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Interconversion of pure Gaussian states requiring non-Gaussian operations

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We analyze the conditions under which local operations and classical communication enable entanglement transformations between bipartite pure Gaussian states. A set of necessary and sufficient conditions had been found [G. Giedke *et al.*, Quant. Inf. Comput. **3**, 211 (2003)] for the interconversion between such states that is restricted to Gaussian local operations and classical communication. Here, we exploit majorization theory in order to derive more general (sufficient) conditions for the interconversion between bipartite pure Gaussian states that goes beyond Gaussian local operations. While our technique is applicable to an arbitrary number of modes for each party, it allows us to exhibit surprisingly simple examples of 2×2 Gaussian states that necessarily require non-Gaussian local operations to be transformed into each other.

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I. INTRODUCTION

Quantum entanglement plays a major role in quantum computation and information theory, where it is a key resource enabling a vast variety of tasks, such as teleportation-based universal quantum computing [1] or blind quantum computing [2], and where it also is a central component of the security analyses of quantum key distribution [3]. More fundamentally, understanding quantum entanglement is at the heart of theoretical physics, with issues ranging from quantum nonlocality and Bell inequalities [4] to novel perspectives on black holes theory [5,6].

A fundamental problem in the theory of quantum entanglement consists in classifying entangled states into different equivalence classes and studying the possibility (or impossibility) of transforming entangled states between each other [7]. In the usual scenario, one considers interconversions between different sets of states that rely on local operations (effected separately by each of the parties sharing the state) supplemented with classical communication (among the different parties sharing the state). These transformations, commonly denoted as LOCC, cannot increase the entanglement between the parties whatever the entanglement monotone that is used to measure it. It is, however, crucial to go beyond that simple fact and be able to determine whether a given entangled state can be reached by applying a LOCC transformation onto another entangled state. A very successful approach to address this question in the case of bipartite pure entangled states in finite dimension has been developed based on the mathematical theory of majorization: the possibility to transform a pure bipartite state into another by a deterministic LOCC protocol is connected to a majorization relation between their corresponding vectors of Schmidt coefficients [8,9].

In this article, we envisage the interconversion between states of the electromagnetic field, and move therefore to an infinite-dimensional Fock space. We focus in particular on the set of Gaussian states, which are of great significance in quantum optics and continuous-variable quantum information theory [10,11]. These states are easy to produce and manipulate experimentally, and at the same time they can be mathematically described by using the first two statistical moments of the quadrature operators in phase space. Consequently, the set of Gaussian states and accompanying Gaussian transformations is particularly relevant for an analysis of entanglement transformations as they well describe a great amount of quantum optical experiments and can be efficiently modeled within the so-called symplectic formalism [12].

The interconversion between Gaussian states has been studied in a few earlier works [13–18], but many problems remain unsolved today. In particular, most works have focused on the entanglement transformations of Gaussian states using Gaussian processes only. Notably, the work by Giedke *et al.* [17] provides a necessary and sufficient condition for the interconversion between pure Gaussian states when restricting to Gaussian local operations with classical communication (denoted as GLOCC), which can be realized in practice by using standard optical components, such as beam splitters, squeezers, phase shifters, and homodyne detectors. The proof starts by exploiting the fact that any $N \times N$ pure Gaussian state, i.e., any bipartite Gaussian state with N modes on each side, can be transformed by a Gaussian local unitary into a tensor product of N two-mode squeezed vacuum states [12], which is completely characterized by the vector of squeezing parameters $\mathbf{r}^{\downarrow}$ conventionally sorted in decreasing order. Then, the problem simplifies to the interconversion between tensor products of two-mode squeezed vacuum states. The authors proved that a pure Gaussian state $|\psi\rangle$ can be transformed into $|\psi'\rangle$ using a GLOCC if and only if $r_i \geqslant r_i'$, $\forall\, i$, or in a more compact notation

$$|\psi\rangle \xrightarrow{\text{GLOCC}} |\psi'\rangle \quad \text{iff} \quad \mathbf{r}^{\downarrow} \geqslant \mathbf{r}'^{\downarrow}, \tag{1}$$

where $|\psi\rangle$ and $|\psi'\rangle$ are, respectively, characterized by their decreasingly ordered squeezing vectors $\mathbf{r}^{\downarrow}$ and $\mathbf{r}'^{\downarrow}$.

The question that we investigate in this work is whether it is possible to achieve, using a non-Gaussian LOCC, transformations between Gaussian states that are otherwise inaccessible by a GLOCC, i.e., do not satisfy condition (1). This possibility was briefly mentioned in Ref. [17] with the example of a couple of Gaussian states that could be connected via a LOCC but could not via a Gaussian process alone. Here, we extend on this idea and develop a systematic approach to explore the possible interconversions between Gaussian states that are not accessible by GLOCC. We broaden the analysis by providing a sufficient condition for the existence of a LOCC transformation between pure bipartite $N \times N$ Gaussian states that generalizes condition (1), at the price of losing its necessary character.

To achieve this goal, we use the theory of majorization, which provides an ideal tool to investigate the conditions for interconverting pure bipartite states using LOCC transformations [8,9]. Specifically, a finite-dimensional bipartite pure state $|\psi\rangle$ can be transformed via a deterministic LOCC into $|\psi'\rangle$ if and only if the vectors of eigenvalues $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ of their respective reduced states ρ and ρ' satisfy a majorization relation, that is,

$$|\psi\rangle \xrightarrow{\text{LOCC}} |\psi'\rangle \quad \text{iff} \quad \boldsymbol{\lambda}' \succ \boldsymbol{\lambda}. \tag{2}$$

Note that the vector of eigenvalues $\boldsymbol{\lambda}$ ($\boldsymbol{\lambda}'$) can be obtained by tracing over either one of the two parties as a result of the Schmidt decomposition of $|\psi\rangle$ ($|\psi'\rangle$). Equivalently to (2), the interconversion $|\psi\rangle \to |\psi'\rangle$ is possible if and only if there exists a bistochastic matrix $\mathbf{D}$ that maps $\boldsymbol{\lambda}'$ onto $\boldsymbol{\lambda}$, i.e., $\boldsymbol{\lambda} = \mathbf{D}\boldsymbol{\lambda}'$ (see Appendix). We will actually use this second condition in this paper, as it is better appropriate to achieve our goal (finding a matrix $\mathbf{D}$ gives a sufficient condition for the existence of a LOCC transformation).

The application of majorization theory in an infinite-dimensional Fock space to explore the interconversion between relevant states in quantum optics is a fertile ground of investigation, with only few known results as of today. In Ref. [19], it was shown that the output states of an optical quantum-limited amplifier that is applied to Fock states satisfy a ladder of majorization relations, while a similar behavior was proven to hold in Ref. [20] for a pure lossy line (a beam splitter with vacuum on the other input port). Aside from these majorization relations intrinsic to generic two-mode Gaussian operations (two-mode squeezer and beam splitter), it was recently shown that the output state resulting from the vacuum state processed through any phase-insensitive Gaussian bosonic channel majorizes the output state corresponding to another input state [21], extending on the proof of the Gaussian minimum entropy conjecture [22]. The results presented here follow this line and illustrate again the power of majorization theory in quantum optics and continuous-variable quantum information theory.

The remainder of the paper is organized as follows. In Sec. II, we outline our main results, illustrated in the simplest case of pure Gaussian states of 2×2 modes. In Sec. III, we give the proof of our main theorem, as well as its generalization to an arbitrary number of modes. Finally, in Sec. IV, we conclude and bring forward some open problems. In the Appendix, we summarize the basics of majorization theory and its connection to entanglement theory.

II. INTERCONVERSION OF GAUSSIAN STATES OF 2 × 2 MODES

We first illustrate our results for the simplest interesting case, namely, pure Gaussian states of 2×2 modes. Indeed, the case of 1×1 modes trivially reduces to the interconversion between two-mode squeezed vacuum states

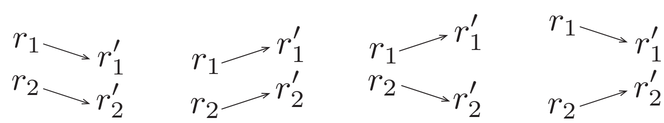

FIG. 1. Four possible evolutions of the squeezing parameters r_1 and r_2 of the two two-mode squeezed vacuum states $|\Phi_{r_1}\rangle|\Phi_{r_2}\rangle$. Note that since the squeezing vectors are sorted in decreasing order ($r_1 \geqslant r_2, r_1' \geqslant r_2'$), the two arrows never cross each other.

$$|\Phi_r\rangle = (1-\gamma^2)^{1/2} \sum_{n=0}^{\infty} \gamma^n |n\rangle|n\rangle \tag{3}$$

with $\gamma = \tanh(r)$ and r being the squeezing parameter. An obvious consequence of condition (1) is that $|\Phi_r\rangle$ can be transformed into $|\Phi_{r'}\rangle$ with a GLOCC iff $r \geqslant r'$. If $r < r'$, the transformation is possible in the opposite direction with a GLOCC, so non-Gaussian transformations are useless both ways. The situation becomes more interesting as soon as two modes are considered on each side. Since any pure Gaussian state of 2×2 modes can be mapped via Gaussian local unitaries onto a pair of two-mode squeezed vacuum states [12], we can assume without loss of generality that Alice and Bob's state $|\psi\rangle$ is already in its normal form $|\Phi_{r_1}\rangle|\Phi_{r_2}\rangle$, characterized by its ordered squeezing vector (r_1,r_2), with $r_1 \geqslant r_2$. Note that the bipartite splitting is not between $|\Phi_{r_1}\rangle$ and $|\Phi_{r_2}\rangle$, but across both states $|\Phi_{r_1}\rangle$ and $|\Phi_{r_2}\rangle$, which are each shared by the two parties. We then study the conditions under which $|\psi\rangle = |\Phi_{r_1}\rangle|\Phi_{r_2}\rangle$ can be transformed under a LOCC into another state $|\psi'\rangle = |\Phi_{r_1'}\rangle|\Phi_{r_2'}\rangle$ with squeezing vector (r_1',r_2') and $r_1' \geqslant r_2'$.

As depicted in Fig. 1, there are four different possibilities for the evolution of the squeezing vector's components when transforming $|\psi\rangle$ into $|\psi'\rangle$. In Fig. 1(a), both squeezing parameters r_1 and r_2 decrease, a transformation that is always achievable using a GLOCC according to condition (1). Figure 1(b) corresponds to the situation where both squeezing parameters r_1 and r_2 increase. It is easy to infer that there cannot exist a LOCC (neither Gaussian nor non-Gaussian) that permits such a transformation. Indeed, Fig. 1(b) can be viewed as the reverse process of Fig. 1(a) and condition (1) permits a GLOCC in the reverse direction. Hence, transformation Fig. 1(b) is forbidden given that majorization is a one-way property. In the latter two cases, the squeezing parameters follow different evolutions, i.e., the two components of the vector $(r_1'-r_1, r_2'-r_2)$ have opposite signs. In Fig. 1(c), r_1 increases and r_2 decreases, while Fig. 1(d) is the converse. Since condition (1) is necessary, it is never possible to transform $|\psi\rangle$ into $|\psi'\rangle$ using a GLOCC in neither of these cases. However, as it turns out, it is nevertheless possible to find a non-Gaussian LOCC that successfully achieves such a transformation when the condition of the following theorem holds:

Theorem 1. Let $|\psi\rangle$ and $|\psi'\rangle$ be two 2×2 pure Gaussian states, respectively, characterized by their decreasingly ordered squeezing vectors (r_1,r_2) and (r_1',r_2') such that the two components of vector $(r_1'-r_1, r_2'-r_2)$ have opposite signs.

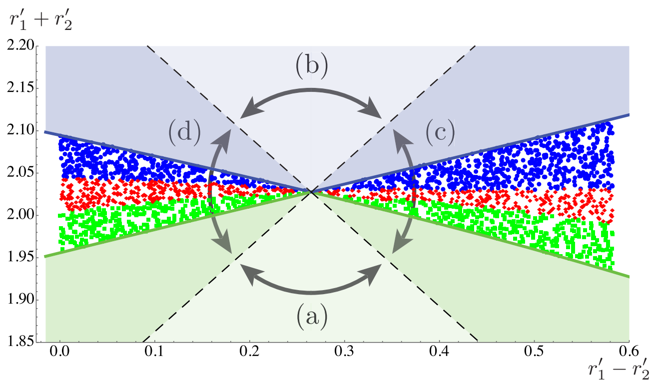

FIG. 2. (Color online) Transformation from state $|\psi\rangle = |\Phi_{r_1}\rangle|\Phi_{r_2}\rangle$ to state $|\psi'\rangle = |\Phi_{r_1'}\rangle|\Phi_{r_2'}\rangle$ for fixed values $r_1 = 1.15$ (9.96 dB), $r_2 = 0.88$ (7.66 dB), and variable (r_1', r_2'). The two diagonal dashed lines divide the plane in four quadrants corresponding to the four situations depicted in Fig. 1. The final states that are reachable with a LOCC transformation are marked in green (green areas and green points), while the final states that are not reachable because a LOCC transformation exists in the opposite direction are marked in blue (blue areas and blue points). The final states that are incomparable with the initial states (no LOCC exists both ways) are marked as red points. The lower quadrant (light green) corresponds to 1(a), while the upper quadrant (light blue) corresponds to Fig. 1(b). Condition (1) implies that a GLOCC exists in the lower quadrant, while a LOCC is ruled out in the upper quadrant since a GLOCC exists in the reverse direction. The right quadrant [Fig. 1(c)] and left quadrant [Fig. 1(d)] correspond to regions where condition (1) is not satisfied both ways, so no GLOCC exists both ways. However, non-Gaussian LOCC are possible and detected by our condition (4) in the dark green region (or nondetected by our condition and represented by green points). Symmetrically, non-Gaussian LOCC are possible in the reverse direction and detected by our condition (4) in the dark blue region (or nondetected by our condition and represented by blue points); this corresponds to cases where no LOCC exists that converts $|\psi\rangle$ into $|\psi'\rangle$. Our criterion, condition (4), is indicated by a solid green line (direct direction $|\psi\rangle \rightarrow |\psi'\rangle$) and blue line (reverse direction $|\psi'\rangle \rightarrow |\psi\rangle$).

Then, $|\psi\rangle$ can be transformed into $|\psi'\rangle$ using a non-Gaussian LOCC if

$$\frac{\sinh(r_1 + r_2) \pm \sinh(r_1 - r_2)}{\sinh(r_1' + r_2') \pm \sinh(r_1' - r_2')} \geqslant 1, \tag{4}$$

where $\pm$ follows the sign of $r_1' - r_1$. [The plus sign corresponds to Fig. 1(c), while the minus sign corresponds to Fig. 1(d).]

Thus, Theorem 1 provides us with a sufficient (but not necessary) condition for the existence of a LOCC that transforms $|\psi\rangle = |\Phi_{r_1}\rangle|\Phi_{r_2}\rangle$ into $|\psi'\rangle = |\Phi_{r_1'}\rangle|\Phi_{r_2'}\rangle$. The usefulness of this result is illustrated in Fig. 2, where we analyze the possibility of the conversion from the initial state $|\psi\rangle$ to the final state $|\psi'\rangle$ for fixed values of the squeezing parameters of the initial state, $r_1 = 1.15$ (9.96 dB) and $r_2 = 0.88$ (7.66 dB). These values correspond simply to a mean photon number of 2 and 1, respectively, when tracing $|\Phi_{r_1}\rangle$ and $|\Phi_{r_2}\rangle$ over Bob's mode. Each possible final state with squeezing parameters r_1' and r_2' is associated with a point of coordinates $(r_1' - r_2', r_1' + r_2')$ in Fig. 2. Thus, we can divide the plane in four quadrants corresponding to the four cases of Fig. 1, where the increasing-diagonal dashed line coincides with $r_2' = r_2$ while the decreasing-diagonal dashed line corresponds to $r_1' = r_1$. The lower quadrant (light green) corresponds to Fig. 1(a), i.e., transformations that are achievable with a GLOCC according to condition (1). The upper quadrant (light blue) corresponds to Fig. 1(b), i.e., transformations that cannot be achieved with a LOCC since condition (1) permits a GLOCC in the reverse direction. The right quadrant corresponds to Fig. 1(c), while the left quadrant corresponds to Fig. 1(d). In both left and right quadrants, the transformations cannot be achieved with a GLOCC as they do not satisfy condition (1), but might be achievable with a non-Gaussian LOCC. Indeed, condition (4) is satisfied in a whole region (dark green) above the lower quadrant, implying that a LOCC exists, which must necessarily be non-Gaussian. Symmetrically, condition (4) is satisfied in the reverse direction $|\psi'\rangle \longrightarrow |\psi\rangle$ in a whole region (dark blue) below the upper quadrant, which means that a (non-Gaussian) LOCC exists in the reverse direction, hence the transformation $|\psi\rangle \longrightarrow |\psi'\rangle$ is impossible to perform with a LOCC, be it non-Gaussian. Thus, the sufficient condition (4) allows us to significantly enlarge the regions where a LOCC is proven to exist (dark and light green) or not to exist (dark and light blue). This is the main outcome of Theorem 1.

Now, the remaining zones in the left and right quadrants can be explored numerically. For each point, we check whether a majorization relation exists between the eigenvalues of the reduced states corresponding to $|\psi\rangle$ and $|\psi'\rangle$ in either direction, or whether the states are incomparable. We find states (marked as green points) where majorization holds, so that a (non-Gaussian) LOCC exists although it is not detected by condition (4), illustrating the fact that this condition is not necessary. Similarly, we find states (marked as blue points) such that majorization holds in the opposite direction (undetected by our condition), implying that a (non-Gaussian) LOCC exists in that direction, hence the transformation $|\psi\rangle \longrightarrow |\psi'\rangle$ is impossible with a LOCC, be it non-Gaussian. Finally, we find states that are incomparable (marked as red points), in which case no deterministic LOCC exists both ways.

III. PROOF AND EXTENSION TO GAUSSIAN STATES OF $N \times N$ MODES

In general terms, we are interested in deterministic LOCC transformations from the pure Gaussian state $|\psi\rangle_{AB}$, shared between Alice having N modes and Bob having N modes, towards the pure Gaussian state $|\psi'\rangle_{AB}$. Using the normal form reduction [12], we can assume without loss of generality that both $|\psi\rangle_{AB}$ and $|\psi'\rangle_{AB}$ are a tensor product of N two-mode squeezed vacuum states characterized by their respective squeezing vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$. So, we seek a sufficient condition on the existence of transformation

$$|\psi\rangle_{AB} = \left|\Phi_{r_1}\right\rangle \cdots \left|\Phi_{r_N}\right\rangle \rightarrow |\psi'\rangle_{AB} = \left|\Phi_{r_1'}\right\rangle \cdots \left|\Phi_{r_N'}\right\rangle \tag{5}$$

under a deterministic LOCC. As shown for example in [19], majorization theory and its connection to entanglement transformations can be adapted to the usual infinite-dimensional states of quantum optics, the only subtlety being that the matrix $\mathbf{D}$ becomes an infinite column-stochastic matrix instead of a double-stochastic matrix (this means that all columns must still sum to one, while the sum of elements in each row must only be $\leqslant 1$). Majorization theory implies that a pure state $|\psi\rangle_{AB}$ can be transformed into $|\psi'\rangle_{AB}$ using a deterministic LOCC if and only if Alice's reduced states $\rho = \mathrm{Tr}_B[|\psi\rangle\langle\psi|_{AB}]$ and $\rho' = \mathrm{Tr}_B[|\psi'\rangle\langle\psi'|_{AB}]$ resulting from tracing the corresponding pure states over Bob's modes satisfy the majorization relation $\rho' \succ \rho$ (see Appendix). Using the normal form of states $|\psi\rangle_{AB}$ and $|\psi'\rangle_{AB}$, this majorization condition can be rewritten as $\Sigma' \succ \Sigma$, where

$$\Sigma = \bigotimes_{i=1}^{N} \sigma_{\nu_i}, \quad \Sigma' = \bigotimes_{i=1}^{N} \sigma_{\nu_i'}, \tag{6}$$

and σ_{ν_i} stands for a thermal state of mean photon number ν_i on the ith mode. Note that the mean photon number ν_i of state σ_{ν_i} is connected to the squeezing parameter r_i of its parent pure state $|\Phi_{r_i}\rangle$ via the relation $\nu_i = \sinh^2(r_i)$. Condition $\Sigma' \succ \Sigma$ means that states Σ and Σ' admit respective vectors of eigenvalues $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ that satisfy the majorization relation $\boldsymbol{\lambda}' \succ \boldsymbol{\lambda}$, which is strictly equivalent to the existence of an infinite column-stochastic matrix $\mathbf{D}$ mapping $\boldsymbol{\lambda}'$ (state Σ') onto $\boldsymbol{\lambda}$ (state Σ), i.e., satisfying $\boldsymbol{\lambda} = \mathbf{D}\boldsymbol{\lambda}'$.

The core of our work consists in finding a systematic way of constructing such matrices $\mathbf{D}$ for different ensembles of thermal states Σ and Σ'. Once such a matrix $\mathbf{D}$ is found, we know that it must be possible to transform Σ' into Σ by using a random mixture of unitaries, i.e.,

$$\Sigma' = \sum_s p_s\, U_s \Sigma U_s^\dagger, \tag{7}$$

where the unitary U_s acts upon N modes (on Alice's side) and is applied with probability p_s. (The sum over s could also be replaced by an integral over a continuous variable and p_s would then be a probability density.) At this point, it is tempting to use standard Gaussian transformations that are well known to map thermal states onto thermal states. For instance, knowing that the mean number of photons of a thermal state is increased by applying a quantum-limited amplifier $\mathcal{A}$, one would be tempted to use $\mathcal{A}$ in order to transform a thermal state of mean photon number ν' into another thermal state of mean photon number ν with $\nu \geqslant \nu'$. Unfortunately, this Gaussian channel $\mathcal{A}$ (as well as the pure-loss channel $\mathcal{L}$, which decreases the mean photon number) can not be written as a mixture of unitaries, so it cannot directly be used for our purposes.

However, the key observation is that some specific tensor products of Gaussian channels $\mathcal{A}$ and $\mathcal{L}$ act on some specific tensor products of thermal states in such a way that the vector of eigenvalues of the state evolves according to a column-stochastic matrix $\mathbf{D}$, i.e., $\boldsymbol{\lambda} = \mathbf{D}\boldsymbol{\lambda}'$. This means that there must exist a mixture of (maybe non-Gaussian) unitaries that behave similarly as the tensor product of Gaussian channels when transforming the tensor product of thermal states Σ' into the tensor product of thermal states Σ. This implies the existence of a (maybe non-Gaussian) LOCC transformation that connects the pure Gaussian states $|\psi\rangle_{AB} \to |\psi'\rangle_{AB}$.

In Sec. III A, we describe our approach in the simplest case of 2×2 modes in the situation of Fig. 1(a), where the mean photon number of the two thermal states decreases (or equivalently both squeezing parameters r_1 and r_2 decrease), recovering the existence condition for a GLOCC as in [17].

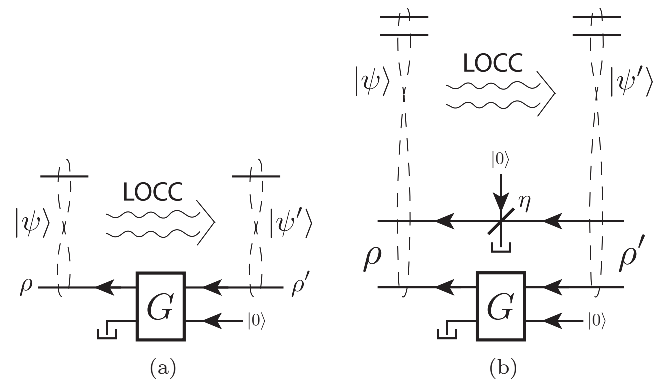

FIG. 3. A bipartite pure state $|\psi\rangle$ can be transformed into $|\psi'\rangle$ with a LOCC if and only if the reduced state ρ' majorizes ρ, or equivalently if the vector of eigenvalues $\boldsymbol{\lambda}'$ of ρ' can be mapped onto the vector of eigenvalues $\boldsymbol{\lambda}$ of ρ with a column-stochastic matrix $\mathbf{D}$. (a) Case of 1×1 modes, where matrix $\mathbf{D}^{\mathcal{A}}$ acts on the vector of eigenvalues of Fock-diagonal states similarly as a quantum-limited amplifier $\mathcal{A}$ of gain G acts on Fock-diagonal states ($\mathbf{D}^{\mathcal{A}}$ is column-stochastic provided that $G \geqslant 1$). (b) Case of 2×2 modes [Fig. 1(c)], where matrix $\mathbf{D}^{\mathcal{L}} \otimes \mathbf{D}^{\mathcal{A}}$ acts on the vector of eigenvalues of tensor product of Fock-diagonal states similarly as the tensor product of a pure loss channel $\mathcal{L}$ of transmittance η with a quantum-limited amplifier $\mathcal{A}$ of gain G acts on tensor products of Fock-diagonal states ($\mathbf{D}^{\mathcal{L}} \otimes \mathbf{D}^{\mathcal{A}}$ is column-stochastic provided that $\eta G \geqslant 1$). Note that $\mathcal{L} \otimes \mathcal{A}$ is a Gaussian map, while the corresponding LOCC is non-Gaussian since condition (1) is not satisfied.

In Sec. III B, we generalize the method to give a sufficient condition for the existence of a LOCC transformation in the situations of Figs. 1(c) and 1(d), where no GLOCC transformations are known to exist, which leads to the proof of Theorem 1. In Sec. III C, the method is generalized to an arbitrary number of modes.

A. Quantum-limited amplifier

As pictured in Fig. 3(a), the action of a quantum-limited amplifier $\mathcal{A}$ on an input state ρ' is equivalent to the unitary interaction between the input mode A and an environmental mode E initially prepared in the vacuum state $|0\rangle$:

$$\mathcal{A}(\rho') = \mathrm{Tr}_E[U_{\mathcal{A}}(\rho' \otimes |0\rangle_E\langle 0|)U_{\mathcal{A}}^\dagger], \tag{8}$$

where $U_{\mathcal{A}} = \exp[\frac{s}{2}(a_A a_E - a_A^\dagger a_E^\dagger)]$ is a two-mode squeezing unitary. When we apply the quantum-limited amplifier to a single-mode phase-invariant state $\rho' = \sum_n \lambda_n' |n\rangle\langle n|$ (where $|n\rangle$ is a Fock state), the output state ρ is also a phase-invariant state, therefore it is also diagonal in the Fock basis, i.e., $\rho = \sum_n \lambda_n |n\rangle\langle n|$. The corresponding vectors of eigenvalues $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ are related through the equation $\boldsymbol{\lambda} = \mathbf{D}^{\mathcal{A}}\boldsymbol{\lambda}'$, where the matrix $\mathbf{D}^{\mathcal{A}}$ reads as

$$\mathbf{D}^{\mathcal{A}} = \begin{pmatrix} (1-\gamma^2) & 0 & 0 & \dots \\ (1-\gamma^2)\gamma^2 & (1-\gamma^2)^2 & 0 & \dots \\ (1-\gamma^2)\gamma^4 & 2(1-\gamma^2)^2\gamma^2 & (1-\gamma^2)^3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

or in a compact form $D^{\mathcal{A}}_{n,m} = Q^{(m-1)}_{n-1} H(n-m)$ with

$$Q^{(i)}_j = \binom{j}{i}(1-\gamma^2)^{i+1}(\gamma^2)^{j-i}, \quad \gamma = \tanh(s)$$

and $H(x)$ being the Heaviside step function defined as $H(x) = 1$ for $x \geqslant 0$ and $H(x) = 0$ for $x < 0$. Here, m is the column index ($m-1$ is the number of input photons), n is the row index ($n-1$ is the number of output photons), so that $n-m$ is the number of photons created by parametric amplification.

An important feature of this transformation $\mathcal{A}$ is that it gives a column-stochastic matrix $\mathbf{D}^{\mathcal{A}}$ in Fock basis. To prove this, notice that the sum of elements of the mth column of $\mathbf{D}^{\mathcal{A}}$ is given by

$$C^{\mathcal{A}}_m = \sum_{n=1}^{\infty} D^{\mathcal{A}}_{n,m} = \sum_{k=m}^{\infty} Q^{(m-1)}_{k-1}. \tag{9}$$

Using the Pascal identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \tag{10}$$

one can show that $C^{\mathcal{A}}_{m+1} = C^{\mathcal{A}}_m$ for all $m \geqslant 1$, which means that $C^{\mathcal{A}}_m = C^{\mathcal{A}}_1 = 1$ for all $m \geqslant 1$. This is consistent with the fact that the elements of $\boldsymbol{\lambda}$ should form a probability distribution if $\boldsymbol{\lambda'}$ is a probability distribution (the map $\mathcal{A}$ conserves the normalization of probabilities). The sum of elements of the nth row of $\mathbf{D}^{\mathcal{A}}$ is given by

$$R^{\mathcal{A}}_n = \sum_{m=1}^{\infty} D^{\mathcal{A}}_{n,m} = \sum_{k=1}^{n} Q^{(k-1)}_{n-1}. \tag{11}$$

Using the formula for the binomial series, it is trivial to see that $R^{\mathcal{A}}_n = 1-\gamma^2 = 1/G$ for all $n \geqslant 1$, where $G = \cosh^2(s)$ is the intensity gain of the amplifier. Since $G \geqslant 1$, it is clear that $R^{\mathcal{A}}_n \leqslant 1$ for all $n \geqslant 1$, so we conclude that $\mathbf{D}^{\mathcal{A}}$ is indeed column stochastic.

This implies that there exists a set of random unitaries that maps ρ' to ρ and behaves exactly the same way as the quantum-limited amplifier $\mathcal{A}$ on Fock-diagonal input states (in particular, on a thermal state σ). As a consequence, the bipartite 1×1 pure state $|\psi\rangle$ can be transformed into $|\psi'\rangle$ by using a deterministic LOCC [see Fig. 3(a)]. More generally, since the tensor product of column-stochastic matrices is itself column stochastic, if each mode of Σ is the output of a quantum-limited amplifier applied to the corresponding mode of Σ' initially prepared in a thermal state, we have found a column-stochastic matrix which maps the vector of eigenvalues of Σ' to that of Σ. This implies that there exists a set of random unitaries that maps Σ' to Σ, hence the bipartite $N \times N$ pure state $|\psi\rangle$ can be transformed into $|\psi'\rangle$ by using a deterministic LOCC.

This reasoning provides a sufficient condition for the existence of a deterministic LOCC protocol in the case of Fig. 1(a), where both squeezing parameters increase. As a corollary, it also implies the nonexistence of a LOCC in the reverse case 1(b). Thus, our approach gives an alternative proof to Ref. [17] for the existence (nonexistence) of a LOCC in the case of Fig. 1(a) [1(b)]. The LOCC achieving such a transformation in Fig. 1(a) is rather simple: Alice combines each mode with vacuum into a beam splitter of transmissivity ν_i/ν'_i followed by a heterodyne detection on the environmental output port, after which Alice and Bob apply displacement operations conditioned on the outcome of the measurement (see Supplemental Material of [19]). Hence, the entanglement transformation is achieved by a Gaussian LOCC, in accordance with condition (1).

B. Combination of quantum-limited amplifier and pure-loss channel

In order to study the cases of Figs. 1(c) and 1(d), which are not covered by condition (1), we will now prove the existence of LOCC protocols by building column-stochastic matrices that act on the vector of eigenvalues of Σ' similarly as the tensor product of a quantum-limited amplifier $\mathcal{A}$ and a pure-loss channel $\mathcal{L}$ would act on Σ'. In contrast with Fig. 1(a), we will see here that the LOCC must be non-Gaussian, even though it is based on the existence of Gaussian underlying maps $\mathcal{A}$ and $\mathcal{L}$.

As shown in Fig. 3(b), the pure-loss channel $\mathcal{L}$ acts on an input state ρ' equivalently as a unitary operation acting on the input mode A and an environmental mode E prepared in the vacuum state $|0\rangle$, i.e.,

$$\mathcal{E}(\rho') = \mathrm{Tr}_E[U_{\mathcal{E}}(\rho' \otimes |0\rangle_E\langle 0|)U^{\dagger}_{\mathcal{E}}], \tag{12}$$

where $U_{\mathcal{E}} = \exp[\theta(a^{\dagger}_A a_E - a_A a^{\dagger}_E)]$ is a beam-splitter unitary. When we apply the pure-loss channel $\mathcal{L}$ to a single-mode phase-invariant state $\rho' = \sum_n \lambda'_n |n\rangle\langle n|$, the vector of eigenvalues $\boldsymbol{\lambda}$ of the resulting state $\rho = \sum_n \lambda_n |n\rangle\langle n|$ is related to the vector of eigenvalues $\boldsymbol{\lambda'}$ of ρ' through the relation $\boldsymbol{\lambda} = \mathbf{D}^{\mathcal{L}}\boldsymbol{\lambda'}$, where the matrix $\mathbf{D}^{\mathcal{L}}$ reads as

$$\mathbf{D}^{\mathcal{L}} = \begin{pmatrix} 1 & (1-\eta) & (1-\eta)^2 & \dots \\ 0 & \eta & 2\eta(1-\eta) & \dots \\ 0 & 0 & \eta^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

or in a compact form $D^{\mathcal{L}}_{n,m} = P^{(m-1)}_{n-1} H(m-n)$, with

$$P^{(j)}_i = \binom{j}{i}\eta^i(1-\eta)^{j-i}, \quad \eta = \cos^2\theta$$

and $H(x)$ being the Heaviside step function. Here again, m is the column index ($m-1$ is the number of input photons), n is the row index ($n-1$ is the number of output photons), so that $m-n$ is the number of photons lost in the environment.

It turns out that the matrix corresponding to this transformation $\mathcal{L}$ is column stochastic in the trivial case $\eta = 1$ only. The sum of elements of the mth column of $\mathbf{D}^{\mathcal{L}}$ is given by

$$C^{\mathcal{L}}_m = \sum_{n=1}^{\infty} D^{\mathcal{L}}_{n,m} = \sum_{k=1}^{m} P^{(m-1)}_{k-1}. \tag{13}$$

Using the binomial series formula, it is straightforward to see that $C^{\mathcal{L}}_m = 1$ for all $m \geqslant 1$. (This is consistent with the fact that the map $\mathcal{L}$ conserves the normalization of probabilities.) However, the sum of elements of the nth row of $\mathbf{D}^{\mathcal{L}}$ is given by

$$R^{\mathcal{L}}_n = \sum_{m=1}^{\infty} D^{\mathcal{L}}_{n,m} = \sum_{k=n}^{\infty} P^{(k-1)}_{n-1}. \tag{14}$$

Using again identity (10), one can prove that $R^{\mathcal{L}}_{n+1} = R^{\mathcal{L}}_n$ for all $n \geqslant 1$, meaning that $R^{\mathcal{L}}_n = R^{\mathcal{L}}_1 = 1/\eta$ for all $n \geqslant 1$. Since the intensity transmittance η of the pure-loss channel satisfies $0 \leqslant \eta \leqslant 1$, the matrix $\mathbf{D}^{\mathcal{L}}$ is column stochastic only if $\eta = 1$.

Despite $\mathbf{D}^{\mathcal{L}}$ being generally not column stochastic, the tensor product of $\mathbf{D}^{\mathcal{L}}$ and $\mathbf{D}^{\mathcal{A}}$ may still provide us with useful column-stochastic transformations, as indicated in the following theorem [see also Fig. 3(b)]:

Theorem 2. Let ρ and ρ' be tensor products of two thermal states. If it is possible to transform ρ' into ρ by applying a tensor product of a pure-loss channel $\mathcal{L}$ of intensity transmittance η and a quantum-limited amplifier $\mathcal{A}$ of intensity gain G such that $\eta G \geqslant 1$, then $\rho' \succ \rho$.

The proof uses the fact that the sum of row elements (column elements) of a tensor product of two matrices is given by the product of a proper selection of sums of row elements (column elements) of the individual matrices. Since the columns of $\mathbf{D}^{\mathcal{L}}$ and $\mathbf{D}^{\mathcal{A}}$ sum to 1, while their rows sum to $1/\eta$ and $1/G$, respectively, it is easy to see that the matrix $\mathbf{D}^{\mathcal{L}} \otimes \mathbf{D}^{\mathcal{A}}$ is such that its columns sum to 1 and its rows sum to $1/(\eta G)$. Therefore, the matrix $\mathbf{D}^{\mathcal{L}} \otimes \mathbf{D}^{\mathcal{A}}$ is column stochastic if $\eta G \geqslant 1$. ■

Now, Theorem 1 can be easily proven by using Theorem 2. As explained in the Appendix, the LOCC transformation $|\psi\rangle \to |\psi'\rangle$ is possible iff the majorization relation $\rho' \succ \rho$ holds. According to Theorem 2, this is the case if we can apply a pure-loss channel $\mathcal{L}$ to one of the modes of ρ' and a quantum-limited amplifier $\mathcal{A}$ to the other mode of ρ' such that condition $\eta G \geqslant 1$ is verified. What is surprising here is that $\mathcal{L} \otimes \mathcal{A}$ is a Gaussian map, but we use it to ensure the existence of a non-Gaussian transformation from ρ' to ρ, guaranteeing in turn the existence of a non-Gaussian LOCC transformation from $|\psi\rangle$ to $|\psi'\rangle$.

Let us assign a subscript η to the mode whose squeezing parameter increases ($r'_\eta > r_\eta$) along the transformation $|\psi\rangle \to |\psi'\rangle$, and a subscript G to the mode whose squeezing parameter decreases ($r'_G < r_G$). When a pure-loss channel is applied on a thermal state of mean photon number ν'_η, it is transformed into another thermal state of mean photon number $\nu_\eta = \eta \nu'_\eta$. Similarly, when a quantum-limited amplifier is applied on a thermal state of mean photon number ν'_G, it is transformed into another thermal state of mean photon number $\nu_G = G\nu'_G + (G-1)$. Condition $\eta G \geqslant 1$ therefore becomes

$$\frac{\nu_\eta(\nu_G+1)}{\nu'_\eta(\nu'_G+1)} \geqslant 1 \Leftrightarrow \frac{\sinh(r_\eta)\cosh(r_G)}{\sinh(r'_\eta)\cosh(r'_G)} \geqslant 1. \qquad (15)$$

Using the properties of the hyperbolic functions, one can easily prove that this is equivalent to

$$\frac{\sinh(r_\eta + r_G) + \sinh(r_\eta - r_G)}{\sinh(r'_\eta + r'_G) + \sinh(r'_\eta - r'_G)} \geqslant 1. \qquad (16)$$

What differentiates Figs. 1(c) and 1(d) is whether it is the squeezing parameter of the first or second mode that increases along transformation $|\psi\rangle \to |\psi'\rangle$. In Fig. 1(c), r_1 increases (r_2 decreases), while the reverse holds in Fig. 1(d). Thus, the roles of the quantum-limited amplifier and pure-loss channel are exchanged between these two cases. In Fig. 1(c), $r_\eta = r_1$ and $r_G = r_2$, so we recover Eq. (4) with the plus sign, consistent with $r'_1 - r_1 \geqslant 0$. In Fig. 1(d), $r_\eta = r_2$ and $r_G = r_1$, so we recover Eq. (4) with the minus sign, consistent with $r'_1 - r_1 < 0$. This concludes the proof of Theorem 1. ■

C. Extension to an arbitrary number of modes

It is easy to see, from the derivation of Theorem 2, that it can be extended to the $N \times N$ case, with N being an arbitrary number of modes. We obtain the following generalization:

Theorem 3. Let ρ and ρ' be tensor products of N thermal states characterized by vectors of mean number of photons $(\nu_1, \ldots, \nu_N)$ and $(\nu'_1, \ldots, \nu'_N)$, respectively. If it is possible to transform ρ' into ρ by applying a tensor product of quantum-limited Gaussian channels $\mathcal{C}_k$, where channel $\mathcal{C}_k$ acting on mode k is either a pure-loss channel or a quantum-limited amplifier, and where the set of channels $\mathcal{C}_k$ is such that

$$\prod_{k=1}^{N} \tau_k \geqslant 1 \qquad (17)$$

with

$$\tau_k = \begin{cases} \eta_k & \text{if } \mathcal{C}_k \text{ is a pure-loss channel,} \\ G_k & \text{if } \mathcal{C}_k \text{ is a quantum-limited amplifier,} \end{cases}$$

then $\rho' \succ \rho$.

The transformation described in Theorem 3 reads as

$$\begin{aligned} \rho' = \sigma_{\nu'_1} &\otimes \sigma_{\nu'_2} \otimes \cdots \otimes \sigma_{\nu'_N} \\ \downarrow \mathcal{C}_1 \ &\downarrow \mathcal{C}_2 \quad \ldots \quad \downarrow \mathcal{C}_n \\ \rho = \sigma_{\nu_1} &\otimes \sigma_{\nu_2} \otimes \cdots \otimes \sigma_{\nu_N} \end{aligned}$$

and the proof goes the same way as in the 2×2 case, exploiting the fact that the sum of row elements (column elements) of a tensor product of matrices is given by the product of a proper selection of sums of row elements (columns elements) of the individual matrices.

If condition (17) is satisfied, then ρ' majorizes ρ, which in turn implies that any purification of ρ (noted $|\psi\rangle$) can be mapped into a purification of ρ' (noted $|\psi'\rangle$) using a deterministic LOCC, for any number N of modes. Just as in the 2×2 case, this theorem generalizes the sufficient condition of Eq. (1) requiring that all squeezing parameters $(r_1, \ldots, r_N)$ decrease, in which case a GLOCC transformation works. Again, for more complicated evolution patterns of the squeezing parameters $(r_1, \ldots, r_N)$ where Eq. (1) precludes the existence of a GLOCC, our Theorem 3 may very well permit a non-Gaussian LOCC to achieve the transformation. We have not systematically explored all possible evolution patterns $(r_1, \ldots, r_N) \to (r'_1, \ldots, r'_N)$, but it is clear that Theorem 3 gives a sufficient condition for LOCC transformations $|\psi\rangle \to |\psi'\rangle$ that encompasses situations that are not covered by condition (1).

IV. CONCLUSION

In this work, we have developed a technique to find existence conditions on entanglement transformations between bipartite pure Gaussian states that go beyond Gaussian local operations and classical communication (GLOCC). We have presented a sufficient criterion for the existence of a deterministic LOCC transforming a pure $N \times N$ Gaussian state into another. This result generalizes Giedke *et al.*'s necessary and sufficient criterion for the existence of a GLOCC relating such pure Gaussian states [17] (while losing the necessary character of the condition, meaning that a LOCC transformation may exist that is not detected by our criterion). Most notably, our criterion guarantees the existence of a non-Gaussian LOCC transformation connecting some pure Gaussian states that cannot be connected otherwise with a GLOCC according to Ref. [17]. In other words, we exhibit situations where pure Gaussian state interconversions can be achieved with non-Gaussian local operations even though Gaussian local operations alone cannot. This is reminiscent of situations where a Gaussian no-go theorem precludes the use of Gaussian resources in order to achieve a task involving Gaussian states, e.g., quantum entanglement distillation [14–16], quantum error correction [23], or quantum bit commitment [24].

Our approach relies on majorization theory (extended to infinite-dimensional spaces) and consists in building explicit column-stochastic matrices $\mathbf{D}$ that map the vector of eigenvalues of state ρ' (whose purification is $|\psi'\rangle$) onto the vector of eigenvalues of state ρ (whose purification is $|\psi\rangle$), hence ensuring that the transformation $|\psi\rangle \rightarrow |\psi'\rangle$ is possible under a LOCC (even if a GLOCC may not suffice). We build our column-stochastic matrices $\mathbf{D}$ by using (tensor products of) Gaussian channels (namely, quantum-limited amplifiers $\mathcal{A}$ and pure-loss channels $\mathcal{L}$) applied to Fock-diagonal states. Thus, remarkably, our approach allows us to infer the existence of non-Gaussian LOCC transformations without leaving the simple mathematical tools developed for Gaussian transformations. Unfortunately, working with an infinite-dimensional state space makes it nontrivial to find the actual set of random unitaries mapping ρ' to ρ, and thus to design the corresponding LOCC protocol (even though $\mathbf{D}$ is known). The tools developed for finite-dimensional spaces cannot easily be applied in the quantum optical scenario considered here, so we leave the problem of designing explicit non-Gaussian LOCCs for further investigation.

To simplify the presentation, we have restricted our numerical analysis to the entanglement transformations between 2×2 pure Gaussian states (see Fig. 2). We observed interesting behaviors in this case already, namely, situations where GLOCCs are precluded by Ref. [17] while non-Gaussian LOCCs are sufficient according to our Theorem 1. Interestingly, the situation considered in Fig. 1(c) can be seen as a way of concentrating the entanglement of two two-mode squeezed vacuum states into a single two-mode squeezed vacuum state (the other one losing its entanglement). Therefore, progress on designing such LOCC protocols could open a way to novel protocols for enhancing the entanglement of Gaussian states through non-Gaussian operations.

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APPENDIX: THEORY OF MAJORIZATION

Majorization is an algebraic theory which provides a mean to compare two probability distributions in terms of disorder or randomness [25,26]. Let $\mathbf{p}$ and $\mathbf{q}$ be two probability distribution vectors of dimension n. If $\mathbf{p}^{\downarrow}$ and $\mathbf{q}^{\downarrow}$ are vectors containing the elements of $\mathbf{p}$ and $\mathbf{q}$ sorted in nonincreasing order, $\mathbf{p}$ majorizes $\mathbf{q}$, i.e $\mathbf{p} \succ \mathbf{q}$, iff

$$\sum_{i=k}^{k} p_i^{\downarrow} \geqslant \sum_{i=k}^{k} q_i^{\downarrow}, \quad k = 1, \ldots, n, \tag{A1}$$

with equality when $k = n$. In this case, one says that $\mathbf{q}$ is more disordered than $\mathbf{p}$. Note that if $\mathbf{p}$ and $\mathbf{q}$ are probability distributions, the equality for $k = n$ is always satisfied. Majorization only provides a preorder, in the sense that if $\mathbf{p} \not\succ \mathbf{q}$, this does not necessarily mean that $\mathbf{p} \prec \mathbf{q}$. When both $\mathbf{p} \not\succ \mathbf{q}$ and $\mathbf{p} \not\prec \mathbf{q}$ are satisfied, $\mathbf{p}$ and $\mathbf{q}$ are said to be incomparable.

In order to understand why majorization allows one to compare probability distributions in terms of disorder, let us introduce an alternative way of detecting majorization. A more intuitive definition is to say that $\mathbf{p}$ majorizes $\mathbf{q}$ iff there exists a set of n-dimensional permutation matrices $\mathbf{\Pi}_n$ and a probability distribution $\{t_n\}$ such that

$$\mathbf{q} = \sum_n t_n \mathbf{\Pi}_n \mathbf{p}. \tag{A2}$$

This last equation clearly shows the relation between disorder and majorization. Indeed, we see that if $\mathbf{p}$ majorizes $\mathbf{q}$, then $\mathbf{q}$ can be obtained by applying random permutations to $\mathbf{p}$, making $\mathbf{q}$ more disordered than $\mathbf{p}$. This definition also allows us to introduce another equivalent way of characterizing majorization in terms of doubly stochastic matrices. These are the matrices whose columns and rows sum to 1. The set of doubly stochastic matrices of a given dimension is convex, and its extremal points are given by permutation matrices of the same dimension. Consequently, any doubly stochastic matrix can be decomposed as a convex combination of permutation matrices. This allows us to introduce the following theorem.

Theorem 4. Given the vectors $\mathbf{p},\mathbf{q} \in \mathbb{R}^d$, $\mathbf{p} \succ \mathbf{q}$ iff

$$\mathbf{q} = \mathbf{D}\mathbf{p} \tag{A3}$$

for some doubly stochastic matrix $\mathbf{D}$.

This theory of majorization beautifully extends to the quantum realm. Indeed, one can compare density matrices by simply comparing their vectors of eigenvalues, whose elements are probability distributions. Thus, given two density matrices ρ and σ whose vectors of eigenvalues are, respectively, given by $\boldsymbol{\lambda}(\rho)$ and $\boldsymbol{\lambda}(\sigma)$, one says that $\rho \succ \sigma$ if $\boldsymbol{\lambda}(\rho) \succ \boldsymbol{\lambda}(\sigma)$. We then naturally have the following theorem [9].

Theorem 5. $\rho \succ \sigma$ iff state σ can be obtained from state ρ by applying a random mixture of unitaries, i.e.,

$$\sigma = \sum_i t_i U_i \rho U_i^{\dagger}, \tag{A4}$$

where $\{t_i\}$ is a probability distribution and the U_i are unitaries for all i.

A very interesting connection between quantum information theory and majorization resides in the fact that one can use the latter in order to compare pure bipartite entangled state or, more precisely, to investigate the possibility to transform a state into another using a LOCC. Suppose Alice and Bob share a pure state $|\psi\rangle$ and want to transform it into a state $|\phi\rangle$. The following theorem investigates such a possibility [8,9]:

Theorem 6. State $|\psi\rangle$ can be converted deterministically into state $|\phi\rangle$ using LOCC iff $\rho_\psi \prec \rho_\phi$, where ρ_ψ is the reduced density matrix of system A, $\rho_\psi \equiv \mathrm{Tr}_B\,(|\psi\rangle\langle\psi|)$ and similarly for ρ_ϕ.

The theory of majorization nicely adapts to the infinite-dimensional case, allowing one to compare Gaussian states in particular. The definitions we stated before stay the same, the only difference residing in the doubly stochastic matrix, which should now be replaced by an infinite-dimensional column stochastic matrix, whose columns still sum to 1, but whose rows sum to a value less or equal to 1 [27]. Note that in the case of Gaussian states, it is difficult to use definition (A1), due to the complexity of the problem of ordering the eigenvalues of a multimode Gaussian state. Verifying that Theorem 4 holds seems an easier task, which is the technique used in our work. Unfortunately, there is no easy algorithm to decide whether a column-stochastic matrix exists that connects the eigenvalues of two infinite sets (and that generates the matrix in case it exists). Therefore, heuristic approaches, such as the one developed in our work, are needed. In this paper, we provide a way to find such a family of column-stochastic matrices, allowing us to use the theory of majorization to compare Gaussian multimode states.

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