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Ultimate classical communication rates of quantum optical channels

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Optical channels, such as fibres or free-space links, are ubiquitous in today's telecommunication networks. They rely on the electromagnetic field associated with photons to carry information from one point to another in space. A complete physical model of these channels must necessarily take quantum effects into account to determine their ultimate performances. Single-mode, phase-insensitive bosonic Gaussian channels have been extensively studied over past decades, given their importance for practical applications. In spite of this, a long-standing unsolved conjecture on the optimality of Gaussian encodings has prevented finding their classical communication capacity. Here, this conjecture is solved by proving that the vacuum state achieves the minimum output entropy of these channels. This establishes the ultimate achievable bit rate under an energy constraint, as well as the long awaited proof that the single-letter classical capacity of these channels is additive.

Since the advent of lasers, the question of how quantum effects must be accounted for in optical communication systems has occupied researchers[1]. The limits imposed by quantum mechanics on the highest possible bit rate achievable through optical channels (the so-called 'classical capacity') have long been studied (for example, in ref. 2), but no complete resolution of the problem was found due to the extreme difficulty of identifying the optimal encodings and decodings. Recent progress in quantum communication theory then moved this question forward by providing a formal expression for the classical capacity of quantum channels[3–5]. Still later, the capacity of a large class of realistic quantum communication models known as phase-insensitive bosonic Gaussian channels (BGCs) was expressed[6] by conjecturing the optimality of Gaussian encodings. These include channels that use the electromagnetic field (or any bosonic field) as an information carrier (for example, optical fibres, optical waveguides or free-space communication lines) and they model the most relevant sources of noise that may affect the transmission line (for example, attenuation noise due to the transferring of signals in dissipative media or the noise associated with their amplification).

The Gaussian conjecture has consequently become one of the most debated conjectures in quantum communication theory over the last decade. It can be rephrased, stating that the minimum von Neumann entropy at the output of a BGC is achieved by Gaussian input states (this is named the 'minimal output entropy' or 'min-entropy' conjecture)[7]. In its simplest form it implies that a bosonic system (say a mode of the electromagnetic field) interacting with a Gibbs thermal state via a quadratic exchange Hamiltonian will attain the minimum possible output entropy value if initially prepared into a coherent state, for example vacuum. This apparently innocuous statement turns out to have profound physical and technological implications[8–15]. Indirect evidence of its validity has accumulated over the years[8–19], yet its proof has remained elusive.

Here, we prove this Gaussian min-entropy conjecture for single-mode phase-insensitive BGCs (a set including, for instance, thermal, additive classical noise, and amplifier channels[6]), which in turn provides the ultimate closed expression for the communication capacity of these channels as prescribed by quantum mechanics (an extended version of this theorem that applies to multi-mode scenarios is given in ref. 20.) The proof restricts the entropy minimization over bounded energy states, but this does not affect the capacities. It also implies that the capacity of phase-insensitive BGCs can be achieved via Gaussian encodings and is additive, which has long been open to question. Furthermore, as sketched in the discussion section, proving the conjecture has broader consequences, allowing us to compute for the first time the entanglement of formation[21] for some non-symmetric Gaussian states or to deduce the optimality of Gaussian measurements in the quantum discord[22] for some Gaussian states[23]. Other major implications can be anticipated.

The Gaussian channel model

BGCs are the quantum counterparts of the Gaussian channels of classical information theory[24]. They describe all physical completely positive trace-preserving (CPT) transformations[25], which, when acting on a collection of bosonic degrees of freedom (continuous-variable quantum systems[26,27]), preserve their Gaussian character[6]. Accordingly, BGCs are the most common noise models that tamper with quantum optical implementations, including those responsible for the attenuation, amplification and squeezing of optical signals[2]. The most physically relevant cases are the phase-insensitive BGCs operating on a single bosonic mode described by the annihilation (resp. creation) operator a (resp. $a^\dagger$), fulfilling the canonical commutation rules $[a,a^\dagger]=1$. Phase-insensitive BGCs include two distinct subclasses: the covariant and contravariant BGCs. A general single-mode covariant BGC transforms the symmetrically ordered characteristic function $\chi(z)=\mathrm{Tr}[\rho D(z)]$ (z being complex) associated with the input state ρ (ref. 28) as $\chi(z)\to\chi(\sqrt{\tau}z)\exp(-y|z|^2/2)$, where $\tau\geq 0$ is a loss/gain parameter and y parametrizes the added noise, while $D(z)=\exp[za^\dagger - z^\star a]$ is the displacement operator. For contravariant (phase-conjugation) channels, we have $\chi(z)\to\chi(-\sqrt{|\tau|}z^*)\exp(-y|z|^2/2)$, where $\tau\leq 0$. The BGC is physical (the map is CPT) provided that y satisfies

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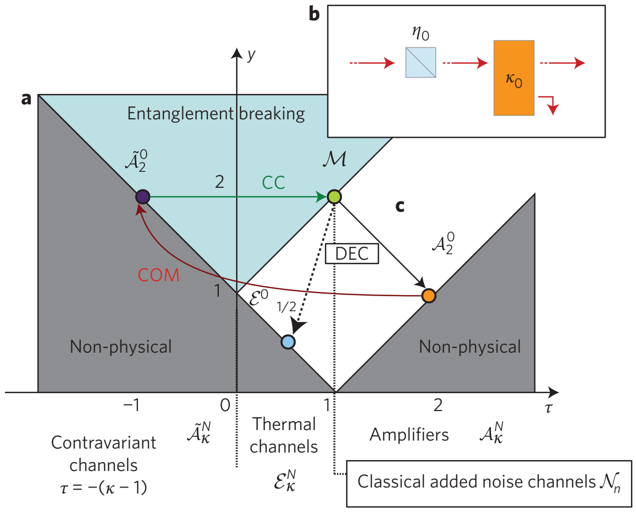

Figure 1 | Compact representation of single-mode phase-insensitive channels. a, Single-mode phase-insensitive bosonic Gaussian channels belong to four families characterized by the loss/gain parameter τ: (i) thermal channel $\mathcal{E}_\tau^N$ with $0 \le \tau \le 1$; (ii) amplifier channel $\mathcal{A}_\tau^N$, where $\tau \ge 1$; (iii) additive classical noise channel $\mathcal{N}_y$ satisfying $\tau = 1$; (iv) (contravariant) phase-conjugation channel, with $\tau \le 0$. All physical channels must satisfy $y \ge |\tau - 1|$ and they are quantum-limited when this inequality is saturated. The channels where $y \ge |\tau| + 1$ can be shown to be entanglement-breaking. **b**, A generic covariant single-mode channel Φ, with parameters τ and y, can be expressed as the concatenation of a quantum-limited lossy channel followed by a quantum-limited amplifier $\Phi = \mathcal{A}_{\kappa_0} \circ \mathcal{E}_{\eta_0}$, where $\tau = \kappa_0 \eta_0$ and $y = \kappa_0(1 - \eta_0) + (\kappa_0 - 1)$. **c**, The proof relies on the following observation, which is applied iteratively. Any quantum-limited amplifier $\mathcal{A}_{\kappa_0}$ has a complementary (COM) channel $\tilde{\mathcal{A}}_{\kappa_0}$ that is phase-conjugating (that is, $\tau \le 0$), and therefore entanglement-breaking. Applying a complex conjugation (CC) to $\tilde{\mathcal{A}}_{\kappa_0}$ results in a new channel $\mathcal{M}$ satisfying $\tau = \kappa_0 - 1$ and $y = \kappa_0$, which can be decomposed (DEC) as $\mathcal{A}_{\kappa_0} \circ \mathcal{E}_{(\kappa_0 - 1)/\kappa_0}$.

$y \ge |\tau - 1|$. The map is called quantum-limited when this latter inequality is saturated. A compact representation of phase-insensitive BGC channels[29] is given in Fig. 1. Among these channels, four fundamental classes can be identified (adopting the notation of refs 7, 9, and 30): the thermal channels $\mathcal{E}_\eta^N$, the amplifier channels $\mathcal{A}_\kappa^N$ and their weak conjugates[30–32] $\tilde{\mathcal{A}}_\kappa^N$, and the classical additive noise channels $\mathcal{N}_n$ (see Fig. 1 and Supplementary Information for a precise definition). Channel $\mathcal{E}_\eta^N$ can be effectively described as the transformation induced by mixing on a beamsplitter of transmissivity $\eta \in [0,1]$ the input state of the system with an external mode B initialized into a Gibbs thermal bosonic state $\rho_G^{(N)} = \left(\frac{N}{N+1}\right)^{b^\dagger b}/(N+1)$ with average photon number $N \in [0,\infty]$. The quantum-limited element of the set $\mathcal{E}_\eta^N$ corresponds to $N = 0$, a purely lossy (attenuator) channel $\mathcal{E}_\eta := \mathcal{E}_\eta^0$ (ref. 33). A well-known property of this channel is that, when applied iteratively on a state, it brings it toward the vacuum state, that is,

$$\lim_{q\to\infty} [\mathcal{E}_\eta]^q(\rho) = |0\rangle\langle 0| \qquad (1)$$

The amplifier channels $\mathcal{A}_\kappa^N$ and their weak-conjugates $\tilde{\mathcal{A}}_\kappa^N$ refer, respectively, to the signal and idler modes one gets at the output of a parametric amplifier that couples (via a two-mode squeezing transformation) the input state with the B mode defined above. These channels are characterized by the gain parameter $\kappa \in [1,\infty]$ and, for $N = 0$, represent quantum-limited transformations ($\mathcal{A}_\kappa := \mathcal{A}_\kappa^0$ and $\tilde{\mathcal{A}}_\kappa := \tilde{\mathcal{A}}_\kappa^0$, respectively). Finally, the classical additive noise channel $\mathcal{N}_n$ is induced by randomly displacing the input states in phase space via a Gaussian probability distribution of variance $n \in [0,\infty]$. Note that $\mathcal{E}_\eta^N$, $\mathcal{A}_\kappa^N$ and $\mathcal{N}_n$ are covariant, while $\tilde{\mathcal{A}}_\kappa^N$ is contravariant under a phase shift of the input state.

The Gaussian channel conjecture

Consider a single-mode BGC effecting map Φ. Following refs 3, 4, 25, 34 and 35, its optimal classical communication rate, or classical capacity[5], can be computed as $C(\Phi) = \lim_{m\to\infty} \frac{1}{m} C_\chi(\Phi^{\otimes m})$, where $\Phi^{\otimes m}$ describes m channel uses (memoryless noise model), while $C_\chi[\Psi]$ is the single-letter or χ-capacity of the map Ψ defined by the expression $C_\chi(\Psi) = \sup_{\mathrm{ENS}}\{S(\Psi(\rho_{\mathrm{ENS}})) - \sum_j p_j S(\Psi[\rho_j])\}$. Here, maximization is performed over the set of input ensembles $\mathrm{ENS} = \{p_j; \rho_j\}$ (p_j are probabilities; ρ_j are density matrices), where $\rho_{\mathrm{ENS}} = \sum_j p_j \rho_j$ is the associated average state and $S(\rho) = -\mathrm{Tr}[\rho \log_2 \rho]$ is the von Neumann entropy of ρ (ref. 25). Input ensembles ENS must respect an average input energy constraint, that is, $\mathrm{Tr}[H^{(m)} \rho_{\mathrm{ENS}}] \le mE$, where $H^{(m)} = \sum_{j=1}^m a_j^\dagger a_j$ is the total photon number operator in the m modes. We also introduce the minimum output entropy quantity $S_{\min}^{(<)}[\Phi^{\otimes m}] := \inf_\psi S(\Phi^{\otimes m}(|\psi\rangle\langle\psi|))$ for the map $\Phi^{\otimes m}$, where the symbol $^{(<)}$ indicates that the minimization is restricted over m-mode states $|\psi\rangle$ having bounded mean input energy, that is, $\mathrm{Tr}[H^{(m)}|\psi\rangle\langle\psi|] < \infty$. This restriction ensures finiteness and the continuity[36] of all the entropy quantities throughout this article.

The min-entropy conjecture then implies that $S_{\min}^{(<)}[\Phi^{\otimes m}]$ should be additive and, if Φ is phase-insensitive, equal to m times the output entropy associated with the vacuum input $|0\rangle$, that is,

$$S_{\min}^{(<)}[\Phi^{\otimes m}] = m\, S_{\min}^{(<)}[\Phi] = m\, S(\Phi(|0\rangle\langle 0|)) \qquad (2)$$

As detailed in the Supplementary Information, the validity of equation (2) allows one to demonstrate a couple of properties first conjectured in ref. 6 and known to hold at least for purely lossy maps $\mathcal{E}_\eta := \mathcal{E}_\eta^0$ (ref. 33). First, for phase-insensitive BGCs, Gaussian states provide optimal ensembles ENS for the classical capacity $C(\Phi;E)$ under the average input energy constraint. Second, C_χ is additive, that is, $C(\Phi;E) = C_\chi(\Phi;E)$. The minimum entropy values $S_{\min}^{(<)}$ as well as the corresponding capacities C for the four classes of Gaussian channels defined in Fig. 1 are provided in the Supplementary Information. Plots of these functions are presented in Fig. 2.

The proof

Following ref. 10, we use the fact that a generic phase covariant (resp. contravariant) single-mode channel Φ can be expressed as the concatenation of a quantum-limited lossy channel followed by a quantum-limited amplifier (resp. quantum-limited contravariant amplifier). Thus,

$$\Phi = \mathcal{A}_{\kappa_0} \circ \mathcal{E}_{\eta_0} \quad (\text{or } \Phi = \tilde{\mathcal{A}}_{\kappa_0} \circ \mathcal{E}_{\eta_0}) \qquad (3)$$

where parameters κ_0 and η_0 are in biunivocal correspondence with the selected channel Φ and are obtained by solving the equations $\tau = \eta_0 \kappa_0$ and $y = \kappa_0(1 - \eta_0) + (\kappa_0 - 1)$ for the covariant case, or $\tau = \eta_0(1 - \kappa_0)$ and $y = (\kappa_0 - 1)(1 - \eta_0) + \kappa_0$ for the contravariant case (see Supplementary Information). Because channels $\mathcal{E}_{\eta_0}$ map the vacuum into the vacuum, it follows that, to prove the min-entropy conjecture for one use of Φ, it is sufficient to prove it for $\mathcal{A}_{\kappa_0}$, or equivalently, for $\tilde{\mathcal{A}}_{\kappa_0}$. These last two channels indeed have the same minimum output entropy as they are conjugate[37,38] (more generally, given a generic pure input state $|\psi\rangle$, density operators $\mathcal{A}_{\kappa_0}(|\psi\rangle\langle\psi|)$ and $\tilde{\mathcal{A}}_{\kappa_0}(|\psi\rangle\langle\psi|)$ have the same non-zero spectra). Applying this single-mode reduction to m channel uses, $\Phi^{\otimes m} = \mathcal{A}_{\kappa_0}^{\otimes m} \circ \mathcal{E}_{\eta_0}^{\otimes m}$, we actually need to prove the min-entropy conjecture for channel $\mathcal{A}_{\kappa_0}^{\otimes m}$ because the product vacuum state is invariant under $\mathcal{E}_{\eta_0}^{\otimes m}$. A further simplification then follows by noting that the conjugate channel $\tilde{\mathcal{A}}_{\kappa_0}$ is entanglement-breaking[39]. This implies that its minimal output entropy is

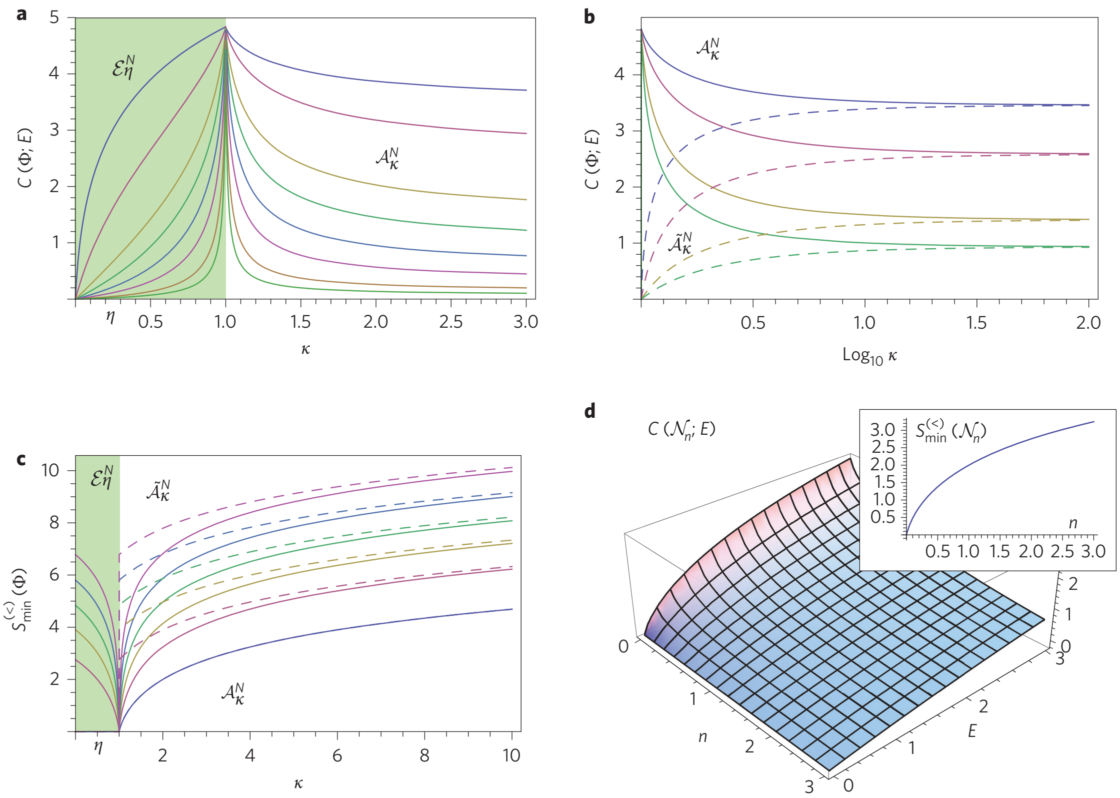

Figure 2 | Plots of the capacities and minimal output entropies for thermal $\mathcal{E}_\eta^N$, classical additive noise $\mathcal{N}_n$, amplifier $\mathcal{A}_\kappa^N$ and contravariant amplifier $\tilde{\mathcal{A}}_\kappa^N$ BGCs. The exact definition of the channels and the analytical expressions for these quantities are provided in the Supplementary Information. **a**, Plot of the capacities of $\mathcal{E}_\eta^N$ (green region) and $\mathcal{A}_\kappa^N$ as a function of parameters η and κ, respectively, for different values of N ($N = 0$, 1, 5, 10, 20, 40, 100 and 200, from top to bottom). For $\mathcal{E}_\eta^N$ the capacity of $N = 0$ coincides with the one given in ref. 33. **b**, Comparison of the capacities of $\mathcal{A}_\kappa^N$ (solid lines) and $\tilde{\mathcal{A}}_\kappa^N$ (dashed lines) as a function of κ for $N = 0$, 1, 5 and 10 (from top to bottom). **c**, Minimum output entropies for $\mathcal{E}_\eta^N$ (green region), $\mathcal{A}_\kappa^N$ (solid lines) and $\tilde{\mathcal{A}}_\kappa^N$ (dashed lines). From bottom to top, $N = 0$, 2, 5, 10, 20 and 40. In all plots $E = 10$. **d**, Plot of the capacity of $\mathcal{N}_n$ as a function of noise parameter n and energy constraint parameter E. Inset: minimum output entropy of $\mathcal{N}_n$.

additive, $S_{\min}^{(<)}[\tilde{\mathcal{A}}_{\kappa_0}^{\otimes m}] = m\, S_{\min}^{(<)}[\tilde{\mathcal{A}}_{\kappa_0}]$, so the same is true for $\mathcal{A}_{\kappa_0}$. Consequently, the minimum output entropy for m uses of channel Φ can always be achieved over separable inputs, and the additivity of C_χ holds provided we can prove the single-mode amplifier conjecture[11]

$$S_{\min}^{(<)}[\mathcal{A}_{\kappa_0}] = S(\mathcal{A}_{\kappa_0}(|0\rangle\langle 0|)) \tag{4}$$

To prove it, we observe that the conjugate channel $\tilde{\mathcal{A}}_{\kappa_0}$ can be expressed as the following measure-and-prepare transformation in the coherent state basis

$$\rho \mapsto \tilde{\mathcal{A}}_{\kappa_0}[\rho] = \int \frac{d^2 z}{\pi} |-\sqrt{\kappa_0} z^*\rangle\langle -\sqrt{\kappa_0} z^*| \langle z|\rho|z\rangle \tag{5}$$

where $|z\rangle = D(z)|0\rangle$ is a coherent state (z being complex). The next step is to notice that, by taking the complex conjugate of $\tilde{\mathcal{A}}_{\kappa_0}[\rho]$ with respect to the Fock basis, one obtains a state that coincides with the output state of an entanglement-breaking covariant channel $\mathcal{M}$ with parameters $\tau = \kappa_0 - 1$ and $y = \kappa_0$ (Fig. 1). Thus, $\tilde{\mathcal{A}}_{\kappa_0}[\rho]$ and $\mathcal{M}[\rho]$ share the same spectrum for a fixed input ρ. Together with the fact that for a pure input state $|\psi\rangle$, $\tilde{\mathcal{A}}_{\kappa_0}(|\psi\rangle\langle\psi|)$ also has the same spectrum as $\mathcal{A}_{\kappa_0}(|\psi\rangle\langle\psi|)$, this implies that there exists a unitary transformation $\mathcal{U}$ (possibly dependent upon $|\psi\rangle$) that fulfils the condition

$$\mathcal{A}_{\kappa_0}(|\psi\rangle\langle\psi|) = [\mathcal{U} \circ \mathcal{M}](|\psi\rangle\langle\psi|) = [\mathcal{U} \circ \mathcal{A}_{\kappa_0} \circ \mathcal{E}_{\eta_0}](|\psi\rangle\langle\psi|) \tag{6}$$

where the second identity follows from equation (3), which allows one to write $\mathcal{M} = \mathcal{A}_{\kappa_0} \circ \mathcal{E}_{\eta_0}$ with $\eta_0 = (\kappa_0 - 1)/\kappa_0$ (Fig. 1). Equation (6) means that, for a pure input state $|\psi\rangle$, the output state of a quantum-limited amplifier $\mathcal{A}_{\kappa_0}$ coincides (up to a unitary transformation that in principle depends upon $|\psi\rangle$) with the one obtained by first applying a lossy channel $\mathcal{E}_{\eta_0}$ to the input and then $\mathcal{A}_{\kappa_0}$, as shown in Fig. 3. Consider next a generic ensemble decomposition of $\mathcal{E}_{\eta_0}(|\psi\rangle\langle\psi|)$, that is, $\mathcal{E}_{\eta_0}(|\psi\rangle\langle\psi|) = \sum_j p_j |\psi_j\rangle\langle\psi_j|$, where $p_j > 0$ are probabilities. Inserting this into equation (6) and iterating the same passage q times, we obtain the identity

$$\mathcal{A}_{\kappa_0}(|\psi\rangle\langle\psi|) = \sum_\ell q_\ell [\mathcal{W}_\ell \circ \mathcal{A}_{\kappa_0}](|\phi_\ell\rangle\langle\phi_\ell|) \tag{7}$$

where $\mathcal{W}_\ell$ are unitaries, $q_\ell > 0$ are probabilities, and $|\phi_\ell\rangle$ are state vectors that provide an ensemble decomposition after q applications of $\mathcal{E}_{\eta_0}$ on the input state $|\psi\rangle$, that is,

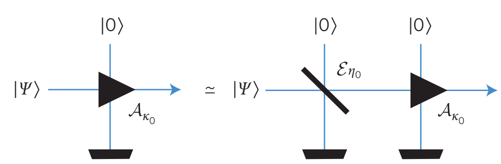

Figure 3 | Graphical representation of equation (6). The symbol '$\simeq$' indicates that applying the quantum-limited amplifier $\mathcal{A}_{\kappa_0}$ to pure input $|\psi\rangle$ or to its evolved counterpart via a lossy channel of transmissivity $\eta_0 = 1 - 1/\kappa_0$ (that is, $\mathcal{E}_{\eta_0}(|\psi\rangle\langle\psi|)$) produces outputs that have the same spectra.

$$\sum_{\ell} q_\ell |\phi_\ell\rangle\langle\phi_\ell| = [\mathcal{E}_{\eta_0}]^q(|\psi\rangle\langle\psi|) \tag{8}$$

(Here, index ℓ refers to a path of j-indices across the q subsequent ensemble decompositions of the output state of $\mathcal{E}_{\eta_0}$; see ref. 20 for details.) Exploiting the concavity of the von Neumann entropy, identity (7) yields

$$S(\mathcal{A}_{\kappa_0}(|\psi\rangle\langle\psi|)) \geq \sum_{\ell} q_\ell\, S(\mathcal{A}_{\kappa_0}(|\phi_\ell\rangle\langle\phi_\ell|)) \tag{9}$$

Recalling property (1), it is intuitively clear at this level that for $q\to\infty$, the ensemble $\{q_\ell, |\phi_\ell\rangle\}$ in equation (8) will contain one single component, namely the vacuum. Equation (9) will then lead us to conclude that the output entropy for input $|\psi\rangle$ is larger than the one associated with the vacuum, and hence to equation (4). To show this rigorously, we start from the inequality

$$\begin{aligned} S(\mathcal{A}_{\kappa_0}(|\phi_\ell\rangle\langle\phi_\ell|)) \geq & -\mathrm{Tr}[\mathcal{A}_{\kappa_0}(|\phi_\ell\rangle\langle\phi_\ell|)\log_2\mathcal{A}_{\kappa_0}(\sigma)] \\ & + \mathrm{Tr}[|\phi_\ell\rangle\langle\phi_\ell|\log_2\sigma] \end{aligned} \tag{10}$$

which holds for an arbitrary state $|\phi_\ell\rangle$ of equation (8) and for a generic state σ, and which can be easily derived from the monotonic decrease[25] of the relative entropy $S(|\phi_\ell\rangle\langle\phi_\ell||\sigma)$ under the action of map $\mathcal{A}_{\kappa_0}$. Setting $\sigma = [\mathcal{E}_{\eta_0}]^q(|\psi\rangle\langle\psi|)$ and inserting into the righthand side of equation (9), we then obtain

$$\begin{aligned} S(\mathcal{A}_{\kappa_0}(|\psi\rangle\langle\psi|)) \geq & S((\mathcal{A}_{\kappa_0}\circ[\mathcal{E}_{\eta_0}]^q)(|\psi\rangle\langle\psi|)) \\ & - S([\mathcal{E}_{\eta_0}]^q(|\psi\rangle\langle\psi|)) \end{aligned} \tag{11}$$

Because $|\psi\rangle$ is of bounded mean energy, the same holds for $[\mathcal{E}_{\eta_0}]^q(|\psi\rangle\langle\psi|)$ and $(\mathcal{A}_{\kappa_0}\circ[\mathcal{E}_{\eta_0}]^q)(|\psi\rangle\langle\psi|)$ (the corresponding energy expectations values being in fact $\eta^q\langle\psi|a^\dagger a|\psi\rangle$ and $\kappa\eta^q\langle\psi|a^\dagger a|\psi\rangle + \kappa(\kappa-1)$). We can then invoke the continuity of the von Neumann entropy[36] and use property (1) for $q\to\infty$ to show that

$$S(\mathcal{A}_{\kappa_0}(|\psi\rangle\langle\psi|)) \geq S(\mathcal{A}_{\kappa_0}(|0\rangle\langle 0|)) \tag{12}$$

and hence equation (4). This proves conjecture (2) and gives the solution for the capacity presented in Fig. 2.

Discussion

Aside from a definitive proof of the classical capacity of the four fundamental Gaussian channels ($\Phi = \mathcal{E}_\eta^N, \mathcal{N}_n, \mathcal{A}_\kappa^N$ or $\tilde{\mathcal{A}}_\kappa^N$), the above approach can be used to prove an identity analogous to (2) for any single-mode channel covariant or contravariant with respect to an arbitrary squeezed complex structure. Furthermore, as explained in the Supplementary Information, when the mean energy E is large enough[29,40], this result can be used to show that the optimality of Gaussian inputs for $C(\Psi;E)$ and the additivity of $C_\chi(\Psi;E)$ hold also for such channels. The solution of the min-entropy conjecture also allows one to extend for the first time the results of refs 41 and 42 by yielding the exact formula for the entanglement of formation (EoF)[21] of some non-permutation-symmetric two-mode Gaussian states. (The EoF is a measure of the cost of generating a given quantum bipartite state ρ.) Consider the two-mode density matrices $\rho(\kappa,N)$ one obtains at the output of a parametric amplifier with gain parameter κ, when injecting the vacuum into one port and a Gibbs thermal state $\rho_G^{(N)}$ into the other, that is, $\rho(\kappa,N) = U_\kappa[\rho_G^{(N)}\otimes|0\rangle\langle 0|]U_\kappa^\dagger$. The symmetric case $\rho(\kappa,0)$ corresponds to the two-mode squeezed vacuum state, whose entanglement of formation is trivial to compute, that is, $\mathrm{EoF}[\rho(\kappa,0)] = g(\kappa-1)$. For a general $\rho(\kappa,N)$ state, equation (4) implies

$$\mathrm{EoF}\big[\rho(\kappa,N)\big] = \mathrm{EoF}[\rho(\kappa,0)] = g(\kappa-1) \tag{13}$$

(Notice that this expression does not depend on the mean photon number N of $\rho_G^{(N)}$.) The proof of this identity follows by first noticing that one can obtain the upperbound $\mathrm{EoF}[\rho(\kappa,N)] \leq g(\kappa-1)$ by generating $\rho(\kappa,N)$ starting from the two-mode squeezed vacuum state $\rho(\kappa,0)$ and applying random correlated displacements (a transformation that only involves local operation and classical communication post-processing) to both modes[42]. To show that this quantity is also a lower bound for $\mathrm{EoF}[\rho(\kappa,N)]$, one can use the fact that the reduced density matrix of $\rho(\kappa,N)$ coincides with the output state $\mathcal{A}_\kappa(\rho_G^{(N)})$ of the quantum-limited amplifier with a thermal state $\rho_G^{(N)}$ input. Exploiting the equivalence relation between the EoF and the minimum output entropy introduced in ref. 43, our proof of conjecture (2) implies equation (13) (see Supplementary Information for detailed proof).

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Author contributions

All authors contributed equally to the research and writing of the manuscript.

Additional information

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Competing financial interests

The authors declare no competing financial interests.