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Majorization preservation of Gaussian bosonic channels

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Abstract

It is shown that phase-insensitive Gaussian bosonic channels are majorization-preserving over the set of passive states of the harmonic oscillator. This means that comparable passive states under majorization are transformed into equally comparable passive states by any phase-insensitive Gaussian bosonic channel. Our proof relies on a new preorder relation called Fock-majorization, which coincides with regular majorization for passive states but also induces another order relation in terms of mean boson number, thereby connecting the concepts of energy and disorder of a quantum state. The consequences of majorization preservation are discussed in the context of the broadcast communication capacity of Gaussian bosonic channels. Because most of our results are independent of the specific nature of the system under investigation, they could be generalized to other quantum systems and Hamiltonians, providing a new tool that may prove useful in quantum information theory and especially quantum thermodynamics.

1. Introduction

Majorization theory (see, e.g., [1]) has long been known to play a prominent role in quantum information theory [2, 3]. When a quantum state ρ majorizes another quantum state σ, denoted as $\rho \succ \sigma$, it means that ρ can be transformed into σ by applying a convex combination of unitary operations, that is $\sigma = \sum_i w_i U_i \rho U_i^\dagger$, with U_i being unitaries, $w_i \geqslant 0$, and $\sum_i w_i = 1$. Thus, $\rho \succ \sigma$ means that σ is more disordered than ρ, and it implies in particular that $S(\sigma) \geqslant S(\rho)$, where S stands for the von Neumann entropy (more generally, it implies a similar inequality for any Shur concave function of ρ). Interestingly, majorization theory also provides a necessary and sufficient condition for the interconversion between pure bipartite states using deterministic Local Operations and Classical Communication (LOCC) [2, 3]. A bipartite pure state $|\psi\rangle$ can be transformed into $|\psi'\rangle$ via a deterministic LOCC if and only if $\rho' \succ \rho$, where $\rho\,(\rho')$ is the reduced state obtained from $|\psi\rangle\,(|\psi'\rangle)$ by tracing over either one of its two parts. Still another application of majorization is related to separability [4]: a separable state ρ_{AB} necessarily obeys $\rho_A \succ \rho_{AB}$ and $\rho_B \succ \rho_{AB}$, which in turn provides a sufficient condition for entanglement that is strictly stronger than the violation of the corresponding entropic conditions $S(\rho_{AB}) \geqslant S(\rho_A)$ and $S(\rho_{AB}) \geqslant S(\rho_B)$ [5, 6]. This result may even be extended to a distillability criterion by noting that any non-distillable (but possibly bound-entangled) state satisfies the same majorization conditions [7].

The importance of majorization theory in continuous-variable quantum information theory was first suggested by Guha in [8], specifically in the context of Gaussian bosonic channels. These channels (defined in section II) are ubiquitous in quantum communication theory as they model most optical communication links, such as optical fibers or amplifiers. Guha was concerned in [8] with the classical capacity of these channels (see [9]), which was known to require the proof of a Gaussian minimum entropy conjecture [10] (now proven in [11]). Denoting an arbitrary Gaussian bosonic channel by $\Phi[.]$, the conjecture is that $S(\Phi[|\psi\rangle]) \geqslant S(\Phi[|0\rangle])$ for any input pure state $|\psi\rangle$, where $|0\rangle$ is the vacuum state. The intuition was that a majorization relation $\Phi[|0\rangle] \succ \Phi[|\psi\rangle]$ might be responsible for the conjectured entropic inequality.

The existence of majorization relations in Gaussian bosonic channels was first proven in [12], where the quantum-limited amplifier $\mathcal{A}[.]$ (defined in section II) was proven to obey an infinite ladder of majorization relations when the input state is an individual Fock state, namely $\mathcal{A}[|k\rangle] \succ \mathcal{A}[|k+1\rangle], \forall k \geqslant 0$. A parametric majorization relation was also proven for varying gain G, namely $\mathcal{A}_G[|k\rangle] \succ \mathcal{A}_{G+\delta G}[|k\rangle]$ if $\delta G \geqslant 0$. Then, in [13], a similar ladder of majorization relations $\mathcal{L}[|k\rangle] \succ \mathcal{L}[|k+1\rangle]$ was shown to hold for a pure loss channel $\mathcal{L}$ (defined in section II). Later on, in [14], the conjectured majorization relation $\Phi[|0\rangle] \succ \Phi[|\psi\rangle]$ was proven for any input state $|\psi\rangle$ and any Gaussian bosonic channel Φ, which generalizes (and implies) the proof [11] of the above Gaussian minimum entropy conjecture $S(\Phi[|\psi\rangle]) \geqslant S(\Phi[|0\rangle])$. Finally, the interconversion between pure Gaussian states was also investigated based on majorization theory [15, 16], which revealed the existence of surprising situations where a non-Gaussian LOCC is required although the states considered are Gaussian.

In this paper, we introduce the notion of Fock-majorization, denoted as $\succ_{\mathrm{F}}$, which induces a novel (pre) order relation between states of a bosonic mode. We show that Fock-majorization has two powerful properties. Firstly, it induces an order relation in terms of the mean energy of the states. Secondly, it coincides with regular majorization for passive states, namely the lowest energy states among isospectral states [17, 18]. Since we focus on the Hamiltonian of an harmonic oscillator, $H = 1/2 + \hat{a}^\dagger\hat{a}$, whose eigenstates are the Fock states, all passive states of a bosonic mode are obviously Fock-diagonal states with decreasing eigenvalues for increasing boson number.

Equipped with this tool, we can prove a new type of intrinsic majorization property in Gaussian bosonic channels, namely the conservation across any channel Φ of a Fock-majorization relation between any two comparable Fock-diagonal states, that is, $\rho \succ_{\mathrm{F}} \sigma \Rightarrow \Phi[\rho] \succ_{\mathrm{F}} \Phi[\sigma]$. This implies in turn that Gaussian bosonic channels preserve regular majorization over the set of passive states of the harmonic oscillator, that is, $\rho \succ \sigma \Rightarrow \Phi[\rho] \succ \Phi[\sigma]$. Finally, we discuss the connection of this result with an open problem related to the broadcast communication capacity of bosonic channels [19].

2. Gaussian bosonic channels

An arbitrary Gaussian bosonic channel, denoted as $\Phi[.]$, is such that if ρ is a Gaussian state, then $\Phi[\rho]$ is a Gaussian state too. In this paper, we restrict to the class of single-mode phase-insensitive Gaussian bosonic channels, in which the two quadrature components ($\hat{x}$ and $\hat{p}$) of the mode operator $\hat{a} = (\hat{x} + i\hat{p})/\sqrt{2}$ are identically transformed under Φ in the Heisenberg picture. A simple example of such a channel is a beam splitter of transmittance η, which linearly couples the input mode with an environment mode in the vacuum state,

$$\hat{a}_{\mathrm{in}} \to \hat{a}_{\mathrm{out}} = \sqrt{\eta}\, \hat{a}_{\mathrm{in}} + \sqrt{1-\eta}\, \hat{a}_{\mathrm{env}}, \tag{1}$$

where $\hat{a}_{\mathrm{in}}$, $\hat{a}_{\mathrm{env}}$, and $\hat{a}_{\mathrm{out}}$ are the bosonic mode operators for the input, environment, and output mode, respectively. This is the so-called pure loss channel $\mathcal{L}_\eta$ of transmittance η, where the Gaussian noise originates from the vacuum fluctuations of the bosonic field in the environment mode. Another basic phase-insensitive Gaussian bosonic channel is a parametric optical amplifier of gain G, which couples the input (or signal) mode with an environment (or idler) mode in the vacuum state according to

$$\hat{a}_{\mathrm{in}} \to \hat{a}_{\mathrm{out}} = \sqrt{G}\, \hat{a}_{\mathrm{in}} + \sqrt{G-1}\, \hat{a}^\dagger_{\mathrm{env}}. \tag{2}$$

In this so-called quantum-limited amplifier channel $\mathcal{A}_G$ of gain G, some Gaussian thermal noise unavoidably affects the output state because of parametric down-conversion. Now if the environment mode is in a thermal state for both cases of a beam-splitter or parametric down-converter, some additional Gaussian noise is superimposed onto the attenuated or amplified output state, giving rise to a noisy version of channels $\mathcal{L}_\eta$ and $\mathcal{A}_G$. More generally, any single-mode phase-insensitive Gaussian bosonic channel Φ may be decomposed as a suitable sequence of channels $\mathcal{L}_\eta$ and $\mathcal{A}_G$ [12, 20].

3. Fock-majorization

We recall the usual definition of the majorization relation between states ρ and σ, namely, $\rho \succ \sigma$ if and only if

$$\sum_{i=0}^{n} \lambda_i^{\downarrow} \geqslant \sum_{i=0}^{n} \mu_i^{\downarrow}, \quad \forall\, n \geqslant 0, \tag{3}$$

where $\lambda_i^{\downarrow}$ and $\mu_i^{\downarrow}$ are the eigenvalues of ρ and σ, respectively, which have been ordered by decreasing value (indicated by the arrow pointing downwards). Furthermore, the two summations in equation (3) should be equal at the limit $n \to \infty$ so the traces of ρ and σ coincide for majorization to hold (this is obviously the case here since ρ and σ are density operators).

Definition. We define that states ρ and σ satisfy the Fock-majorization relation denoted as $\rho \succ_{\mathrm{F}} \sigma$ if and only if

$$\mathrm{Tr}(P_n \rho) \geqslant \mathrm{Tr}(P_n \sigma), \quad \forall\, n \geqslant 0, \tag{4}$$

with $P_n = \sum_{i=0}^{n} |i\rangle\langle i|$ being a projector onto the space spanned by the $(n+1)$ first Fock states $|i\rangle$. This yields a distinct (pre)order relation in state space, which only depends on the diagonal elements of ρ and σ (or their eigenvalues if the states are Fock-diagonal). In contrast with regular majorization, the diagonal elements are not ordered by decreasing values, but instead by increasing boson number. Such a relaxed definition of majorization without prior sorting is sometimes called 'unordered majorization' [1]; it makes sense only when there exists a natural way of ordering the elements (in the present case, it is the energy). To our knowledge, the notion of Fock-majorization (where the elements are ordered by increasing energy) has never been defined nor exploited in the context of Gaussian bosonic channels or more generally continuous-variable quantum information (see also section 6).

Note that any two Fock states $|n\rangle$ and $|m\rangle$ satisfy the Fock-majorization relation $|n\rangle\langle n| \succ_{\mathrm{F}} |m\rangle\langle m|$ only if $n \leqslant m$, whereas they are always equivalent (isospectral) in terms of usual majorization. Another feature of Fock-majorization is that if $\rho \succ_{\mathrm{F}} \sigma$ and $\sigma \succ_{\mathrm{F}} \rho$ both hold, then $\mathrm{diag}(\rho) = \mathrm{diag}(\sigma)$. By comparison, for regular majorization, if $\rho \succ \sigma$ and $\sigma \succ \rho$ both hold, then the states are equivalent (isospectral).

Interestingly, Fock-majorization implies an energy order relation between comparable states, namely

$$\rho \succ_{\mathrm{F}} \sigma \implies \mathrm{Tr}(\rho\,\hat{n}) \leqslant \mathrm{Tr}(\sigma\,\hat{n}), \tag{5}$$

where $\hat{n} = \hat{a}^{\dagger}\hat{a}$ is the number operator. Although equation (5) holds in general, we only give its proof for Fock-diagonal states here because we only need to consider these states (especially passive states of the harmonic oscillator) in the following. Take two Fock-diagonal states $\rho = \sum_{i=0}^{N} r_i\,|i\rangle\langle i|$ and $\sigma = \sum_{i=0}^{N} s_i\,|i\rangle\langle i|$ whose support is the space spanned by $\{|0\rangle, \cdots |N\rangle\}$. (If their support have unequal sizes, we take the largest size for N.) We assume that $\rho \succ_{\mathrm{F}} \sigma$, that is

$$\sum_{i=0}^{n} r_i \geqslant \sum_{i=0}^{n} s_i \implies \sum_{i=n}^{N} r_i \leqslant \sum_{i=n}^{N} s_i, \qquad \forall\, n \text{ s.t. } 0 \leqslant n \leqslant N. \tag{6}$$

Summing this expression over n and interchanging the two summations gives

$$\sum_{i=1}^{N}\sum_{n=1}^{i} r_i \leqslant \sum_{i=1}^{N}\sum_{n=1}^{i} s_i \implies \sum_{i=1}^{N} i\; r_i \leqslant \sum_{i=1}^{N} i\; s_i. \tag{7}$$

By taking the limit $N \to \infty$, we conclude that the mean energy of ρ is lower than that of σ, which proves equation (5). Note that the converse of equation (5) is not true.

Finally, it is straightforward to see that Fock-majorization $\rho \succ_{\mathrm{F}} \sigma$ coincides with regular majorization $\rho \succ \sigma$ over the set of passive states. By definition, passive states are diagonal in the energy eigenbasis of the harmonic oscillator (i.e., Fock basis of a bosonic mode) and their eigenvalues are non-increasing with respect to energy, that is,

$$\rho = \sum_{i=0}^{\infty} r_i\;|i\rangle\langle i|, \qquad \text{with } r_i \geqslant r_{i+1}, \;\; \forall\, i \geqslant 0, \tag{8}$$

for a passive state ρ [17, 18]. Hence, when restricting to passive states, the Fock-majorization condition (4) becomes equivalent to the regular majorization relation (3). Otherwise, $\rho \succ_{\mathrm{F}} \sigma$ and $\rho \succ \sigma$ are distinct order relations (in section 6, we discuss some implications between them).

Before coming to the main results of this paper (sections 4 and 5), we first introduce the following two lemmas (proven in appendix A), which state fundamental Fock-majorization relations in phase-insensitive Gaussian bosonic channels.

Lemma 1. *The pure loss channel $\mathcal{L}_\eta$ of arbitrary transmittance η exhibits a ladder of Fock-majorization relations*

$$\mathcal{L}_\eta[|k\rangle\langle k|] \succ_{\mathrm{F}} \mathcal{L}_\eta[|k+1\rangle\langle k+1|], \quad \forall\, k \geqslant 0. \tag{9}$$

Lemma 2. *The quantum-limited amplifier $\mathcal{A}_G$ of arbitrary gain G exhibits a ladder of Fock-majorization relations*

$$\mathcal{A}_G[|k\rangle\langle k|] \succ_{\mathrm{F}} \mathcal{A}_G[|k+1\rangle\langle k+1|], \quad \forall\, k \geqslant 0. \tag{10}$$

4. Fock-preserving and passive-preserving channels

A channel Φ is called Fock-preserving when it is such that if ρ is a Fock-diagonal state, then $\Phi[\rho]$ is also a Fock-diagonal state. Phase-insensitive Gaussian bosonic channels are well known to be Fock-preserving channels since they map Fock states onto mixtures of Fock states [21]. A stronger condition, which we need here, is that a Fock-preserving channel Φ is passive-preserving, i.e., it maps passive states onto passive states. In order to show that phase-insensitive Gaussian bosonic channels are indeed passive-preserving[1], we need to prove the following theorem, which provides a key to determine whether any channel Φ is passive-preserving.

Theorem 1. *A channel Φ is passive-preserving if and only if its adjoint $\Phi^\dagger$ obeys the ladder of Fock-majorization relations*

$$\Phi^\dagger[|k\rangle\langle k|] \succ_{\mathrm{F}} \Phi^\dagger[|k+1\rangle\langle k+1|], \quad \forall k \geqslant 0. \tag{11}$$

Proof. Equation (11) is equivalent to

$$\mathrm{Tr}(P_n\Phi^\dagger[|k\rangle\langle k|]) \geqslant \mathrm{Tr}(P_n\Phi^\dagger[|k+1\rangle\langle k+1|]), \ \forall n \geqslant 0, \tag{12}$$

where $P_n = \sum_{i=0}^{n} |i\rangle\langle i|$. Using the definition of the adjoint of a channel, we get

$$\mathrm{Tr}(|k\rangle\langle k| \ \Phi[P_n]) \geqslant \mathrm{Tr}(|k+1\rangle\langle k+1| \ \Phi[P_n]), \ \forall n \geqslant 0. \tag{13}$$

Now, assume that the input of channel Φ is a passive state (of the harmonic oscillator)

$$\rho = \sum_{n=0}^{\infty} r_n \ |n\rangle\langle n|, \qquad \text{with } r_n \geqslant r_{n+1}, \ \forall n \geqslant 0. \tag{14}$$

It can also be rewritten as

$$\rho = \sum_{n=0}^{\infty} e_n P_n, \qquad \text{with } e_n = r_n - r_{n+1}. \tag{15}$$

where $e_n \geqslant 0, \forall n \geqslant 0$, since ρ is passive. Then, we may take the convex combination of inequalities (13) with weights e_n and n going from 0 to ∞, resulting in

$$\mathrm{Tr}(|k\rangle\langle k| \ \Phi[\rho]) \geqslant \mathrm{Tr}(|k+1\rangle\langle k+1| \ \Phi[\rho]). \tag{16}$$

Hence, the output state $\Phi[\rho]$ is passive, so that channel Φ is indeed passive-preserving. Conversely, it is trivial to see that Φ being passive-preserving implies equation (13) since P_n is (proportional to) a passive state, hence it implies equation (11). □

Corollary 1. *Phase-insensitive Gaussian bosonic channels are passive preserving.*

Using lemmas 1 and 2 together with theorem 1, we obtain that the pure loss channel $\mathcal{L}_\eta$, whose adjoint is $1/\eta$ times the quantum-limited amplifier $\mathcal{A}_{1/\eta}$, as well as the quantum-limited amplifier $\mathcal{A}_G$, whose adjoint is $1/G$ times the pure-loss channel $\mathcal{L}_{1/G}$, are both passive preserving. Then, the corollary follows from the fact that any phase-insensitive Gaussian bosonic channel Φ can be expressed as the concatenation of a pure loss channel $\mathcal{L}$ and a quantum-limited amplifier $\mathcal{A}$, i.e., $\Phi = \mathcal{A} \circ \mathcal{L}$ [12, 20], and that passive-preservation is transitive over channel composition.

5. Fock-majorization preserving channels

A Fock-preserving channel Φ is called Fock-majorization preserving provided it is such that if $\rho \succ_{\mathrm{F}} \sigma$, with ρ and σ being Fock-diagonal states, then $\Phi[\rho] \succ_{\mathrm{F}} \Phi[\sigma]$. In order to prove that phase-insensitive Gaussian bosonic channels are Fock-majorization preserving, we need to prove the following theorem.

Theorem 2. *A channel Φ is Fock-majorization preserving if and only if it obeys the ladder of Fock-majorization relations*

$$\Phi[|k\rangle\langle k|] \succ_{\mathrm{F}} \Phi[|k+1\rangle\langle k+1|], \quad \forall k \geqslant 0. \tag{17}$$

Proof. We start with two Fock-diagonal states

$$\rho = \sum_{i=0}^{N} r_i \ |i\rangle\langle i|, \qquad \sigma = \sum_{i=0}^{N} s_i \ |i\rangle\langle i|, \tag{18}$$

[1] This property is also proven in [22], where it is called Fock-preserving, but we give an independent simple proof here.

whose supports is the space spanned by $\{|0\rangle, \cdots|N\rangle\}$. (If their supports have unequal sizes, we take the largest size for N.) We assume that we have a Fock-majorization relation between two states at the input of the channel, that is

$$\rho \succ_{\mathrm{F}} \sigma \qquad \Longleftrightarrow \quad R_n \geqslant S_n, \forall\, n \geqslant 0, \tag{19}$$

where

$$R_n = \mathrm{Tr}(P_n \rho) = \sum_{i=0}^{n} r_i, \quad S_n = \mathrm{Tr}(P_n \sigma) = \sum_{i=0}^{n} s_i. \tag{20}$$

We want to prove that the same Fock-majorization relation holds at the output,

$$\Phi[\rho] \succ_{\mathrm{F}} \Phi[\sigma] \qquad \Longleftrightarrow \quad A_n \geqslant B_n, \forall\, n \geqslant 0 \tag{21}$$

where

$$A_n = \mathrm{Tr}(P_n \Phi[\rho]) = \sum_{i=0}^{N} r_i P_n^{(i)}, \quad B_n = \mathrm{Tr}(P_n \Phi[\sigma]) = \sum_{i=0}^{N} s_i P_n^{(i)}, \tag{22}$$

with

$$P_n^{(i)} = \mathrm{Tr}\,(P_n \Phi[|i\rangle\langle i|]). \tag{23}$$

Now, we define the quantities

$$\alpha_n^{(k)} = R_k P_n^{(k)} + \sum_{i=k+1}^{N} r_i P_n^{(i)}, \qquad k = 0, \cdots N, \tag{24}$$

where the second term in the right-hand side is taken equal to zero when $k = N$, so that $\alpha_n^{(N)} = R_N P_n^{(N)}$. Similarly, we define

$$\beta_n^{(k)} = S_k P_n^{(k)} + \sum_{i=k+1}^{N} s_i P_n^{(i)}, \qquad k = 0, \cdots N, \tag{25}$$

with the convention $\beta_n^{(N)} = S_N P_n^{(N)}$. The Fock-majorization relation we need to prove, equation (21), is equivalent to

$$\alpha_n^{(0)} \geqslant \beta_n^{(0)}, \qquad \forall\, n \geqslant 0 \tag{26}$$

corresponding to $k = 0$. We will now prove

$$\alpha_n^{(k)} \geqslant \beta_n^{(k)}, \qquad \forall\, n \geqslant 0 \tag{27}$$

by recurrence in k, starting from $k = N$ and ending at $k = 0$. We have trivially $\alpha_n^{(N)} \geqslant \beta_n^{(N)}, \forall\, n \geqslant 0$, since $R_N = S_N = 1$. Now, we assume that

$$\alpha_n^{(k+1)} \geqslant \beta_n^{(k+1)}, \qquad \forall\, n \geqslant 0 \tag{28}$$

which can be rewritten as

$$R_{k+1} P_n^{(k+1)} + \sum_{i=k+2}^{N} r_i P_n^{(i)} \geqslant S_{k+1} P_n^{(k+1)} + \sum_{i=k+2}^{N} s_i P_n^{(i)}. \tag{29}$$

Using $R_{k+1} = R_k + r_{k+1}$ and $S_{k+1} = S_k + s_{k+1}$, we reexpress it as

$$R_k P_n^{(k+1)} + \sum_{i=k+1}^{N} r_i P_n^{(i)} \geqslant S_k P_n^{(k+1)} + \sum_{i=k+1}^{N} s_i P_n^{(i)}, \tag{30}$$

which is equivalent to

$$R_k (P_n^{(k+1)} - P_n^{(k)}) + \alpha_n^{(k)} \geqslant S_k (P_n^{(k+1)} - P_n^{(k)}) + \beta_n^{(k)}, \tag{31}$$

or simply

$$\alpha_n^{(k)} - \beta_n^{(k)} \geqslant (R_k - S_k)(P_n^{(k)} - P_n^{(k+1)}). \tag{32}$$

Since ρ Fock-majorizes σ by hypothesis (equation (19)), we have $R_k - S_k \geqslant 0, \forall\, k \geqslant 0$. If $\Phi[|k\rangle\langle k|]$ Fock-majorizes $\Phi[|k+1\rangle\langle k+1|]$, which means that $P_n^{(k)} - P_n^{(k+1)} \geqslant 0, \forall\, n \geqslant 0$, then the right-hand side of equation (32) is greater than zero. Thus, equation (28) implies equation (27), which concludes the recurrence in k and proves equation (26), hence equation (21). Conversely, it is trivial to see that Fock-majorization preservation for channel Φ implies the ladder of Fock-majorization relations since individual Fock states satisfy the Fock-majorization relation $|n\rangle\langle n| \succ_{\mathrm{F}} |n+1\rangle\langle n+1|, \forall\, n \geqslant 0$. □

Corollary 2. *Phase-insensitive Gaussian bosonic channels are Fock-majorization preserving.*

We use again the fact that any phase-insensitive Gaussian bosonic channel Φ can be expressed as the concatenation $\Phi = \mathcal{A} \circ \mathcal{L}$ and that Fock-majorization preservation is transitive over channel composition.

Corollary 3. *Phase-insensitive Gaussian bosonic channels are majorization-preserving over the set of passive states.*

As a consequence of the equivalence between Fock-majorization and regular majorization over the set of passive states, a Fock-majorization preserving channel is necessarily also majorization-preserving over the set of passive states provided it is passive-preserving. Since phase-insensitive Gaussian bosonic channels are passive-preserving (corollary 1) and Fock-majorization preserving (corollary 2), we conclude that they preserve regular majorization over the set of passive states.

6. Discussion and conclusion

We have introduced the notion of Fock-majorization, which induces a novel (pre)order relation between states of the harmonic oscillator and coincides with regular majorization for passive states, namely the lowest energy states among isospectral states. As a notable application of this tool, we have shown that phase-insensitive Gaussian bosonic channels preserve majorization over the set of passive states. This property nicely complements the one very recently found in [22]. There, it was shown that among all isospectral states ρ at the input of a phase-insensitive Gaussian bosonic channel Φ, the passive state, denoted as $\rho^{\downarrow}$, produces an output state that majorizes all other output states, namely $\Phi[\rho^{\downarrow}] \succ \Phi[\rho]$. Here, we consider instead two input states that have different spectra but are both passive, $\rho^{\downarrow}$ and $\sigma^{\downarrow}$, and have demonstrated that $\rho^{\downarrow} \succ \sigma^{\downarrow}$ implies $\Phi[\rho^{\downarrow}] \succ \Phi[\sigma^{\downarrow}]$. This reflects the fact that Gaussian bosonic channels exhibit quite a wide variety of majorization properties, going well beyond what was originally expected in [8].

As a matter of fact, our main result may be combined together with that of [22], giving what can be viewed as a *fundamental majorization-preservation property*

$$\rho^{\downarrow} \succ \sigma \implies \Phi[\rho^{\downarrow}] \succ \Phi[\sigma], \tag{33}$$

valid for any phase-insensitive Gaussian bosonic channel Φ. Interestingly, this property (unlike the one of [22]) is transitive if we concatenate several passive-preserving channels. In particular, it means that proving it for an infinitesimal channel (e.g., using the Lindbladian) suffices to prove it for any concatenated channel. To be complete, let us also mention some implications between Fock-majorization and regular majorization relations. First, it is clear from their respective definitions that

$$\rho \succ \sigma \iff \rho^{\downarrow} \succ_{\mathrm{F}} \sigma^{\downarrow} \tag{34}$$

where $\rho^{\downarrow}$ ($\sigma^{\downarrow}$) is the passive state with the same spectrum as ρ (σ). Furthermore, we have obviously $\rho^{\downarrow} \equiv \rho$ in terms of regular majorization, while $\rho^{\downarrow} \succ_{\mathrm{F}} \rho$ in terms of Fock-majorization. This yields the following implication from regular to Fock-majorization

$$\rho^{\downarrow} \succ \sigma \implies \rho^{\downarrow} \succ_{\mathrm{F}} \sigma, \tag{35}$$

where we have used $\rho^{\downarrow} \equiv \rho$ and $\sigma^{\downarrow} \succ_{\mathrm{F}} \sigma$. Conversely, we have the implication from Fock-majorization to regular majorization

$$\rho \succ_{\mathrm{F}} \sigma^{\downarrow} \implies \rho \succ \sigma^{\downarrow}, \tag{36}$$

where we have used $\rho^{\downarrow} \succ_{\mathrm{F}} \rho$ and $\sigma^{\downarrow} \equiv \sigma$. Combining these various results may help pave the way to solving some of the open problems in the field of Gaussian bosonic channels.

For example, the fundamental majorization-preservation property (33) may be a key element to solve the broadcast communication capacity of phase-insensitive Gaussian bosonic channels Φ, which is known to rely on proving the conjecture that $S(\Phi[\rho]) \geqslant S(\Phi[\tau])$ for any input state ρ satisfying the entropy constraint $S(\rho) = S$, where τ is the (Gaussian) thermal state with the same entropy $S(\tau) = S$ [19]. Indeed, if all states were always comparable under majorization, a consequence of property (33) would be that the passive state at the input $\rho^{\downarrow}$ that gives the lowest output entropy necessarily majorizes all other input states σ, namely $\rho^{\downarrow} \succ \sigma$ implies $\Phi[\rho^{\downarrow}] \succ \Phi[\sigma]$, which in turn implies $S(\Phi[\rho^{\downarrow}]) \leqslant S(\Phi[\sigma])$. Then, using implication (35), we would know that $\rho^{\downarrow} \succ_{\mathrm{F}} \sigma$. Next, from the energy order imposed by Fock-majorization (equation (5)), we would deduce that the optimal input state $\rho^{\downarrow}$ satisfying the entropy constraint $S(\rho^{\downarrow}) = S$ should also have minimum energy. Since the thermal state τ has the lowest possible energy for a given entropy S, we would conclude that $\rho^{\downarrow} = \tau$, thereby proving the conjecture. Unfortunately, majorization is a preorder (instead of a full order) relation, which means that there exist pairs of incomparable states that neither satisfy $\rho \succ \sigma$ nor $\sigma \succ \rho$. Hence, the previous argument is not conclusive, despite providing further evidence of the conjecture being true. It also reflects that understanding the properties of states that are incomparable to the thermal state under majorization is a crucial step in solving the above conjecture.

For completeness, let us mention that our results can also be extended to the set of phase-conjugate Gaussian bosonic channels, which can be expressed as a concatenation of a pure loss channel $\mathcal{L}_\eta$ and a quantum-limited phase-conjugate channel $\tilde{\mathcal{A}}_G$. The latter corresponds to the complementary channel of the quantum-limited amplifier $\mathcal{A}_G$, when it is represented using its Stinespring dilation [23]. The interested reader is referred to appendix B.

Finally, we would like to stress that all proofs in this work, except for lemmas 1 and 2, are independent of the specific nature of the system (i.e., the harmonic oscillator Hamiltonian for a bosonic mode). Therefore, we believe that the application of Fock-majorization could be extended to other quantum systems and arbitrary Hamiltonians, yielding a general tool that could prove very useful in quantum information theory, more specifically in quantum thermodynamics. As a matter of fact, Fock-majorization bears some similarity to a relation called 'upper-triangular majorization' that has been introduced in [24]. There, the authors show that when two states obey such a relation, one can be transformed into the other via a so-called 'cooling map', resulting from the coupling of the system with a zero-temperature reservoir with an energy-conserving unitary. Instead, Fock-majorization can be interpreted as a relation indicating the existence of a 'heating' or 'amplifying' map between the two states (it actually corresponds to a lower-triangular majorization)[2]. It may thus be quite fruitful to investigate the thermodynamical consequences of the existence of Fock-majorization, just as it was done for upper-triangular majorization in the context of cooling maps. The latter maps happen to be a special case of the so-called 'thermal maps', which result from the coupling with a finite-temperature heat bath and are linked to another type of majorization relation, called 'thermo-majorization' [25]. Since these various thermal operations provide a suitable model in the study of thermodynamical processes for microscopic systems, we anticipate that our results on Fock-majorization will find interesting applications in the field of quantum thermodynamics.

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Appendix A. Proof of lemmas 1 and 2

Proof of Lemma 1. The pure loss channel $\mathcal{L}$ of arbitrary transmittance exhibits a ladder of Fock-majorization relations

$$\mathcal{L}[|k\rangle\langle k|] \succ_{\mathrm{F}} \mathcal{L}[|k+1\rangle\langle k+1|], \quad \forall\, k \geqslant 0.$$

It is known that a similar relation holds when replacing Fock-majorization with majorization [13]. Here, we will adapt this proof in order to derive a Fock-majorization relation. We have

$$\rho^{(k)} \equiv \mathcal{L}[|k\rangle\langle k|] = \sum_{n=0}^{k} r_n^{(k)}\, |n\rangle\langle n| \tag{A.1}$$

where

$$r_n^{(k)} = \binom{k}{n}\eta^n(1-\eta)^{k-n} \tag{A.2}$$

and η is the transmittance of channel $\mathcal{L}$. Majorization was proven in [13] based on the recurrence relation

$$r_n^{(k+1)} = \eta\, r_{n-1}^{(k)} + (1-\eta)\, r_n^{(k)}, \qquad \forall\, k \geqslant 0, \forall\, n \geqslant 0, \tag{A.3}$$

where the first term in the r.h.s. is taken equal to zero for $n = 0$. We can rewrite it as

$$r_n^{(k)} - r_n^{(k+1)} = \eta(r_n^{(k)} - r_{n-1}^{(k)}), \tag{A.4}$$

[2] It can be checked that a quantum-limited amplified $\mathcal{A}_G$ acting on Fock-diagonal states gives rise to a Fock-majorization relation, while a pure loss channel $\mathcal{L}_\eta$ acting on Fock-diagonal states gives rise to upper-triangular majorization [16].

Hence,

$$\sum_{i=0}^{n} r_i^{(k)} - \sum_{i=0}^{n} r_i^{(k+1)} = \eta\, r_n^{(k)} \geqslant 0 \tag{A.5}$$

which gives the Fock-majorization relation $\rho^{(k)} \succ_{\mathrm{F}} \rho^{(k+1)}$ in addition to the majorization relation $\rho^{(k)} \succ \rho^{(k+1)}$ of [13].

Proof of Lemma 2. The quantum-limited amplifier $\mathcal{A}$ of arbitrary gain exhibits a ladder of Fock-majorization relations

$$\mathcal{A}[|k\rangle\langle k|] \succ_{\mathrm{F}} \mathcal{A}[|k+1\rangle\langle k+1|], \quad \forall\, k \geqslant 0.$$

We also use the related majorization property for an amplifier as proven in [12]. We have

$$\sigma^{(k)} \equiv \mathcal{A}[|k\rangle\langle k|] = \sum_{n=0}^{\infty} s_n^{(k)}\ |n+k\rangle\langle n+k| = \sum_{n=k}^{\infty} s_{n-k}^{(k)}\ |n\rangle\langle n|, \tag{A.6}$$

where

$$s_n^{(k)} = \binom{n+k}{n} t^n (1-t)^{k+1}, \tag{A.7}$$

and $t = \tanh^2(r)$ is related to the gain $G = 1/(1-t)$ of amplifier $\mathcal{A}$, with r being the squeezing parameter. Majorization was proven in [12] by using the recurrence relation

$$s_n^{(k+1)} = t\, s_{n-1}^{(k+1)} + (1-t)\, s_n^{(k)}, \qquad \forall\, k \geqslant 0, \forall\, n \geqslant 0, \tag{A.8}$$

where the first term in the r.h.s. is taken equal to zero for $n = 0$. We can rewrite it as

$$s_n^{(k)} - s_n^{(k+1)} = (G-1)(s_n^{(k+1)} - s_{n-1}^{(k+1)}). \tag{A.9}$$

The differences between the cumulated sums of eigenvalues are given by

$$\sum_{i=k}^{n} s_{i-k}^{(k)} - \sum_{i=k+1}^{n} s_{i-(k+1)}^{(k+1)} \geqslant \sum_{i=k}^{n} (s_{i-k}^{(k)} - s_{i-k}^{(k+1)}). \tag{A.10}$$

Using equation (A.9), we have

$$\sum_{i=k}^{n} s_{i-k}^{(k)} - \sum_{i=k+1}^{n} s_{i-(k+1)}^{(k+1)} \geqslant (G-1)\, s_{n-k}^{(k+1)} \geqslant 0, \tag{A.11}$$

giving the Fock-majorization relation $\sigma^{(k)} \succ_{\mathrm{F}} \sigma^{(k+1)}$ in addition to the majorization relation $\sigma^{(k)} \succ \sigma^{(k+1)}$ [12].

Appendix B. Majorization preservation for the phase-conjugate channel

Lemma 3. *The quantum-limited phase-conjugate channel $\tilde{\mathcal{A}}_G$ of arbitrary gain G exhibits a ladder of Fock-majorization relations*

$$\tilde{\mathcal{A}}_G[|k\rangle\langle k|] \succ_{\mathrm{F}} \tilde{\mathcal{A}}_G[|k+1\rangle\langle k+1|], \quad \forall\, k \geqslant 0. \tag{B.1}$$

Proof of Lemma 3.

We have

$$\omega^{(k)} \equiv \tilde{\mathcal{A}}[|k\rangle\langle k|] = \sum_{n=0}^{\infty} s_n^{(k)}\ |n\rangle\langle n|, \tag{B.2}$$

where the $s_n^{(k)}$ are defined in equation (A.7). Note that the diagonal of $\mathcal{A}[|k\rangle\langle k|]$ in fact corresponds to the diagonal of $\tilde{\mathcal{A}}[|k\rangle\langle k|]$, shifted by an index k. The differences between the cumulated sums of eigenvalues are given by

$$\sum_{i=0}^{n} s_i^{(k)} - \sum_{i=0}^{n} s_i^{(k+1)} = (G-1)\, s_n^{(k+1)} \geqslant 0, \tag{B.3}$$

where we used equation (A.9) again. This gives the Fock-majorization relation $\omega^{(k)} \succ_{\mathrm{F}} \omega^{(k+1)}$.

Using lemma 3 together with theorem 1, we obtain that the quantum-limited phase-conjugate channel $\tilde{\mathcal{A}}_G$, whose adjoint is $1/(G-1)$ times the quantum-limited phase-conjugate channel $\tilde{\mathcal{A}}_{G/(G-1)}$, is passive preserving. Since any phase-conjugate Gaussian bosonic channel Φ can be expressed as the concatenation of a pure loss channel $\mathcal{L}$ and a quantum-limited phase-conjugate channel $\tilde{\mathcal{A}}$, i.e., $\Phi = \tilde{\mathcal{A}} \circ \mathcal{L}$, and since passive-preservation is transitive over channel composition, we deduce (following the reasoning of corollaries 1, 2 and 3) that phase-conjugate Gaussian bosonic channels are passive preserving, Fock-majorization preserving, and majorization-preserving over the set of passive states.

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