of covariance matrix γ out of a two-mode squeezed vacuum state $\rho(\kappa, 0)$ of covariance matrix γ_0 and applying random correlated displacements (LOCC post-processing) to both modes. This proves an upperbound to the entanglement of formation of such state, as the optimal decomposition can only have lower cost in terms of entanglement, i.e.,

$$\mathrm{EoF}\left[\rho(\kappa, N)\right] \leq \mathrm{EoF}\left[\rho(\kappa, 0)\right] = g(\kappa - 1), \tag{23}$$

where we used the fact that $\rho(\kappa, 0)$ is a two-mode squeezed vacuum state and pure.

The lower bound To show that this quantity is also a lower bound for $\mathrm{EoF}[\rho(\kappa, N)]$ one can use the fact that the reduced density matrix of $\rho(\kappa, N)$ coincides with the output state $\mathcal{A}_\kappa(\rho_G^{(N)})$ of the quantum-limited amplifier to the thermal state $\rho_G^{(N)}$. Therefore exploiting the equivalence relation introduced in [15] one has that

$$\begin{aligned} \mathrm{EoF}[\rho(\kappa, N)] &= \inf_{\{p_j;|\psi_j\rangle\}} \sum_j p_j S(\mathcal{A}_\kappa(|\psi_j\rangle\langle\psi_j|)) \\ &\geq S(\mathcal{A}_\kappa(|0\rangle\langle 0|)) \ , \end{aligned} \tag{24}$$

where the minimization being performed over the pure state ensembles $\mathcal{E} = \{p_j; |\psi_j\rangle\}$ of $\rho_G^{(N)}$ (i.e. $\sum_j p_j |\psi_j\rangle\langle\psi_j| = \rho_G^{(N)}$) and where the last inequality follows from Eq. (4) of the main text. Notice that since $\rho_G^{(N)}$ has finite mean energy all the vectors $|\psi_j\rangle$ entering the ensemble are elements of $\mathcal{B}^{(<)}$.

The combination of the upperbound and the lowerbound proves the equality

$$\mathrm{EoF}\left[\rho(\kappa, N)\right] = \mathrm{EoF}\left[\rho(\kappa, 0)\right] = \kappa \log\left[\kappa\right] - (\kappa - 1) \log\left[\kappa - 1\right]. \tag{25}$$

[1] D.F. Walls & G.J. Milburn, Quantum Optics (Springer 1994).

[2] V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, & J. H. Shapiro, Minimum output entropy of bosonic channels: A conjecture, Phys. Rev. A **70**, 032315 (2004).

[3] F. Caruso & V. Giovannetti, Degradability of Bosonic Gaussian channels, Phys. Rev. A **74**, 062307 (2006).

[4] R. García-Patrón, C. Navarrete-Benlloch, S. Lloyd, J. H. Shapiro, & N. J. Cerf, Majorization theory approach to the Gaussian channel minimum entropy conjecture, Phys. Rev. Lett. **108**, 110505 (2012).

[5] J. Schäfer, E. Karpov, R. García-Patrón, O. V. Pilyavets, & N. J. Cerf, Equivalence Relations for the Classical Capacity of Single-Mode Gaussian Quantum Channels, Phys. Rev. Lett. **111**, 030503 (2013).

[6] V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, & J. H. Shapiro, Minimum bosonic channel output entropies, AIP Conf. Proc. **734**, 21 (2004).

[7] R. König & G. Smith, Classical capacity of quantum thermal noise channels to within 1.45 Bits. Phys. Rev. Lett. **110**, 040501 (2013).

[8] R. König & G. Smith, Limits on classical communication from quantum entropy power inequalities, Nature Photon. **7**, 142 (2013).

[9] V. Giovannetti, S. Lloyd, L. Maccone, & J. H. Shapiro, Electromagnetic channel capacity for practical purposes, Nature Photon. **7**, 834 (2013).

[10] A. S. Holevo & R. F. Werner, Evaluating capacities of bosonic Gaussian channels, Phys. Rev. A **63**, 032312 (2001).

[11] A. S. Holevo, One-mode quantum Gaussian channels: Structure and quantum capacity, Probl. Inf. Transm. **43**, 1-11 (2007).

[12] F. Caruso, V. Giovannetti, & A. S. Holevo, One-mode Bosonic Gaussian channels: a full weak-degradability classification, New J. Phys. **8**, 310 (2006).

[13] V. Giovannetti, A. S. Holevo, S. Lloyd, & L. Maccone, Generalized minimal output entropy conjecture for one-mode Gaussian channels: definitions and some exact results, J. Phys. A **43**, 415305 (2010).

[14] C. Lupo, O. V. Pilyavets, & S. Mancini, Capacities of lossy bosonic channel with correlated noise, New J. Phys. **11**, 063023 (2009).

[15] K. Matsumoto, T. Shimono, & A. Winter, Remarks on Additivity of the Holevo Channel Capacity and of the Entanglement of Formation, Commun. Math. Phys. **246**, 427-442 (2004).