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Processes with indefinite causal structure in quantum theory

The Multi-Round Process Matrix

Mémoire présenté en vue de l'obtention du diplôme d'Ingénieur Civil Physicien
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"What would be the first thing you would think of trying to do if you have a photon interacting with its past self ? [...] Would you have it buy its former self a beer, which would be a nice thing to do? No! Of course, we have it to try to kill itself!"

Seth Lloyd [1]

"Well, to say that no photon were harmed in the course of this experiment would be an exaggeration."

Seth Lloyd, later in the talk

[1] Transcription of a guest lecture at the Institute for Quantum Computing, University of Waterloo. Recorded on Nov. 4, 2010. The entire lecture is entitled "Sending a Photon Backwards in Time."

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Abstract

École Polytechnique de Bruxelles
Centre for Quantum Information and Computation (QuIC)

Master en Ingénieur Civil Physicien

Processes with indefinite causal structure in quantum theory

by Timothée HOFFREUMON

A quantum theory compatible with the general theory of relativity is expected to have a global causal structure that is no longer compatible with a pre-defined causal order. The process formalism of Oreshkov, Costa, and Brukner offers such a formulation of the theory that is describing quantum mechanics without assuming an *a priori* fixed global causal structure. It studies the physical processes as locally abiding by the laws of quantum theory, but not necessarily embedded in a well-defined global causal structure. The causal relations between the parties are then described by the most general object compatible with locally valid quantum mechanics, called process matrix (PM). In this thesis an extension of this formalism, in which the parties linked together by the process matrix are allowed to perform more than one operation during the process, is explored. The new object that keep track of the most general causal structure possible with this requirement, named *multi-round process matrix* (MPM), is defined. A mathematical characterisation of the MPM is then provided.
This thesis first offers a review of the theoretical background needed to understand the MPM. Notably, the concept of an *operational theory*, *process* and *process matrix*, *quantum comb* and *quantum network*, as well as a condensed linear algebra and quantum theory reminders, are presented.
Then to achieve the characterisation, the concept of positive, normalised and projective conditions of validity for process matrix is used and expanded. Furthermore the properties of the projector to the space of maximally mixed state, which is at the heart of these conditions, are assessed and rigorously proved because they are shown to play a capital role in the derivation of valid process matrix, and as it turn out, in the derivation of valid quantum comb as well. An equivalent to these conditions are indeed also found for the quantum combs. Taken together this allows us to show that the MPM is a mathematical object whose domain of definition lies in between the quantum comb and the process matrix. Finally the definition of causal non-separability, *i.e.* the character of a process to be impossible to be explained without dropping the assumption of fixed global causal structure, is modified from its definition for the PM case to the case of multi-round process matrix because we show that there is a possible new kind of activation of non-separability that can be made through the memory of the parties in between each round.

Keywords : Process Matrix, indefinite causal structure, causal separability, operational quantum theory, Quantum Combs, LOCC paradigm

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Résumé

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par Timothée HOFFREUMON

Il est attendu qu'une théorie quantique compatible avec la théorie de la relativité générale doit avoir une structure causale qui n'est plus compatible avec un ordre causal prédéfini. Le formalisme procédé d'Oreshkov, Costa, et Brukner offre une telle formulation de la théorie qui décrit la mécanique quantique sans supposer de structure causal globale fixée *a priori*. Elle étudie les processus physiques en respectant localement les lois de la théorie quantique, mais pas nécessairement dans une structure causale globale bien définie. Les relations de causalité entre les parties sont ensuite décrites par l'objet compatible avec la mécanique quantique localement valide le plus général qu'il soit, appelée *Process Matrix* (PM). Dans cette thèse, une extension de ce formalisme, dans lequel les parties liées par la process matrix sont autorisées à effectuer plus d'une opération au cours du processus, est explorée. Le nouvel objet qui garde la trace de la structure causale la plus générale possible avec cette exigence, nommé *Multi-round Process Matrix* (MPM), est défini. Une caractérisation mathématique du MPM est ensuite fournie.
Cette thèse propose d'abord un passage en revue des bases théoriques nécessaires à la compréhension de la MPM. Notamment, le concept de textit théorie quantique opérationnelle, *processus* et *Process Matrix*, textit Quantum Comb et *Quantum Network*, ainsi qu'un rappel condensé d'algèbre linéaire et de théorie quantique, sont présentés.
Ensuite, pour réaliser la caractérisation, le concept de conditions de validité positives, normalisées et projectives pour la *Process Matrix* est utilisé et étendu. En outre, les propriétés du projecteur dans l'espace des états maximalement mixés, qui est au cœur de ces conditions, sont évaluées et rigoureusement prouvées, car il est prouvé qu'elles jouent un rôle capital dans l'élaboration de la *Process Matrix* valide, et ce rôle est aussi constaté dans la dérivation de la validité des *Quantum Comb*. Un équivalent à ces conditions est en effet également trouvé pour les peignes quantiques. Pris ensemble, cela nous permet de montrer que le MPM est un objet mathématique dont le domaine de définition se situe entre le *Quantum Comb* et la matrice de processus. Enfin, la définition de la non-séparabilité causale, c'est à dire le caractère d'un processus à être impossible à expliquer sans abandonner l'hypothèse de structure causale globale fixée, est modifiée de sa définition pour le cas PM au cas du MPM car nous montrons qu'il est possible d'observer un nouveau type d'activation de la non-séparabilité grâce à la mémoire des parties entre chaque round.

Mots-clef : Process Matrix, structure causale indéfinie, séprabilité causale, théorie quantique opérationelle, Quantum Comb, paradigme LOCC

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Written using LaTeX. The thesis template was made using Steve Gunn[4] and Sunil Patel[5] template made available by Vel on Overleaf[6].

[2] As long as he was not sleeping over the answer.
[3] Especially about his love for a particular french violinist.
[4] `http://users.ecs.soton.ac.uk/srg/softwaretools/document/templates/`
[5] `http://www.sunilpatel.co.uk/thesis-template/`
[6] `https://www.latextemplates.com/template/masters-doctoral-thesis`

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List of Abbreviations

CJ	**C**hoi-**J**amiołkowski
CP	**C**ompletely **P**ositive
CPTP	**C**ompletely **P**ositive **T**race **P**reserving
CTC	**C**losed **T**imelike **C**urve
GGB	**G**eneralised **G**ell-mann **B**asis
LOCC	**L**ocal **O**perations and **C**lassical **C**ommunication
MPM	**M**ulti-Round **P**rocess **M**atrix
OPT	**O**perational **P**robabilistic **T**heory
OCB	**O**reshkov, **C**osta, and **B**rukner (see [1])
ONB	**O**rtho**N**ormal **B**asis
PD	**P**ositive **D**efinite
PM	**P**rocess **M**atrix
POVM	**P**ositive **O**perator-**V**alued **M**easure
PSD	**P**ositive **S**emi-**D**efinite

Chapter 1

Introduction

From Newtonian mechanics to quantum field theory in curved spacetime, the causal structure is always assumed to be fixed. However, general relativity teaches us that this causal structure can become dynamical, while quantum theory tells us that dynamical variables are subject to quantum indefiniteness. Theses two ideas together hint that a theory unifying quantum physics with general relativity would feature causal structure that present quantum uncertainty, and that consequently there can be *indefinite* causal structures in the theory [2–4]. To work with this idea, several frameworks for quantum theory in which, contrary to standard formulation, the fixed background spacetime used to define the causal structure is not presumed *a priori* were proposed in the recent years [1, 5–11]. Among which the process framework of Oreshkov, Costa, and Brukner (OCB) [1], which will be the framework used for this thesis, was mathematically shown to allow for such new indefinite processes in which two operations are neither causally ordered nor in a probabilistic mixture of different but definite causal orders.

The process framework is considering the most general causal structure that can connect together several local laboratories inside which the standard formulation of quantum mechanics is supposed valid. The only assumption for the causal structure represented by the process outside the laboratories is that it cannot create logical paradoxes such as a causal loop allowing agents to interact with their own past. This framework uses a mathematical object called the *process matrix* (PM) to encode the causal structure between these local laboratories. When the process matrix is accounting for correlations that cannot be understand with a causal (possibly dynamical though) scenario, we say that it is *causally non-separable*. Experimental implementation of this causal non-separability have already been observed. There is already a huge amount of possible applications for the theoretical aspects as well as for the new capabilities in information processing and quantum computing. We are presenting these possibilities in greater details in the chapter introducing the formalism.

1.1 Scope of the work

The PM formalism is thus a relatively new framework whose features and possible applications will be explored in this work. In particular, the possibility of certain process matrix to be the representation of a realistic communication protocol led us to consider a new extension of the theory. The basic question that we wish to address is 'what happens when we extend the formalism by allowing the parties to have multiple rounds of exchanging input and output systems ?' as in usual communication protocols were the agents are not restricted to sending only one message. The basic object of this extension will be called the *multi-round process matrix* (MPM) and is basically encoding how the multiple rounds of communication of each party are distributed to the others.

In this thesis we show that with some mathematical reformulation of the validity conditions for the process matrix, one can obtain a very nice way to treat the MPM as a generalisation of PM and that it can be obtained through a similar procedure. This reformulation will require to investigate the properties of a specific linear application that we will refer to as *depolarising superoperator*, first introduced without name in [12] and whose characteristics are thoroughly studied in this work. With this procedure we will provide a full mathematical characterisation of the MPM. We then look into the new correlations made possible within the framework of MPM. Our conclusion is that there is nothing new compared to process matrix formalism than a new way of *activating* causal non-separability. This last notion will be explained in the main text and as a consequence will lead us to a new definition of *causal non-separability* for the MPM (*i.e.* how one designates an MPM that does not always lead to correlations that admit a causal explanation).

1.2 Organisation of the Thesis

This thesis is split into two parts. In the first part, we will review the theoretical prerequisites needed to understand this work in its full extent. In a first chapter, chapter 2, the basic notions in quantum theory will be addressed. In particular it will be shown how to shift from the usual formulation of quantum mechanics to its expression in an operational framework and how the mathematics follow accordingly. Notions in causality will have to be reviewed for the definitions to be unambiguous. The second chapter in the theoretical reminders will present the two frameworks used in the thesis. The third chapter is about the process matrix formalism alongside another formalism called *quantum network* itself relying on an object called the *quantum comb*.

The second part of the thesis is the results part. Firstly, the research motivations will be explained (chapter 4). The depolarising superoperator will be introduced in chapter 5, we will demonstrate some of its characteristics and use them to reformulate the validity conditions of both quantum comb and process matrix. In chapter 6, the reformulated validity conditions will be used to characterise the multi-round process matrix. We will end this chapter by looking into the new correlations that can be obtained with the MPM formalism. Finally in chapter 7, we end the thesis with a discussion about the findings and the possible paths of further research, as well as a conclusion.

Part I

Theoretical Background

Chapter 2

Fundamental Notions

In this section we will review the basics concepts that will be used through the thesis. In a first section, the field of research in which this thesis fits will be defined, namely quantum theory. We will bring basic presentation of the main topics which are of concern : quantum mechanics, information and computation. To close this section we will define the kind of theory in which the framework of quantum mechanics will be expressed : an operational probabilistic one. In the next section, basics of quantum mechanics will be reviewed. The notions of state and transformation will be presented, followed by the different ways to mathematically transcript them. As quantum mechanics is an inherently linear theory, the development will be accompanied by reminders in linear algebra. The goal is to see how quantum theory is formulated in the language of an operational probabilistic theories (OPT), and to this end the notion of higher-order map and Choi-Jamiołkowski isomorphism will have to be covered. In the last section we will lay on solid grounds the notion of causality, and mathematically characterise it in the context of OPT, in order to be able to unequivocally describe processes with indefinite causal structure (this latter notion will be covered in chapter 3).

2.1 Quantum Theory

The framework in which will be expressed the used theories is called *quantum theory*. It is a set of rules and axioms that determine the constraints on what is possible or not. In its conventional formulation, it relies on the accepted idea that the whole mathematical structure stems from the theory of Hilbert space and Hilbert-Schmidt operators [13–15].

Quantum Mechanics is a physical theory that relies on quantum theory to explain transformations, or evolution when the concept of time is available, between the states. It is well-suited to explain phenomena happening at atomic and subatomic scale, where classical mechanics is no longer precise enough to describe nature. We will have the occasion to explore this theory in more details below.

Quantum computation and quantum information is the study of the information processing tasks that can be accomplished using quantum mechanical systems [16, 17]. Quantum information plays a particular role because originally it was a computer science theory that was translated in the context of quantum computation to help improve the computational capabilities, but some parallel discoveries in a more fundamental picture, like Laundauer principle or Bekenstein bound have led to the conclusion that *information is physical* [18]. From a tool, the study of how information behaves inside a quantum system becomes a fundamental principle. A recent trend in quantum theory, initiated in the early 2000's by Hardy [19], Fuchs [20, 21], Caves [22], and Brassard [23], aims to place information theory as a fundament for quantum theory (and every physical theory actually). This is motivated by the fact that such a formulation of quantum theory relies on axioms coming from intuitive concepts in information theory -like continuity or causality- [19, 24, 25] from which the abstract Hilbert space structure follows as a consequence, rather than the usual formulation that postulates Hilbert space as an axiom [13].

There exist more than one framework to express quantum theory. The one used in this text is an *operational probabilistic theory* (OPT). These two aspects -operational and probabilistic- have been motivated by the characteristic one could expect of a theory that can conciliate (general) relativity and quantum mechanics [2, 3].

The theory is said to be *operational* because it uses this new perspective that prefer to focus on the operations a system undergo during an experiment rather than on abstract quantities such as speed or momentum. Therefore, the physical quantities of interest are the instrument setting and the outcome of measurement [1, 2, 26]. One then thinks of a physical system as a black box which is probed by an input and responds with an output[1].

The other notion is that the theory is fundamentally probabilistic as it is known since the early days of the quantum theory [13], and which is related to the Kochen-Specker theorem [27] which rules out the possibility of predicting experimental outcome with certainty. This probabilistic characteristic also imply that there is a notion of *free randomness* [28]. It is itself an underlying consequence of the theorem. By that we mean that there are outcome that cannot be known with certainty before a measurement is performed in an experiment.

2.2 Quantum Mechanics and Linear Algebra

For this review of the quantum mechanics needed we will assume that the reader already have an undergraduate background in the subject and know how the Dirac *bra-ket* formulation of quantum mechanics works. If it is not the case, the reader is invited to read appendix A.1 before this section.

Here, we will review how to represent states and transformation mathematically. Several formulations will be explored[2] as it is needed to illustrate how one shifts from the paradigm of usual quantum theory to the operational probabilistic theory. An assumption that will be made through the whole thesis is that we restrict ourselves to finite-dimensional systems. This is a mathematical convenience which is very often done in the fields of quantum computation and information [15, 25, 29, 30]. The infinite dimensional generalisation being of interest for the continuous limit and the shift to classical mechanics. Note that not everything that can be proven in finite dimensional case is automatically valid in infinite dimensional extension. Mathematically, this assumption translates to

We consider only quantum systems which can be described with the aid of a finite-dimensional Hilbert space $\mathcal{H}$ of dimension $d \in \mathbb{N}_0$.

2.2.1 Von Neumann picture

The Dirac picture, as elegant as it is, is unfortunately not suited to describe every element of physical reality, and suffer from an absence of interpretation for what the amplitude *is*[3]. Suppose you don't know your state with certainty, for example your state models electrons emitted from radioactive decay but there is a 50/50 chance that it have +1 or -1

[1] Concretely, we describe all *physical quantities*, *i.e.* outputs, as function of inputs. This will be presented in the next section

[2] Among which the Dirac picture that was left as an appendix.

[3] Moreover there is an awkward extra degree of freedom in the theory : every ket are defined up to a global phase which can be arbitrarily chosen.

spin state[4]. Here the state is not in a coherent superposition of spin + and spin -, it is either one or the other. We talk about a *statistical mixture* (opposed to a *pure state*). The trouble is that there is no way of representing it with a ket, since kets are only well suited to describe *pure* states *i.e.* states that are known with certainty (although possibly entangled or superposed which will lead to randomised outcomes when measured) [16, 18].

To overcome this difficulty and provide a more general description, the density operator formalism was developed [13, 25]. The highlight of this formulation is that since observables are a particular set of linear operators on the Hilbert space, the states lies on the dual of this set. So for an observable A, its expectation value, given a state ρ is then simply given by the inner product of the two

$$\langle A \rangle \equiv (\rho \,| A\,) \tag{2.1}$$

the notation is in parenthesis to emphasise the fact that the inner product is no longer over $\mathcal{H}$ but $\mathcal{L}(\mathcal{H})$. We will now provide a short introduction to the mathematics of such a space.

The space of linear operators on a Hilbert space is also a complex vector space that possesses an inner product, so it is itself a Hilbert space[5]. To differentiate $\mathcal{L}(\mathcal{H})$ from $\mathcal{H}$, we will follow the convention of [29] and call the former a *Liouville* space, or sometimes a Hilbert-Schmidt space. Its elements are called *Hilbert-Schmidt operators*[6] [14]. A vector in that Liouville space can be decomposed using a basis of d^2 dyads on $\mathcal{H}$, hence the dyadic decomposition of an operator (A.16) is now the decomposition into basis elements

$$\begin{aligned}
\hat{A} &= \sum_{i=1}^{d}\sum_{j=1}^{d} A_{ij} \,|i\rangle\langle j| \\
\hat{A} &\rightarrow |A) \\
\sum_{i=1}^{d}\sum_{j=1}^{d} A_{ij} &\rightarrow \sum_{k=1}^{d^2} A_k \\
\sum_{i=1}^{d}\sum_{j=1}^{d} |i\rangle\langle j| &\rightarrow \sum_{k=1}^{d^2} |k) \\
|A) &= \sum_{k=1}^{d^2} A_k \,|k)
\end{aligned} \tag{2.2}$$

Let $|A)$ and $|B)$ elements of the Liouville space $\mathcal{L}(\mathcal{H})$, with matrix representation A and B, then the adjoint operation is also represented by a Hermitian conjugation

$$|A)^{\dagger} = (A| \tag{2.3}$$

The inner product of this space is called the *Hilbert-Schmidt* inner product, and represented by a trace in matrix formulation.

$$(A\,|B\,) = \mathrm{Tr}\left\{A^{\dagger}B\right\} \tag{2.4}$$

[4]By this we mean $\sigma_z = \pm\hbar/2$, but again for the sake of conciseness all the constants that are not relevant for the presentation will be set to one without further notice.

[5]In the finite dimensional case only, the full story is of course way more complicated than that, and rigorously we should be talking about Banach's C*-algebra defined on a Hilbert space representation, see [31, 32].

[6]Which imply that the operators are bounded, again this is mathematical subtlety left for the reference books.

which induces the *Hilbert-Schmidt norm*,

$$\||A)\| = \sqrt{(A\,|A\,)} = \sqrt{\mathrm{Tr}\{A^\dagger A\}} \tag{2.5}$$

It is also possible to define an outer product on the space which will be a linear mapping from the Liouville to itself, so a linear operator:

$$|A\,)(\,B| \in \mathcal{L}\left((\mathcal{L}\,(\mathcal{H})) \rightarrow (\mathcal{L}\,(\mathcal{H}))\right) \tag{2.6}$$

these operators will be called *superoperators* (again following [29]) to avoid confusion with the ones defined on $\mathcal{H}$; $\mathcal{L}\left((\mathcal{L}\,(\mathcal{H})) \rightarrow (\mathcal{L}\,(\mathcal{H}))\right)$ designates the set of linear map from $\mathcal{L}\,(\mathcal{H})$ to $\mathcal{L}\,(\mathcal{H})$. Superoperators are thus a particular kind of linear mapping from one space to itself. They will also be referred to as *maps* in general, when the precision on their operator characteristic is not needed. Maps in general will be referred to with calligraphic letters, with the exclusion of $\mathcal{A}, \mathcal{H}, \mathcal{L}$ and $\mathcal{P}$ which are used for other purpose in the text.

This sets the stage, now for the play we require that the set of observables is still given by hermitian elements A taken from an *algebra*[7] $\mathcal{A}$. This algebra is represented by *bounded* linear operators, so the sets that we have been using like $\mathcal{L}\,(\mathcal{H})$ will all be considered bounded from now on. The states are, by essence, linear function that maps the observable into a real number -its expectation value-, which imply that states are elements of the dual $\mathcal{A}^*$ of the algebra $\mathcal{A}$. This, together with the probabilistic mixtures argument presented below, motivates the von Neumann picture that models states with a special kind of operator called *density operator* (also referred to as *density matrix*, for their representation as a matrix), noted ρ, to represent states as a probabilistic mixture of projectors onto pure states

$$\text{state} \equiv \rho := \sum_i p_i\, |\psi_i\rangle\langle\psi_i| \tag{2.7}$$

where by probabilistic mixture it is implied that the convex coefficients respect $p_i > 0\,\forall i$ and $\sum_i p_i = 1$. It is important to notice that density operators defined like that form a *convex set*[8] and moreover they are hermitian.

So **states are density operators**, and because of the conditions of normalisation of pure states (A.8), one can show that density matrices obey the conditions

$$\mathrm{Tr}\{\rho\} = 1 \tag{2.8a}$$

$$\rho \geq 0 \tag{2.8b}$$

the expectation value of A given some ρ is then given as the inner product (2.1), $(\mathcal{A}, \mathcal{A}^*) \rightarrow \mathbb{C}$, represented by the Hilbert-Schmidt inner product (2.4) [31]

$$\langle A\rangle = \mathrm{Tr}\{\rho A\} \tag{2.9}$$

In general this expectation value will decompose as $\sum_\alpha a_\alpha p_\alpha$, where a_α are the eigenvalues of the observable and p_α the associated probability of measurement. One can link it to POVM : the probabilities p_α are calculated by assigning a positive operator $0 \leq E_\alpha \leq \mathbb{1}$ to each outcome so that

$$p_\alpha = \mathrm{Tr}\{\rho E_\alpha\} \tag{2.10}$$

[7] An *algebra* is a set which is closed under multiplication and addition as well as under multiplication with scalars [31], intuitively this is the requirement that if you are able to plug several measurement data points into a calculator and do operations with them, nature can do it as well.

[8] That is a set in which every mixture of elements in the set are also in the set [33].

,where E_α are *effect* operators. As probabilities sum to one, the effects form a POVM (A.26) : $\sum_\alpha E_\alpha = \mathbb{1}$. The link between measurement and an observable is now clearer : the spectral decomposition of an observable A gives

$$A = \sum_\alpha a_\alpha E_\alpha \tag{2.11}$$

where the a_α, its eigenvalues, are the outcomes and $\{E_\alpha\}$ is a set of POVM that are orthogonal so one can show that actually they are related to the von Neumann measurement operators (A.23) P_α : $E_\alpha = P_\alpha^\dagger P_\alpha$, for example if ρ was a pure state $\rho = |\psi\rangle\langle\psi|$, then

$$\mathrm{Tr}\{\rho E_\alpha\} = \mathrm{Tr}\Big\{|\psi\rangle\langle\psi|\, P_\alpha^\dagger P_\alpha\Big\} = \mathrm{Tr}\Big\{P_\alpha\, |\psi\rangle\langle\psi|\, P_\alpha^\dagger\Big\} = P(\alpha)$$

Any other decomposition of the operator than the spectral one, so any decomposition in which the effects E_α are not orthogonal to each other but sum up to unit operator, will form a particular POVM (with an infinity of representation for each, depending on the choice of basis) [31].

In this picture, the linear maps like those defined in (2.6) are the *quantum operation*. In the Schrodinger picture, which is the point of view taken by this text, transformations are applied to the states so a general quantum operation have the form

$$\text{transformation} \equiv \rho' = \mathcal{E}(\rho) \tag{2.12}$$

with ρ' the output system and ρ the input. Projective measurement, as we just have seen, is a special case of transformation, which lead to the updated system, given an outcome m, (see the discution below equation (A.23))

$$\mathcal{E}_m(\rho) = P_m \rho P_m^\dagger \tag{2.13}$$

Closed dynamics is also a particular kind of evolution, as one can infer from (A.10) : let U be a unitary operator, then the output state resulting of the application of U to some initial state ρ is given by

$$\mathcal{E}(\rho) = U\rho U^\dagger \tag{2.14}$$

Actually, the dynamics that can be represented using the notion of quantum operation is much more richer than just unitary evolution and measurement allowed in Dirac's braket formulation. Maps don't need to be operators, they can map to bigger Hilbert spaces (*isometries*, and preparations) as well to smaller Hilbert space (measurement essentially), this higher versatility makes it possible to consider open dynamics as well. Ultimately, we will see that this picture allows one to treat everything in a same unified manner through *quantum channel* formalism.

Finally, von Neumann's picture have a notion of composition of subsystems that is also based on tensor product: let A and B be two parties with associated Hilbert spaces $\mathcal{H}^A$ and $\mathcal{H}^B$ that composes the whole system. The global system is thus defined on a Hilbert space of the form $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$. The state of the system is given by a density matrix $\rho^{AB} \in \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$. If there is no entanglement, that is we can describe the states of the system of A and of B separately, then the state is in a product state

$$\rho^{AB} = \rho^A \otimes \rho^B \tag{2.15}$$

where the tensor product is still the Kronecker product when the operators are represented by matrices and ρ^A, ρ^B are *reduced* density operators, accessible through the partial trace operation

$$\rho^A = \mathrm{Tr}_B\left[\rho^{AB}\right] \qquad \rho^B = \mathrm{Tr}_A\left[\rho^{AB}\right] \tag{2.16}$$

which consists of taking the trace over a single subsystem like, in braket notation :

$$\begin{aligned}\mathrm{Tr}_B\left[\rho^{AB}\right] &= \mathrm{Tr}_B\left[\left(\sum_{i,j}\rho^A_{i,j}\left|i\right\rangle\!\left\langle j\right|\right)^A \otimes \left(\sum_{k,l}\rho^B_{k,l}\left|k\right\rangle\!\left\langle l\right|\right)^B\right] \\ &= \left(\sum_{i,j}\rho^A_{i,j}\left|i\right\rangle\!\left\langle j\right|\right)^A \times \left(\sum_{p}\sum_{k,l}\rho^B_{k,l}\left\langle p|k\right\rangle\left\langle k|l\right\rangle\right)^B\end{aligned}$$

where the dyadic decomposition in arbitrary basis for both subsystems have been made, ρ^{AB} was taken as general as possible in this particular example, and the equation can be simplified through the orthonormality conditions of the basis $\langle i|j\rangle = \delta_{i,j}$. On can see that the rightmost sum (the one over p) when simplified, will give 1 as a result because ρ^B is itself a valid density operator and must have unit trace (2.8a) and so we're left with dyadic decomposition of ρ^A, as claimed in (2.16). Physically, taking the partial trace over a subsystem consists on doing a destructive measurement on it, without selecting a particular outcome so basically it's throwing away this subsystem. Mathematically, it correspond to mapping the subsystem to the trivial 1 dimensional space composed of the number 1 alone, so trivially factoring it out. This will be covered in more details in the next section.

To sum up, in the von Neumann picture, called *density matrix representation*, the states form a particular $\mathcal{A}^*$ algebra on the space of linear operators on the Hilbert space, whose elements are called *density operators*. This algebra is the dual to the algebra of observable $\mathcal{A}$. Both are hermitian and admit a hermitian matrix representation. Transformations are maps from a valid density operator to another. And measurements are a particular mapping -the Hilbert-Schmidt inner product- that links an observable and a density matrix to a set of real number: the associated probabilities of each measurement outcomes.

2.2.2 Circuit formalism extended : quantum channels and instruments

With simple axiomatic considerations, based on physical ground, one can derive constraints on the most general transformation allowed by the mapping of one state to another in the density matrices representation (2.12) [16]. Such constrained mapping will be referred to as *quantum physical evolution*, or evolution in short and the mapping will then be called a *quantum channel*. An important assumption to clarify at the outset is that we are viewing a quantum physical evolution as a "black box", meaning that a party can prepare any state that she wishes before the evolution begins, including pure states or mixed states. Critically, we even allow her to input one share of an entangled state [34].

Definition 1. *A* ***quantum operation*** *is represented by a map from one set of density operator on* $\mathcal{L}\left(\mathcal{H}^A\right)$ *to an other set of density operator* $\mathcal{L}\left(\mathcal{H}^B\right) : \mathcal{E} \in \mathcal{L}\left(\left(\mathcal{L}\left(\mathcal{H}^A\right)\right) \to \left(\mathcal{L}\left(\mathcal{H}^B\right)\right)\right)$ *that obey the three following constraints :*
1) Convex linearity : for any probabilistic ensemble of valid density operators $\sum_i p_i\rho_i$, $\sum_i p_i = 1$, $p_i \geq 0 \forall i$,

$$\mathcal{E}\left(\sum_i p_i\rho_i\right) = \sum_i p_i\mathcal{E}\left(\rho_i\right) \tag{2.17}$$

2) Complete Positivity (CP) : the map is positive, that is it outputs a semi-positive definite operator for all input PSD operators

$$\mathcal{E}(\rho) \geq 0 \quad , \forall\rho \geq 0 \tag{2.18}$$

FIGURE 2.1: Representation of a quantum channel, with closed dynamics to the left and open to the right. Figure from [16].

and it is completely positive meaning that if the map acts on a subsystem of a bigger PSD operator, then the output is also PSD, let there be $\rho^{AR} \in \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^R\right)$*, and* $\rho^{BR} \in \mathcal{L}\left(\mathcal{H}^B \otimes \mathcal{H}^R\right)$*, then*

$$\rho^{BR} = \left(\mathcal{E}^A \otimes \mathcal{I}^R\right)\left(\rho^{AR}\right) \geq 0 \quad ,\forall \rho^{AR} \geq 0 \tag{2.19}$$

where $\mathcal{I}^R$ *is the identity mapping on the space* $\mathcal{L}\left(\mathcal{H}^R\right)$*.*
3) Trace-preserving (TP) : the map preserves the trace

$$\mathrm{Tr}\{\mathcal{E}(\rho)\} = \mathrm{Tr}\{\rho\} \quad ,\forall \rho \tag{2.20}$$

*Linear CPTP maps are then called **quantum channels**.*

These maps, in a circuital sense, keep track of the flow of information along a wire and how it is exchanged with the other wires during the mappings. The interpretation of linear CP character is that because this information have a probabilistic nature, it is necessary for the maps to be positive, completely positive and linear for the proper normalisation of the associated probabilities to hold through the maps. And because information must be conserved at all time, the maps must be trace-preserving (TP), since the probabilities are accessed through the trace operation [30].

A CPTP map can then be conveniently represented in matrix formulation by the *Kraus*, or *Operator-Sum*, representation [34, 35] based on the following theorem, called *Choi-Kraus theorem* [35]

Theorem 1. *(Choi-Kraus) A map* $\mathcal{E}$ *in* $\mathcal{L}\left(\left(\mathcal{L}\left(\mathcal{H}^A\right)\right) \rightarrow \left(\mathcal{L}\left(\mathcal{H}^B\right)\right)\right)$ *is linear CP if, and only if, it has a Choi-Kraus representation as follows*

$$\mathcal{E}(\rho) = \sum_{i=1}^{d} E_i \rho E_i^\dagger \tag{2.21}$$

where $1 \leq d \leq dim(\mathcal{H}^A) dim(\mathcal{H}^B)$ *and* $E_i \in \mathcal{L}\left(\left(\mathcal{L}\left(\mathcal{H}^A\right)\right) \rightarrow \left(\mathcal{L}\left(\mathcal{H}^B\right)\right)\right)$*. And the map is CPTP if and only if the Kraus operator* E_i *of the decomposition verify the condition*

$$\sum_{i=1}^{d} E_i^\dagger E_i = \mathbb{1}^A \tag{2.22}$$

When the dimension is preserved and there is only one Kraus operator, the channel is called *unitary channel*, as it implements an unitary evolution (2.14)

$$\mathcal{U}(\rho) = U\rho U^\dagger \qquad , \tag{2.23}$$

when the output dimension is bigger than the input (dilatation), but there is still only one Kraus operator, we say that the channel is *isometric*, and call it *an isometry*

$$\mathcal{V}(\rho) = V\rho V^\dagger \tag{2.24}$$

where $\mathcal{V} \in \mathcal{L}\left(\left(\mathcal{L}\left(\mathcal{H}^A\right)\right) \to \left(\mathcal{L}\left(\mathcal{H}^B\right)\right)\right)$ with $\dim(\mathcal{H}^A) \leq \dim(\mathcal{H}^B)$, the special case where the dimension match being a unitary channel.

The concept of CPTP maps have been introduced, but there is still an extra layer of abstraction to add. Indeed, the dynamics of CPTP maps are well-suited to represent any deterministic operation one party could apply on the system, but this cannot render the cases where the operations yield different possible outcomes at random.

As an example consider the following scenario : there is one party, Alice, that is preparing a 0 qubit out of nothing. Per example she is using a single photon source and a z-polariser to generate this qubit as the polarisation state of the single photon. Her operation will be represented as follow : her input space will be the trivial unit operator on one dimensional Hilbert space[9] ; we wish her output space to be a 2 dimensional linear operator that is represented in Alice's z basis as the $|0\rangle$ state, hence the $|0\rangle\langle 0|$ dyadic operator. To do so, Alice's operation will be represented by a CPTP map that is an isometry. But what if Alice was not always preparing a 0 ? What if she's flipping a coin and preparing a 0 or a 1 depending on if she got head or tails ? Then the result of the coin flipping is her setting and her operations are now a probabilistic mixture of 50% her preparing a 0 and 50% a 1. The map is no longer a CPTP map but two CP trace-non-increasing maps, in this particular case each map will decrease the trace from a value of 1 to 0.5, since the trace is the probability of the event happening. The 2 maps taken together must lead back to a map that is CPTP as the initial information is only shared between the 2 possibilities. We call this ensemble of maps a *quantum instrument* [36].

Definition 2. *A **quantum instrument** is a collection of n CP maps $\{\mathcal{M}_i\}_{i=1}^n$ such that their sum $\mathcal{M}$*

$$\mathcal{M} = \sum_{i=1}^{n} \mathcal{M}_i \tag{2.25}$$

is a CPTP map.

In a sense, a quantum instrument is to a quantum channel what a POVM is to a deterministic measurement[10]. Remember that we introduced this concept in order to have the most general way of representing the operations of one party. Note that the CP maps constituting a quantum instrument can themselves be dilated into a bigger space so their Kraus operators reduce to unique matrices, this is *Stinespring's dilatation theorem* [15, 30].

Theorem 2. *(Stinespring) Let $\mathcal{M}$ be a linear CP map on $\mathcal{L}\left(\left(\mathcal{L}\left(\mathcal{H}^A\right)\right) \to \left(\mathcal{L}\left(\mathcal{H}^B\right)\right)\right)$, there exists a **Stinespring dilatation** $V : \mathcal{H}^A \to \mathcal{H}^B \otimes \mathcal{H}^E$ such that the inverse map $\mathcal{M}^*$ can be expressed as*

$$\mathcal{M}^*(\sigma^B) = V^\dagger \left(\sigma^B \otimes \mathbb{1}^E\right) V \tag{2.26}$$

*for all $\sigma^B \in \mathcal{L}\left(\mathcal{H}^B\right)$. If the map is TP, then V is an isometry : $V^\dagger V = \mathbb{1}$, (2.24) and the Stinespring dilatation is called **isometric extension**. A corrolary to this theorem is that the CP map can be*

[9] This is a mathematical representation of the classical (d=1) nothing in order to conserve probability since Tr(1) = 1, this is done here just to be strictly mathematically consistent, in the rest of the text we will just say 'trivial input' as the dimension and the form of this kind of input don't really matter in the formalism.

[10] *Deterministic measurement* is a special case of von Neumann measurement in which the output system is discarded and only remains the outcome [17].

represented as

$$\sigma^B = \mathcal{M}\left(\rho^A\right) = Tr_E\left[V\rho^A V^\dagger\right] \tag{2.27}$$

for $\rho^A \in \mathcal{L}\left(\mathcal{H}^A\right)$.

This is a way of mathematically transform your ensemble into something pure on a higher Hilbert space [31, 34]. To carry on with the analogy, Stinspring dilatation of linear CP maps as higher-dimensionnal CP maps with only one Kraus operator is in the same spirit as Neumark dilatation of POVM into von Neumann measurement : one raises the dimension of the space to retrieve pure state behaviour.

2.2.3 The Choi-Jamiłokowski picture

But how do we reach the operational formulation ? *I.e.* where the whole *process*, or *quantum network*, is itself represented as a big black box in which individual parties act on their own part of the black box, which will depend on the parties' settings and give party's outcomes in return. Again with a map. Per example if the experiment involve 2 parties, Alice and Bob, then it is represented as a whole as an again linear map but a map that takes in the quantum instruments of Alice and Bob and outputs a probability of happening. This map is a mapping of CP maps to a probability. which is linear in both A and B's input and output spaces so it is *bi-linear*. We call it a *bilinear supermap*, echoing to the superoperator name. These supermaps will be the central tool for the representation of the formalisms that will be presented in the next chapter, so in this section we will outline an isomorphism that is very convenient for handling them.

The trouble now is that this kind of representation, for N parties rely on (bilinear) supermaps, *i.e.* linear maps of N maps between input and output space, which are not easy to manipulate objects, and that they cannot be represented by Kraus operators. But as we claimed in the beginning of this chapter, every mathematical object we are dealing with can be reduced to matrices, which are easier to work with.

The mathematical trick we use to do so is called the *Choi-Jamiołkowski isomporphism* [37, 38], that basically states that every bilinear supermap is isomorph to a linear operator in a higher dimensional Hilbert space, attentive reader would have guessed that the Choi-Kraus theorem 1 is a particular case of this isomorphism. And actually purification, Naimark and Stinesping dilatation are all consequences of this isomorphism.

Definition 3. *For two Hilbert spaces* $\mathcal{H}^X, \mathcal{H}^Y$*, let* $\mathcal{M}^X$ *be a linear mapping from the space of linear operator on the first space* $\mathcal{L}\left(\mathcal{H}^X\right)$ *to the second* $\mathcal{L}\left(\mathcal{H}^Y\right)$*, i.e.* $\mathcal{M}^X \in \mathcal{L}\left(\left(\mathcal{H}^X\right) \to \left(\mathcal{H}^Y\right)\right)$*, and let* M^{XY} *be an operator on the space of linear operators on the tensor product of the two spaces* $M \in \mathcal{L}\left(\mathcal{H}^Y \otimes \mathcal{H}^X\right)$ *(the superscript are used to emphasise which spaces the objects are related to). Then, the bijective correspondence between the two* $\mathfrak{C} : \mathcal{M}^X \to M^{XY}$ *defined through*

$$M^{XY} = \mathfrak{C}\left(\mathcal{M}^X\right) := \left[\mathcal{I}^X \otimes \mathcal{M}^X\left(|I_{\mathcal{H}^X}\rangle\rangle\langle\langle I_{\mathcal{H}^X}|\right)\right]^T \tag{2.28}$$

where $\mathcal{I}_{\mathcal{L}(\mathcal{H}^X)}$ *is the identity map on* $\mathcal{L}\left(\mathcal{H}^X\right)$, $|I_{\mathcal{H}^X}\rangle\rangle \in \mathcal{H}^X \otimes \mathcal{H}^X$ *is the maximally entangled vector defined as* $|I_{\mathcal{H}^X}\rangle\rangle = \sum_{i=1}^{d_X} |i\rangle|i\rangle$*, with* $\{|i\rangle\}$ *a basis of* $\mathcal{H}^X$ *and* d_X *the dimension of this space. Such correspondence is called the* ***Choi-Jamiołkowski isomorphism*** *[37, 38]. The inverse map,* $\mathfrak{C}^{-1} : M^{XY} \to \mathcal{M}^X$ *transforms the superoperator* M^{XY} *into the map* $\mathcal{M}^X$ *that acts on any operator* $\rho^X \in \mathcal{L}\left(\mathcal{H}^X\right)$ *as follows*

$$\mathcal{M}^X\left(\rho^X\right) = \mathfrak{C}^{-1}\left\{M^{XY}\right\}\left(\rho^X\right) := \left[Tr_X\left[\left(\rho^X \otimes \mathbb{1}^Y\right)\cdot M^{XY}\right]\right]^T \tag{2.29}$$

with $\mathrm{Tr}_X[\cdot]$ *denoting the partial trace over the subsystem X i.e.* $\mathrm{Tr}_X\left[M^{XY}\right] := \sum_{i=1}^{d_X} \langle i| M^{XY} |i\rangle$, *again with* $\{|i\rangle\}$ *an arbitrary basis of* $\mathcal{H}^X$.

Note that the definition 3 is slightly modified compared to the original one : there is an extra overall transposition of M. This is a convenience first adopted in the original definition of the process matrix framework [1], and that have been since then adopted in every subsequent work using the Process Matrix (PM) formalism[11]. The reasons why will be made clear in chapter 3. Please notice that because of this transposition, every definition taken from the Quantum Comb formalism that will be presented in chapter 3 will be modified accordingly from their original definitions in the sources [9, 10, 39–43] because they are all based on the non-transposed version of the CJ isomorphism, refer to appendix B.1 for the mathematical implications.

If the map $\mathcal{M}$ in the definition 3 is actually a CPTP map, *a quantum channel*, then M is called the *Jamiolkowski state*. Otherwise, operator M is called either the *Choi matrix*, *Choi-Jamiołkowski matrix* or *CJ matrix* in short [31]. Also, it should be clear by now that all these matrices are a fixed basis representation of operators, therefore the terms 'matrix' and 'operator' will be abusively interchanged in the text without further care. Moreover, for conciseness, we will omit the superscripts whenever they are not essential to the understanding of an equation.

The motivation to work with CJ matrices instead of maps is because a series of theorems and lemma actually show that not only the operator M encodes the complete positivity character of the associated map $\mathcal{M}$, but it actually encodes every property the map could have. Here we will summarised the 3 most important properties, for the demonstrations, the reader is invited to refer to the original papers [37, 38] or, for a more structured accounting to a reference book such as [32]. Let $\mathcal{M}^X$ be a map, M^{XY} its CJ matrix, $\mathbb{1}^X$ the unit matrix on subsystem X and d_X the dimension associated to the Hilbert space of the system X, every CJ representation have the following properties (superscript will be omitted when no confusion is possible) **Hermiticity Preserving** : the map is hermitian preserving, *if and only if* its CJ matrix is hermitian

$$\mathcal{M}\left(\rho^{\dagger}\right) = \left[\mathcal{M}\left(\rho\right)\right]^{\dagger} \iff M^{\dagger} = M \tag{2.30}$$

Complete Positivity : the map is completely positive (CP) *iff* its CJ matix is semi-positive defined, *i.e.* the eigenvalues of M^{XY} are all greater or equal to zero. Mathematically let there be an extra system Z and let ρ^{XZ} be defined on $\mathcal{L}\left(\mathcal{H}^X \otimes \mathcal{H}^Z\right)$,

$$\mathcal{M}^X \otimes \mathcal{I}^Z\left(\rho^{XZ}\right) \geq 0, \forall \rho^{XZ} \iff M^{XY} \geq 0 \tag{2.31}$$

Trace Preserving : the map is trace preserving *iff* its CJ operator enjoys have a unit partial trace over the output subsystem

$$\mathrm{Tr}\left\{\mathcal{M}^X\left(\rho^X\right)\right\} = \mathrm{Tr}\left\{\rho^X\right\} \iff \mathrm{Tr}_Y\left[M^{XY}\right] = \mathbb{1}^X \tag{2.32}$$

Finally, now that we have the tools to mathematically encode an operational probabilistic theory, and that we have found how to represent it as matrices, a last thing will be needed for the mathematics to be easy to manipulate : a good basis. The full discussion is left as an appendix A.1, here are the highlights : Motivated by the form of the Hilbert-Schmidt inner product (2.4), this good basis will be taken as having only one element that

[11] Up to the best knowledge of the author.

have non-zero trace. So what is done is to restrict the dyadic formulation of basis element for operators $\{|i\rangle\langle j|\}_{i,j=1}^{d}$ to a particular linear combination of orthonormal basis elements such that the new basis elements $\{\sigma_i\}_{i=1}^{d^2}$ enjoy the following properties (A.33) :

$$\sigma_0 \equiv \mathbb{1} \tag{2.33a}$$

$$\mathrm{Tr}\{\sigma_i\} = 0 \quad ,\forall i > 0 \tag{2.33b}$$

$$\sigma_i^\dagger = \sigma_i \quad ,\forall i \tag{2.33c}$$

$$\mathrm{Tr}\{\sigma_i \sigma_j\} = d\delta_{i,j} \tag{2.33d}$$

We see that only the element σ_0 is of non zero trace (2.33a) and (2.33b), that they are hermitian (this is also a mathematical convenience) and the elements are orthogonal in the sense of Hilbert-Schmidt inner product. The particular choice of an hermitian basis is called *Generalised Gell-Mann Basis* (GGB). In the thesis we will often refer to the GGB as traceless basis although not all traceless bases are hermitian. Explicit correspondence between braket formulation and GGB basis elements $\{\sigma_i\}_i$ can be found in [44]. An intuitive example of traceless basis in the two-dimensional case is the set formed by the unit matrix and the 3 Pauli matrices $\sigma_x, \sigma_y, \sigma_z$, from which the notation by greek sigma letter is inherited.

As an example of decomposition in this basis, consider the CJ matrix as defined in equation (2.28) M^{XY} :

$$M^{XY} = \sum_{i=1}^{d^2} \sum_{j=1}^{d^2} m_{ij}\, \sigma_i^X \otimes \sigma_j^Y \tag{2.34}$$

where m_{ij} are coefficients in $\mathbb{C}$ and, if M^{XY} is hermitian, are real coefficients $\left(M^{XY}\right)^\dagger = M^{XY} \iff m_{ij} \in \mathbb{R}\,; \forall i, j$.

2.3 Notion of causality

In this work, the notion of causality is to be understood as in probability theory. We say that a random variable x is correlated to a random variable y if y plays a role its probability distribution; otherwise it is not. If the conditional probability function of x given y, $P(x|y)$ is not equal to the probability density function of x, $P(x) = \sum_y P(x|y)$ then x is correlated to y.

In the OPT context, let 2 parties Alice, A, and Bob, B, act on a physical system. For both the part of the system they are acting on is represented as a black box around their local operations, which is a CP trace non-increasing map between an input state of the system (and the possibly shared ancilla) and the output state. They will probe this black box with classical inputs, the settings, that select the CP map a party will apply on the system. And the black box will subsequently respond with classical outputs, the outcomes. The physical quantities, *i.e. the observables* are the association of this outcome with the output state of the black box, and it will depend on the inputs. A party, say A, is then described by a set of inputs $\{s_i^A\}$ chosen by A, and by the set of outputs $\left\{o_j^A\right\}$, the systems and the map applied, her operation, being 'hidden' in the black box.

The causal structure is how the outcomes are correlated to the settings. In the non-trivial cases, outcome of a party depend on her settings, this is the local causal structure. The global causal structure arises from correlations of outcomes with settings from different parties. If party A have an output that depends on the settings of B : $P\left(o_j^A\right) =$

$P\left(o_j^A|s_i^A, s_k^B\right)$, then we say that B can signal to A. If he does so, *i.e.* if his signalling correlations are non-trivial so that Alice can retrieve information about s_k^B in her outcome, we say that o_j^A *causally depends* on s_k^B. The possibility to signal implies that Bob is in the causal past of Alice (resp. Alice is in the causal future of Bob), and we note this situation as $B \preceq A$. The negation of this relation is noted $A \npreceq B$. If neither Bob nor Alice can signal to each other, then we note it $A \npreceq\nsucceq B$. We will assume that there is only *unidirectional* signalling : if A can signal to B, then B cannot signal to A [12]. We will also assume that there is no loop in the causal structure, again for logical consistency : we want to avoid situations where an agent signal to himself in the past. These relations are the causal order between A and B. The ensemble of all the causal orders between a group of parties is called their *causal structure*.

Physically the *no-signalling correlations*, *i.e.* those obtainable when $A \npreceq\nsucceq B$, arise by local measurement on a shared quantum state, and signalling correlations arise by encoding information into a quantum system that is subsequently sent to another party via a quantum channel. Causal order can be understood using the terms from the theory of relativity, under the assumption that there is a background space-time, see below. If $A \preceq B$, then Bob is in the future space cone of Alice and they are time-like separated. If $A \npreceq\nsucceq B$ then they are space-like separated[13]. We are now able to define what we mean by *predefined* and *indefinite* causal structure (or order).

Definition 4. *A theory with N parties $A^{(1)}, A^{(2)}, A^{(3)}, ...$ with settings noted as $\left\{s_i^{A^{(1)}}, s_j^{A^{(2)}}, ...\right\}$ (shorthand $\vec{s}$) and outcomes noted $\left\{o_k^{A^{(1)}}, o_l^{A^{(2)}}, ...\right\}$ (shorthand $\vec{o}$), is said to be compatible with* ***predefined causal order*** *if all achievable probability distribution $P(\vec{o}|\vec{s})$ can be written*

$$P(\vec{o}|\vec{s}) = \sum_{q=0}^{N-1} Pr\left(\alpha_q \cap \neg\alpha_O \cap ... \cap \neg\alpha_{q-1}\right) Pr\left(\vec{o}|\vec{s}, A^{(q)} \textit{ is first}\right) \tag{2.35}$$

where α_q is the event that party $A^{(q)}$ is first i.e. each party $A^{(x\neq q)}$ either is in her causal future $A^{(q)} \preceq A^{(x\neq q)}$ or have no causal relation with her $A^{(q)} \npreceq\nsucceq A^{(x\neq q)}$. The symbol $\neg$ is the usual negation of an event, $\cap$ the intersection and the term $Pr\left(\vec{o}|\vec{s}, A^{(q)} \textit{ is first}\right)$ is a convex mixture of distributions compatible with the causal order between the remaining parties. If the relation (2.35) *don't hold with say that the theory have an* ***indefinite causal structure****.*

A remark on the notion of *free* will in the choice of setting : in usual quantum theory this is justified by the structure of space-time [46, 47] but in the theory of process matrix, which is central to this text and will be introduced in the next chapter, we refrain from assuming a global space-time as given *a priori*. Therefore the free will is a concept that must be taken as fundamental in the theory. We thus postulate that the parties are *free* in their choice of settings [28].

Another remark is that this notion of causal structure is the one defined in the works of Baumeler, Wolf and associates [28, 48, 49]. But this definition based on signalling and correlation is not exactly the definition of causal structure that is agreed upon in the theory of classical and quantum causal models. In general, a causal structure is described by a

[12] Two way signalling can always be represented by multiplying the parties to represent them at several place of the causal structure, like $A \preceq\succeq B \rightarrow A^{(1)} \preceq B \succeq A^{(2)}$. But in physical theories it is an uneasy assumption as it often allow for one party to interact with its past self for example, which is impossible, except if you use Closed Timelike Curve (CTC) [45] but we wont consider that case [26].

[13] Although it is not correct to talk about space- or time-like separation to express causal ordering when there is no background space time, it is so frequently done in the field that we won't try to avoid making the mistake.

directed acyclic graph (DAG) of causal relations [50] , see [51]. This imprecision in the definition however will not impact the content of the present thesis.

2.4 Summary

In this section we have reviewed the basic notions that will be used in the thesis. We started with a definition of the theory used, quantum theory, and showed how it is expressed in the operational framework. Recall that this is a point of view in which a set of experimenters, or parties, are given settings that makes them interact with a black box that give them back an outcome in return as well as a probability of seeing this particular situation happen. The quantum mechanics necessary to represent such a scenario were introduced accordingly. We have reviewed how one goes from the traditional Dirac bra-ket formulation to the very abstract Choi-Jamiołkowski picture that allows to represent the quantum channel formulation as matrices. The key point to remember is that parties' actions are conditioned on their settings, which makes them choose a particular quantum instrument that they will give as input in the supermap representing the quantum network, and the supermap will output a probability of the situation to be observed. The CJ isomorphism can be used to represent both the quantum instrument and the supermap as matrices, which are then easier to work with. These notions of quantum networks as being represented by a supermap will be the starting point of the introduction to the quantum comb formalism in the upcoming chapter. Finally, a note on the notion of causality, and especially fixed causal order was given in order to give a precise definition of what is meant by processes with indefinite causal structure.

Chapter 3

The Comb and Process Formalisms

3.1 Quantum Comb Formalism

Quantum Comb formalism is intended to treat quantum networks and all possible transformations thereof, including as special cases all possible manipulations of quantum states, measurements, and channels, such as e.g. cloning, discrimination, estimation, and tomography [40]. It is motivated by the development of the different dilatation theorems (Stinespring and Neumark, see *e.g.* [34]) and representation theorems (Kraus [35] and Choi [38]) that can be used to represent any dynamics, open or closed, in terms of density matrix, POVM and channels only.

The quantum instrument formalism [36], as evoked will treat preparation and POVM on the same footing as a collection of CP maps that output, or takes as input, density operators. Everything else that happens between the 2 operations will be represented by CPTP maps, the quantum channels. What the quantum comb formalism shows is that because of CJ isomorphism, all these objects will be operators defined on the tensor product of several Hilbert spaces, thus matrices[1], and that the composition of several operations follows simple matrix manipulation rules.

The main point of the formalism is that all the states and operations can be represented by a general object called a *quantum comb* [39, 52], which, in the CJ picture, is but a semi-positive definite operator presenting special characteristics as we will see. Quantum comb are the representation of all the transformations that can happen to a *quantum network* which appears to be the most general object in OPT we were looking for. Composition of quantum networks is made possible by a new operation called the *link product* and that is everything needed to represent everything in the theory. The axiomatic approach showed a very strong principle which is called *the universality of quantum memory channel* : any admissible transformation of quantum network, *i.e.* any valid quantum comb, can be realised by a suitable sequence of memory channels[2] [40].

In the rest of this section we will subsequently introduce the notions of quantum comb and link product. Then the section will be closed by showing that there exist physically realisable situations for which the quantum network framework cannot give a proper description, which will motivate the introduction of an even more general framework, *the process matrix formalism*, in the next section.

[1]Remember that because we restrict the discussion to finite dimensional case, these are indeed matrices as the d-dimensional Hilbert spaces can be though of as $\sim \mathbb{C}^d$ and the associated linear operators space can be seen as the linear mappings $\mathcal{L}(\mathcal{H}) \sim \mathbb{C}^d \to \mathbb{C}^d \sim \mathbb{C}^{d^2}$, mathematically speaking we're working with a Banach C^*-algebra $\mathcal{A}$ [15, 31]

[2]Basically the memory channels are the quantum channels that can be physically realised with the help of an ancillary side-channel called memory, see [53] for details.

3.1.1 The quantum comb

Mathematically, a quantum comb is the most general admissible map between a set of input states to a set of output states, when a global causal structure exists. By admissible, it means that the map takes in valid density operator and outputs valid density operators. What changes fundamentally from quantum channel formalism is that everything is through the same object, whereas the information is classical (which will correspond to a situation where every matrices can be simultaneously diagonalised [28]), or quantum.

A comb is characterised by its number of teeth, which correspond to, roughly, the number of operations one party makes. We say of a comb that possesses n teeth that it is a n-comb. Each tooth have an associated input and output Hilbert space, and the tooth takes as input an operator on the input Hilbert space and output one in the output space. A quantum channel will be the simplest object : a deterministic quantum 1-comb. A party A applying a CPTP map (quantum channel) is represented as a deterministic 1-comb in this formalism : the tooth is simply the CJ matrix associated to the channel, and the *comb conditions* are the translation of the validity condition of the map through the CJ isomorphism. Using equations (2.30), (2.32), (2.31), one can define a quantum 1-comb (refer to [41] for the full developments of the theory of quantum network) :

Definition 5. *Let there be a system A, to this system is associated two Hilbert spaces $\mathcal{H}^{A_I}$, the input space and $\mathcal{H}^{A_O}$, the output space. A* ***deterministic quantum 1-comb*** *on $\left(\mathcal{H}^{A_I}, \mathcal{H}^{A_O}\right)$ is the Choi-Jamiołkowski operator of a quantum channel from $\mathcal{H}^{A_I}$ to $\mathcal{H}^{A_O}$*

The word *deterministic* in the definition is here to account for the fact that the issue is certain, but since a $1 - comb$ is but a CPTP map, attentive reader could have guessed that probabilistic 1-comb will naturally be a CP trace-non-increasing maps.

Now comes the generalisation : the idea is to consider more than one tooth, so several input and output spaces. The rule is that the teeth will be ordered, per example if we associate to A a n-comb, it will have n teeth whose order will be reminded by a superscript, *e.g.* $A^{(4)}$ refers to the fourth tooth of the comb associated to A and $A_I^{(4)}$ specifically to the input of this particular tooth. By order what is meant is that inside and outside the comb the causal order of the teeth is fixed, so there can be no signalling from a tooth to another one with a lower superscript. Both signalling terms inside the operator associated with the whole comb, like $M^{A^{(j)} \preceq A^{(i)}} \in M^A$, $i < j$, and outside the operator, like plugging a signalling channel from $A_O^{(j)}$ to $A_I^{(i)}$, $i < j$, are forbidden. In addition to this rule, n-combs are normalised and CP(TP) between each teeth so that probabilities associated with it are well defined (and sum up to 1 in the TP case). Physically, the quantum comb can be thought of as a fragment of a quantum circuit, and the gaps between the teeth of the comb are where one can plug other combs, so other circuit fragment to 'complete' the circuit.

With these few considerations, it can be shown that the main notions of the formalism arises subsequently (again, the reader is invited to consult the original articles [39–41, 52] for the details)

Definition 6. *For $n \geq 2$, a* ***quantum*** ***n-comb*** *on an ordered set of n parties with $2n$ associated Hilbert spaces $\left\{\mathcal{H}^{A_I^{(1)}}, \mathcal{H}^{A_O^{(1)}}, \mathcal{H}^{A_I^{(2)}}, \dots, \mathcal{H}^{A_O^{(n)}}\right\}$ is the Choi-Jamiołkowski operator of an* admissible *n-map, i.e. a linear completely positive map transforming $(n-1)$-combs on $\left\{\mathcal{H}^{A_I^{(1)}}, \mathcal{H}^{A_O^{(1)}}, \mathcal{H}^{A_I^{(2)}}, \dots, \mathcal{H}^{A_O^{(n)}}\right\}$ into 1-combs on $\left\{\mathcal{H}^{A_I^{(1)}}, \mathcal{H}^{A_O^{(n)}}\right\}$*

Using this definition, one can adapt it to the deterministic case :

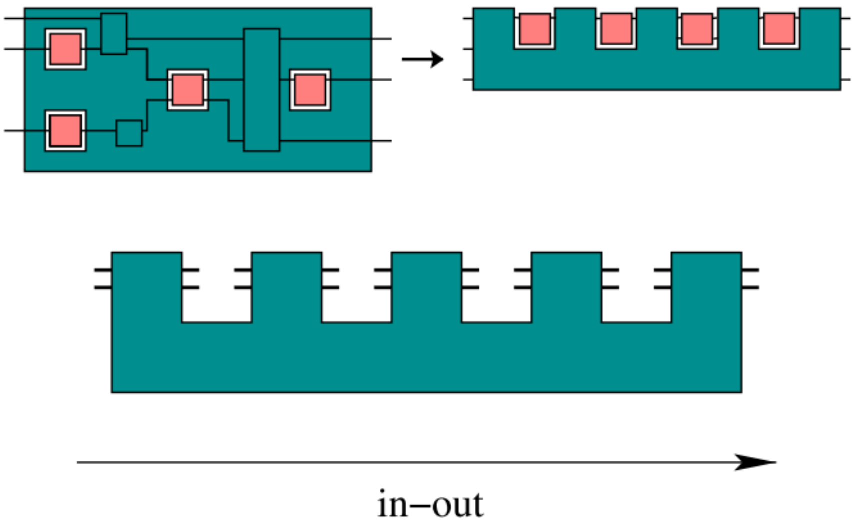


FIGURE 3.1: Illustration of what a comb physically corresponds to : a fragment of a complete quantum circuit (green) is a quantum comb with 1 more teeth that it have holes to be filled, and the holes are filled by fragments of quantum circuits, hence quantum combs as well (here in red, those are quantum 1-combs). (Figure from [39].)

Definition 7. *A deterministic n-comb $S^{(n)}$ is the CJ operator of a deterministic n-map, i.e. a map $\mathcal{S}^{(n)}$ that transforms deterministic $(n-1)$-combs into deterministic 1-combs.*

As well as the probabilistic :

Definition 8. *A **probabilistic n-comb** on $\left\{\mathcal{H}^{A_I^{(1)}}, \mathcal{H}^{A_O^{(1)}}, \mathcal{H}^{A_I^{(2)}}, \ldots, \mathcal{H}^{A_O^{(n)}}\right\}$ is a positive operator $R^{(n)} \in \mathcal{L}\left(\bigotimes_{i=1}^{n}\left(\mathcal{H}^{A_I^{(i)}} \otimes \mathcal{H}^{A_O^{(i)}}\right)\right)$ such that $R^{(n)} \leq S^{(n)}$ for some deterministic n-comb $S^{(n)}$ defined on the same set.*

And the generalisation of quantum instrument as a collection of probabilistic combs follows naturally

Definition 9. *An n-**instrument** $I^{(n)}$ is a set of probabilistic n-combs $\left\{S_i^{(n)}\right\}$ such that $\sum_i S_i^{(n)}$ is a deterministic n-comb.*

Here is a small example for illustration : let there be two parties, Alice and Bob. The scenario we wish to represent is the following : Alice is preparing a state, she sends it to Bob, Bob encodes information on this state by applying an unitary on it, and then sends it back to Alice who will measure it. For simplicity, we will assume that Alice's preparation and Bob's operation are both deterministic. In that case, Alice's operations are represented as a 2-comb : $S_A^{(2)}$ on $\left\{\mathcal{H}^{A_I^{(1)}}, \mathcal{H}^{A_O^{(1)}}, \mathcal{H}^{A_I^{(2)}}, \mathcal{H}^{A_O^{(2)}}\right\}$ with the particularity that the first input space and the last output space are both trivial *-there is no information getting in or out-*. Bob's operation is a 1-comb $S_B^{(1)}$ on $\left\{\mathcal{H}^{B_I^{(1)}}, \mathcal{H}^{B_O^{(1)}}\right\}$[3]. Here the 2 comb of Alice will map the 1-comb of Bob into a 1-comb on $\left\{\mathcal{H}^{A_I^{(1)}}, \mathcal{H}^{A_O^{(2)}}\right\}$, but this is a particular comb since these 2 spaces are trivial, *i.e.* of dimension 1. The deterministic character of the whole process will

[3]Note that this is an even more particular case since the operation is a unitary so it is a CPTP map with homomorphic input and output spaces, so the CJ operator can be decomposed in the Kraus representation [35], and Bob's operations is just the application of the Kraus operator onto the state Alice has sent to him.

therefore imply that this ouput 1-comb have the following interpretation : we start from the only deterministic properly-normed linear operator on a 1-dimensional Hilbert space, which is the number 1 (Alice's always doing something, with probability one), and we are mapped to a similar output space, which is also 1 (Something between Bob and Alice's operations is always happening).

This was a very trivial example, what could be done to enhance it is to allow Alice and Bob to use quantum n-instruments instead of deterministic combs. In that case, Bob has access to a collection of maps that will correspond to the way he's encoding the information, an example will be that he receive a qubit from Alice and he chooses to apply either the $+\sigma_z$ or $-\sigma_z$ to it depending if he wants to send a 0 or a 1, the choice between the two could be motivated by information exterior to the system, like Bob flipping a coin to decide which bit to send. Alice is then measuring in the z basis and her measuring operations will be the 2 quantum instruments corresponding to either case, namely the POVM $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ (where here, $|0\rangle$ and $|1\rangle$ are the eigenvectors of the Pauli's Z matrix, with eigenvalues +/-1, by convention [16]).

3.1.2 Mathematical characterisation of Quantum Combs

A series of theorems in [40] give a set of conditions an operator must obey in order to be a valid n-comb. Here we will summarise them in one theorem, and then give explicit characterisation of simple combs, as it will be useful for the developments of part 2.

Theorem 3 (Quantum Comb [40]). *A positive operator $S^{(n)}$ on $\mathcal{L}\left(\bigotimes_{i=1}^{n}\left(\mathcal{H}^{A_I^{(i)}} \otimes \mathcal{H}^{A_O^{(i)}}\right)\right)$ is a deterministic n-comb if and only if the following n identities are verified :*

$$Tr_{A_O^{(i)}}\left[S^{(i)}\right] = \mathbb{1}^{A_I^{(i)}} \otimes S^{(i-1)}, \quad 2 \leq i \leq n \tag{3.1a}$$

$$Tr_{A_O^{(1)}}\left[S^{(1)}\right] = \mathbb{1}^{A_I^{(1)}} \tag{3.1b}$$

where $S^{(j)}$, $1 \leq j \leq n$ are deterministic j-combs.

Two important corollaries of this theorem are that on one hand, the set of probabilistic n-combs on $\mathcal{L}\left(\bigotimes_{i=1}^{n}\left(\mathcal{H}^{A_I^{(i)}} \otimes \mathcal{H}^{A_O^{(i)}}\right)\right)$ is the whole *cone*[4] of positive operators. And, on the other hand, that a deterministic n-comb is the CJ operator of a n-partite *memory channel* [53]. It is thus tempting to say that the only relevant cone in quantum mechanics is the one of positive operators, and that every deterministic map is physically implementable by the use of a n-partite memory channel. But while the latter statement is proven to be true [39], the former is not, as this will be explained at the end of this section.

Applying theorem 3 to operators represented in a traceless basis, one can fully characterise the most general form of quantum combs in such a basis. Here we do it for a 1-comb and 2-comb, as it will be useful in latter discussions.

[4] A cone is a nonempty set of vectors $C \subseteq \mathbb{R}^d$ that with any finite set of vectors also contains all their linear combinations with nonnegative coefficients. For conciseness considerations, convex analysis notions will be used without further introduction as their understanding is not necessary to follow the arguments presented in this thesis. Interested readers can refer to *e.g.* [33].

For a single party A, with associated input and output Hilbert spaces $\mathcal{H}^{A_I}, \mathcal{H}^{A_O}$, the basis expansion for the most general deterministic 1-comb reads

$$M^A = \frac{1}{d_{A_O}} \left(\mathbb{1}^{A_I} \otimes \mathbb{1}^{A_O} + \sum_{i=0}^{d_{A_I}^2-1} \sum_{j=1}^{d_{A_O}^2-1} m_{ij} \sigma_i^{A_I} \otimes \sigma_j^{A_O} \right), \qquad m_{ij} \in \mathbb{R} \forall i,j; M^A \geq 0 \tag{3.2}$$

Notation conventions Now we introduce a few conventions to shorten the equations. The most important one is that we make the tensor product implicit where no confusion is possible, *e.g.* $\sigma_i^{A_I} \otimes \sigma_j^{A_O} := \sigma_i^{A_I}\sigma_j^{A_O}$ and we avoid repeating tensor product of the unit matrix *e.g.* $\mathbb{1}^{A_I} \otimes \mathbb{1}^{A_O} := \mathbb{1}^{A_I A_O}$. Also when some terms in a tensor product are the unit matrix, they will be omitted like $\sigma_0^{A_I} \otimes \sigma_j^{A_O} \equiv \mathbb{1}^{A_I} \otimes \sigma_j^{A_O} := \sigma_j^{A_O}$. Also the Greek letter sigma σ will always be used to represent traceless matrix elements of a basis and we will often omit to note the upper bound in the sums as it should be understood that there is always d^2 basis elements associated to a linear space of operators on a Hilbert space. Finally, we shorten the superscripts when they apply to both input and output space like : $X_I X_O := X$ for all party X, for example : $\mathbb{1}^{A_I A_O} := \mathbb{1}^A$.

With these considerations, equation (3.2) reduces to

$$M^A = \frac{1}{d_{A_O}} \left(\mathbb{1}^{A_I A_O} + \sum_{j>0} m_{0j}\, \sigma_j^{A_O} + \sum_{i>0} \sum_{j>0} m_{ij}\, \sigma_i^{A_I} \sigma_j^{A_O} \right), \quad m_{ij} \in \mathbb{R} \forall i,j; M^A \geq 0 \tag{3.3}$$

The interpretation of the terms is the following : the $\mathbb{1}$ matrix is always present because of the normalisation and trace-preservation conditions; the central term correspond to situations where some state(s) is(are) outputted no matter what was incoming, and the last term correspond to the state(s) that are outputted conditionally on getting a particular $\sigma_i^{A_I}$ state as input.

Now for the explicit formulation of a quantum 2-comb, here with tooth named A and B[5], and with causal order $A \preceq B$, in a traceless basis :

$$\begin{aligned} M^{A \preceq B} =& \frac{1}{d_{A_O} d_{B_O}} \left(\mathbb{1} + M^{A_I \prec A_O} + M^{A \prec B} \right) \qquad , \\ M^{A_I \prec A_O} :=& \mathbb{1}^A + \sum_{j>0} m_{0j}^{(1)}\, \sigma_j^{A_O} + \sum_{\substack{i>0\\ j>0}} m_{ij}^{(1)}\, \sigma_i^{A_I} \sigma_j^{A_O} \\ M^{A \preceq B} :=& \sum_{\substack{i,j,k \geq 0\\ l>0}} m_{ijkl}^{(2)}\, \sigma_i^{A_I} \sigma_j^{A_O} \sigma_k^{B_I} \sigma_l^{B_O} \end{aligned} \tag{3.4}$$

$m_{ij}^{(1)}, m_{ijkl}^{(2)} \in \mathbb{R}$. We see that in the characterisation terms have been grouped to highlight their significance : $M^{A_I \prec A_O}$ is the nontrivial part of the 1-comb one obtains when the second tooth is traced out, while $M^{A \prec B}$ is the vanishing part under trace over the subsystem B. It can be thought of as the non trivial part of a memory channel between A and B *i.e.* the signalling terms [53].

[5]This could have been $A := A^{(1)}$ and $B := B^{(2)}$, we use different names for the tooth in anticipation for the comb combination rules introduced in next section.

3.1.3 The link product

The quantum comb is a very useful tool regarding the fact that it treat channels, states, state preparation and measurement the same way, but the quantum instrument formalism was already doing that. What is added by this formalism is that we are thinking about quantum circuit as a network of operations, in which each node, a quantum teeth, correspond to a local operation and each vertex correspond to a quantum channel (which are hidden in the handle of the comb). An even more practical property is still to be introduced : because every object is represented by an operator (a matrix when expressed in some basis) one could expect that there is a very general operation used to link combs together, *e.g.* how to plug the red fragments in the green one in figure 3.1. This is *the link product*. It can be used to represent the tensor product of combs acting on different spaces, composition of combs over some common space, up to the combination of an n-comb with a corresponding $(n-1)$-comb to get back to the 1-comb, *i.e.* what we did in the example (which is the inner product of the space when the target 1-comb is of trivial input and output dimensions).

Hence, the link product should have the tensor and inner product as limit cases, and in general gives the CJ operator associated with resulting map when composing two (or more) maps together. It should be noted that the definitions introduced here are slightly different from the original ones because of the way we have defined the CJ isomorphism (2.28), see section B.1 in the appendix B.

Definition 10. *Consider two linear maps*

$$\mathcal{M} \in \mathcal{L}\left(\left(\mathcal{L}\left(\mathcal{H}^A\right)\right) \to \left(\mathcal{L}\left(\mathcal{H}^B\right)\right)\right)$$

and

$$\mathcal{N} \in \mathcal{L}\left(\left(\mathcal{L}\left(\mathcal{H}^B\right)\right) \to \left(\mathcal{L}\left(\mathcal{H}^C\right)\right)\right) \quad ,$$

*with associated CJ operators $M \in \mathcal{L}\left(\mathcal{H}^B \otimes \mathcal{H}^A\right)$ and $N \in \mathcal{L}\left(\mathcal{H}^C \otimes \mathcal{H}^B\right)$. Then, the Choi-Jamiołkowski operator $S \in \mathcal{L}\left(\mathcal{H}^C \otimes \mathcal{H}^A\right)$ of the composition of the maps $\mathcal{S} = \mathcal{N} \circ \mathcal{M} \in \mathcal{L}\left(\left(\mathcal{L}\left(\mathcal{H}^A\right)\right) \to \left(\mathcal{L}\left(\mathcal{H}^C\right)\right)\right)$ is given by the **link product** of the operators M and N :*

$$C = N * M := \left\{ Tr_B \left[\left(N^{BC} \otimes \mathbb{1}^A \right) \cdot \left(\mathbb{1}^C \otimes \left(M^{AB} \right)^{T_B} \right) \right] \right\}^T \tag{3.5}$$

where T_B denotes the partial trace over subsystem B.

This was the particular case of composition of maps that have corresponding output and input spaces, so this is the combination of 2 1-comb into another 1-comb. The generalisation of the definition consider the cases where composition of a n-comb with a m-comb can give any j-comb with $1 \leq j \leq n+m$ depending on the Hilbert spaces that are common to both combs. This more general definition reads :

Definition 11. *Let there be two CJ operators $M \in \mathcal{L}\left(\bigotimes_{i\in\mathfrak{M}} \mathcal{H}^i\right)$ and $N \in \mathcal{L}\left(\bigotimes_{j\in\mathfrak{N}} \mathcal{H}^j\right)$ where $\mathfrak{M}, \mathfrak{N}$ are arbitrary sets of subsystems constituting the tensor Hilbert space of M and N, respectively, these sets may or may not share common elements in general. The **link product** between the two is given as an operator $S \in \mathcal{L}\left(\mathcal{H}^{\mathfrak{N}\setminus\mathfrak{M}} \otimes \mathcal{H}^{\mathfrak{M}\setminus\mathfrak{N}}\right)$, where $\mathcal{H}^{\mathfrak{N}\setminus\mathfrak{M}}$ is the shorthand for $\bigotimes_{j\in\mathfrak{N}} \mathcal{H}^j \setminus \bigotimes_{i\in\mathfrak{M}} \mathcal{H}^i$, defined as*

$$S = N * M := \left\{ Tr_{\mathfrak{M}\cap\mathfrak{N}} \left[\left(N \otimes \mathbb{1}^{\mathfrak{M}\setminus\mathfrak{N}} \right) \cdot \left(\mathbb{1}^{\mathfrak{N}\setminus\mathfrak{M}} \otimes (M)^{T_{\mathfrak{M}\cap\mathfrak{N}}} \right) \right] \right\}^T \tag{3.6}$$

The main properties of the link product are that [41]

1. The link product is commutative : $M * N = N * M \quad , \forall M, N$
2. The link product is associative if and only if the parties don't all share a common element : $(M * N) * Q = M * (N * Q) \iff \mathfrak{M} \cap \mathfrak{N} \cap \mathfrak{Q} = \emptyset$
3. If M and N are hermitian, so is their link product
4. If M and N are positive semidefinite, so is their link product.

Using these properties one can deduce the following theorem :

Theorem 4. *The link product of 2 quantum combs is also a quantum comb, provided an admissible matching between the teeth (adapted from theorem 5 in [40], refer to this source for the subtleties implied by 'admissible'). If both comb were deterministic, so will be the resulting one.*

Finally, there is two limit cases for this definition : consider M and N defined on a set of subsystems $\mathfrak{M}$ and $\mathfrak{N}$, respectively. Then

$$N * M = \text{Tr}\left\{M^T \cdot N\right\} \qquad , \mathfrak{M} = \mathfrak{N} \tag{3.7}$$

and

$$N * M = N \otimes M \qquad , \mathfrak{M} \cap \mathfrak{N} = \emptyset \tag{3.8}$$

3.1.4 Limitation of quantum network : The Quantum Switch

The trouble with quantum comb formalism is that it is not broad enough to really account for everything one experimenter can do in her lab. The same year that the group lead by Chiribella formalised and axiomatised the concept of *supermap* and *quantum comb* (2009), they found a counterexample to the formalism. There was an operation that was not possible to be represented as combs, called *the quantum switch* [43].

The idea behind the quantum switch is to coherently control the order of the operations applied on a system. Two parties, Alice and Bob, get as input a target system $|\psi\rangle_t$ and a control system $|\psi\rangle_c$. Then, depending on the value of the control, Alice either apply her local operation first on the target system or it is Bob who do. When the control system is in a pure state, the circuit formalism hold on : the operations of Alice and Bob are blackboxes applied in a particular order or another depending on this control bit. It also hold when the control is in a probabilistic mixture, as a natural consequence of the convexity of the space of density operators[6]. But when the control bit is in a superposed state, like $|\psi\rangle_c = |+\rangle \equiv \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, it breaks down, even when extended by the quantum comb ideas. The causal structure of the circuit appears to be in an entangled state. In the paper [43], this is formulated as a no-go theorem (refer to it for details) :

> "The [SWITCH supermap] cannot be computed deterministically by a circuit in which the two unknown oracles [i.e. the operations of Alice and Bob] are called a single time in a fixed causal order."

It was already know at that time that the way around this no-go theorem is to drop the assumption of a fixed causal order. The motivation for such a radical step was known before the example of the quantum switch, since a quantum theory compatible with gravitation should admit indefinite causal structure, as we talked about in the introduction. The causal structure must then itself be a variable that present quantum characteristics : it can

[6] Although *stricto sensu* this is already a case outside of the initial quantum comb formalism, but the tools are adapted to represent it as well and the situation can be purified [41].

be in a coherent superposition of several states, and consequently there can be uncertainty on it, we then talk about *non-fixed causal structure* [2].

This motivates the next section, where we will introduce another formalism, called *Process Matrix* formalism. Its original motivation was actually quite different, whereas quantum combs were intended to implement general tools for analysing the properties of every physically possible quantum circuit, *process matrices* were aimed to probe more fundamental aspects of nature, as needed in the search for a theory of quantum gravity per example. It aimed to provide a more general framework to quantum theory that could, according to Hardy (and others)'s ideas, treat the causal structure as *non-fixed*, which comes from the foundations of quantum theory as we have just explained, but also as *dynamical* which is motivated by the theory of general relativity [3].

A few remarks to conclude this section : first, and as pointed out in the introduction, these two theories are -by far- not the only possible generalisations of the framework of quantum mechanics in OPT context, see *e.g.* [4–8, 11]. This work is using PM formalism as its main framework, but it is extending it using concepts from quantum comb theory, hence the need to provide an introduction to both of them. Another thing to note is that quantum comb formalism was subsequently refined and extended by Bisio and Perinotti in a very more general theory based on the axiomatisation of quantum supermaps (briefly put : a channel is a lower ranking supermap than a comb, that is itself a lower ranking supermap compared to a class of objects from which process matrices are coming from as special cases, and the hierarchy of admissible supermap then continue *ad infinitam*) [9, 10]. We are acknowledging this work here as there are findings made in this formalism that are similar to some of the results of this work, to the best knowledge of the author.

3.2 Process Matrix Formalism

Despite the fact that they are different ways of representing the same thing, an operational quantum theory, the motivation, philosophy and genesis of process matrix and quantum combs are very different. The quantum process framework is itself a particular theory within a general operational framework of pre-selected processes [54]. It relies on certain assumptions about the local operations of the parties and the joint probabilities of their outcomes, which will be reviewed next section. While, as shown in the previous section, quantum comb is a generalisation of quantum instruments with an aim of transcribing and analysing concrete physical experiments, process matrix formalism is more of a generalisation of density matrix with an aim of going beyond the scope of relativistic causality that focused on non-signalling correlations [55, 56].

In this section, the notion of process will be introduced. We will see how the quantum process matrix is built on that notion, and how it is used to formulate a quantum theory. With this tool introduced, some important concepts will be explored like the *causal (non)-separability*, which tells whether or not a process matrix actually have a causal explanation, *i.e.* if it is possible to find an implementation with circuit formalism and probabilistic control of the order of the gate or not; the *causal inequalities*, which are an analogue to the Bell inequalities [57] but in the context of superposition of causal orders; and multipartite generalisation of the notions. The section will be closed with a state of the art in PM formalism literature, in order to give the reader a global picture.

3.2.1 The process framework and matrix

The process framework can be though of as a possible generalisation of any circuit-based operational probabilistic theory. It relies on the abandon of the assumption of the existence of an *a priori* fixed background space-time (or any definite causal structure in which the circuit is embedded). The process framework thus describes probabilities for the outcomes of local experiments[7] associated with different parties -the usual Alice, Bob, Charlie, ...etc-, performed in abstract circumstances defined without assuming the existence of a global causal order between the experiments, but only a local order of the event within each of the parties [54].

Each local experiment are described based on two variables : their *setting* s^A ("how Alice is preparing and measuring her experiment") and *outcome* o^A ("What result she is measuring")[8]. To these 2 variables, intrinsic to each party, a third one $w^{ABC...}$ depicting how the different parties get correlated to each other is added.

Definition 12. *A* ***process*** $\mathcal{W}^{A,B,...}$ *is defined for a set of parties (or local experiments)* $\mathcal{S} = \{A, B, ...\}$ *as the collection of conditional probabilities*

$$\mathcal{W}^{A,B,...} \equiv \left\{ P\left(o^A, o^B, ...|s^A, s^B, ...w^{AB...}\right)\right\} \tag{3.9}$$

for all the possible settings and outcome, when and for a fixed value of w.

To particularise this process framework to the quantum process matrix formalism, one must do a few assumptions on the structure of the theory. The first one is that since the global space-time is not assumed *a priori*, one cannot defined the free randomness upon consideration of the space-time structure like in [46, 47] but have to take it as fundamental [28].

The two next assumptions come from the underlying structure of an operation in the context of operational probabilistic theory, as argued in [6, 7]. On one hand, the input and output systems of an operation (not to be confused with the settings and outcome) are the only mean of information exchange between the parties and are thus responsible for the correlations between their outputs. As a consequence, when the parties and the systems that they exchange are taken as fundamental, and that a notion of causal ordering exists[9], then the outcome of Alice can per example only be correlated with the settings of Bob if and only if he is in her causal past. This assumption is called *closed laboratories* [1] or *closed-boxes* [54][10]

On the other hand, it is assumed that the settings of a party can be known with certainty before the interaction of the party with the input system unconditionally on any event in the future. This makes sense with the local temporal sequence of events within each local laboratory, as circuit picture is locally valid. Also outside the local laboratories the variable $w^{AB...}$ will be assumed to be obtainable without post-selection. Per example it is acting on each local experiment before anyone receive an input system, or it has been fixed by LOCC protocol[11] between the parties before the process. This is the *no-post-selection*

[7] We also abusively talk about the local parties, or the local laboratory of a party to say local experiment.

[8] Remember that superscript indicate the party to which a variable is associated.

[9] Again this is a simplification, and it is not rigorously correct as the notion that we used in the thesis does not match perfectly with the one used by the sources that we are referring to in this paragraph. See also the remark at the bottom of section 2.3.

[10] Note that this removes the possibility of two-way and self-signalling but let the possibility for the operations to be one-way signalling (also called semicausal in [55]), as announced.

[11] *Local Operations and Classical Communication*, we will talk about it in chapter 4 see [58] for the introduction of this paradigm and *e.g.* [59] for a review.

criterion and the processes are called *pre-selected processes* consequently. Note that dropping this assumption to extend the formalism by including post-selection is discussed in [7].

The last assumption is that of *local quantum mechanics*. It basically says that the local operations of the parties are described by the standard space-time formulation of quantum mechanics. Therefore each operation performed by a party on the system is represented by a quantum instrument: local CP maps between the input and output systems, and the systems are themselves described as operators living in the input and output Hilbert space. Therefore the selection of a particular quantum instrument will be the settings and the probability associated with a process of fixed $w^{AB...}$ will then reduce to $P\left(o_i^A, o_j^B, o_k^C, ...| \{\mathcal{M}_i^A\}, \left\{\mathcal{M}_j^B\right\}, \{\mathcal{M}_k^C\}, ...\right)$, where the outcomes depend on the set of quantum instrument that will lead to them, themselves depending on the settings and input systems.

On top of these assumptions comes the *noncontextuality* of the joint probabilities, which states that this probability is independent of any variable concerning the concrete implementation of the local CP maps. For example the probability for a particular choice of maps $\mathcal{M}_i^A, \mathcal{M}_j^B, ...$ should not depend on the particular set of instruments $\{\mathcal{M}_1^A, \mathcal{M}_2^A, \mathcal{M}_3^A ...\}$ associated with Alice's operation. This implies

$$P\left(o_i^A, o_j^B, o_k^C, ...| \left\{\mathcal{M}_i^A\right\}, \left\{\mathcal{M}_j^A\right\}, \left\{\mathcal{M}_k^A\right\}, ...\right) = \omega\left(\mathcal{M}_i^A, \mathcal{M}_j^B, \mathcal{M}_k^C, ...\right) \tag{3.10}$$

The requirement that local procedures agree with standard quantum mechanics imply that the function ω is a linear on every map (see [1] for the details).

We are left with a function that maps a set of CP map to probabilities, thus *omega* is itself a n-linear map. Both ω and the CP maps it takes as input $\mathcal{M}$ are linear maps, so we can represent everything as matrices in the Choi-Jamiołkowski picture (2.28). For the latter it's easy, we know from last section that a quantum instruments is represented by its associated CJ matrices under the form of a quantum 1-comb (3.3): Alice map is encoded in matrix $M^{A_I A_O} \in \mathcal{L}\left(\mathcal{H}^{A_I} \otimes \mathcal{H}^{A_O}\right)$, Bob's map in $M^{B_I B_O} \in \mathcal{L}\left(\mathcal{H}^{B_I} \otimes \mathcal{H}^{B_O}\right)$, ...etc. For the former, it will be a matrix, called *process matrix*, $W^{A_I^{()}, A_O^{()}, B_I^{()}, ...}$ defined on the dual space where the matrices representing the quantum instruments are defined $W^{A_I^{()}, A_O^{()}, B_I^{()}, ...} \in \mathcal{L}\left(\mathcal{H}^{A_I} \otimes \mathcal{H}^{A_O} \otimes \mathcal{H}^{B_I} \otimes ...\right)$. As the supermap will be represented by the bilinear mapping $(W, M^{A_I A_O} \otimes M^{B_I B_O} \otimes ...) \to [0;1]$, we can infer that because it map to real numbers only, the process matrix must be hermitian [1, 60]. Then, considerations about the positivity and normalisation of the probabilities, as well as the fact that if the input combs are deterministic (ergo CPTP maps) the probability of the process must be necessary 1 (if the process takes in CPTP maps as input, something is happening with 100% certainty, the unit trace is preserved), will lead to the following theorem [54] :

Theorem 5 (Process Matrix [1]). *The probability associated to a certain process can be computed in the CJ picture as*

$$\begin{aligned} P\left(o_i^A, o_j^B, o_k^C, ...| \left\{\mathcal{M}_i^A\right\}, \left\{\mathcal{M}_j^A\right\}, \left\{\mathcal{M}_k^A\right\}, ...\right) \\ = \\ \mathrm{Tr}\left\{W^{A_I A_O B_I B_O C_I C_O ...}\left(M_i^{A_I A_O} \otimes M_j^{B_I B_O} \otimes M_k^{C_I C_O} \otimes ...\right)\right\} \end{aligned} \tag{3.11}$$

where the M_x^X's are the CJ operators of the x-th CP map $\mathcal{M}_x^X$ of party X and $W^{A_I A_O B_I B_O C_I C_O ...} \in \mathcal{L}\left(\mathcal{H}^{A_I} \otimes \mathcal{H}^{A_O} \otimes \mathcal{H}^{B_I} \otimes \mathcal{H}^{B_O} \otimes \mathcal{H}^{C_I} \otimes \mathcal{H}^{C_O} \otimes ...\right)$ is a linear semi-definite positive operator. This operator is well defined, i.e. leads to correct probability if and only if the following two conditions *are met : first,*

$$W^{A_I A_O B_I B_O C_I C_O \dots} \geq 0 \tag{3.12}$$

which is necessary to ensure non-negative probabilities. Second, and for any choice of CPTP maps $M^{A_I A_O}, M^{B_I B_o}, M^{C_I C_O}, \dots$ *taken by the parties we must have*

$$\mathrm{Tr}\left\{W^{A_I A_O B_I B_O C_I C_O \dots}\left(M^{A_I A_O} \otimes M^{B_I B_O} \otimes M^{C_I C_O} \otimes \dots\right)\right\} = 1 \tag{3.13}$$
$$\forall M^{A_I A_O}, M^{B_I B_o}, M^{C_I C_O}, \dots \geq 0$$
$$Tr_{A_O}\left[M^{A_I A_O}\right] = \mathbb{1}^{A_I},\ Tr_{B_O}\left[M^{B_I B_O}\right] = \mathbb{1}^{B_I},\ Tr_{C_O}\left[M^{C_I C_O}\right] = \mathbb{1}^{C_I}, \dots$$

which is necessary to ensure the normalisation of probabilities. When both conditions are met, such an operator is called (a valid) ***process matrix****.*
Remark - It is assumed that all the party can always share arbitrary (possibly entangled) ancillary states independent of the process, and use them in their local operation.

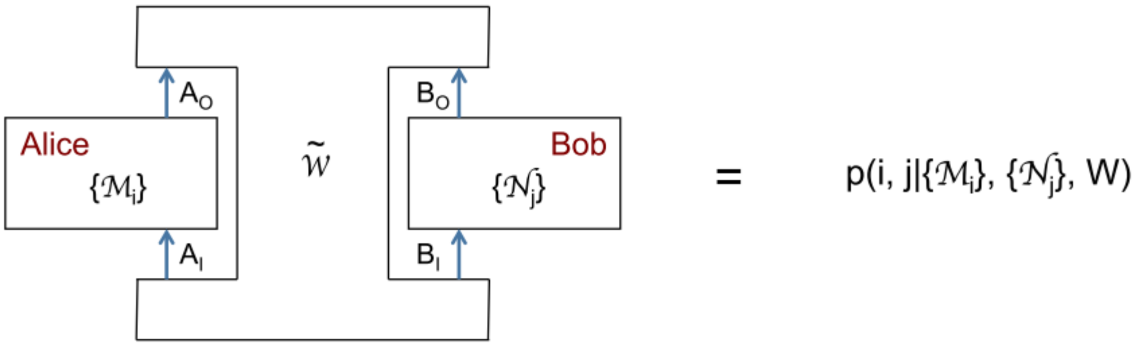


FIGURE 3.2: Graphical representation of a two parties process matrix (Figure from [61]).

In a sense, whereas the combs are a generalisation of quantum instruments representing the transformations between states, the process matrix is a generalisation of density matrix representing the states.

One can notice that the process matrix defined as such (3.11) have led to a Gleason-type theorem [32, 62]: for every set of *effects* in our theory, here the *effects* will be the quantum (1-)instruments of each party, there is a *frame function* that links these effects to the unit interval. Then, for every frame function $f : \mathcal{E}_d \to [0,1]$, there exists a unique unit-trace positive operator W such that $f(E) = (W, E) = \mathrm{Tr}\{W \cdot E\}$. This is not accidental [50]. Subsequently, we often call the equation (3.11) "generalised Born rule". In simpler terms : for all possible operations made by the parties, there is a function that links these operations to the associated probabilities (in $[0,1]$). This function have a unique representation by a particular operator W and the associated probability for this particular event to happen, conditionally on the operations performed by the parties, is obtained by taking the inner product of the effects with this operator.

3.2.2 Mathematical characterisation of the process matrix

Like in the section about quantum combs, we will transcribe the full characterisation of 1- and 2-partites process matrix in a given traceless basis here, as they will be useful in the next part.

The explicit characterisation of a valid 1-partite process matrix in traceless basis can be found in [1, 63], it is to be noted that it was also derived in the context of quantum combs by Chiribella and collaborators [52].

$$W^{A_I A_O} = \frac{1}{d_{A_I}} \left(\mathbb{1}^{A} + \sum_{i>0} w_i \, \sigma_i^{A_I} \right) \tag{3.14}$$

$w_i \in \mathbb{R}$.
Remember that the same conventions as those introduced in subsection 3.1.2 are in application, and they will be through the whole thesis. The interpretation of the 1-partite PM is clearer if equation (3.14) is written like

$$W^{A_I A_O} = \frac{1}{d_{A_I}} \left(\mathbb{1}^{A_I} + \sum w_i \sigma^i_{A_I} \right) \otimes \left(\mathbb{1}^{A_O} \right)$$

, we see that it allows any (positive and normalised) incoming state to the party A $\left(\mathbb{1}^{A_I} + \sum w_i \sigma^i_{A_I} \right)$, hence the choice of this term is the *pre-selection* of the process, but forbids to choose a particular measurement result for the reception of output system by the process matrix, hence the imposed $\mathbb{1}^{A_O}$ term, which illustrates the *no-post-selection* criterion.

For the explicit formulation of a valid 2-partite process matrix in traceless basis, refer to [1] :

$$\begin{aligned} W^{A_I A_O B_I B_O} &= \frac{1}{d_{A_I} d_{B_I}} \left(\mathbb{1} + \sigma^{A \preceq B} + \sigma^{B \preceq A} + \sigma^{A \npreceq \nsucceq B} \right) \qquad , \\ \sigma^{A \preceq B} &:= \sum_{\substack{j>0 \\ k>0}} w_{0jk0} \, \sigma_j^{A_O} \sigma_k^{B_I} + \sum_{\substack{i>0 \\ j>0 \\ k>0}} w_{ijk0} \, \sigma_i^{A_I} \sigma_j^{A_O} \sigma_k^{B_I} \\ \sigma^{B \preceq A} &:= \sum_{\substack{i>0 \\ l>0}} w_{i00l} \, \sigma_i^{A_I} \sigma_l^{B_O} + \sum_{\substack{i>0 \\ k>0 \\ l>0}} w_{i0kl} \, \sigma_i^{A_I} \sigma_k^{B_I} \sigma_k^{B_O} \\ \sigma^{A \npreceq \nsucceq B} &:= \sum_{i>0} w_{i000} \, \sigma_i^{A_I} + \sum_{k>0} w_{00k0} \, \sigma_k^{B_I} + \sum_{\substack{i>0 \\ k>0}} w_{i0k0} \, \sigma_i^{A_I} \sigma_j^{B_I} \qquad . \end{aligned} \tag{3.15}$$

$w_{ijkl} \in \mathbb{R}$
The interpretation of the bipartite PM is a bit more complicated, so without entering in the details we already have grouped the terms together to illustrate the common features of each. Therefore, terms grouped in $\sigma^{A \preceq B}$ ($\sigma^{B \preceq A}$) are those that can be used to signal from A to B (B to A), and those in $\sigma^{A \npreceq \nsucceq B}$ are those that don't allow signalling

3.2.3 Notions in indefinite causal structure

A quick comparison between (3.4) and (3.15) shows that the process matrix framework allow a greater variety of possible scenarios[12]. Actually, as long as there is 2 party or more, the process matrix is always richer, in terms of the achievable correlations between the parties, than anything possible with quantum combs with the same number of parties, even if it is a probabilistic mixture of several combs with different causal orderings [1, 48, 63, 64] and the structure get richer for increasing number of parties [48, 64].

[12] To be formally exact, one should compare a 3-comb with trivial first input and last output teeth to the 2-partite PM, but for now the discussion is still heuristic, the exact comparison will be undertaken in later chapters.

Obviously one possible thing to do with process matrix that cannot be done with quantum combs is to provide a description and characterisation of the quantum Switch supermap, as it is how we motivated the introduction of this formalism. Quantum Switch can actually be represented as a 4-partite process matrix [61, 65]. But this is only a speck of what this formalism offer. As stated before, what this formalism is useful for is to describe correlations between parties when the causal structure becomes dynamical and/or coherently superposed. We then talk about *causally non-separable process matrix* for PM that have correlations that are not compatible with a theory with *predefined causal order*. The main finding of [1] was to show, for the 2-partite case, that one could achieve correlations that cannot be understood in terms of definite causal order. These correlations violate a *causal inequality* that must be satisfied by all correlations obtained when assuming pre-existing space-time.

In a sense, there is a formal resemblance between entanglement and causal non-separability. A fair amount of important developments of the theory came by transposing concepts encountered in the theory of entangled states[13] into the theory of non-separable process matrix. Just like it is necessary for a state to be entangled in order to be non-local and hope to beat Bell inequalities [57] or in general a CHSH inequality [67], a process matrix must necessarily be causally non-separable to beat a causal inequality [1]. However there is a difference in the cases involving more than 2 parties : when going to the classical limit, *i.e.* when (local) quantum theory is replaced by probability theory, it is impossible to obtain non-local correlations no matter the formalism used [59], but conversely PM formalism shows that it is possible to obtain non-causal correlations in the classical limit [28].

Regardless of this difference, the analogy can be pushed forward : in the same fashion there exist a entangled states that cannot beat a Bell inequality, there is process matrix that are proven to be unable to beat any causal inequality. The now classical example of such a PM is the Switch process matrix [43] whose inability to do better the inequality has been showed in [12, 54]. It is in this work that the concept of *causal witness* have been introduced [68]. It is a tool that allow to detect causal non-separability in the same fashion as an entanglement witness can be used to detect entanglement [69].

To close this section, we will introduce the mathematical definition of causal non-separability, because it will needed it in the results part. Whilst this notion seems obvious in the 2 partite case, there was a lot of difficulties in defining it for more than 2 parties -which we will refer to multipartite, or 'N>2', case- which led to several definitions [12, 54]. In this thesis we will use the most recent definition that aims to clarify and conciliate every point of view [70].

The definition is recursively built from the definition causal separability. It is indeed obvious that for one party, the process matrix can always have a causal explanation since, as revealed by equation (3.14), it correspond to preparing a state, giving it to the party, then measuring its output without post-selecting a particular outcome, and what the party is doing follows local quantum mechanics so it have implied fixed causal structure. For more than one party, especially when $N > 2$, causal separability is based on a recursive unravelling of every possible casual scenario : "Take a party as being first in the causal structure, consider every operation he could make, then see if the remaining process matrix shared by the $N-$ other parties is itself a valid causally separable process matrix. If you can do that for every party being first, then your PM is causally separable." Formally [70] :

Definition 13. *For $N = 1$ any process matrix is causally separable. For $N \geq 2$, a $N-$partite process matrix W among N parties $\mathfrak{N} := \left\{ A^{(1)}, A^{(2)}, ..., A^{(N)} \right\}$ is said to be* ***causally separable***

[13] For a review of entanglement, see *e.g.* [59, 66].

if and only if, for any extension $A^{\mathfrak{N}}_{I'}$ of all the parties' input space and any ancillary quantum system in it $\rho \in \mathcal{L}\left(\bigotimes_{i=1}^{N} \mathcal{H}^{A^{(i)}_{I'}}\right) := \mathcal{L}\left(\mathcal{H}^{A^{(\mathfrak{N})}_{I'}}\right)$, the resulting process matrix $W \otimes \rho$ can be decomposed as

$$W \otimes \rho = \sum_{k \in \mathfrak{N}} q_k \, W^{\rho}_{(k)} \tag{3.16}$$

with $q_k \geq 0$, $\sum_k q_k = 1$, and where for each k, $W^{\rho}_{(k)} \in \mathcal{L}\left(\mathcal{H}^{A^{(1)}_I} \otimes ... \otimes \mathcal{H}^{A^{(N)}_O} \otimes \mathcal{H}^{A^{(\mathfrak{N})}_{I'}}\right) := A^{(\mathfrak{N})}_{II'O}$ is a process matrix compatible with party $A^{(k)}$ acting first, and is such that for any CP map $M^{A^{(k)}} := M^k \in A^{(\mathfrak{N})}_{II'O}$ applied by party $A^{(k)}$, the conditional $(N-1)$-partite process matrix $\left(W^{\rho}_{(k)}\right)_{|M^k} \equiv Tr_{A^{(k)}}\left[\left(M^k \otimes \mathbb{1}^{\mathfrak{N}\setminus A^{(k)}}\right) \cdot W^{\rho}_{(k)}\right]$ is itself causally separable.

A few convention have been introduced to shorten the equations, first an ensemble of parties is designated by a gothic letter, here $\mathfrak{N}$, its cardinal by the italic letter associated to it , N. As for the parties, they're designated by the same letter with a superscript[14] $A^{((i))}$. The linear space where the CJ operators live also have its shortcut notation :

$$\mathcal{L}\left(\mathcal{H}^{A^{(1)}_I} \otimes ... \otimes \mathcal{H}^{A^{(N)}_O} \otimes \mathcal{H}^{A^{(\mathfrak{N})}_{I'}}\right) := A^{(\mathfrak{N})}_{II'O}$$

when it is not possible to cause (more) confusion[15].

3.3 Conclusion

In this chapter, we have introduced both the formalism of quantum combs (also called quantum networks) and process matrix. The first one was developed as a way to generalise circuit formalism using the CJ isomorphism. In this formalism both states and transformations are quantum combs, which are represented by semi-definite positive operators following a particular set of rules (3.1) and one compose them together using the link product (3.6). The generalisation of quantum instrument is also given as an ensemble of probabilistic combs which sum to a deterministic one. The quantum comb formalism, despite its merits, fails to be able to encompass all the physically possible processes, because of certain realisations such as the quantum switch which rely on a coherent control of the causal order.

Process matrix formalism was introduced as the most general way of keeping track of correlations between a set of parties that act locally on a system. It differs from quantum comb regarding the fact that it does not assume that there is a global space time, only that quantum mechanics is locally valid within each party's laboratory. It relies on an object that generalise the notion of state, the process matrix (3.11), and the transformations within each laboratory are represented with the usual notion of quantum instrument, or, using comb formalism, by a collection of probabilistic 1-combs that sum to a deterministic 1-comb. This formalism can be used to represent that quantum switch the combs couldn't, but it is not its main interest. This interest is based on the study of the widest variety of obtainable correlations, which can even be used to do more than what is achievable

[14] Remark that compared to the quantum comb case, the superscripts don't necessary imply the causal structure : one can have $A^{(n)} \preceq A^{(n-1)}$ for example. It will be precised in the text if it happen to be the case otherwise.

[15] This will be the last conventions introduced, the author wish to apologise to the reader for the very complex notation used, although the convention mostly follow [70] (it will be re-explained in different places of the text, when needed). Note however that there exist easier 'tensor-like' notation, first used by Hardy [4, 71] and adapted to the PM formalism in [72], but it is not widespread.

within the theories with predefined causal structure. When this is the case, we say that the process is violating a causal inequality.

The PM formalism have led to its own thriving field of study, there is a lot of exciting directions to be explored, like the possibility to use the advantage conferred by the violation of a causal inequality to do better computations and algorithms than it is normally possible within the usual circuit framework. In appendix B.2, we provide an extensive review of the field for the interested readers.

Part II

Results

Chapter 4

Research Motivation

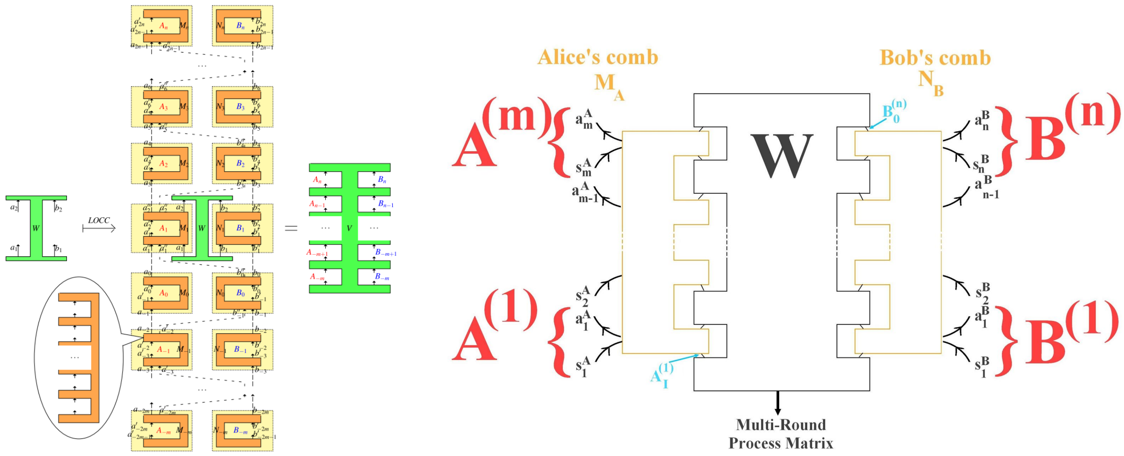
As we have seen in the conclusion of last chapter, the field of quantum correlations with undefined causal structure have been very active lately. In this thesis we wanted to look into the possibilities that a particular extension of the process formalism had to offer. The extension consists of allowing the parties linked together by a process matrix to act more than once. So instead of just receiving and sending a quantum system, the parties could be allowed to have multiple rounds of communications, where they receive and send one quantum system each time. This results in a scenario where each party possesses several 'slots' that can be used one after the other to exchange input and output systems with a common central object. This object will be referred to as *multi-round process matrix* (MPM) because it is no longer *stricto sensu* a process matrix as it allows for some communication to take place outside of it. This situation, as we will see, would be equivalent to splitting each party into a subset of slots linked together by an extra side-channel that allow them to signal to each other outside of the process matrix and that would represent the 'memory' of this subset between the different rounds of communication, so the subset is in fact representing one party acting several times.

This topic of research is motivated by the LOCC paradigm in entanglement theory[1] [58], that is interested in the channels that are achievable when only local operations by a party on a system and classical communication is allowed between the parties. Pre-shared entangled ancillas between the parties are nonetheless also permitted. In PM language this notion of local operations will be translated into the multiple rounds of communications the agents may have at their disposal during a communication protocol. The subset of parties linked together by side channels represent the many stages of a local laboratory of an agent, for example this is a fixed point in space taken at different time, or the side channel is representing the memory of the agent itself between each round. Each operation in the subset will then be representing her local actions at those different rounds, with an associated input and output system that is exchanged by the *multi-round process matrix* (MPM). The MPM being the communication protocol. A first interesting question in that representation is to assess how the presence of the side channel influence the causal order, of its absence, of the global system.

Note that there is a previous occurrence of research made in this direction in [72], but this is not exactly in the same underlying spirit. In this work it was considered that there is only one 'round' in which the agents can have access to a common process matrix and they each can only input one general operation (1-comb) into it. The other rounds were then restricted to local operation on their systems and one-way classical communication between the agents. Here we explore a different point of view : at each round the agents receive and send a part of their whole quantum system in the MPM, they act locally on

[1] For a review see *e.g.* [59] or refer to [15].

their systems but can transmit either classical or quantum information through the side-channels.



(A) Jia's LOCC : parties (yellow) input their instruments in quantum combs (orange) at each rounds, between the rounds one way classical communication is allowed (dashed lines), except during one round during which the combs are connected through a smaller process matrix W (green). This results in a overall process matrix V (also green). Figure from [72].

(B) Multi-round Process Matrix : Bob and Alice each do local operation and communications between each rounds, which will be represented by quantum combs (yellow). At each round they get a setting s, they do an operation (quantum comb tooth, red letters) and get back an outcome o. Also at each round they exchange quantum subsystems with the MPM (black wires).

FIGURE 4.1: Difference between Jia's point of view and the one adopted here

The work that will be presented here is concerned with the possibility that these allowed side-channel between the parties of a process matrix lead to new, unforeseen dynamics between the parties. The objectives are to mathematically define the MPM then assess the new capabilities given by it, in terms of communications, and observe whether or not there is a possibility of an *activation*, *i.e.* the sudden passage from a process that could not violated causal inequalities to one that can by the addition of a side-channel [54]. Less extreme cases of activation would be the augmentation of the bound up to which the process allows the violation of a causal inequality, if it was violating one beforehand, or the activation of causal non-separability from a PM that was causally separable.

This sets the main goal of the thesis : characterising the multi-round process matrix. It will be done gradually in chapter 6. First we will look at the effect of a side channel between the two parties of a bi-partite PM, then we will augment the number of parties, each time looking for an activation. Finally we will establish rules similar to theorem 5 for building valid MPM for any number of parties divided into arbitrary subsets.

There will also be an extra research topic introduced in appendix E. The tools that will be developed in next chapters to study the MPM have revealed to bring insightful hints on the characterisation of the linear subspace of valid PM. This appendix presents a theorem as well as ensuing preliminary findings and corollaries of it that could potentially help toward a better characterisation of it.

Chapter 5

Validity Constraints seen as projectors

In this chapter we develop the mathematical tools that will be the crucial in our characterisation of the MPMs. They are base on an alternative approach to the process matrices validity conditions (3.11), first introduced in [12]. The idea is to reformulate the conditions as *positivity*, *normalisation* and *subset restriction* (or *projective conditions*). The first two conditions are easy to understand as they are only there for the PM to output properly normalised probabilities. The last condition is in fact projective constraints to restrict the space where the operator is defined into the subspace of valid PM. This physically correspond to restrict the set of valid PM to avoid the ones that allow logical paradoxes like those made possible with CTC [45] (like an agent going back to the past to kill his former self) and also to forbid post-selection. These projective conditions can be expressed using a certain map that will be introduced in the next section. Through the thesis, we will take the liberty to refer to this map as *depolarising superoperator*, since no name have been given to it yet.

Following [12] ideas, we will see how one can build validity conditions for PM through the Hilbert-Schmidt inner product (2.4). However, we will slightly modify the original idea because there is a nice connection with the comb formalism that can be made. To do so, a reformulation of comb conditions in this form will have to be made beforehand. Then, the result of this chapter will be to show that both are very similar, almost dual to each other - An observation that will turn out to be quite important as we will see in chapter 6 -.

Once quantum comb and process matrix conditions have been build in this fashion, we will show recurrence relations on building them for increasing number of parties. While being very helpful in general, they will be capital for the proofs of the theorems in the auxiliary result to the thesis E.

5.1 The depolarising superoperator

Suppose an operator F acting on some Hilbert space $\mathcal{H} = \mathcal{H}^X \otimes \mathcal{H}^Y$ with dimension $d = d_X d_Y$. The *depolarising superoperator over* X is defined as a map acting on the Hilbert-Schmidt space $\mathcal{L}(\mathcal{H})$. The map have a non-trivial part acting on subsystem $X : \mathcal{P}^X(\cdot) : \mathcal{L}(\mathcal{H}^X) \to \mathcal{L}(\mathcal{H}^X)$ and is the identity mapping on all the other subsystems. The depolarising superoperator then acts on arbitrary elements of the Liouville space, $F \in \mathcal{L}(\mathcal{H}^X \otimes \mathcal{H}^Y)$, as a mapping $(\mathcal{P}^X \otimes \mathcal{I}^Y) : \mathcal{L}(\mathcal{H}^X \otimes \mathcal{H}^Y) \to \mathcal{L}(\mathcal{H}^X \otimes \mathcal{H}^Y)$, such that

$$\left(\mathcal{P}^X \otimes \mathcal{I}^Y\right)\{F\} \equiv \frac{\mathbb{1}^X}{d_X} \otimes \mathrm{Tr}_X[F] :=_X F \tag{5.1}$$

where the shorthand notation using prescript is introduced in order to use it more easily. This map, which is a superoperator, will be a central element in the mathematics of this part for reasons that will appear clear soon. To the best of our knowledge this map was first used in the context of PM formalism in [12]. In appendix C.1, we prove that this map is actually an orthogonal projector onto a subspace of a Liouville space and we develop on some of its properties introduced without proof in [12, 70, 73] in order to build a quick notation for the mathematics to come in the next chapters. We also prove its positive TP character.

The characteristics of the map, as just evoked, and proven in the appendix C.1, are that it is *Linear* (C.3) ; *Idempotent* (C.6) ; *Positive* (C.9) ; *Completely positive* if it acts on Hermitian operators only (C.10) ; *Trace Preserving* (C.11) ; *Hermitian Preserving* (C.13) ; and *Self-Dual* (C.14) . The idempotentcy together with hermitian preservation are enough to prove self-duality and the fact that this is actually an orthogonal projector, see the appendix. We can physically interpret this map as, when acting on a channel, replacing a part of it by a completely depolarising channel [72], as the operation preserve the characteristics of the CJ matrices but replace the subsystem X it has been applied on by the maximally mixed state $\frac{\mathbb{1}^X}{d_x}$ which is trivially deterministic, *i.e.* which has only one basis element, and is of unit trace.

The linearity property will be very often used, actually we will directly write the linear coefficient and the maps in the subscript itself to lighten the equations, for example (C.4) :

$$a\ {}_{X}F + b\ {}_{Y}F + c\ {}_{XY}F \equiv {}_{aX+bY+cXY}F \tag{5.2}$$

with a, b, c arbitrary scalars and F arbitrary operator. Also the multiplication of prescripts can be made when they act on different subsystems, or if they act one after the other on the same operator (C.8)

$$\left({}_{X}F^{XY} \otimes \mathbb{1}^{Z}\right)\left(\mathbb{1}^{X} \otimes_{Z} G^{YZ}\right) = {}_{XZ}\left(\left(F^{XY} \otimes \mathbb{1}^{Z}\right)\left(\mathbb{1}^{X} \otimes G^{YZ}\right)\right) \tag{5.3}$$

And depolarising superoperators commute with each other (C.5) :

$${}_{XY}F =_{YX} F \tag{5.4}$$

All together this imply some sort of basic algebraic properties of the prescripts. Technically, one can defined this algebra as

The algebraic relations between the depolarising superoperators acting on a same space (but not necessary the same subsystem) form a boolean ring.

see the appendix C.1.1 for more details. Here is an example of what can be done using this algebra

$${}_{(aX+bY)(cX+dZ)}F = {}_{ac\ X+ad\ XZ+bc\ XY+bd\ YZ}F$$

with lowercase elements being arbitrary coefficients and F an operator on $\mathcal{L}\left(\mathcal{H}^X \otimes \mathcal{H}^Y \otimes \mathcal{H}^Z\right)$, this will be often used to simplify the calculations.

5.2 Quantum Comb projective validity conditions

Using the tool provided by the depolarising superoperator one can reformulate the theorem 3 as a set of 3 conditions an operator has to verify. The superoperator is a central element in the reformulation since we can use it to define a projector to the linear subset of quantum combs $\mathcal{L}_C$ through the following theorem :

Theorem 6 (Projector to the quantum comb subspace). *For N parties, with causal order in numerical order $A^{(1)} \preceq ...A^{(N-1)} \preceq A^{(N)}$, it is given through the recurrence relation*

$$\mathcal{P}_C^N = \left(\mathcal{I}^{\mathfrak{N}} - \left(\mathcal{I}^{\mathfrak{N}\backslash A^{(N)}} \otimes \mathcal{P}^{A_O^{(N)}}\right)\right) + \left(\mathcal{P}^{A^{(N)}} \otimes \mathcal{P}_C^{(N-1)}\right) \tag{5.5}$$

where $\left(\mathcal{I}^{\mathfrak{N}\backslash A^{(N)}} \otimes \mathcal{P}^{A_O^{(N)}}\right)$ *is the depolarising superoperator over* $A_O^{(N)}$*, i.e.* ${}_{A_O^{(N)}}(\cdot)$*;* $\mathcal{I}^X$ *is the identity mapping over a(n ensemble of) subsystem(s) X and* $\mathcal{P}_C^{(N-1)}$ *is the projector onto the linear subset of deterministic quantum combs for* $N-1$ *causally ordered parties. In the language of depolarising supermaps, this equation is equivalent to*

$$\mathcal{P}_C^N = {}_{\left(1-A_O^{(N)}\right)}(\cdot) + {}_{A^{(N)}}\left(\mathcal{P}_C^{N-1}(\cdot)\right) \tag{5.6}$$

As a consequence, for $N = 1$ parties, the projector onto the linear subspace of quantum comb $\mathcal{L}_C^{(1)}$ is given by the superoperator (C.26c) :

$$\mathcal{P}_C = {}_{1-A_O+A}(\cdot) \tag{5.7}$$

See the appendix C.2 for the proof of the theorem, as well as how the following new definition was built :

Definition 14 (Quantum Combs reformulated). *Let there be a matrix*

$$M \in \mathcal{L}\left(\mathcal{H}^{A_I^{(1)}} \otimes \mathcal{H}^{A_O^{(1)}} \otimes \mathcal{H}^{A_I^{(2)}} \otimes \ldots \otimes \mathcal{H}^{A_O^{(n)}}\right)$$

This matrix is the CJ representation of a part of a deterministic quantum network, a ***deterministic quantum n-comb*** *between n systems (the teeth)* $\left\{A^{(1)}, A^{(2)}, ..., A^{(n)}\right\}$ *with causal order between the teeth of* $A^{(1)} \preceq A^{(2)} \preceq \ldots \preceq A^{(n)}$ *if and only if it satisfies the following set of conditions*

$$M \geq 0 \tag{5.8a}$$

$${}_{A^{(1)}A^{(2)}...A^{(n)}}M = \frac{\mathbb{1}^{A^{(1)}A^{(2)}...A^{(n)}}}{\prod_{i=1}^{n} d_{A_O^{(i)}}} \tag{5.8b}$$

$$\mathcal{P}_C^{A^{(1)} \preceq A^{(2)} \preceq ... \preceq A^{(n)}}(M) = M \tag{5.8c}$$

where $\mathcal{P}_C^{A^{(1)} \preceq A^{(2)} \preceq ... \preceq A^{(n)}}$ *is a projector from the Liouville space onto the linear subspace* $\mathcal{L}_C$ *defined through recurrence relation in equation* (5.6)*. The probabilistic quantum combs are still defined through definition 8.*

The interpretation of the 3 conditions (5.8) is that the first condition (5.8a) impose that the combs CJ matrices must be positive because of the CP character of the quantum combs that ensures that probabilities always yields a positive number (2.31). The second condition, (5.8b), which can be rewritten as $\text{Tr}\{M\} = \prod_i d_{A_I^{(i)}}$ is the translation of trace preservation of the map (2.32), which is necessary for the normalisation of probabilities. The last condition, (5.8c) is a restriction to a subspace of the whole space of Hilbert-Schmidt operators in order to avoid unwanted situations like post-selection within the combs. Notice that equation (5.8a) restricts the set to the cone of positive operators and together with (5.8c) they define the convex cone $\mathcal{C} = \mathcal{P} \cap \mathcal{L}_C$ of (nonnormalised) valid quantum combs, where $\mathcal{P}$ is the cone of semi-definite positive matrices.

5.3 PM projective validity conditions

Starting from the original validity conditions derived in chapter 3,

$$\mathrm{Tr}\Big\{W^{A_IA_OB_IB_OC_IC_O...}\left(M^{A_IA_O}\otimes M^{B_IB_O}\otimes M^{C_IC_O}\otimes ...\right)\Big\}=1 \tag{3.13}$$
$$\forall M^{A_IA_O}, M^{B_IB_o}, M^{C_IC_O}, ... \geq 0$$
$$\mathrm{Tr}_{A_O}\left[M^{A_IA_O}\right]=\mathbb{1}^{A_I},\ \mathrm{Tr}_{B_O}\left[M^{B_IB_O}\right]=\mathbb{1}^{B_I},\ \mathrm{Tr}_{C_O}\left[M^{C_IC_O}\right]=\mathbb{1}^{C_I}, ...$$

with

$$W^{A_IA_OB_IB_OC_IC_O...}\geq 0 \tag{3.12}$$

We will now derive a general procedure to link these conditions to the conditions when expressed as a projector like in [12] and all the subsequent works (*e.g.* [70] for the most detailed one on the matter) :

$$W^{A^{(1)}A^{(2)}...A^{(n)}}\geq 0 \tag{5.9a}$$
$$\mathrm{Tr}\left[W^{A^{(1)}A^{(2)}..A^{(n)}}\right]=\prod_{i=1}^{n}d_{A_O^{(i)}} \tag{5.9b}$$
$$W^{A^{(1)}A^{(2)}...A^{(n)}}=\mathcal{P}_V\left(W^{A^{(1)}A^{(2)}...A^{(n)}}\right) \tag{5.9c}$$

where $\mathcal{P}_V$ is the projector onto the linear subspace of process matrices, $\mathcal{L}_V$, that was defined as follows : for an N partite set of parties $\mathfrak{N}=\{A^{(1)}A^{(2)}...A^{(n)}\}$, let $\mathcal{X}$ represent all 2^N-1 non-empty subsets of parties $\in\{A^{(1)}A^{(2)}...A^{(n)}\}$, i.e.

$$\mathcal{X}\in\{\{A^{(1)}\},\{A^{(2)}\},...\{A^{(n)}\},\{A^{(1)}A^{(2)}\},\{A^{(1)}A^{(3)}\},...,\{A^{(n-1)}A^{(n)}\},...\{A^{(1)}A^{(2)}...A^{(n)}\}\} \quad ,$$

the projector $\mathcal{P}_V$ is

$$\mathcal{P}_V\{\cdot\}:={}_{1-\prod_{i\in(\mathcal{X})}\left(1-A_O^{(i)}\right)\prod_{j\in\mathcal{X}\setminus\mathcal{X}}A_I^{(j)}A_O^{(j)}}\cdot, \qquad \mathcal{X}\subseteq\mathfrak{N},\mathcal{X}\neq\emptyset \tag{5.10}$$

Again, condition (5.9a) restricts the set to the cone of positive operators, and its intersection with the subspace of Process Matrices $\mathcal{L}_V$ (obtained through projector (5.9c)) is the convex cone $\mathcal{W}=\mathcal{P}\cap\mathcal{L}_V$ of (nonnormalised) valid process matrices, where $\mathcal{P}$ is the cone of positive operators [70].

Up to [12], this projector was derived in an *ad hoc* way. In this paper, they noticed the process matrix validity conditions were actually coming from the form of the operator one plugs into the bi-linear mapping with the PM to obtain probabilities. The trick is to remark that the equation (3.13) is an inner product of an object with a tensor product of operators which conditions of validity can be expressed as projectors using the depolarising supermap. In this thesis we extend the trick by noticing that the projective conditions imposed on the operators were actually deterministic 1-comb condition like (C.26c) [61, 72], which is logical when one expects the objects to be plugged into a process matrix to be the most general ones possible with local causal order. Because the constraints on the combs are expressed as projectors, and that a projector is self-dual (C.14), it is possible to send the projective constraints from the combs to the process matrix, so the 'imprint' of comb conditions on the process matrix gives you the process matrix validity conditions. For clarity we will apply this idea in details for a 2-partite process matrix in the appendix C.3, here we will show it in the general case.

The procedure that was just explained in the text above, and explicited for 2 parties in the appendix, works for any number of parties. We just showed that condition (3.13) yielded conditions (5.9b) and (5.9c) for the case with N=2, we now formally derive the conditions for an arbitrary N.

Starting from (3.13), we use property (5.3) to rewrite it as

$$\left(W \Bigg|_{\prod_{i=1}^{n}(1-A_O^{(i)}+A_I^{(i)}A_O^{(i)})} \left(\bigotimes_{i=1}^{n} M^{A_I^{(i)}A_O^{(i)}} \right) \right) = 1$$

Then we factor out the part on which the depolarising map is over the whole set of systems. To do so, define $\mathcal{X}_k$ the subset of cardinality k in $\mathfrak{N} = \{1, 2, ..., n\}$ e.g. $\mathcal{X}_1 = \{\{1\}, \{2\}, ..., \{n\}\}$, $\mathcal{X}_3 = \{\{1, 2, 3\}, \{1, 2, 4\}, ..., \{n-2, n-1, n\}\}$, etc... and hence

$$\begin{aligned}\prod_{i=1}^{n}\left(1 - A_O^{(i)} + A_I^{(i)}A_O^{(i)}\right) = & \prod_{i=1}^{n}\left(1 - A_O^{(i)}\right) \\ & + \prod_{i\in(\mathfrak{N}\setminus\mathcal{X}_1)}\left(1 - A_O^{(i)}\right)\prod_{j\in\mathcal{X}_1} A_I^{(j)}A_O^{(j)} \\ & + \prod_{i\in(\mathfrak{N}\setminus\mathcal{X}_2)}\left(1 - A_O^{(i)}\right)\prod_{j\in\mathcal{X}_2} A_I^{(j)}A_O^{(j)} \\ & + ... \\ & + \prod_{i\in(\mathfrak{N}\setminus\mathcal{X}_{n-1})}\left(1 - A_O^{(i)}\right)\prod_{j\in\mathcal{X}_{n-1}} A_I^{(j)}A_O^{(j)} + \prod_{i\in\mathfrak{N}} A_I^{(i)}A_O^{(i)}\end{aligned}$$

which can be simplified by defining

$$\mathcal{X} = \bigcup_{i=1}^{n-1} \mathcal{X}_i \tag{5.11}$$

$$\prod_{i=1}^{n}\left(1 - A_O^{(i)} + A_I^{(i)}A_O^{(i)}\right) = \prod_{i\in(\mathfrak{N}\setminus\mathcal{X})}\left(1 - A_O^{(i)}\right)\prod_{j\in\mathcal{X}} A_I^{(j)}A_O^{(j)}$$

, now the factoring reads

$$\begin{aligned}&\left(W \Bigg|_{\prod_{i=1}^{n}(1-A_O^{(i)}+A_I^{(i)}A_O^{(i)}} \left(\bigotimes_{i=1}^{n} M^{A_I^{(i)}A_O^{(i)}} \right) \right) = \\ &\left(W \Bigg|_{\prod_{i\in(\mathfrak{N}\setminus\mathcal{X})}\left(1-A_O^{(i)}\right)\prod_{j\in\mathcal{X}} A_I^{(j)}A_O^{(j)}} \left(\bigotimes_{i=1}^{n} M^{A_I^{(i)}A_O^{(i)}} \right) \right) + \left(W \Bigg|_{\prod i\in\mathfrak{N} A_I^{(i)}A_O^{(i)}} \left(\bigotimes_{i=1}^{n} M^{A_I^{(i)}A_O^{(i)}} \right) \right)\end{aligned} \tag{5.12}$$

The treatment is the same as the 2 partite case: using (C.26b) and the possibility of having all M's equal to the unit matrix times a constant, the rightmost term gives a normalisation constraint

$$\text{Tr}\{W\} = \prod_{i=1}^{n} d_{A_O^{(i)}}$$

that is exactly the one expected (5.9b). Whereas the leftmost inner product shall always be zero, which is imposed for any M's by using the self-duality property of the depolarising superoperator (C.14)

$$
\begin{aligned}
&\left(W \middle| {}_{\prod_{i\in(\mathfrak{N}\setminus\mathcal{X})}\left(1-A_O^{(i)}\right)\prod_{j\in\mathcal{X}} A_I^{(j)}A_O^{(j)}} \left(\bigotimes_{i=1}^{n} M^{A_I^{(i)}A_O^{(i)}} \right) \right) \\
&= \left({}_{\prod_{i\in(\mathfrak{N}\setminus\mathcal{X})}\left(1-A_O^{(i)}\right)\prod_{j\in\mathcal{X}} A_I^{(j)}A_O^{(j)}} W \middle| \left(\bigotimes_{i=1}^{n} M^{A_I^{(i)}A_O^{(i)}} \right) \right) \\
&\iff \mathcal{P}_{V^\perp}\{W\} := {}_{\prod_{i\in(\mathfrak{N}\setminus\mathcal{X})}\left(1-A_O^{(i)}\right)\prod_{j\in\mathcal{X}} A_I^{(j)}A_O^{(j)}} W = 0
\end{aligned}
$$

which is another formulation of the projective constraints (5.9c) because

$$\mathcal{P}_{V^\perp}\{\cdot\} = 1 - \mathcal{P}_V\{\cdot\}$$

as the $\perp$ in the symbol of the projector suggested.

To close this section, we will formulate an equivalent of theorem 6 but for the projector onto the linear subspace of process matrices. This is actually a reformulation of the definition of the process matrix that is recursive and derived the same way as it was done for quantum combs.

Theorem 7. *The projector onto the linear subspace of process matrices for one party A is*

$$\mathcal{P}_V^{(1)} = \left(\mathcal{I}^{A_I} \otimes \mathcal{P}^{A_O}\right) := {}_{A_O}(\cdot) \tag{5.13}$$

For a N-partite PM the projector is obtained through the recurrence relation

$$\mathcal{P}_V^{(n+1)} = {}_{\left(1-A_O^{(n+1)}+A_{IO}^{(n+1)}\right)}\mathcal{P}_V^{(n)} + {}_{\left(A_O^{(n+1)}-A_{IO}^{(n+1)}\right)}(\cdot) - {}_{\left(1-A_O^{(n+1)}\right)\prod_{i=1}^{n} A_{IO}^{(i)}}(\cdot) \tag{5.14}$$

The proof is left as an appendix (app. C.3.2). By plugging right side of equation (5.14) into its left side, one can see that the formula for the projector found in [70], (5.10), arises naturally in the recurrence relation.

5.4 Summary

In this section, we introduced the *depolarising superoperator* (5.1) first used in the context of PM in [12] and explicitly proved its properties it has, since it have not been done before. We proved that this map is hermitian preserving and TP (therefore CPTP when only acting on CJ operators) and acts as an orthogonal projector in the space of operators. We also noticed that the relations between the depolarising superoperators could be seen as a specific algebra in which the elements are all idempotent called *a boolean ring*. Using this superoperator, we reformulated the validity conditions of a deterministic quantum comb (definition 14) into a formulation similar to the current process matrix validity conditions (5.9). This formulation is based on 3 conditions : a normalisation, positivity, and projective conditions *i.e.* an orthogonal projector to restrict the space of linear operator to the subspace of admissible operators. This lead us to the main result of the section which is the realisation that the PM conditions expressed like that are in fact obtained as a consequence of the form of the operators we take the inner product of the process matrix with, which are nothing else than deterministic 1-combs. And this allowed us to show a new way of getting from the 'old' PM validity conditions of OCB [1] to the 'new' ones of Araújo *et al.*

[12]. Finally, we also introduced recursive relations for the projectors in the projective conditions of the quantum combs (theorem 6) and of the process matrices (theorem 7) which will be useful in the upcoming chapters.

Chapter 6

The Multi-round Process Matrix

Suppose that Alice is in a spaceship afar from earth while Bob have stayed on earth. They communicate by exchanging quantum systems, travelling in the space between them. We postulate that their actions in their own frame of reference can be represented operationally by local quantum mechanics. It is a scenario like the one in figure 4.1b page 38 : Alice's first round of communication, $A^{(1)}$ consists on her choosing a quantum instrument based on her settings s_1^A, and receive a first outcome o_1^A based on the measurement reading, her second round, $A^{(2)}$, is also based on her next setting s_2^A, and output her second outcome o_2^A, etc. The only difference at each successive operation of Alice is that she possesses a memory of the previous steps. Then the succession of the m operations performed by Alice can be represented as a quantum m-comb, as they possess an underlying causal structure and it is the most general representation of a series of quantum operations sharing a memory. The same goes for Bob, whose operations will be an n-comb. So by construction, both parties have their operations represented by combs, and we want to find a general way to represent the communication between their combs.

If we assume that the exchange of messages obey a global causal order, then the theory of combs offer an answer through the tensor product of combs [40] : join Alice and Bob's teeth into a set forming your quantum network. The underlying causal structure of the universe implies that there must be a global causal ordering in the set, defined between each tooth[1]. When the global structure is known, the two combs are linked together through the link product (see definition 10) which will encode the global causal structure as it is applied, *i.e.* which tooth comes after which.

But suppose that Alice is actually a massive body that obey quantum mechanics, and so is Bob. And that she happens to be in the vicinity of a black hole, or any massive enough body for her to feel quantum effects alongside gravitational effects. Moreover suppose that Alice is in a state that is spatially undetermined. With these considerations the global causal structure can become undetermined, as Alice will feel the distortion of time in different manners depending on how close she's of the black hole, but her exact position is not determined. This is actually a simplification of the gedanken experiment considered in [74], the point being that it is possible that the underlying global structure between the parties can become undetermined[2], whilst their local frame of reference still obey the laws of quantum mechanics. This indeterminacy makes the link product unfit to represent the communication between them, as it cannot render a superposition of causal orders. The natural choice would be to go for the Process Matrix formalism, but we would spoil the information about the local causal structure within Alice and Bob's frame of reference that is already known.

[1] Or at least a probabilistic mixture of several global causal orders.
[2] And also possibly dynamical as evoked in the theoretical part.

This motivates the introduction of the **Multi-Round Process Matrix** (MPM) in this section. This is an operator similar to the process matrix but that allows side-communication between subsets of its parties[3]. This side-communication naturally fixes the causal structure inside each subset associated to a party. In a sense, the MPM is a particular kind of PM that allows its parties to input CPTP maps that may be bigger deterministic quantum combs than a tensor product of 1-combs.

In this chapter, the multi-round process matrix will be introduced. In a first section, it will be shown how to obtain its validity conditions like it was done for the PM and quantum comb. Then the object will be characterised, we will finally see what are the new allowed correlations it allows and how the definition of causal separability for PM is adapted for the MPM.

6.1 Constructive approach

The large mathematical detour that we did in the last chapters is now paying off. As the operations of the parties are deterministic n-combs, a point that will be discussed in the next section and the appendix D.1, we will see in this section that the MPM is exactly build like we did last chapter with PM.

6.1.1 One party MPM

The multi-round process matrix for one party is of course a special case. We wish to implement an object that is the most general way to represent the communications that take place between the several local operations of a party, Alice. We will refer to such an object as $W^{\mathfrak{N}}$ or simply W, with $\mathfrak{N} \equiv \left\{A^{(n)}\right\}$, being the set of Alice's n operations in numerical order : $\left\{A^{(1)} \preceq A^{(2)} \preceq \ldots \preceq A^{(n)}\right\}$. Obviously the 1-slot, 1 party MPM $W^{\left\{A^{(1)}\right\}}$ is equivalent to the 1-slot process matrix (3.14).

For the case when more than one operation performed by Alice is allowed we can seek our answer either in comb or PM formalism. The former brings a simple answer : Alice's n operations are a n-comb, and the world outside of Alice's local laboratory will be a $(n+1)$-comb with trivial first input and last output so that it maps Alice's comb to a probability.

It is equivalent to treating this scenario with process matrix formalism. When an n-slots process matrix gets the causal order between all the slots fixed in an absolute manner, that is we impose a definite global causal order, the resulting process matrix is by definition a deterministic comb taking the n 1-combs plugged into its slots to the trivial channel. It is possible to show that this ensemble of 1-combs to be plugged into a big $n+1$ PM is mathematically equivalent to a n-comb formed with all the parties to be plugged into a smaller (in dimension) $n+1$ comb that is the process matrix with definite causal structure, through some mathematical reformulation of the objects. An example is provided in appendix D.1.1 for a 2-slot scenario, leading to lemma 1, and the generalisation from this case is straightforward, here we will do it explicitly as it will help for the generalisation. The idea is that if we are given a process matrix $\tilde{W} \in \mathcal{L}\left(\bigotimes_{i=1}^{n} \mathcal{H}^{\tilde{A}_I^{(i)}} \otimes \mathcal{H}^{\tilde{A}_O^{(i)}}\right)$ for a set of operations represented by the n 1-combs $\{M^{A^{(i)}}\}_{i=1}^{n}$ whose causal order is known and

[3]Vocabulary point: the MPM partites are referred to as *slots* and the ordered subsets of *slots* as *parties*. For the above example, the MPM is (m+n)-slot and is shared among 2 parties, Alice and Bob, which have respectively m and n CP maps (or teeth, that will be plugged into the slots) acting on their local subsystems, and are to be plugged in the MPM. Whereas in PM all the slots are associated to an unique party.

determined *a priori* with certainty to be $\left\{A^{(1)} \preceq A^{(2)} \preceq \ldots \preceq A^{(n)}\right\}$, then it is possible to show that the process matrix can be decomposed like

$$\tilde{W} = W \otimes \left(\bigotimes_{j=1}^{n-1} V^{(j)} \right) \tag{6.1}$$

where $W \in \mathcal{L}\left(\bigotimes_{i=1}^{n} \mathcal{H}^{A_I^{(i)}} \otimes \mathcal{H}^{A_O^{(i)}}\right)$; the $V^{(j)}$'s are operators on $\mathcal{L}\left(\mathcal{H}^{\bar{A}_O^{(j)}} \otimes \mathcal{H}^{\bar{A}_I^{(j+1)}}\right)$ with $\mathcal{H}^{\tilde{A}^{(i)}} = \mathcal{H}^{A^{(i)}} \otimes \mathcal{H}^{\bar{A}^{(i)}}$, and $\mathcal{H}^{\bar{A}_O^{(n)}} = \mathcal{H}^{\bar{A}_I^{(1)}} = 1$. The $V^{(i)}$'s obey 1-comb conditions from the *output* system of a slot to the *input* system of the next slot in the causal ordering with the exception of a different kind of normalisation (remark : this different normalisation was observed elsewhere in the context of transformation of PM into other PM [73]). With such a decomposition one can construct a well-defined deterministic n-comb C on $\mathcal{L}\left(\bigotimes_{i=1}^{n} \mathcal{H}^{A_I^{(i)}} \otimes \mathcal{H}^{A_O^{(i)}}\right)$, see figure D.1, by combining these factored out pieces of PM with the M's via the link product (3.5)

$$C = M^{\tilde{A}^{(1)}} \underset{\bar{A}_O^{(1)}}{*} V^{(1)} \underset{\bar{A}_I^{(2)}}{*} M^{\tilde{A}^{(2)}} \underset{\bar{A}_O^{(2)}}{*} \ldots \underset{\bar{A}_I^{(n)}}{*} M^{\tilde{A}^{(n)}} \tag{6.2}$$

with the subscripts below the link product symbols indicating over which system the link product is taken.

Such a reformulation leads to the same result in terms of probabilities, as shown in lemma 1, which implies

$$\mathrm{Tr}\left\{\tilde{W} \cdot \left(M^{\tilde{A}} \otimes N^{\tilde{B}}\right)\right\} = \mathrm{Tr}\left\{C^T \cdot W\right\} \tag{6.3}$$

where the inner product in the PM point of view becomes a link product[4] in the comb point of view, but the 2 formulations are equivalent. In an abstract notation this is

$$\left(\tilde{W} \middle| M^{\tilde{A}} \otimes N^{\tilde{B}}\right) = W * C \tag{6.4}$$

which proves that a causally ordered process matrix can be proven mathematically equivalent to a quantum comb with trivial input and output spaces. This motivates the MPM as being a particular kind of process matrix compatible with a n-quantum comb as an input. This is actually a general result, that can be expressed through the following theorem

Theorem 8. *Fully causally ordered N-slot process matrix are 1 party MPM, which in turn are themselves N+1 deterministic quantum combs with trivial first tooth input and last tooth output systems.*

Therefore, as soon as the causal order in uniquely determined between all the parties in a PM, one can find a factoring that allows one to make all the singles parties to be unified into a single big deterministic comb. However, this is mostly a trivial statement as the tensor product of n 1-combs is already a n-comb. Here what is explicited is that it is possible to mathematically make this comb appear, since we can always find a set of side channels V_i to be factored out of the PM, although most of its members will often be combs of unit input and output dimension. The end of discussion and the proof of this theorem are left as an appendix because they bring no new elements to the argument and are rather technical, see app. D.2.3.

[4]The overall transpose in the definition (3.5) have been dropped as the link product is taken over the whole space. Note however that although the OCB convention of CJ isomorphism makes the transpose disappear in the generalised Born's rule (left side of the equation (6.3)), it is back when one consider input operators bigger than 1-combs (right side).

Remark : In a work of Feix, Araújo and Brukner [75], they noticed that there exist 2 parties process matrices that are causally separable but become non-separable when a partial trace is applied on one of the subsystems, in the same spirit of the PPT criterion and Werner states in entanglement theory [59]. The interpretation of causally ordered PM as quantum combs could bring a new way to see this behaviour : the partial transpose operation don't modify the result of the inner product (6.4) (as it conserves the CPTP characteristic of the represented maps), but it does change the nature of a quantum comb (partial transpose can affect the positivity of a matrix for example) and can even make it non-valid. The process matrix that they consider is a weighted sum of 2 process matrix with defined causal order that admit a non-trivial decomposition as we just presented, so there must be a way to decompose the process matrix and the operations being applied to it as 2 different products of combs that are in a convex sum. Then an ill-definition of the comb because of the application of the partial trace on it could be an explanation of what is happening from a mathematical point of view when the PM is shifting from causally separable to non-causally separable, and the link between quantum comb and memory channel could maybe lead to a physical interpretation. This can be a possible path for further research.

One could also take the other point of view, and wonder what is happening when directly plugging a quantum comb into a valid process matrix. This the scenario explored in appendix D.1.2, which lead to the conclusion that the PM should have the same causal structure of the comb

All these considerations drive the definition of the one party MPM :

Definition 15 (1 party Multi-round Process Matrix). *A 1 party multi-round process matrix is the most general object that represent the correlations outside the local laboratory of a single party.*

Again this is very obvious and theorem 8 already teaches us that the 1 party MPM is equivalent to a 2-comb and a specific 2 parties PM.

This definition can be reformulated using the tools presented in the chapter 5, and because of theorem 8, we can infer the definition as being a PM whose input CP map is a comb, as wanted. Let W be the MPM of Alice and M her n-comb, the generalised born rule reads

$$\mathrm{Tr}\left\{M^T \cdot W\right\} = 1 \tag{6.5}$$

The bi-linear map leading to a real number regardless of the extensions made on the input or output spaces imply that $W \geq 0$ [1]. This is the positivity condition, now we make the projective constraints on the comb appear in the product

$$\mathrm{Tr}\left\{\left(\mathcal{P}_C^{(n)}\left\{M\right\}\right)^T \cdot W\right\} = 1$$

once again using linearity, self duality and the fact that the depolarising superoperator commutes with the trace operation we can modify this equation into

$$\begin{aligned}
\mathrm{Tr}\left\{\mathcal{P}_C^{(n)}\left\{M^T\right\} \cdot W\right\} &= 1 \\
= \mathrm{Tr}\left\{\left(\mathcal{P}_C^{(n)} + {}_{\prod_i A^{(i)} - \prod_i A^{(i)}}(\cdot)\right)\left\{M^T\right\} \cdot W\right\} \\
= \mathrm{Tr}\left\{\left(\left(\mathcal{P}_C^{(n)} - {}_{\prod_i A^{(i)}}\right)\left\{M^T\right\} + {}_{\prod_i A^{(i)}}M^T\right) \cdot W\right\} \\
= \mathrm{Tr}\left\{\left(\mathcal{P}_C^{(n)} - {}_{\prod_i A^{(i)}}\right)\left\{M^T\right\} \cdot W\right\} + \mathrm{Tr}\left\{{}_{\prod_i A^{(i)}}M^T \cdot W\right\} \\
\mathrm{Tr}\left\{M^T \cdot \left(\mathcal{P}_C^{(n)} - {}_{\prod_i A^{(i)}}\right)\left\{W\right\}\right\} + \mathrm{Tr}\left\{{}_{\prod_i A^{(i)}}M^T \cdot W\right\} &= 1
\end{aligned}$$

From which we can deduce the normalisation condition

$$\mathrm{Tr}\Big\{{}_{\prod_i A^{(i)}}M^T \cdot W\Big\} = 1 \iff {}_{\prod_i A^{(i)}}W = \frac{\mathbb{1}^{\bigotimes_i A^{(i)}}}{\prod_{i=1}^n d_{A_I^{(i)}}} \iff \mathrm{Tr}\{W\} = \prod_{i=1}^n d_{A_O^{(i)}}$$

and the projective one

$$\mathrm{Tr}\Big\{M^T \cdot \Big(\mathcal{P}_C^{(n)} - {}_{\prod_i A^{(i)}}(\cdot)\Big)\{W\}\Big\} = 0 \iff \Big(\mathcal{P}_C^{(n)} - {}_{\prod_i A^{(i)}}(\cdot)\Big)\{W\} := \mathcal{P}_{M^\perp}^{(n)}\{W\} = 0$$

where we have defined a symbol for the projector

$$\mathcal{P}_{M^\perp}^{(n)} \equiv \mathcal{P}_C^{(n)} - {}_{\prod_i A^{(i)}} \tag{6.6}$$

$$\mathcal{P}_M^{A^{(1)} \preceq A^{(2)} \preceq \ldots \preceq A^{(n)}} \equiv \mathcal{P}_M^{(n)} = \Big(1 - \mathcal{P}_{M^\perp}^{(n)}\Big) \equiv {}_1(\cdot) - \mathcal{P}_C^{(n)} + {}_{\prod_i A^{(i)}}(\cdot) \tag{6.7}$$

The recursive definition of this projector is proven in the appendix D.2.2, it yields

$$\mathcal{P}_M^{(1)} = {}_{A_O^{(1)}}(\cdot) \tag{6.8a}$$

$$\mathcal{P}_M^{(n+1)} = {}_{\left(A_O^{(n+1)} - A_{IO}^{(n+1)}\right)}(\cdot) + {}_{A_{IO}^{(n+1)}}\mathcal{P}_M^{(n)} \tag{6.8b}$$

where the subscripts IO were reintroduced to make it clearer. Notice that the projector for a single party acting once is the same as the one for PM, $\mathcal{P}_V$, again a consequence of theorem 8.

Hence the following reformulation of the definition

Theorem 9 (One party multi-round process matrix)**.** *An operator W defined on a Liouville space*

$$W \in \mathcal{L}\left(\bigotimes_{i=1}^n \Big(\mathcal{H}^{A_I^{(i)}} \otimes \mathcal{H}^{A_O^{(i)}}\Big)\right)$$

*is a **one party, n-slots multi-round process matrix** if it is positive, normalised and belong to the subset of 1 party MPM's. Consequently, it must obey*

$$W \geq 0 \tag{6.9a}$$

$$\mathrm{Tr}\{W\} = \prod_{i=1}^n d_{A_O^{(i)}} \iff {}_{\prod_i A^{(i)}}W = \frac{\mathbb{1}^{\bigotimes_i A^{(i)}}}{\prod_{i=1}^n d_{A_I^{(i)}}} \tag{6.9b}$$

$$\mathcal{P}_M^{(n)}\{W\} = W \tag{6.9c}$$

with the projector in equation (6.9c) *defined recursively as* (6.8).

6.1.2 Multiple parties MPM

Now for the extension of the definition to the case of interest *i.e.* when more than one party is present. For the notation the parties will be referred to alphabetically, *i.e.* first party will be Alice with N_A operations $\left\{A^{(N_A)}\right\}$, which forms an N_A-comb M^A on a space

$$M^A \in \mathcal{L}\left(\mathcal{H}^{A_I^{(1)}} \otimes \mathcal{H}^{A_O^{(1)}} \otimes \ldots \mathcal{H}^{A_O^{(N_A)}}\right)$$

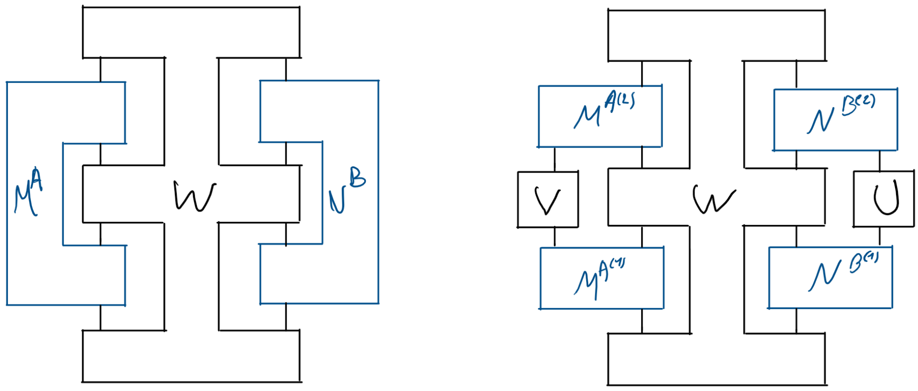


FIGURE 6.1: 4 slot, 2 party MPM with corresponding process matrix

then comes Bob with N_B operations $\left\{B^{(N_B)}\right\}$ that also forms a comb M^B, then Charlies' N_B operations $\left\{C^{(N_B)}\right\}$, etc... So all these parties will get several teeth quantum combs and we want to find the most general object that will link them together, that is to keep track of the correlations between each combs, without assuming a global causal structure:

Definition 16 (n parties MPM)**.** *For a set of quantum combs of n different parties $M^A, M^B, M^C, \ldots$, the most general n-linear mapping of these combs to a probability is represented in the Choi-Jamiołkowski picture by the multi-round process matrix W defined on the Hilbert-Schmidt space of linear operators on the tensor product of all the Hilbert spaces the combs are defined to act upon :*

$$W \in \mathcal{L}\left(\mathcal{H}^{A_I^{(1)}} \otimes \mathcal{H}^{A_O^{(1)}} \otimes \ldots \mathcal{H}^{A_O^{(N_A)}} \otimes \mathcal{H}^{B_I^{(1)}} \otimes \mathcal{H}^{B_O^{(1)}} \otimes \ldots \mathcal{H}^{B_O^{(N_B)}} \otimes \mathcal{H}^{C_I^{(1)}} \otimes \ldots\right) \quad (6.10)$$

Again, we know that the object is some kind of restricted process matrix since the parties can always choose not to use the side-channels, so the n-combs are retrograded to a tensor product of 1-combs, which brings the situation back to a PM. The argument about the side channels as being elements in tensor product with the MPM to form a regular PM can be made again. But this time there is several local subsystems that are not linked together by side channels. Figure 6.1 (left) is an example : It's a 4-slot MPM in which 2 parties are each plugging a 2-comb. There is a side channel inside the subsystems, for example from Alice's first to second tooth, but not between Bob and Alice. Generalised Born's rule is derived the same way as we did in last section. We suppose that the whole thing is an actual valid 4-slot process matrix $\tilde{W}$ which can be factored so the side-channels and the MPM appear in a tensor product, this is always possible to do when the dimension of the side-channels can be taken arbitrarily (figure 6.1, right). The connection between the link product and the Hilbert-Schmidt inner product is again made using trace properties. As we hinted in the discussion below lemma 1, the absence of side-channel can still be interpreted as a special kind of link product where both systems being linked shared no element in common. The link product properties tells us that this special case reduce to regular tensor product (3.8). Therefore, if we take the situation represented in the figure as an example the equivalence reads

$$
\begin{gathered}
W * M^A * N^B = \left(\tilde{W} \middle| \left(M^{A^{(1)}} \otimes M^{A^{(2)}} \otimes N^{B^{(1)}} \otimes N^{B^{(2)}}\right)\right) \\
\mathrm{Tr}\left\{W \cdot \left(M^A \otimes N^B\right)^T\right\} = \mathrm{Tr}\left\{\tilde{W} \cdot \left(M^{A^{(1)}} \otimes M^{A^{(2)}} \otimes N^{B^{(1)}} \otimes N^{B^{(2)}}\right)\right\} \\
\tilde{W} = V \otimes W \otimes U \\
M^A = M^{A^{(1)}} * V * M^{A^{(2)}} \\
N^B = N^{B^{(1)}} * U * N^{B^{(2)}}
\end{gathered}
\tag{6.11}
$$

The new condition can be enforced straightforwardly using the method developed in last chapter. The existing partial order among subset of parties will translate to n-comb conditions which in turn will be imprinted onto the MPM by self-duality of the projector. When no causal relation is known between some (subset of) parties, they get linked through a tensor product and thus their projective condition multiply with each other (5.3). The situation of figure 6.1 is given as an example in an appendix (app. D.3), here we skip the development directly to the theorem as it almost the same as the one we did in last section for the one party case.

Theorem 10 (Several parties multi-round process matrix)**.** *An operator W defined on a Liouville space*

$$
W \in \mathcal{L}\left(\bigotimes_{i=1}^{N_A}\left(\mathcal{H}^{A_I^{(i)}} \otimes \mathcal{H}^{A_O^{(i)}}\right) \otimes \bigotimes_{j=1}^{N_B}\left(\mathcal{H}^{B_I^{(j)}} \otimes \mathcal{H}^{B_O^{(j)}}\right) \otimes ...\right)
$$

*is a $(N_A + N_B + ...)$**-slot multi-round process matrix** between parties A, B, ... if it obeys the conditions*

$$
W \geq 0 \tag{6.12a}
$$

$$
\mathrm{Tr}\{W\} = \prod_{i=1}^{N_A} d_{A_O^{(i)}} \prod_{j=1}^{N_B} d_{B_O^{(j)}} ... \iff {}_{\prod_i A^{(i)} \prod_j B^{(j)} ...}W = \frac{\mathbb{1}^{\otimes_i A^{(i)} \otimes \otimes_j B^{(j)} \otimes ...}}{\prod_{i=1}^{N_A} d_{A_I^{(i)}} \prod_j^{N_B} d_{B_O^{(j)}} \cdots} \tag{6.12b}
$$

$$
\mathcal{P}_M^{(N_A)(N_B)...}\{W\} = W \tag{6.12c}
$$

where the projector to the linear subspace of corresponding multi-round process matrix is given as

$$
\mathcal{P}_M^{(N_A)(N_B)...} := \mathcal{I} - \mathcal{P}_C^{(N_A)} \mathcal{P}_C^{(N_B)} ... + {}_{\prod_i A^{(i)} \prod_j B^{(j)} ...}(\cdot) \tag{6.13}
$$

with $\mathcal{I}$ the identity mapping and $\mathcal{P}_C^{(N_A)} = \mathcal{P}_C^{A^{(1)} \preceq A^{(2)} \preceq ... \preceq A^{(N_A)}}$, $\mathcal{P}_C^{(N_B)} = \mathcal{P}_C^{B^{(1)} \preceq B^{(2)} \preceq ... \preceq B^{(N_B)}}$, ... are the comb projectors defined as in (5.6)*.*

Remark that the definition blends in nicely between the quantum comb and process matrix formalism, as the process matrix formalism is obtained as the limit case where all the parties act only once (so the global causal order is totally undefined). Whereas the quantum comb formalism is obtained as the other limit case : when there is only one party that act as many times as there is slots in the MPM so the global causal order is totally pre-defined, as we have seen last section.

A nice thing is that this way of extending the process matrix framework and the procedure to derive the validity conditions for a given case can be used to represent a wide variety of applications. Because of the Choi-Jamiołkowski isomorphism, one can use this procedure for defining the most general object that takes in combs and output a probability to consider even more higher-order transformation, like from comb to comb, or to process matrix to process matrix. An interesting path of further research would be to retrieve the validity conditions for these two kinds of transform as they were derived in [10] and [73], respectively.

6.2 Properties of the MPM

Now that the definition of the MPM have been established, we move on to investigate the properties of the new object and look into the possible correlations that can be achieved with it. We already know that the set of correlations will at least contain the set of those that admit a causal explanation, since as discussed in the previous section, the MPM admits the quantum comb as a limit case when all the slots have been grouped to form one only comb. Similarly we know that the set will be included in the set of what is achievable with a corresponding process matrix with as much slots as there is in the MPM, since an MPM with as much slots as parties is a PM. Between these 2 sets lies the set of the correlations one can reach with a MPM for which we will now prove various features of the MPM.

6.2.1 Achievable correlations for a fixed number of parties

The first thing that we will show is that a MPM is not equivalent to a PM with an equivalent number of parties. So it is not possible to *coarse-grain* all the actions taken by a party into a single operation within a regular PM. To understand properly the implications, we will need a point of vocabulary [76] : for a N-slot process matrix, the set of correlations achievable when no partial causal ordering between any partitions in the set of parties exist at all is called *genuinely N-slot noncausal* correlations. In this set one can define the 2-causal polytope, which is the set of all the correlations obtainable when it is possible to split the process as a convex sum of every way of splitting the slots into 2 partitions, so that one partition can always be shown to be in the causal past of the other. To each facet of that polytope is associated an inequality that a process is *violating* if it can be used to obtain correlations outside this facet. Within this polytope lies a smaller polytope, the 3-causal polytope, which is the correlations obtainable when the process can be divided into a convex sum of all possible 3 partitions such that there is a causal order between the three, and within it lies a the 4-causal polytope, *et cetera* up to the N-causal polytope which is the polytope formed by the correlations of a probabilistic sum of every processes that admit a fully causal explanation *i.e.* for which there is a definite global causal structure between all the slots [64].

There is a few subtleties for the MPM object taken alone to be clarified. So in this section, we will only consider the MPM as if it was a particular PM, this will help us build insight for the next sections. And to do so, we will use an example. Consider the 3-slot, 2 parties MPM where Alice acts twice and Bob once.

A point to understand is that the local causal order between Alice's operations is absolute. By this we mean that the projective conditions of validity on the MPM are a condition that is more stringent than just forbidding signalling in the wrong direction. Consider the following : if the condition for validity of the MPM was *'compatible with the parties local causal order'* only, it would imply that all the formalism is asking for is that the quantum instruments of the parties cannot be used to signal from their causal future to their causal past, like *e.g.* this kind of requirement : $A^{(2)} \not\preceq A^{(1)}$. But this is not what is considered here. If it was the case, there would be some terms in W that would allow to signal from Alice' second operation to Bob's first as well as from Bob's second to Alice's first operation as long as Bob is not using the side-channel to signal to his future : $A^{(2)} \preceq B^{(1)} \not\preceq\not\succeq B^{(2)} \preceq A^{(1)}$, $A^{(1)} \not\preceq\not\succeq A^{(2)}$. It would be a correct global causal structure in the PM formalism but here the formalism explicitly forbid this. If neither Alice or Bob can pass on information from their causal past to their causal future, it does not mean that Alice can signal to the past of Bob while he can signal to hers at the same time. This can be rephrased in layman's terms as "Just because Alice have forgotten that she received a message from Bob yesterday does not mean that the message could come from today". Actually the formalism forbids to consider such a scenario, as this kind of global causal structure would allow Alice and Bob to maintain a global causal loop as long as their signalling to their own future averages to zero, like in some of the scenarios considered in [77].

To see why such pathological cases are forbidden in the MPM formalism, a direct interpretation of the projective conditions for the examples can provide the insight [5]. Using definition 6.12, the projector to the subspace of MPM taking in Alice's 2-comb and Bob's 1-comb is

$$\mathcal{P}_M^{(2)(1)}(W) \equiv {}_{A_O^{(2)}+B_O-A_O^{(2)}B_O-\left(1-A_O^{(1)}+A^{(1)}\right)A^{(2)}(1-B_O)-\left(1-A_O^{(2)}+\left(1-A_O^{(1)}\right)A_O^{(2)}\right)B}(W) \tag{6.14}$$

We see a first term that restricts the set to operators with either trivial output system of Alice's second operation $A_O^{(2)}$ or Bob's B_O since (C.18)

$$\mathcal{P}^{A_O^{(2)}} \cup \mathcal{P}^{B_O} = {}_{A_O^{(2)}+B_O-A_O^{(2)}B_O}(\cdot)$$

This means that either Alice's second operation or Bob have to be last in the causal sequence[6]. This condition forbids the PM to present terms that allow non transitive causal relation like $A^{(2)} \preceq B \preceq A^{(1)}$ when a party is not using the MPM nor the side-channels to signal to her future : $A^{(2)} \npreceq\nsucceq A^{(1)}$. Systematic analysis of MPM projectors show that because the projector is inferred from comb conditions, the multiplication of comb projectors together always leads to a causal structure in which the only slots that can be last, when not trivial, must be the last tooth of some party's comb. In this regard this is similar to [54]'s definition of valid process matrix which states that at least one of the partie's output system must be trivial.

Therefore, the MPM causal structure is more constrained than simply being compatible with the local order of the combs : it must be in an absolute sense and so it must imply the transitivity of the causal relations between the different operations; The different global causal structures the MPM allows are the ones that respect the local causal order of the combs and are just all the possible shuffling between the different subset of ordered teeth, this was already observed in [9] when generalising the link product to higher-order combs. See also the discussion in appendix D.1.2 for other motivations than this one that made us choose this more narrow definition of MPM.

Another thing to notice is that the subset of N-slot MPM is always smaller or equal to the one of N-slot PM, with the equality being verified if and only if we are in the already discussed special case of when there is as much parties as there is slots. It is also always bigger or equal to the one of $N+1$ slot combs with trivial first input and last output, the equality verified if there is only one party, as already mentioned in this chapter. This can

[5]See also the discussion at the end of appendix D.3, which provide the same interpretation but for the 2 parties acting twice each example

[6]The two other terms in the projector are there to forbid post-selecting when one of the parties in the causal future is trivial, and can be interpreted in the same manner as we are doing now : although they are not signalling among them, the parties must conserve their local ordering, no matter how many slots of the other parties happen in between the slots of one party, her slots will always appear keep their local ordering within the global causal structure : $A^{(1)} \preceq ... \preceq A^{(2)} \preceq ... \preceq A^{(3)} \preceq ....$

be seen by comparing projectors (5.14) with (6.13). It is again a consequence of the duality of the definition, when you allow bigger combs in the MPM the dynamic inside the comb is more varied but it restrict the allowed terms inside the MPM as it must follow the same partial causal ordering as the comb.

Finally, MPM with N parties could be expected to be reducible to a -maybe bigger in dimension- PM with the same number of parties but only one slot for each. This is not the case, the succession of different operations performed by a party cannot be reduced to a 1-comb in a bigger process matrix, in stark contrast with the Jia's point of view of LOCC in which they could [72]. To see it, we can show that there is correlations that would require N parties PM to be obtained that can be obtained with less parties in a N slot MPM. Formally, we say that an N-slot MPM with n parties, $N > n$, is not restricted to correlations being at most n-causal, but can achieve genuinely N-partite causal non-separability as defined in [76], and violate associated causal inequalities. Genuine N-partite causal non-separability mean that the MPM causal structure cannot be reduced to some subsets that have defined causal structure between them but undefined inside of them. To violate a N-causal inequality require that superposition of the causal structure between the slots is at least N-causal. To prove this claim we provide an explicit counter-example of a 3-slots MPM but with only 2 parties that can however violate a 3-causal inequality. Here the example is for the case where Alice has 2 operations, Bob only one and where we have set the dimensions of all the subsystems to 2. Consider the MPM

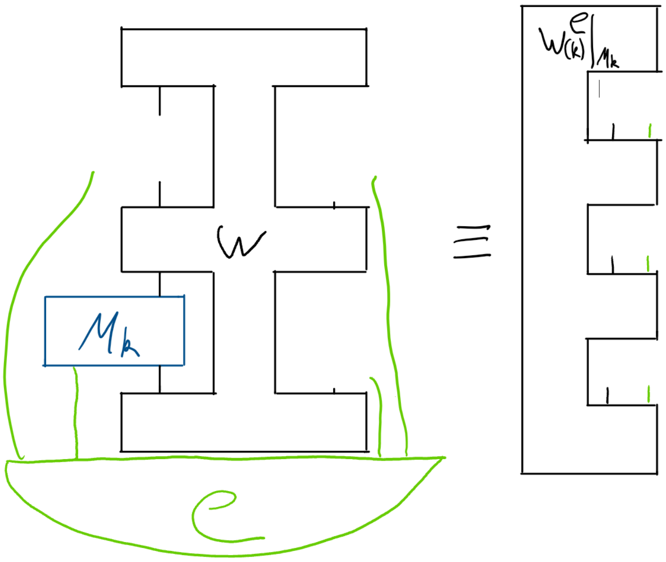
$$W = \frac{1}{8}\left(\mathbb{1}^{\otimes 6} + \frac{1}{\sqrt{5}}\left(\sigma_x\sigma_y\sigma_y\mathbb{1}\sigma_z\sigma_x + \mathbb{1}\sigma_y\sigma_y\mathbb{1}\sigma_y\sigma_y + \mathbb{1}\sigma_y\sigma_y\sigma_x\sigma_x\mathbb{1}\right)\right) \tag{D.18}$$

where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices. The matrix, when considered as a PM is obviously causally non-separable as in definition 13 since one cannot even get to the necessary condition of splitting the terms into a convex sum of those that don't share the same first party, as the 3 coefficients of $\sqrt{5}^{-1}$ don't add up to 1. In appendix D.4.1, we show that it can be used to do an OCB-like game and violate the 3-partite causal inequality in that case. Nonetheless, this MPM cannot be used to achieve **all** the possible genuine N-slot noncausal correlations that could have been obtained with a PM, for example the matrix presented in [78] cannot be realised with the MPM since it's requiring a term in which $A^{(1)}$ is last without $A^{(2)}$ being trivial. So we see that, in terms of correlations, the N-slot MPM with n parties ($N > n$) is a constrained version of the equivalent N parties process matrix, but it is not reducible to a n parties PM by coarse-graining the local operation into one big operation.

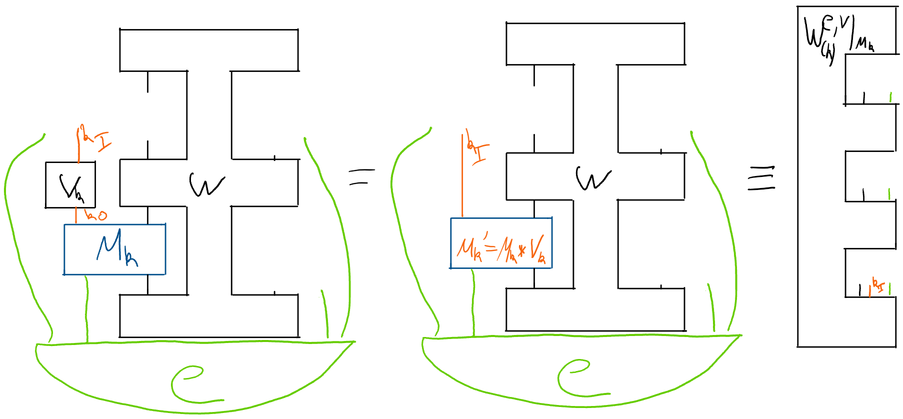
6.2.2 Causal separability and activation

Or can it ? So far we just treated the MPM as being a constrained PM without using the extra resource provided by the side-channels. Here we assess the possibility to observe some *activation* when the side channels are used. For the characterisation to make sense we must first rethink the notion of causal separability, definition 13, to see if it makes sense within the MPM framework.

Once again the 2 limit cases are helpful : when the MPM only takes in one party, its global causal structure is the local causal structure of the party so they are always causally separable. And when the MPM have as many parties as slots, the definition is back to the original one since it is a process matrix. But in between these two, there is indeed the presence of the side channels that can lead to an activated causally non-separable multi-round process matrix when used. Our definition of causal separability is motivated by the one for PM, because as we have shown in the last section the MPM together with the combs it takes in



(A) Illustration of how to obtain the reduced PM



(B) Illustration of how to obtain the reduced MPM

FIGURE 6.2: Interpretation of the reduced element in the definition of causal non-separability for PM and MPM

can be reinterpreted as an overall process matrix by decomposing the combs into 1-combs and side-channels like in *e.g.* equation (D.2). Therefore our proposed definition of causal separability for an MPM will work as the one for PM but with the addition that the output systems of the local operation of a party can be extended by an arbitrary side-channel to the input system of another operation in the causal future of this party.

For example, in the PM definition of causal separability (def. 13), a process matrix was causally separable if one could split it into a convex sum of process matrices $W_{(k)}$ where one party k was first, and then the reduced split elements $\left(W^{\rho}_{(k)}\right)_{|M_k}$ have to remain causally separable for any CP map M_k applied to this party and any extension of all the parties input by an arbitrary entangled system ρ, see figure 6.2a. Then the MPM definition is similar : we also require that the MPM can be split into a convex sum of valid MPM compatible with slot k acting first (remark that if it's not the first action of a party it implies that it will be in a no-signalling causal relation with every other slot of the party that were meant to be in its past so it can be interpreted as first), and that for all CP map acting on k input and output system as well as any channel $V^{A^{(k)} \preceq A^{(j)}}$ extending the output space of k the input system $A^{(j)}_I$ of the same party's causal future, and as well as any arbitrary extension of the input system by a state ρ of all the slots of the MPM, it remains causally separable (figure 6.2b, left). The trick is that the arbitrary side channels V can always be blended in the CP map by the properties of the link product (see theorem 4) because as we have seen they are themselves 1-combs (figure 6.2b, centre). These considerations motivate our definition of causal separability of an MPM.

Definition 17 (Causal Separability of an N-slot MPM with m parties)**.** *For $n = 1$, any MPM is causally separable. For $m \geq 2$, let the m parties be referred to alphabetically $A, B, C, ...$, the set of their subsystems referred with a gothic letter e.g. $\mathfrak{A} = \left\{A^{(1)}, A^{(2)}, ...A^{()}_{N_A}\right\}$, with N_X the cardinality of the set $\mathfrak{X}$ associated with party X. The set of all slots is noted as $\mathfrak{N} = \mathfrak{A} \cup \mathcal{B} \cup ...$, with cardinality $N = \sum_X N_X$. A multi-round process matrix W between the n parties*

$$W \in \mathcal{L}\left(\left(\bigotimes_{i=1}^{N_A} \mathcal{H}^{A_I^{(i)}} \otimes \mathcal{H}^{A_O^{(i)}}\right) \otimes \left(\bigotimes_{j=1}^{N_B} \mathcal{H}^{B_I^{(j)}} \otimes \mathcal{H}^{B_O^{(j)}}\right) \otimes ...\right) := \mathcal{W}_{IO}^{\mathfrak{N}}$$

*is said to be **causally separable** if and only if, for any extension $\mathcal{R}_{I'}^{\mathfrak{N}}$ of the parties' incoming systems and for any ancillary quantum state ρ in this extension*

$$\rho \in \mathcal{L}\left(\left(\bigotimes_{i=1}^{N_A} \mathcal{H}^{A_{I'}^{(i)}}\right) \otimes \left(\bigotimes_{j=1}^{N_B} \mathcal{H}^{B_{I'}^{(j)}}\right) \otimes ...\right) := \mathcal{R}_{I'}^{\mathfrak{N}}$$

$W \otimes \rho$ can be decomposed as

$$W \otimes \rho = \sum_{\mathfrak{X} \subset \mathfrak{N}} \sum_{k \in \mathfrak{X}} q_k W_{(k)}^{\rho} \tag{6.15}$$

with $q_k \geq 0$, $\sum_k q_k = 1$, and where for each slot k of a party X, $W_{(k)}^{\rho} \in \mathcal{W}_{IO}^{\mathfrak{N}} \otimes \mathcal{R}_{I'}^{\mathfrak{N}}$ is a process matrix compatible with the k-th action of party X, $X^{(k)}$, being first in the global causal structure. And the decomposition is made such that for any extension of the output subsystem of k, towards the input subsystems of all the slots in the causal future of its party X,

$$\mathcal{L}\left(\mathcal{H}^{X_{O''}^{(k)}} \otimes \left(\bigotimes_{i>k}^{N_X} \mathcal{H}^{X_{I''}^{(i)}}\right)\right) := \mathcal{V}_{O''I''}^{\mathfrak{X} \setminus X^{(i<k)}}$$

and for all CP map M_k defined on the MPM plus its extensions

$$M_k \in \mathcal{W}_{IO}^{\mathfrak{N}} \otimes \mathcal{R}_{I'}^{\mathfrak{N}} \otimes \mathcal{V}_{O''I''}^{\mathfrak{X} \setminus X^{(i<k)}} \tag{6.16}$$

the conditional $(N-1)$-slot MPM

$$\left(W_{(k)}^{\rho}\right)_{|M^k} \equiv Tr_{X_{IOI'I''O''}^{(k)}}\left[\left(M^k \otimes \mathbb{1}^{\mathfrak{N} \setminus X^{(k)}}\right) \cdot \left(W_{(k)}^{\rho} \otimes \mathbb{1}^{\mathcal{V}_{O''I''}^{\mathfrak{X} \setminus X^{(i<k)}}}\right)\right]$$

is itself causally separable.

This definition is made to avoid the kind of activation like in the case considered in [54] : they showed that there exist process matrix that couldn't lead to non-causal correlations between the parties no matter the CP maps the parties applied, but when pre-shared entanglement was used they could obtain non-causal correlations. Hence they argued that the definition must also take into account the possibility of extending the input systems by shared ancillas. In [70], they proved that this was the only coherent definition of causal non-separability hence definition 13. This is in the same spirit that the side-channels are added alongside with pre-shared entangled ancillas to the possible thing that can extend the parties' input and output subsystems, see appendix D.4.2 for an example of such an activation by side channel.

Remark that the above definition is equivalent to imposing that the overall constrained process matrix, $\tilde{W} = W \otimes (\bigotimes_i V_i)$ is itself causally separable for all possible set of channels V_i between all the slots of all the parties.

6.2.3 Achievable correlations for a fixed number of slots

But is there an advantage provided by the presence of the side channel ? Or are all the non-causal correlations coming from the PM behaviour of the central MPM only ?

What is certain is that not all valid PM that are compatible with MPM conditions can be obtain by the transformations applied by the parties' combs. Indeed if for example the MPM (D.13) can be factored in a tensor product between A and B like $W = W^{A^{(1)}A^{(2)}} \otimes W^{B^{(1)}B^{(2)}}$, no matter the operations of Alice and Bob they will never be able to transform the MPM into a PM $\bar{W}$ that allow signalling between them [73, 79]. Formally, all the operations the parties can perform on a MPM can be taken back to an equivalent PM situation by the kind of transformation considered in figure 6.3 (here represented without the possibility of pre-shared ancillas). That is, here for a 2 parties example, for all quantum instruments of Alice and Bob, $\{C_i^A\}$ and $\left\{C_j^B\right\}$ on can decompose the elements into probabilistic 1-combs and 2-combs so that, for a 2-slots example $C_i^A = M_i^{A^{(2)}} * \tilde{M}_i * M_i^{A^{(1)}}$, $C_i^A \in \mathcal{L}\left(\mathcal{H}^{A^{(1)}} \otimes \mathcal{H}^{A^{(2)}}\right)$, $M_i^{A^{(2)}} \in \mathcal{L}\left(\mathcal{H}^{A_{I'}^{(2)}} \otimes \mathcal{H}^{A_O^{(2)}}\right)$, $\tilde{M}_i \in \mathcal{L}\left(\mathcal{H}^{A_O^{(1)}} \otimes \mathcal{H}^{A_{O'}^{(1)}} \otimes \mathcal{H}^{A_I^{(1)}} \otimes \mathcal{H}^{A_{I'}^{(1)}}\right)$, $M_i^{A^{(1)}} \in \mathcal{L}\left(\mathcal{H}^{A_I^{(1)}} \otimes \mathcal{H}^{A_{O'}^{(1)}}\right)$. The example of the figure correspond to the situation

$$W \underset{A_O^{(1)}A_I^{(2)}}{*} \tilde{M}_i \underset{B_O^{(1)}B_I^{(2)}}{*} \tilde{N}_j = \bar{W} \tag{6.17}$$

where $\bar{W}$ is a valid PM

$$\bar{W} \in \mathcal{L}\left(\mathcal{H}^{A_I^{(1)}} \otimes \mathcal{H}^{A_{O'}^{(1)}} \otimes \mathcal{H}^{A_{I'}^{(2)}} \otimes \mathcal{H}^{A_O^{(2)}} \otimes \mathcal{H}^{B_I^{(1)}} \otimes \mathcal{H}^{B_{O'}^{(1)}} \otimes \mathcal{H}^{B_{I'}^{(2)}} \otimes \mathcal{H}^{B_O^{(2)}}\right)$$

The situation we considered as an example of what the MPM is not able to do is when :

$$\bar{W} = W^{A^{(1)}A^{(2)}} \otimes W^{B^{(1)}B^{(2)}} \underset{A_O^{(1)}A_I^{(2)}}{*} \tilde{M}_i \underset{B_O^{(1)}B_I^{(2)}}{*} \tilde{N}_j$$

$$\bar{W} = \left(W^{A^{(1)}A^{(2)}} \underset{A_O^{(1)}A_I^{(2)}}{*} \tilde{M}_i\right) \otimes \left(W^{B^{(1)}B^{(2)}} \underset{B_O^{(1)}B_I^{(2)}}{*} \tilde{N}_j\right)$$

$$\forall \tilde{M}_i \, , \forall \tilde{N}_j$$

where we have used the fact that the link product is associative, distributive and have the tensor product as a special case. This trivially mean that if the MPM does not allow communication between the two parties, no matter what the parties will do locally they cannot enable a signalling between them.

Passed this obvious example to show that not all valid PM that respect the MPM conditions can be obtained from any MPM, one can infer that the signalling between the parties cannot be activated by a party alone. It can only be present *a priori* in the MPM and all the parties can do is to use their local operations and channels to activate it. To see why it is true, remark that to activate a term like a signalling from Alice's first output to Bob's

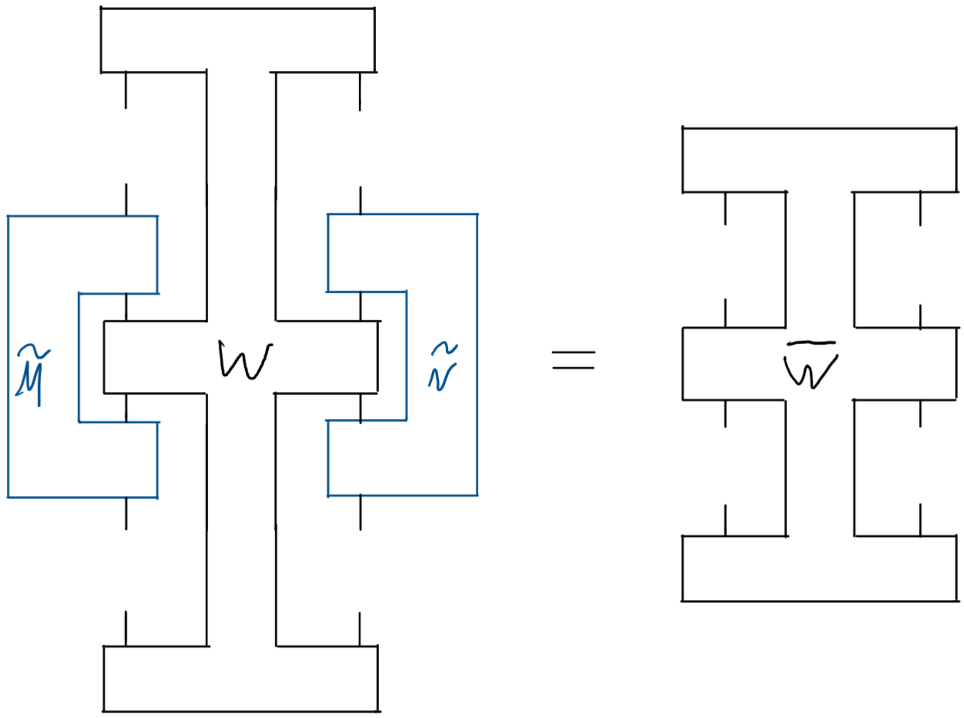


FIGURE 6.3: Reduction of the MPM into a PM

second input, if there was no term in the expansion of the MPM that allowed so, for example if the MPM is $W = \mathbb{1} + \sigma_x^{A_I^{(2)}} \sigma_x^{B_O^{(2)}}$ and we want to activate new communications between Alice's first operation and Bob's second, the operators $\tilde{M}$ and $\tilde{N}$ must be equal to something similar to

$$\tilde{M}_i \otimes \tilde{N}_j = \mathbb{1} + \sigma_y^{A_{I'}^{(1)}} \sigma_y^{B_{O'}^{(1)}}$$

which is not a product state. This has a simple interpretation, the only mean for Alice and Bob side channels to enable a non-existing communication between them is that the side channels are in fact themselves signalling to each other. Technically we want the joint partial operations plus the side channels, *i.e.* everything in blue in figure D.1, to be a *non-localisable* operator [55], but this can only be realised if there is at least one communication between the two.

One can ask if all the new correlations brought by the side-channels can be though only as side channels, *i.e.* new wires inside a party local operations as we have seen. But the only situation where they cannot be understood as a decomposition in side channels and pre-shared entanglement correspond to the one where the partial actions of the parties on the MPM are semi-localisable, which mean that there was signaling between them. Thus the only new things allowed by side-channels, although they may increase the bound of violation of some causal inequality, is only due to the extra resources which can be interpreted causally as the addition of an extra 'wire' between the local operations of a party. For a scenario to present genuinely new correlations compared to a regular process matrix, we must allow the side channels to be able to signal to the side channels of other parties, which is contrary to the locality postulate used as a starting point of the formalism.

Chapter 7

Discussion and Conclusion

Through this thesis we have presented an extension of the formalism of process matrices of Oreshkov, Costa, and Brukner called the Multi-Round Process matrix. This extension was motivated by the LOCC paradigm and looked at what happened when the parties in a process matrix were authorised to act more than once during the process in order to represent a concrete communication protocol where several rounds of messages are exchanged between the parties. The main conclusion is that there is no new arising kind of correlation that cannot be explained when the full situation is seen as a bigger process matrix that encompass the memory of the parties. This was expected as we have proven the MPM to be the kind of object that lies in between the quantum comb and the process matrix formalisms, from which it is, respectively, a less and a more constrained version in the space of linear operator acting on the parties input and output systems.

These results could not be understood without the first part of the thesis, in which we presented how to reinterpret quantum mechanics in the operational framework. In this part, we presented two seemingly different formalisms : the quantum combs, representing how one can compose fragments of quantum circuit together, and the process matrix, representing the most general way of keeping track of the correlations between a set of local experimenters for whom the validity of quantum theory is only assumed inside their local laboratories. The common ground for these two objects being that they both relies on the Choi-Jamiołkowski isomorphism to be mathematically represented.

In the second part of the thesis we built various results on this isomorphism towards the formulation of the MPM. We first developed a mathematical tool introduced in the field in [12], that we named *depolarising superoperator*, definition 5.1, and thouroughly studied its properties since it hasn't been done before. In particular we showed that this map, when simplified in its prescript formulation could be easily manipulated by the algebraic rules of a *boolean ring*. Then we followed the ideas of [12, 70, 73, 79] to reformulate the validity conditions of process matrix in terms of this superoperator together with a normalisation and positivity condition. We proved that it could be done for quantum combs -theorem 6- and process matrices -theorem 7-. We noted that the validity conditions of the latter were nothing else than the imprint left on it by the ones of the former through the Hilbert-Schmidt inner product. The reason is due to the self-dual character of the depolarising superoperator, as it is an orthogonal projector. This method for deriving conditions of validity turn out to be very general, since to any object that can be expressed in projective formulation we can find the validity conditions of an other object mapping it to a probability via the Hilbert-Schmidt inner product. This paved the road for a generalisation: while for the process matrix the imprint are the conditions of validity of 1-combs, for the sought multi-round process matrix we proved that the validity conditions were the imprint on the object of bigger combs, theorem 10.

This theorem provided a characterisation of the MPM, which was the objective of this thesis. In its one party variation, given by theorem 9, it was notably used to show the equivalence between the formalism of process matrices, quantum combs, and multi-round process matrices. This theorem 8 is but a proof of the fact that when the global causal structure is totally determined, the 3 objects are equivalent. To further explore the features of the MPM we showed that if we used the definition of causal separability of process matrices on the MPM alone, this could result in scenarios where an activation of causal non-separability by allowing side-channels in an MPM was possible, this motivated a new definition that considered the ensemble formed by the tensor product of all the possible side channels and the MPM as a bigger process matrix on which the regular definition of causal separability made sense. This yielded the new definition 17 for the causal separability of an MPM. In its essence the MPM was observed to always corresponds to a bigger, but constrained, process matrix. Finally we looked into the new correlations made possible for the MPM and we argued that there is nothing the MPM could do that could not be explained as something that was equivalent to a bigger process matrix.

This still leave room for further research, as new correlations could arise if the side-channels could became entangled together, but this would require to extend the formalism again since the MPM framework was not defined to be able to consider these kind of non-localised side channels. However with the method and the tools developed, there is no reason why such an object should not be derived in quite the same manner as we did for the combs, process matrices and multi-round process matrices. This is not the only use of the algebra of superoperators that can be made. Actually, and as we mentioned below theorem 10, another related path of research would be to retrieve the theory of dynamics [73] and the generalisation of combs as a hierarchisation of supermaps [9, 10] using the algebra of the depolarising superoperator and its self duality with respect to the inner product. This would provide an interesting way of treating various concepts and theories on the same footing.

Another, yet related consequence of this reformulation is that since we can reinterpret certain process matrix as a multi-round process matrix and vice-versa, it could be insightful to always try to reformulate the PM as an MPM where all the side-channels that could be factored out of the PM are incorporated in the combs to be plugged into the corresponding MPM. The idea would be to use the fact that, compared to the PM, the combs always have a physical way of being implemented, which could be a step in the effort to find a way of physically interpreting the PM. An insight of why this could be a helpful reformulation was done in the remark p. 50 were we noticed that there could be a link between the application of a partial transpose on the operation of a party that suddenly lead to a causally non-separable PM [75] and the interpretation of this PM as an MPM taking in 2-combs and for which the partial transpose could make the combs invalid.

Finally, the characterisation of the MPM also opens the door for a generalisation of the LOCC paradigm. Now that we have the tool, it would be interesting to express equivalents of the results of this paradigm in this language. An example thereof would be to characterise the full extend of the processes obtainable by an MPM and those who are not. This would require some generalisation of the notion of entanglement, as well as localisable and causal operations [55]. A first step in this direction would be to consider whether the generalisation proposed by Jia [72] is adapted for the MPM formalism. Another link to explore is the one that can be made between the MPM and the attempt to formulate a resource theory for the absence of pre-defined causal structure [80].

To conclude with this thesis, we can say that the theory of the MPM has now been established as being a midway between the quantum combs and the process matrices. The object itself and the mathematics developed to tackle the problem have both opened several paths for further research as we just have reviewed. A more general open question raised incidentally would be whether or not it is possible to demonstrate connections between the different formalisms and physical transformations whose representation rely on the utilisation of Choi-Jamiołkowski isomorphism, as we just did with the combs and the process matrix.

Part III

Appendices

Appendix A

Appendices to chapter 2

A.1 Quantum Mechanics Reminder

Quantum theory tells us that the systems are described by *states* living in an associated Hilbert space. These states are accessible by making a *measurement*. There lies the difference with classical theory : in Newtonian mechanics, measurement results are points in the phase space while in quantum mechanics, they are traditionally one-dimensional projectors on a Hilbert space. This is the reason of Kochen-Specker theorem : point in the phase space are uniquely determined, there is one point that is "true" which correspond to the result of measurement and all the other are false, while it is impossible to assign a truth-falsity nature to a projector so that there is always only one result of a measurement [14, 32, 62]. Now the question is how does one state becomes another ?

Simply put, this is the actual goal of quantum mechanics : "given two states X and Y, find what is the *probability* of A *transforms* into B ?". In the operational framework, there is the notion of a party : "Alice, the party A, is given a state X, for measurement settings s^A what's the probability that she gets outcome o^A and output state Y ?". To achieve this end, another fundamental theorem in quantum theory, Gleason theorem, or more precisely *Gleason-type* theorems, is here to prescribe the permitted probabilistic theory that links X with Y, depending on how the state is represented (this is of course a simplification, see [62] for an example of Gleason-type derivation, and refer to [32] for a rigorous treatment).

Quantum mechanics is then itself a mathematical framework, or a set of rules, for the construction of physical theories. For example one can build the theory of quantum electrodynamics (QED) which describes with fantastic accuracy the interaction of light with matter using the framework. The subtlety is that the theories built using quantum mechanics contains specific rules that are not determined by quantum mechanics. To paraphrase [16] : if quantum mechanics is the operating system of a computer, then the physical theories like QED are its softwares. This thesis won't go that far nonetheless. As stated in the introduction, we are interested in exploring the features of a particular formulation of the theory and its associated formalism.

A.1.1 Dirac Pitcure

Since the framework is built on finite dimension Hilbert space, the mathematics can essentially be reduced to matrix manipulation and thus plain linear algebra. In dimension d, a Hilbert space $\mathcal{H}$ is a linear complex vector space where a scalar product is defined, and can therefore be represented as : $\mathcal{H} \sim \mathbb{C}^d$. We will now show the mathematical structure of this representation without further justification, for more details see *e.g.* the following textbooks [16, 29] or lectures notes [18, 31].

The basic way of representing the states and transformation of quantum theory is through Dirac's bra-ket formulation. The **states are vectors** in the Hilbert space, which are noted with a *ket* :

$$\text{state} \equiv |\psi\rangle \in \mathcal{H} \tag{A.1}$$

The *dual* of the states, obtained through the adjoint operation †, are given as bra :

$$\text{state in dual space} \equiv (|\psi\rangle)^\dagger = \langle\psi| \in \mathcal{H}^* \tag{A.2}$$

In the representation, the adjoint operation correspond to a hermitian conjugation, so applying the transpose and complex conjugation : $(\cdot)^\dagger = (\cdot^*)^T$. The *inner product* (often called *scalar*) between element is then a linear application $(\mathcal{H}, \mathcal{H}^*) \to \mathbb{C}$

$$\text{inner product} \equiv \langle\psi| \cdot |\phi\rangle = \langle\psi|\phi\rangle = c \tag{A.3}$$

where $|\phi\rangle \in \mathcal{H}$, $\langle\psi| \in \mathcal{H}^*$ and $c \in \mathbb{C}$ is a complex number called the *amplitude*. If two vectors have a zero inner product, they are said to be *orthogonal*

$$\langle\psi|\phi\rangle = 0 \iff |\psi\rangle \perp |\phi\rangle \tag{A.4}$$

Note that the inner product induces a *norm* on the Hilbert space as

$$\text{norm of a vector} \equiv \| |\psi\rangle \| = \sqrt{\langle\psi|\psi\rangle} \tag{A.5}$$

also, because the space is conventionally normalised, the d dimensional states, called *qudits*, are a *linear superposition* of *orthonormal basis elements* (ONB elements) of the space, defined like a set of d vectors $\{|i\rangle\}_{i=1}^d$ that are normalised

$$\langle i|i\rangle = 1 \tag{A.6}$$

and orthogonal $\langle a|b\rangle = \delta_{a,b}$, $|a\rangle, |b\rangle \in \{|i\rangle\}_{i=1}^d$. The decomposition then write [34]

$$\text{state decomposition} \equiv |\psi\rangle = \sum_{i=1}^{d} \alpha_i |i\rangle \tag{A.7}$$

where the coefficients of the superposition, α_i, conventionally follow the normalisation condition

$$\text{normalisation} \equiv \sum_{i=1}^{d} |\alpha_i|^2 = 1 \tag{A.8}$$

The *outer product* (sometimes called *dyadic*) between two states gives a linear operator on the space $(\mathcal{H}, \mathcal{H}^*) \to \mathcal{L}(\mathcal{H})$:

$$\text{outer product} \equiv |\phi\rangle \otimes \langle\psi| = |\phi\rangle\langle\psi| = \hat{O} \tag{A.9}$$

where $\hat{O} \in \mathcal{L}(\mathcal{H})$ is an operator in the space of linear operators on the space $\mathcal{H}$. Notice that this space is also a Hilbert space which admits a representation as a complex matrix space (or a complex vector space of dimension d^2) : $\mathcal{L}(\mathcal{H}) \sim \mathcal{H} \otimes \mathcal{H}^* \sim \mathbb{C}^{d\times d}$. Operators in general link states to other states, they will then be how the transformations are represented in Dirac picture *i.e.* **transformations are linear operators** defined on the Hilbert space

$$\text{transformation} \equiv \hat{O} |\psi\rangle = |\phi\rangle \tag{A.10}$$

$|\psi\rangle, |\phi\rangle \in \mathcal{H}$. The probability associated with the transformation of a state $|\psi\rangle$ into another $|\phi\rangle$ is given by the **Born's rule** :

$$\Pr\big(|\psi\rangle \to |\phi\rangle := \hat{O}\,|\psi\rangle\,|\hat{O}\big) \equiv |\langle\psi|\phi\rangle|^2 \tag{A.11}$$

which state that the probability of a transformation is the squared modulus of the associated amplitude.
Vectors that are not modified up to a scalar factor by an operator are said to be its *eigenvectors*. The scalar factor is called the *eigenvalue*.

$$\text{eigenvector} \equiv \hat{O}\,|\psi\rangle = o\,|\psi\rangle \tag{A.12}$$

In equation (A.12), $|\psi\rangle$ is an *eigenvector* of $\hat{O}$ with *eigenvalue* $o \in \mathbb{C}$. Note that since states are vectors in Dirac picture, one can say *eigenstate* to designate the vector or ensemble of vectors associated with a particular eigenvalue o. The subspace spanned by the set of eigenvectors with the eigenvalue o is called the *eigenspace* of o.

The states admit a dyadic decomposition into an ONB of the space, $\{|i\rangle\}_{i=1}^d \equiv \langle a|b\rangle = \delta_{a,b}$, $|a\rangle, |b\rangle \in \{|i\rangle\}_{i=1}^d$ like

$$\text{state decomposition} \equiv \psi = \sum_{i=1}^{d} |i\rangle\,\langle i|\psi\rangle \tag{A.13}$$

, where the **closure relation** has been used $\sum_{i=1}^d |i\rangle\,\langle i| = \mathbb{1}$, with $\mathbb{1}$ the identity operator, defined as the neutral of transformations :

$$\mathbb{1}\,|\psi\rangle = |\psi\rangle \tag{A.14}$$

Remark that, by comparison of equation (A.7) and (A.13), we have that $\alpha_i = \langle i|\psi\rangle$. This is the reason why coefficient α_i are often referred as *amplitudes of the decomposition*.

As for the operators, they also admit a *bi-orthogonal decomposition* [29], let there be 2 orthonormal basis $\{|r_i\rangle\}_{i=1}^d, \{|s_i\rangle\}_{i=1}^d$ with $|r_i\rangle \neq |s_i\rangle$ in general,

$$\text{operator decomposition} \equiv \hat{O} = \sum_i \lambda_i\,|r_i\rangle\langle s_i| \tag{A.15}$$

with $\lambda_i \in \mathbb{C}\,\forall i$. The representation of an operator in a particular basis is given by using the closure relation twice

$$\text{dyadic decomposition} \equiv \hat{O} = \sum_{i,j} |i\rangle\langle i|\,\hat{O}\,|j\rangle\langle j| = \sum_{i,j} \langle i|\hat{O}|j\rangle\,|i\rangle\langle j| := \sum_{i,j} O_{ij}\,|i\rangle\langle j| \tag{A.16}$$

where the shorthand notation have been used : $\{|r_i\rangle\}_{i=1}^d := \{|i\rangle\}$,$\{|r_j\rangle\}_{j=1}^d := \{|j\rangle\}$. This is the *dyadic decomposition* of this operator. When there exist a base in which an operator admits a diagonal representation, we say that this operator is *normal*

$$\text{normal operator} \equiv \hat{O} = \sum_i \lambda_i\,|i\rangle\langle i| \tag{A.17}$$

again with $\{|i\rangle\}_{i=1}^d$ an orthonormal basis. Notice that the vectors in the dyadic decomposition are in fact the eigenvectors of the operator and the coefficients its eigenvalues.

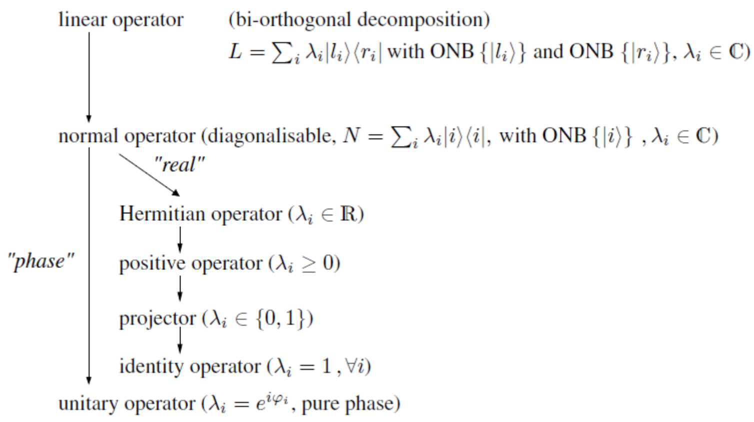


FIGURE A.1: Hierarchy of operators. Characterisation of operators through their dyadic decomposition. The characteristics of the eigenvalues are in parenthesis. This figure comes from [29], pay attention to the fact that the term 'projector' in the figure should have been 'orthogonal projector', see text.

Therefore one can characterise an operator by characterising its eigenvalues and eigenvectors. An important theorem in linear algebra states that an operator is diagonalisable, or *admits a spectral decomposition*, if and only if it is commuting with its adjoint :

$$\hat{O} \text{ is normal } \iff \left[\hat{O}, \hat{O}^\dagger\right] \equiv \hat{O}\hat{O}^\dagger - \hat{O}^\dagger\hat{O} = 0 \tag{A.18}$$

There is an associated lemma that states that two operators $\hat{A}$ and $\hat{B}$ are *simultaneously diagonalisable in a basis* if they happen to commute : $\left[\hat{A}, \hat{B}\right] = 0 \iff$ there exist a base in which both $\hat{A}$ and $\hat{B}$ admit a diagonal representation (A.17). A normal operator is *hermitian* if he is self-adjoint

$$\text{Hermitian operator} \equiv \hat{O}^\dagger = \hat{O} \iff \lambda_i \in \mathbb{R} \forall i \quad ; \tag{A.19}$$

positive (semi-)definite (PD (PSD)) if all its eigenvalues are greater (or equal) to zero $\lambda_i > (\geq)0$; and unitary if

$$\text{unitary operator} \equiv \hat{O}\hat{O}^\dagger = \hat{O}^\dagger\hat{O} = \mathbb{1} \iff \lambda_i = e^{i\varphi_i},\ \varphi_i \in \mathbb{R} \tag{A.20}$$

where operator that only respect one of the two conditions, $\hat{O}^\dagger\hat{O} = \mathbb{1}$ or $\hat{O}\hat{O}^\dagger = \mathbb{1}$ are called, respectively, *isometry* and *coisometry*. With these considerations, one can show that, in *closed* quantum system, *i.e.* systems that don't exchange information with their environment, information preservation imply that **the evolution of the system from a state to another is descried by unitary operators**.

A special kind of linear operator $\hat{P} \in \mathcal{L}(\mathcal{H})$ is called *projector* [13]. A projector is *idempotent*

$$\text{idempotent operator} \equiv \hat{P}^2 = \hat{P} \tag{A.21}$$

If, moreover, the projector is self-adjoint *i.e.* Hermitian, then it is called an *orthogonal projector*[1]. An orthogonal projector then only have eigenvalues in $\{0, 1\}$. The rank of a projector is its number of non-zero eigenvalues. The outer product of a state with itself gives an

[1]It is common to only deal with orthogonal projectors in Quantum Mechanics so most people often omit the word orthogonal, which is often implicitly implied when talking about projectors.

example of rank 1 orthogonal projector.

$$\text{rank 1 orthogonal projector} \equiv |\psi\rangle\langle\psi| \tag{A.22}$$

Remark that the only operator that is hermitian, unitary and idempotent is the identity operator, whose eigenvalues are all 1.

The need for the physical theory to output real quantities will have as an implication that the observables quantities -*the observables*-, which in this picture will be operators, must be self-adjoint, hence the observables are hermitian operators. In the Dirac picture, this implies that **an observable is a sum of orthogonal projectors**, often called *projective* or *von Neumann* measurement [16]. Let there be M a projective measurement, which is also a linear operator on the space, since it's a linear combination of linear operators, it has spectral decomposition

$$\text{von Neumann Measurement} \equiv M = \sum_m \lambda_m \hat{P}_m \tag{A.23}$$

where $P_m = |m\rangle\langle m|$ are orthogonal projectors that project onto the eigenspace[2] of M. They obey $P_i P_j = \delta_{ij} P_i$. The eigenvalues λ_m are the measurement outcomes associated with an output system state of $\hat{P}_m |\psi\rangle$, where $|\psi\rangle$ was the state of input system. The probability of measuring a particular outcome λ_m is given, using Born's rule, by

$$P(m) = |\langle m|\psi\rangle|^2 = \langle\psi|m\rangle \langle m|\psi\rangle = \langle\psi|\hat{P}_m|\psi\rangle \tag{A.24}$$

the expectation value of an observable, given a state $|\psi\rangle$, is thus given by

$$E(M) = \sum_m \lambda_m P(m) = \langle\psi|M|\psi\rangle \tag{A.25}$$

whose shorthand notation is $\langle M\rangle$. However, projective measurement tend to be too restrictive in how one can define the set of measurement outcomes to be observe and they are mathematically less convenient than general unitary operators. The reason of that rigid structure is because measurement operation must lead to updated state that are well defined and normalised. This difficulty is overcome by *Positive Operator-Valued Measure (POVM) formalism*, which is not concerned by the output system but only in probabilities linked with observables [16]. The only conditions one have to impose on POVM is that they are a set of operators[3] $\{E_m\}$ that are positive and satisfy

$$\sum_m E_m = \mathbb{1} \quad , E_m \geq 0\, \forall m \tag{A.26}$$

in order for the outcome probabilities associated with each *POVM element* (or *effect*)

$$P(m) = \langle\psi|E_m|\psi\rangle \tag{A.27}$$

to be properly defined (positive and normalised) for every vector $|\psi\rangle$ in the space. This may seems like a mathematical convenience, which it is, but Nairmark's[4] theorem states that POVM are actually a special case of projective measurement onto a *bigger* space. So any POVM $\{E_m\}$ in the Hilbert space $\mathcal{H}$ can be *dilated* to a set of projective measurement

[2] Or eigenstate in the particular case of rank 1 projector.

[3] Notice that we are progressively getting rid of the hats on operators as they are no longer relevant for the story.

[4] Or Neumark, depending on the Cyrillic transcription.

$\{M_m\}$ in a larger Hilbert space $\mathcal{H}'$, $\dim(\mathcal{H}') > d$ [32]. This imply that POVM are physically implementable using extra system extending the original system, called *ancillary subsystem* or *ancilla*. We will come back to this notion in the next section.

To summarise, in the Dirac picture states are vector in a Hilbert space (A.1) and transformations are linear operators on this space (A.10). Admissible transformations, -evolution- require that the operators are unitary (A.20). As for the measurements they are represented by a special kind of unitary operation that are linear combination of orthogonal projectors (A.23).

A.1.2 Circuit formalism

The circuit formalism is a concept that comes from the quantum computation field of research [16, 18]. It is motivated by its intuitive formulation, close to the classical circuit formalism (like electrical circuit representation). In quantum computing, the wires represent quantum states, and the boxes are operations on it. A special kind of box that takes in quantum state and output classical information, the outcome, is representing the von Neumann (destructive) measurements. the circuit formalism makes use of the fact that the global state of a system composed of several non-entangled states can be factored as a tensor product of subsystems.

For example, suppose Alice and Bob each possesses a two-level quantum harmonic oscillator (called *qubit* in quantum information and computation) in their own laboratories. These two apparatus are set up locally and are not connected in any way. Then the Hilbert space of the whole situation, of dimension $\dim(\mathcal{H}) = 2 \times 2 = 4$, can be factored into a **tensor product**[5] of two smaller Hilbert space of dimension 2, each one representing the local 2-level system of Alice and Bob. We then say that the global state is in a product state where $|\psi\rangle^{AB} \in \mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$,

$$\text{product state} \equiv |\psi\rangle^{AB} = |\phi\rangle^{A} \otimes |\chi\rangle^{B} \tag{A.28}$$

with $|\phi\rangle^A$ is the local state of Alice, thus the state of her harmonic oscillator, and $|\chi\rangle^B$ is the one of Bob.Often the tenso product is omited for conciseness when no confusion is possible, like

$$|\psi\rangle^{AB} = |\phi\rangle^{A} |\chi\rangle^{B}$$

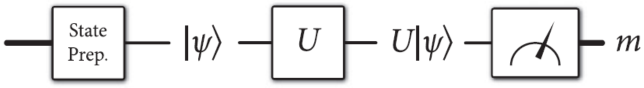
and one can show that every such $|\psi\rangle^{AB}$ vector, there exist orthonormal bases $\{|r_i\rangle\} \in \mathcal{H}^A$ and $\{|p_j\rangle\} \in \mathcal{H}^B$ such that we can decompose the state as

$$\text{Schmidt decomposition} \equiv |\psi\rangle^{AB} = \sum_{k=1}^{d'} \sqrt{\lambda_k}\, |r_k\rangle\, |p_k\rangle \tag{A.29}$$

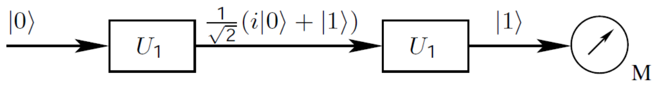
with $d' = \min\left(\dim\left(\mathcal{H}^A\right), \dim\left(\mathcal{H}^B\right)\right)$. This is the *Schmidt decomposition* and the coefficients λ_k are the *Schmidt coefficients*, verifying the following properties : $\lambda_k \geq 0$, $\forall k$; $\sum_k \lambda_k = \| |\psi\rangle \|^2$. The number of non-zero Schmidt coefficient is called *Schmidt rank*.

The particularity of quantum mechanics is that because of the vector character of states they can be in a *coherent superposition*, which is a well-normalised convex combination of

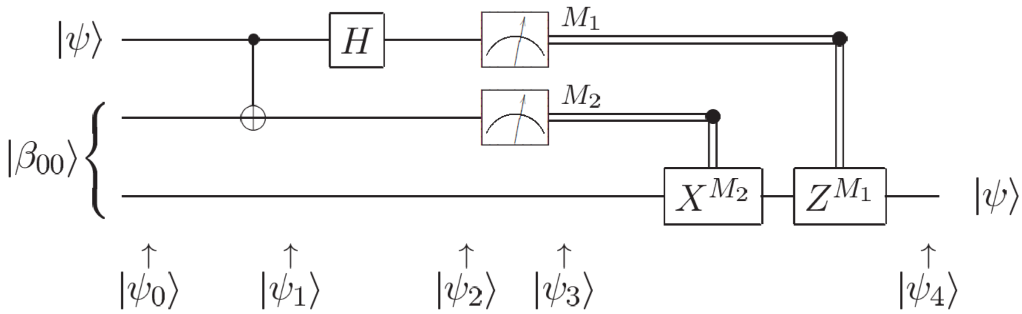
[5] The reason why it is in a tensor product and not a direct sum is that because the global description have dimension that is the product of the dimensions of its part, and not the sum. To see why, consider 3 level harmonic oscillators (*qutrits*): if Alice and Bob both have one, the number of possible (pure) states, that is orthogonal states, we can have to describe is given by <Alice possibilities> times <Bob's>, hence 3×3 and not plus.



(A) General structure of a quantum circuit : from classical information (bold font wire in the left) a quantum state is prepared, then is subject to unitary evolution, before finally undergoing a deterministic measurement where the quantum system is tossed after measurement and we get back a new classical information from it (bold font wire to the right). Figure from [34].



(B) Basic quantum circuit : a qubit initialised in state $\langle 0|$ (left) undergoes two unitary operations (the boxes, taking in an input state by its left and with the resulting state getting out on its right) then is measured (here the M circle, usually measurement is represented with a gauge in its pictogram). Figure from [29].



(C) More advanced quantum circuit (*quantum teleportation of a qubit*). Notice how the outcome of a measurement (on the right of the gauge boxes) is represented with 2 wires (as in electrical circuits) and can be used to act on the unitary applied on the remaining qubit. For a full explanation and the source of this figure, the reader is invited to consult [16] pp.26-28.

FIGURE A.2: Illustrations of quantum circuit formalism

several states like,

$$\text{superposition of a qubit} \equiv |\phi\rangle^A = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \tag{A.30}$$

where $\{|0\rangle, |1\rangle\}$ are the basis vectors of Alice' subspace $\mathcal{H}^A$, so the two possible states of her 2-level harmonic oscillator. This is an inherently quantum phenomenon, which has no equivalent in classical mechanics. It is not to be confused with a probabilistic mixture of two pure states (this will be considered in the von Neumann picutre). An other particularity is that sometimes states can be entangled, which mathematically correspond to a global state that cannot be written as a product state in any basis, or to a state that have a Schmidt rank above 1. Entangled states are then also a particularity of quantum mechanics. Their particular behaviour require that their state can only be described as a whole since one cannot find a basis in which to factor the state as a tensor product. Often the two subsystems are entangled while not being at the same place, like per example a pair of photons coming out of a matter-antimatter annihilation. One can show that entangled states can present particular correlations that cannot be explained when each subsystem is considered as a local system, but only when the full entangled state is considered. Such correlations are called *non-local*. An example of such a state is called the *maximally entangled state* between two subspaces $\mathcal{H}^A$ and $\mathcal{H}^B$

$$\text{maximally mixed state} \equiv |I_{\mathcal{H}^{A\otimes B}}\rangle\rangle := \frac{1}{\sqrt{d'}} \sum_{i=1}^{d'} |i\rangle\,|i\rangle\,, \tag{A.31}$$

with again $d' = \min\left(\dim\left(\mathcal{H}^A\right), \dim\left(\mathcal{H}^B\right)\right)$, and shorthand notation $HilbA \otimes B = \mathcal{H}^A \otimes \mathcal{H}^B$. This is a state that possess maximal Schmidt rank.

As we will see in the main text, section 2.2.1, there are more objects than pure states and projective measurement that one can wish to represent in circuit formalism, like POVM for instance. The way of doing it is to *purify*, which consists on adding extra *ancillary* states, called the *ancillas*, to express the state on a higher dimensional space, in which it can be expressed as pure states and projective measurements. Usually this procedure is only a mathematical artifice. When augmenting the dimension of a system to this end, we say that we are *dilating the system*.

A.2 Matrix representation in a particular basis

Since every mathematical object in the theory can ultimately be represented as a matrix, one benefits to find a basis where the matrices are easy to manipulate. As explained above, every density operator on n quantum systems (we will refer to them as $A^{(1)}, A^{(2)}, ..., A^{(n)}$) can be expressed as a matrix in a particular basis. We call this the Hilbert-Schmidt decomposition of the density matrix.

Definition 18. *Hilbert-Schmidt decomposition [30] Let $\rho \in \mathcal{L}\left(\bigotimes_{i=1}^n \mathcal{H}^{A^{(i)}}\right)$ with $d_{A^{(i)}}$ the dimension of the $i-th$ Hilbert space $\mathcal{H}^{A^{(i)}}$. For each Hilbert space $\mathcal{H}^{A^{(i)}}$ in the tensor product, let us associate an orthonormal basis noted as $\left\{\sigma_j^{A^{(i)}}\right\}_{j=0}^{d_j^2-1}$, where the superscript refer to the space the matrix is a basis of, while the subscript refer to the number of the basis element. Then, the Hilbert-Schmidt decomposition of ρ is given by*

$$\begin{aligned} \rho &= \bigotimes_{i=1}^{n} \sum_{j=0}^{d^2_{A^{(i)}}-1} r_j^{(A^{(i)})} \sigma_j^{A^{(i)}} \qquad , r^{(A^{(i)})} \in \mathbb{C}, \forall i, j \\ \rho &= \sum_{j=0}^{d^2_{A^{(1)}}-1} \sum_{k=0}^{d^2_{A^{(2)}}-1} \dots \sum_{l=0}^{d^2_{A^{(n)}}-1} \tilde{r}_{jk...l}\, \sigma^j_{A^{(1)}} \otimes \sigma^k_{A^{(2)}} \otimes \dots \otimes \sigma^l_{A^{(n)}} \qquad , \tilde{r}_{jk...l} = r_j^{(A^{(1)})} r_k^{(A^{(2)})} \dots r_l^{(A^{(n)})} \end{aligned} \tag{A.32}$$

As a consequence of Gleason's theorem, the inner product in the space of linear operators on an Hilbert space is represented by a trace (see equations (2.4), (2.9)) [14]. A natural choice of basis is then one that is traceless in every basis element but one, so the traces are quicker to compute. One define this traceless basis for a d-dimensional Liouville space as a particular Hilbert-Schmidt basis represented by an ensemble of d^2 basis elements $\{\sigma_i\}_{i=0}^{d^2-1}$ that span the whole space and that are chosen such that

$$\sigma_0 \equiv \mathbb{1} \tag{A.33a}$$

$$\mathrm{Tr}\{\sigma_i\} = 0 \quad , \forall i > 0 \tag{A.33b}$$

$$\mathrm{Tr}\{\sigma_i \sigma_j\} = d\ \delta_{i,j} \tag{A.33c}$$

where $\delta i, j$ is the Kronecker symbol and d the dimension of the space. If we add the supplementary constraint that the basis element must all be hermitian, $\sigma_i^\dagger = \sigma_i\, \forall i$, this is a particular basis called Generalised Gell-Mann Basis.

Since any matrix in the Hilbert-Schmidt space of dimension d can be expanded in such a base, the density matrix expansion will admit a decomposition in that fashion. This is the

Bloch vector expansion of the density matrix

$$\rho = \frac{1}{d}\mathbb{1} + \vec{b}\cdot\vec{\Gamma} \tag{A.34}$$

where $\vec{b}\cdot\vec{\Gamma}$ is a linear combination of all matrices $\{\sigma_i\}$ and the vector $\vec{b} \in \mathbb{R}^{d^2-1}$, defined like $b_i = \langle\Gamma_i\rangle = \mathrm{Tr}\{\rho\Gamma_i\}$ is called *Bloch vector* [16]. The $b_0 = \frac{1}{d}$ factor is here because of the fact that every density operator have unitary trace[6].

For a density matrix on several subsystems $A^{(1)}, A^{(2)}, ...A^{(n)}$, the Bloch vector version of the general Hilbert-Schmidt expansion (18) reads

$$\rho \;=\; \frac{1}{d_{A^{(1)}}d_{A^{(2)}}...d_{A^{(n)}}}\mathbb{1}^{A^{(1)}}\otimes\mathbb{1}^{A^{(2)}}\otimes...\mathbb{1}^{A^{(n)}} + \sum_{j=1}^{d^2_{A^{(1)}}-1}\sum_{k=1}^{d^2_{A^{(2)}}-1}...\sum_{l=1}^{d^2_{A^{(n)}}-1}\tilde{b}_{jk...l}\;\sigma^j_{A^{(1)}}\otimes\sigma^k_{A^{(2)}}\otimes...\otimes\sigma^l_{A^{(n)}} \tag{A.35}$$

where we have again grouped the coefficient into one :

$$\tilde{b}_{jk...l} = b_j^{(A^{(1)})}b_k^{(A^{(2)})}...b_l^{(A^{(n)})} \in \mathbb{R}$$

A.2.1 Generalised Gell-Mann basis

As it have been implied, the restrictions (A.33) on the choice of basis for practical Hilbert-Schmidt decomposition still leave one extra degree of freedom in the choice of basis. In this work we will assume a particular choice of basis, the one where basis elements are all hermitian, called the generalised Gell-Mann Basis (GGB) [44]. The reason for this choice is double : on one hand, it is an intuitive choice for physicists, as it reduce, respectively, to the well-known Pauli basis and the Gell-Mann basis in dimension 2 and 3. On the other hand, this particular basis have extra properties following from its hermitian character and due to the fact that it is the standard generators for SU(N) algebras [32], which can help simplify the computation by using group theory[7].

The explicit formulation of the basis elements in Dirac bra-ket notation can be found in [44]. Here We summarise the important characteristics of such a basis, for a d-dimensional Hilbert space $\mathcal{H}^X$ (therefore a Liouville space of dimension d^2). There is d^2 elements among which :

1. $\sigma_0 = \mathbb{1}$, the unit matrix.
2. $\frac{d(d-1)}{2}$ symmetric basis elements (*e.g.* Pauli's σ_x matrix in $d = 2$);
3. $\frac{d(d-1)}{2}$ antisymmetric basis elements (*e.g.* Pauli's σ_y matrix in $d = 2$);
4. $(d-1)$ diagonal elements (*e.g.* Pauli's σ_z matrix in $d = 2$).

[6] For CP trace non-increasing maps that lead to a linear operator with trace inferior to one, the convention is to represent it by a trace one density operator like (A.34) multiplied by some factor in $[0;1]$, so the trace of the ensemble have the correct value.

[7] See *e.g.* how the proofs in [28] are simplified by using the properties of the tensor product of several SU(2) groups.

And their properties are the following

$$\mathrm{Tr}\{\sigma_i\} = 0 \quad ,\forall i > 0 \tag{A.36a}$$

$$\sigma_i^X \sigma_j^X = \delta_{ij} \mathbb{1}^X + i\, \epsilon_{ijl}\, \sigma_l^X \quad ,\forall i, j, l \tag{A.36b}$$

$$\det\left\{\sigma_i^X\right\} = 1 \quad ,\forall i \tag{A.36c}$$

$$\text{Eigenvalues of } \sigma_i^X \in \{-1, 1\} \tag{A.36d}$$

Appendix B

Appendices to chapter 3

B.1 Quantum network in OCB convention of Choi-Jamiołkowski isomorphism

When the Choi-Jamiołkowski ismorphism (2.28) was introduced in chapter 2, subsection 2.2.3, it was emphasised that it was not the regular definition of the isomorphism. Indeed, an extra transpose was introduced in the definition, following OCB's convention [1] rather than the usual one introduced in [38] :

$$M^{XY} = \mathfrak{C}\left(\mathcal{M}^X\right) := \mathcal{M}^X \otimes \mathcal{I}^X \left(|I_{\mathcal{H}^X}\rangle\rangle \langle\langle I_{\mathcal{H}^X}|\right) \tag{B.1}$$

The motivation behind the extra transpose added by OCB is a convenience in that it gets rid of the transpose inside the inner product. In this section we will show how this convention influence the equations that were defined with the original Choi-Jamiołkowski isomorphism. Because most of the literature about the undefined causal structure field of research is using PM formalism, most of the formulas that are required to be modified by the convention change were those coming from of quantum comb formalism and from work anterior to 2012, with a few notable exceptions in recent literature like, *e.g.*, [11, 72, 80].

B.1.1 Quantum Combs in OCB convention

Because the transpose conserve the semi-definite positive character of a matrix, don't change its trace and commutes with the partial trace operation, it is straightforward to see that the validity conditions of a deterministic quantum comb (3.1) are unchanged. Consequently, if the conditions on an operator to be a deterministic comb are unchanged, and because the transposition don't affect the Hilbert-Schmidt norm nor the positivity of a matrix, the probabilistic combs of definition 8 are unaffected as well.

B.1.2 Link product in OCB convention

Here we show that formulae (3.5) and (3.6) follows from the original definition of [40] when one applies the OCB convention of transposing the Choi-Jamiołkowski matrix (2.28). Let the maps $\mathcal{M} \in \mathcal{L}\left(\left(\mathcal{L}\left(\mathcal{H}^0\right)\right) \to \left(\mathcal{L}\left(\mathcal{H}^1\right)\right)\right)$ and $\mathcal{N} \in \mathcal{L}\left(\left(\mathcal{L}\left(\mathcal{H}^1\right)\right) \to \left(\mathcal{L}\left(\mathcal{H}^2\right)\right)\right)$, with associated CJ operators $M \in \mathcal{L}\left(\mathcal{H}^1 \otimes \mathcal{H}^0\right)$ and $N \in \mathcal{L}\left(\mathcal{H}^2 \otimes \mathcal{H}^1\right)$ given through (2.28). We know that to the composition of the maps correspond a link product in the CJ picture :

$$\mathcal{C} = \mathcal{N} \circ \mathcal{M} \iff C = \tilde{N} * \tilde{M} \tag{B.2}$$

where the tilde notation is here to emphasise that these operators don't follow the definition of CJ matrices used in this text. Translating in to the OCB convention we obtain

$$\tilde{N} * \tilde{M} = N^T * M^T \qquad ,$$

which gives

$$\tilde{N} * \tilde{M} \equiv \mathrm{Tr}_1 \left[\left(\mathbb{1}^2 \otimes \left(M^T \right)^{T_1} \right) \cdot \left(\left(N^T \right) \otimes \mathbb{1}^0 \right) \right] \tag{B.3}$$

where T_1 denotes the partial transpose over 1. Using (F.4a), one successively obtains

$$\begin{aligned} \tilde{N} * \tilde{M} &= \mathrm{Tr}_1 \left[\left(\mathbb{1}^2 \otimes (M)^{T_1} \right)^T \cdot \left(N \otimes \mathbb{1}^0 \right)^T \right] \\ &= \mathrm{Tr}_1 \left[\left(N \otimes \mathbb{1}^0 \right) \cdot \left(\mathbb{1}^2 \otimes (M)^{T_1} \right) \right]^T \\ &= \left\{ \mathrm{Tr}_1 \left[\left(N \otimes \mathbb{1}^0 \right) \cdot \left(\mathbb{1}^2 \otimes (M)^{T_1} \right) \right] \right\}^T \end{aligned}$$

which then motivates the definition (3.5) adopted in the text. The general case, defined like

$$\tilde{N}^B * \tilde{M}^A = \mathrm{Tr}_{A\cap B} \left[\left(\mathbb{1}^{B\setminus A} \otimes \left(\tilde{M}^A \right)^{T_{A\cap B}} \right) \cdot \left(\tilde{N}^B \otimes \mathbb{1}^{A\setminus B} \right) \right] \tag{B.4}$$

gives, by the same reasoning as above,

$$N^B * M^A = \left[\mathrm{Tr}_{A\cap B} \left[\left(N^B \otimes \mathbb{1}^{A\setminus B} \right) \cdot \left(\mathbb{1}^{B\setminus A} \otimes \left(M^A \right)^{T_{A\cap B}} \right) \right] \right]^T \tag{B.5}$$

where the $\cap$ and $\setminus$ symbols have the same significance than in section 3.1.3. This motivates formula (3.6) in the main text, chapter 3.

B.2 Extended state of the art for the process matrix

In this appendix we proposed a more technical and extended review of the field of quantum theory without fixed causal structure. We first take an historical approach, mainly inspired by [25] and then we present the most recent trends in the field. The reader is also invited to consult the progress article by Č. Brukner [26] for an alternative introduction to the subject.

B.2.1 A brief history

In 1928, John von Neumann published his work on the mathematical formulation of quantum theory. Since then, it has become the canonical text about the math needed for the theory[1]. The theory is based on a set of axioms one can now find in any book related to quantum mechanics or quantum information theory [2]. But 90 years later, the interpretation of the very axioms on which rely the theory is still unclear. Contrary to the axioms of classical mechanics or relativity, those of quantum theory appear to ask, per example, for Hilbert space *simply because it gives good experimental predictions* and that's it for the underlying reason motivating this choice of axioms. People carried on with the problem open for a long time, but when the field of quantum information began to thrive in the 2000's, renewed interest in the foundations of quantum theory came with it. As evoked in chapter 2, Hardy [19], Fuchs [20, 21], Caves [22], and Brassard [23] initiated a program

[1] The theoretical reminders of this work itself are partly based on the new print of this book [13].

[2] *e.g.* they appear in the "reference" textbook about quantum computation [16]

that consisted on rebuilding the quantum theory from axioms based on principles from information theory. The feature of such theories was their operational formulation, i.e. that they function in intuitive terms of preparation and measurement of observable quantities in a laboratory instead of the more classical concepts like position and momentum.

Among the proposed way of doing so, Lucien Hardy [19], pushed on the program further and proposed to address the question of including undetermined causal structure into the framework of quantum theory [2, 3]. This is motivated by a will to combine causal properties of quantum and relativity theories for situations where effects from both theory become relevant. Note that, as it was also pointed out by Brukner, this is a way to answer -or at least go around- what is referred to as *the problem of time* because it provides a no longer background dependent notion of causality [81]. The argument for a quantum framework with undetermined causal structure can be summarised as follow: on one hand, quantum mechanics is known to be formally incompatible with physical observable that have pre-existing value independently of the measurement context [27]. In the light of this consideration, it is reasonable to question the absolute character of the causal structure. Shouldn't it be also tainted with uncertainty ? On the other hand, general relativity teaches us that it is possible to have dynamical causal structure, per example for observers in the vicinity of a massive body. One can thus conceive a scenario where a massive particle is in a superposition of locations that are more or less near a much more massive body. Because the particle don't see the same geometry of space-time in both its superposed paths, the causal structure will be itself in a superposition [74]. Therefore, the framework must have room for both dynamical and undetermined causal structure.

The search for an operational quantum theory that would treat correlation in time variable and in spatial variables in a unified way and arise from quantum information principles gave birth to a wide variety of funky-named theories. Hardy himself first proposed a theory based on a mathematical object called the *causaloid* [3], that he later refined by the introduction of the *duotensor* [4]. Other proposed formalism included, chronologically, *process matrix* [1], *causaly neutral* theory [5], theory without predefined time [6, 7], Oeckl's theory [8], *higher order quantum computation*[3] [9, 10], and Portmann *et al.*'s *causal boxes* [11]. Note that all these theories were not all specifically introduced in order to provide an quantum theory with no defined causal structure. Their similarities lies mainly in the fact that they are all operational and they rely on the mathematical tool of Choi-Jamiołkowski isomorphism which we will introduce in chapter 2 and that allows to treat transformation and higher-order transformations (i.e. transformations of transformations, this notion will also be explained in chapter 2) in a similar manner.

B.2.2 Current developments

As the theory of indefinite causal structure is similar to entanglement theory on many aspects, like evoked in the main text, a lot of the progress realised in the field is driven by the search for finding the PM equivalent to concepts in entanglement. One such example is the bounding and to measure how much a process matrix can violate a given causal inequality. This is currently an active topic of research, using the causal witness and Semidefinite programming (SPD) people could classify and derive bounds on simplified cases [64, 75, 82, 83]. An analogue of the Tsirelson bound, which gives the upper bound on how much an inequality can be violated [84], have also been theorised [1], but have been only proven for a restricted class of operations [85]. Other work showed that the set of what

[3] The one base on the quantum comb framework.

is feasible in terms of correlations with the PM is equivalent to a particular kind of causal processes for which you allow some post-selection [86].

Although there is experimental proof of the feasibility of a causally non-separable process matrix [65, 87–89], as well as the possibility of superposing temporal order [90], be aware that the only PM that was realised in a lab so far was the quantum switch and upon which all these experiment rely. The thing with the switch is, as we just pointed out, that it cannot violate any causal inequality. So far, all that was demonstrated was that superposition of causal order is possible with photons. Moreover there is still debate about how to interpret the experiments [61]. In any case, doubt exist on whether or not there actually is a physically achievable process that can violate any causal inequality. Process matrix formalism have been used to prove that this is impossible for 1-partite processes [1, 63]. An argument based on dynamics of these process matrix [73], as well as one based on an analogue procedure of mixed state purification (see *e.g.* [16]) that purify a causally non-separable process matrix into a causally separable one of higher dimension [79], give reasonable evidence on the impossibility for a 2-partite matrix that violate a causal inequality to be physical. For a higher number of parties, this actually seems to be possible in the formalism, and surprisingly even for classical processes [28, 78] but such processes are more thought of an artefact of the formalism than something actually physically implementable. Keep in mind however that these arguments are still postulates and that their veracity is thus to be taken with carefulness.

Actually, when the number of parties in the process is bigger than 2, there is conceptual difficulties that are still being addressed, like what is genuine N-partite causal non-separability and what is just a 'classical' extension of some non-separable process with a lower number of parties [76]. The principal problem lying in the fact that the definition of causal separability does not easily get generalised because of the possibility for some pre-shared entangled ancillas to *activate* causal non-separability. The original definition given in [1] was adapted accordingly in [54] were the concept of activation was defined and further refined very recently in [70], where they argued that talking about causal separability without ruling out the possibility of activation should not be allowed.

Composition of several process matrices together is also a topic of interest as it could be used in an extended quantum Shannon theory. Clues on how to do this composition are given in [9, 10, 73] but, as Jia and Sakharwade shown there exists an inherent problem of creating causal loops when linking two process matrices together [91]. This problem have lead people to think that there might not actually exist such composition rules [92]. This will also be evoked in the chapter 6 of this thesis. Other related directions of research are the one toward finding a correspondence between all process matrices and physical implementation [93], as well as toward a resource theory for causal non-separability [80]

Nonetheless, possible applications of this new resource, particularly for the quantum switch are already beginning to flourish. Several tasks and algorithms for quantum computation in which there is a speedup compared to the classical case have been found. Among these one can find examples communication complexity [94–96], information processing [42, 97, 98], query complexity [42, 99, 100], and even computation [48] per example. It was even thought for a while that it was possible to use this superposition of causal structure to allow perfect signalling through noisy channel [101] (and related works [102, 103]) but this have been very recently proven to be a misinterpretation and that the perfect signalling was not coming from the absence of causal structure itself [97, 98].

Appendix C

Appendices to chapter 5

C.1 Properties of the depolarising superoperator

Here we will show the properties of the depolarising superoperator (5.1) introduced in section 5.1. Recall that it is defined as a map acting on a subspace X of a Hilbert space like $\mathcal{P}^X(\cdot) : \mathcal{L}\left(\mathcal{H}^X \otimes \mathcal{H}^Y\right) \to \mathcal{L}\left(\mathcal{H}^X \otimes \mathcal{H}^Y\right)$ such that, for $F \in \mathcal{L}\left(\mathcal{H}^X \otimes \mathcal{H}^Y\right)$,

$$\left(\mathcal{P}^X \otimes \mathcal{I}^Y\right)\{F\} \equiv \frac{\mathbb{1}^X}{d_X} \otimes \mathrm{Tr}_X\left[F\right] :=_X F \tag{5.1}$$

These properties will be used extensively in the result part. It is helpful to see what the action of the superoperator is doing in traceless basis, *e.g.* suppose $F^X \in \mathcal{L}\left(\mathcal{H}^X\right)$, $F^{XY} \in \mathcal{L}\left(\mathcal{H}^X \otimes \mathcal{H}^Y\right)$

$${}_XF^X = \frac{\mathbb{1}^X}{d_X} \otimes \mathrm{Tr}_X\left[\sum_i f_i/d_X\, \sigma_i^X\right] = f_0\, \mathbb{1}^X \tag{C.1}$$

and thus

$${}_XF^{XY} = \frac{\mathbb{1}^X}{d_X} \otimes \mathrm{Tr}_X\left[\sum_{ij} f_{ij}/d_X\, \sigma_i^X \sigma_j^Y\right] = \sum_j f_{0j} \mathbb{1}^X \sigma_j^Y \tag{C.2}$$

remark that we have factored the dimension of $\mathcal{H}^X$ out of the coefficients as pure mathematical convenience, we will often omit to do so when it is not important.

C.1.1 Linear properties of the depolarising superoperator

For the illustrating purposes of this subsection, we will need an Hilbert space $\mathcal{H} = \mathcal{H}^X \otimes \mathcal{H}^Y \otimes \mathcal{H}^Z$ with dimension $d = d_X d_Y d_Z$. We define on it 2 arbitrary operators F and G as well as two hermitian operators H and K. Let there also be the coefficients a, b, c and d $\in \mathbb{R}$.

Most of the trivial properties can be deduced from the properties of the tensor product as well as the (partial) trace. Among those we have that the map is linear

$${}_X(aF + bG) = a\ {}_XF\ +\ b\ {}_XG \tag{C.3}$$

The linearity property will be very often used, actually we will directly write the linear coefficient and the maps in the subscript itself to lighten the equations, for example :

$$a\ {}_XF + b\ {}_YF + c\ {}_{XY}F \equiv {}_{aX+bY+cXY}F \tag{C.4}$$

Another property is that depolarising superoperators commute with each other when acting on different subsystems

$$_{XY}F = \frac{\mathbb{1}^X \otimes \mathbb{1}^Y}{d_X d_Y} \otimes \mathrm{Tr}_X \left[\mathrm{Tr}_Y \left[F\right]\right] = \frac{\mathbb{1}^Y}{d_Y} \otimes \mathrm{Tr}_Y \left[\frac{\mathbb{1}^X}{d_X} \otimes \mathrm{Tr}_X \left[F\right]\right]$$

because the partial trace is itself commutative when acting on different subsystems, therefore

$$_{XY}F =_{YX} F \tag{C.5}$$

And when the map are acting on the same subsystems, they're obviously the same and applying the depolarising superoperator twice is the same as once

$$\begin{aligned} _X(_X(F)) = (_{X^2}(F)) &= \frac{\mathbb{1}^X}{d_X} \otimes \mathrm{Tr}_X \left[\frac{\mathbb{1}^X}{d_X} \otimes \mathrm{Tr}_X F\right] \\ &= \frac{\mathbb{1}^X}{d_X} \otimes \left(\mathrm{Tr}_X \left[\frac{\mathbb{1}^X}{d_X}\right] \otimes \mathrm{Tr}_X F\right) \\ &= \frac{\mathbb{1}^X}{d_X} \otimes \left(\frac{d_X}{d_X}\ \mathrm{Tr}_X F\right) \\ &= \frac{\mathbb{1}^X}{d_X} \otimes \mathrm{Tr}_X F \\ &=_X F \end{aligned}$$

We have thus proven that the map is idempotent :

$$_{X^2}F =_X F \tag{C.6}$$

A property that will be needed is how the maps behave with tensor product. First, depolarising superoperators acting on different elements of a tensor product can be obviously be factored out because of partial trace relations (F.7) and distributivity of tensor product (F.5d)

$$\left(_X F^X \otimes {_Y G^Y}\right) = {_{XY}}\left(F^X \otimes G^Y\right) \tag{C.7}$$

It is less trivial that this is also true for when they act on a product of operators that have been trivially extended using tensor product. Consider the following link-ish[1] product of 2 operators acting on not totally equivalent systems, if each one have a projector acting upon the non-common subsystem, one can factorise out the projectors as a product of them two i.e.

$$\left(_X F^{XY} \otimes \mathbb{1}^Z\right)\left(\mathbb{1}^X \otimes_Z G^{YZ}\right) = {_{XZ}}\left(\left(F^{XY} \otimes \mathbb{1}^Z\right)\left(\mathbb{1}^X \otimes G^{YZ}\right)\right) \tag{C.8}$$

Proof :
We use GGB decomposition :

$$\begin{aligned} \left(_X F^{XY} \otimes \mathbb{1}^Z\right)\left(\mathbb{1}^X \otimes_Z G^{YZ}\right) &= {_X}\left(\sum_{ij} f_{ij}\ \sigma_i^X \sigma_j^Y \mathbb{1}^Z\right) \cdot {_Z}\left(\sum_{kl} g_{kl}\ \mathbb{1}^X \sigma_k^Y \sigma_l^Z\right) \\ &= \left(\sum_j f_{0j}\ \mathbb{1}^X \sigma_j^Y \mathbb{1}^Z\right) \cdot \left(\sum_k g_{k0}\ \mathbb{1}^X \sigma_k^Y \mathbb{1}^Z\right) \\ &= \sum_{jk} f_{0j} g_{k0}\ \mathbb{1}^X \left(\sigma_j^Y \sigma_k^Y\right) \mathbb{1}^Z \end{aligned}$$

[1]Compare with (3.5).

Which is the same expression as

$$
{}_{XZ}\left(\left(F^{XY}\otimes \mathbb{1}^{Z}\right)\left(\mathbb{1}^{X}\otimes G^{YZ}\right)\right) = {}_{XZ}\left(\left(\sum_{ij} f_{ij}\ \sigma_i^X\sigma_j^Y\mathbb{1}^Z\right)\cdot\left(\sum_{kl} g_{kl}\ \mathbb{1}^X\sigma_k^Y\sigma_l^Z\right)\right)
$$
$$
= {}_{XZ}\left(\sum_{ij}\sum_{kl} f_{ij}g_{kl}\ \sigma_i^X\left(\sigma_j^Y\sigma_k^Y\right)\sigma_l^Z\right)
$$
$$
= \sum_{jk} f_{0j}g_{k0}\ \mathbb{1}^X\left(\sigma_j^Y\sigma_k^Y\right)\mathbb{1}^Z
$$

These properties together define an algebra of the prescripts. This algebra is associative, commutative, distributive and all its elements are idempotent this is thus a *Boolean ring* [60]. Here's an example of an application of the algebra for the simplification of depolarising superoperators without expanding them :

$$
{}_{(aX+bY)(cX+dZ)}F = {}_{ac\ X+ad\ XZ+bc\ XY+bd\ YZ}F
$$

lowercase letter being coefficients and uppercase the maps.

Finally, the depolarising superoperator happen to be Positive Trace-Preserving[2] and CPTP if the operator it is acting on is hermitian. Indeed when acting on the whole space of the operator the depolarising superoperator map any operator to the unit matrix times, which have all eigenvalues equal to 1, multiplied by a positive constant, so the output operator is always positive. For illustration purpose we will use the calligraphic notation $\mathcal{P}^X(\cdot) \equiv_X (\cdot)$, to emphasise the fact that this is a map. The positivity reads

$$
{}_XF^X = \mathcal{P}^X\left(F^X\right) \geq 0 \quad ,\forall F^X \geq 0 \tag{C.9}
$$

The complete positivity for H^{XY} hermitian is then given as

$$
{}_XH^{XY} = \left(\mathcal{P}^X\otimes\mathcal{I}^Y\right)\left\{H^{XY}\right\} \geq 0 \quad ,\forall H^{XY} \geq 0 \tag{C.10}
$$

Proof :
Using GGB, one have that if H^{XY} is PSD, then all its eigenvalues are in $[0,+\infty[$ so this means that

$$
\text{Eigenval}\left(H^{XY}\right) = \text{Eigenval}\left(\sum_{ij} h_{ij}\ \sigma_i^X\sigma_j^Y\right) \geq 0
$$

Using the fact that the eigenvalues of a tensor product is equivalent to all the pairwise products of the eigenvalues of its members, and that the coefficients can be factored out, we have

$$
\text{Eigenval}\left(H^{XY}\right) = h_{ij}\text{Eigenval}\left(\sigma_i^X\right)\text{Eigenval}\left(\sigma_j^Y\right) \geq 0\,,\forall i,j
$$

which are all positive reals or zero. Particularly we have

$$
h_{0j}\text{Eigenval}\left(\sigma_0^X\right)\text{Eigenval}\left(\sigma_j^Y\right) \geq 0 \iff h_{0j}\text{Eigenval}\left(\sigma_j^Y\right) \geq 0\,,\forall j
$$

[2]People familiar with operator-sum formalism will be pleased as, compared to Audretsch's denomination [29], a superoperator usually imply that the map is trace-preserving [55, 56], so we are making no mistake by calling the map a superoperator.

since the eigenvalues of $\sigma_0^X = \mathbb{1}^X$ are all 0. Since ${}_X H^{XY} = \sum_j h_{0j} \sigma_0^X \sigma_j^Y$, using equation (C.2), we have that

$$\text{Eigenval}\left({}_X H^{XY}\right) = h_{0j}\text{Eigenval}\left(\sigma_0^X\right) \text{Eigenval}\left(\sigma_j^Y\right)$$

which we just have proven to be always positive or null. Therefore ${}_X H^{XY} \geq 0$ if $H^{XY} \geq 0$. The trace preservation property then imply

$$\text{Tr}\left\{F^{XY}\right\} = \text{Tr}\left\{{}_X F^{XY}\right\} \tag{C.11}$$

Proof :

$$\begin{aligned}
\text{Tr}\left\{F^{XY}\right\} &= \text{Tr}_Y\left[\text{Tr}_X\left[F^{XY}\right]\right] \\
\text{Tr}\left\{{}_X F^{XY}\right\} &= \text{Tr}_Y\left[\text{Tr}_X\left[\frac{\mathbb{1}^X}{d_X} \otimes \text{Tr}_X\left[F^{XY}\right]\right]\right] \\
&= \text{Tr}_Y\left[\text{Tr}_X\left[\frac{\mathbb{1}^X}{d_X}\right] \otimes \text{Tr}_X\left[F^{XY}\right]\right] \\
&= \text{Tr}_Y\left[\text{Tr}_X\left[F^{XY}\right]\right]
\end{aligned}$$

C.1.2 The depolarising superoperator as a projector

It is now time to realise that the map is an orthogonal projector [12] because on the one hand we have already proven that it is idempotent (C.6). On the other hand it is self-dual [12], which means that the fact that it is acting either onto the direct or dual basis inside an inner product (2.4) is the same, i.e. : $(F|{}_X G) = ({}_X F|G)$. In order to show this, a first thing to notice is that the map commutes with hermitian conjugation :

$$\begin{aligned}
({}_X F)^\dagger &= \left(\frac{\mathbb{1}^{X\dagger}}{d_X} \otimes \text{Tr}_X F^\dagger\right) \\
&= \left(\frac{\mathbb{1}^X}{d_X} \otimes \text{Tr}_X F^\dagger\right)
\end{aligned}$$

where the fact that $\mathbb{1}$ is hermitian was used, hence

$$({}_X F)^\dagger = {}_X\left(F^\dagger\right) \tag{C.12}$$

This will imply that the map is hermitian preserving for an hermitian operator H.

$$({}_X H)^\dagger = {}_X H \tag{C.13}$$

We now have all the elements to prove self duality for the depolarising superoperator. In the following derivation superscripts will indicate to which subsystem operator acts upon, i.e. $F = F^X \otimes F^{YZ}$, $F^X \in \mathcal{H}^{\mathcal{X}}$ and $F^{YZ} \in \mathcal{H}^Y \otimes \mathcal{H}^Z$, and we suppose a product state.

$$
\begin{aligned}
(F|_X G) &= \mathrm{Tr}\left[F^\dagger \left(\frac{\mathbb{1}^X}{d_X} \otimes \mathrm{Tr}_X\, G\right)\right] \\
&= \mathrm{Tr}_{YZ}\left[\mathrm{Tr}_X\left[(F^{X\dagger} \otimes F^{YZ\dagger}) \left(\frac{\mathbb{1}^X}{d_X} \otimes \mathrm{Tr}_X\left(G^X \otimes G^{YZ}\right)\right)\right]\right] \\
&= \mathrm{Tr}_{YZ}\left[\mathrm{Tr}_X\left[F^{X\dagger}\left(\frac{\mathrm{Tr}_X\, G^X}{d_X}\right) \otimes F^{YZ\dagger} G^{YZ}\right]\right] \\
&= \mathrm{Tr}_{YZ}\left[\frac{1}{d_X}\left(Tr_X F^{X\dagger}\, \mathrm{Tr}_X\, G^X\right) \otimes F^{YZ\dagger} G^{YZ}\right] \\
&= \mathrm{Tr}_{YZ}\left[\mathrm{Tr}_X\left[\mathrm{Tr}_X\left[\frac{1}{d_X}\left(Tr_X F^{X\dagger} G^X\right)\right] \otimes F^{YZ\dagger} G^{YZ}\right]\right] \\
&= \mathrm{Tr}_{YZ}\left[\mathrm{Tr}_X\left[\left(\frac{\mathrm{Tr}_X\, F^{X\dagger}}{d_X}\right) G^X \otimes F^{YZ\dagger} G^{YZ}\right]\right] \\
&= \mathrm{Tr}_{YZ}\left[\mathrm{Tr}_X\left[\left(\frac{\mathbb{1}^X}{d_X} \otimes \mathrm{Tr}_X\left((F^{X\dagger} \otimes F^{YZ\dagger}\right)\right) \left(G^X \otimes G^{YZ}\right)\right]\right] \\
&= \mathrm{Tr}_{YZ}\left[\mathrm{Tr}_X\left[\left(\frac{\mathbb{1}^X}{d_X} \otimes \mathrm{Tr}_X\left((F^{X} \otimes F^{YZ}\right)\right)^\dagger \left(G^X \otimes G^{YZ}\right)\right]\right] \\
&= \mathrm{Tr}\left[({}_X F)^\dagger \left(G^X \otimes G^{YZ}\right)\right]
\end{aligned}
$$

Most of the algebra used to go from one line to the other rely on the distribution of the tensor product, its linearity and the fact that for every operator $F = \mathbb{1}F = F\mathbb{1}$. The passage from the ante-penultimate to the penultimate lines used equation (C.12). This was a proof in the special case where the operators were in product state. The general proof follows from linearity: it can be shown by the same reasoning but with operators decomposed in basis elements that this hold in the general case as partial trace is linear with addition and tensor product, the proof won't appear here however because of its length and limited relevance, in any case the map have been proven to be idempotent and hermitian preserving, which is enough to prove that it is an orthogonal projector on Hilbert spaces and thus always self-dual [13, 25]. This will imply that

$$(F|_X G) = ({}_X F|G) \tag{C.14}$$

Taken together, these two properties show that ${}_X\cdot$ is actually a projector [13]. The orthogonal complement of this projector is naturally obtained through the map ${}_{(1-X)}\cdot$, with prescript 1 being the identity mapping

$${}_1 F = \mathcal{I}(F) = F \tag{C.15}$$

, since

$$
\begin{aligned}
\left({}_{(1-X)}F|_X G\right) &= (F|_X G) - ({}_X F|_X G) \\
&= (F|_X G) - (F|_{X^2} G) \\
&= (F|_X G) - (F|_X G) \\
\left({}_{(1-X)}F|_X G\right) &= 0
\end{aligned} \tag{C.16}
$$

where the properties (C.3), (C.14) and (C.6) where used in lines 1, 2 and 3. Hence the orthogonal complement to projector ${}_X\cdot$ has been proven to be ${}_{(1-X)}\cdot$, which was what we expected from an orthogonal projector [13].

C.1.3 Projector identities

Thus the depolarising superoperator is an orthogonal projector to some linear subspace of some space of Hilbert-Schmidt operators. In this last subsection of mathematical characterisation, we are concerned by the composition of projectors to build projector onto the intersection or the union of the subspace they project onto. Here we will present the needed formulas to do so. In projective terms, union and intersection of subset are build with the product and sum of different projectors. The identities that will be presented here without proof come from the article "Constructing Projections on Sums and Intersections" by Piziak *et al.* [104]. Suppose $\mathcal{M}$ and $\mathcal{N}$ two subspace included in a bigger space $\mathcal{S}$, if we note the projector onto these subspace by $\mathcal{P}_{\mathcal{M}}$ and $\mathcal{P}_{\mathcal{N}}$, and the property $\mathcal{P}_{\mathcal{M}}\mathcal{P}_{\mathcal{N}} = \mathcal{P}_{\mathcal{N}}\mathcal{P}_{\mathcal{M}}$ is true, then the following identities hold

$$\mathcal{P}_{\mathcal{M}\cap\mathcal{N}} = \mathcal{P}_{\mathcal{M}}\mathcal{P}_{\mathcal{N}} \tag{C.17}$$

$$\mathcal{P}_{\mathcal{M}\cup\mathcal{N}} = \mathcal{P}_{\mathcal{M}} + \mathcal{P}_{\mathcal{N}} - \mathcal{P}_{\mathcal{M}}\mathcal{P}_{\mathcal{N}} \tag{C.18}$$

Where $\mathcal{M}\cap\mathcal{N}$ means the intersection of both subset and $\mathcal{M}\cup\mathcal{N}$ their union. Moreover, if the two projectors were orthogonal, then the formulas simplify

$$\mathcal{P}_{\mathcal{M}\cap\mathcal{N}} = 0 \tag{C.19}$$

$$\mathcal{P}_{\mathcal{M}\cup\mathcal{N}} = \mathcal{P}_{\mathcal{M}} + \mathcal{P}_{\mathcal{N}} \tag{C.20}$$

This result can be extended to case with more subspace by recursive application of the properties. For a n elements set of subspaces $\{\mathcal{M}_i\}, 1 \leq i \leq n$ of some space $\mathcal{S}$, whose projectors all commute with each other i.e. $\mathcal{P}_{\mathcal{M}_i}\mathcal{P}_{\mathcal{M}_j} = \mathcal{P}_{\mathcal{M}_j}\mathcal{P}_{\mathcal{M}_i} \forall 0 \leq i, j \leq n$, we have

$$\mathcal{P}_{\bigcap_{i=0}^{n}\mathcal{M}_i} = \prod_{i=0}^{n}\mathcal{P}_{\mathcal{M}_i} \tag{C.21}$$

$$\mathcal{P}_{\bigcup_{i=0}^{n}\mathcal{M}_i} = \sum_{\forall\mathcal{X}\subseteq\{1,\ldots n\}} (-1)^{1+|\mathcal{X}|}\prod_{i\in\mathcal{X}}\mathcal{P}_{\mathcal{M}_i} \tag{C.22}$$

C.2 Reformulation of valid quantum comb as projective constraints

Here we reformulate the deterministic quantum combs conditions of theorem 3 using the depolarising superoperator (5.1) to obtain definition 14. We then show the recursive relation for the projector onto the subspace of quantum combs presented in theorem 6.

For a given party A, her successive local actions can be represented by a fragment of the quantum circuit, or network, she possesses. Let the corresponding CJ representation of it, the deterministic quantum n-comb, be the matrix $M^{A^{(n)}} \equiv M^{(n)}$ with the set $\{A^{(1)}, A^{(2)}, \ldots, A^{(n)},\}$ of subsystems associated with each tooth, $A^{(1)}$ being the first tooth and $A^{(n)}$ being the last, n-th, tooth. We also define $M^{A^{(i)}} \equiv M^{(i)}, \forall 0 < i < n$ a smaller i-comb obtained when every action after i has been traced out i.e. $M^{(i-1)} = \mathrm{Tr}_{A_{IO}^{(i)}} M^{(i)}$.

For such an operator $M^{(i)}$ acting on the Hilbert space $\mathcal{H}_{A_I^{(1)}} \otimes \mathcal{H}_{A_O^{(1)}} \otimes \mathcal{H}_{A_I^{(2)}} \otimes \ldots \otimes \mathcal{H}_{A_O^{(n)}}$ to be a **valid** deterministic $i - comb$ i.e. to be a Choi-Jamiołkowski matrix representing the operations the party A will perform in causal order $A^{(1)} \preceq A^{(2)} \preceq \ldots \preceq A^{(i)}$, it must obey theorem (3).

C.2.1 For 1-comb

First, let a positive semi-definite operator on some space of one party only $\mathcal{H}^A = \mathcal{H}_{A_I} \otimes \mathcal{H}_{A_O}$, and $M^A \in \mathcal{L}(\mathcal{H}_{A_I} \otimes \mathcal{H}_{A_O})$ be a deterministic 1-comb on that space. If we expand it in a Hilbert-Schmidt traceless basis,

$$M^A = \sum_{i=0}^{d_{A_I}^2-1} \sum_{j=0}^{d_{A_0}^2-1} m_{ij}\, \sigma_i^{A_I} \sigma_j^{A_O},$$

where d_X is the dimension of the space $\mathcal{H}_X$ and m_{ij} some constants $\in \mathbb{R}$ (the operator is Hermitian), the comb conditions (3.1b) yield the following relation

$$\begin{aligned}
\mathrm{Tr}_{A_O}\left[M^A\right] &= \sum_{i=0}^{d_{A_I}^2-1} m_{i0}\, \sigma_i^{A_I} \mathrm{Tr}_{A_O}\left[\mathbb{1}^{A_O}\right] + 0 \\
&= d_{A_O} \sum_{i=0}^{d_{A_I}^2-1} m_{i0}\, \sigma_i^{A_I} \\
\mathrm{Tr}_{A_O}\left[M^A\right] &\equiv \mathbb{1}^{A_I}
\end{aligned}$$

where we have used the traceless property for basis element of non-null index (A.36a) to simplify the first line. This impose that $m_{i0} = 0\, i > 0$ and $m_{00} = \frac{1}{d_{A_O}}$. If we redefine the m coefficient to factor out m_{00} : $m_{ij} \equiv m_{ij}/m_{00}$, we retrieve formula (3.3)

$$M^A = \frac{1}{d_{A_O}} \left(\mathbb{1}^{A_I A_O} + \sum_{j>0} m_{0j}\, \sigma_j^{A_O} + \sum_{i>0} \sum_{j>0} m_{ij}\, \sigma_i^{A_I} \sigma_j^{A_O} \right), \quad m_{ij} \in \mathbb{R} \forall i, j; M^A \geq 0 \qquad (3.3)$$

We can rephrase the conditions to reach (3.3) as the fact that M[3] must belong to the convex cone $\mathcal{P}$ on $\mathcal{H}_{A_I} \otimes \mathcal{H}_{A_O}$ of positive operators. This will be the **positivity condition**.

Transforming the trace condition (3.1b) into a depolarising superoperator by applying $\frac{\mathbb{1}^{A_O}}{d_{A_O}} \otimes$ on the left of both side of the equation we get

$$_{A_O}M = \frac{\mathbb{1}^{A_I A_O}}{d_{A_O}} \qquad (C.23)$$

if we carry on by applying the depolarising superoperator in A_I on both side we reach the **trace condition**

$$_{A}M = \frac{\mathbb{1}^{A}}{d_{A_O}} \qquad (C.24)$$

where the convention $A := A_I A_O$ have been used. Explicit decomposition can show why it is a trace :

$$\frac{\mathbb{1}^A}{d_A} \otimes \mathrm{Tr}_A[M] = \frac{\mathbb{1}^A}{d_{A_O}} \iff \mathrm{Tr}_A[M] = \mathrm{Tr}\{M\} = d_{A_I}$$

Direct inspection of equations (C.23) and (C.23) imply that

$$_{A_O}M = {_A}M \qquad (C.25)$$

[3]We are dropping the superscript for conciseness.

It is actually a consequence of the **projective condition**. By inspection of (3.3) we see that every term that is not the unit matrix belongs to the space

$$\left\{\mathcal{L}\left(\mathcal{H}^{A_I}\right)\otimes \mathrm{TL}\left(\mathcal{H}^{A_O}\right)\right\}$$

where 'TL(·)' means "the traceless part of $\mathcal{L}(\cdot)$" as shown in [9]. Remark that this subspace is orthogonal to the subspace of matrices with unit basis element in A_O, because of the orthogonality condition (2.33d). Since this subspace can be accessed through projector $\mathcal{I}^{A_I}\otimes\mathcal{P}^{A_O} \equiv {}_{A_O}(\cdot)$, the non-unit matrix part of (3.3) is accessible by projector onto the orthogonal complement ${}_{(1-A_O)}(\cdot)$ of this subspace. When the normalisation condition is ignored, the whole subspace of operators of the form (3.3) is obtained by the direct sum of the non-unit matrix part with the unit matrix i.e. the linear subspace defined by

$$\mathcal{L}_C^{(1)} \equiv \left\{\mathrm{Span}\{\mathbb{1}\}\oplus\left[\mathrm{Herm}\left(\mathcal{H}^{A_I}\right)\otimes \mathrm{TL}\left(\mathcal{H}^{A_O}\right)\right]\right\}$$

it is then quite straightforward to show that this projector can be expressed in the language of the ${}_X\cdot$ maps as ${}_{1-A_O+A}(\cdot)$ since the direct sum is translated as a sum in projective terms [33]. This is how we obtain the **projective condition** : M must belong to a linear subspace whose orthogonal projector is ${}_{1-A_O+A}(\cdot)$ so

$${}_{1-A_O+A}M = M$$

Finally, the 1-comb conditions becomes, in that formulation,

$$M \geq 0 \tag{C.26a}$$

$${}_AM = \frac{\mathbb{1}^A}{d_{A_O}} \iff \mathrm{Tr}M = d_{A_I} \tag{C.26b}$$

$$\mathcal{P}_C(M) :={}_{1-A_O+A}\, M = M \tag{C.26c}$$

Note that we will use $\mathcal{P}_C$ to designate the projector onto the n-comb subspace in general, *i.e.* when there is no confusion possible on the number of parties. When appropriate, we will add a superscript to precise the number of parties $\mathcal{P}_C^{(n)}$ if the subsystems are in normal causal order *i.e.* $A^{(1)} \preceq A^{(2)} \preceq \ldots \preceq A^{(n)}$. And the superscript will indicate the causal structure when it is not normally ordered like, *e.g.* $\mathcal{P}_C^{A^{(3)}\preceq A^{(1)}\preceq A^{(2)}}$ is the projector onto the space of deterministic 3-combs with causal order between the teeth of $A^{(3)} \preceq A^{(1)} \preceq A^{(2)}$.

Remark : the equation (C.26c) is written like what is usually made in the literature (e.g. [70, 73, 75, 79, 83]). But it could have been given in the form $\mathcal{P}_C^{\perp}(X) = 0$ where $\mathcal{P}_C^{\perp}$ is a projector onto the orthogonal complement of the linear subspace of 1-combs $\mathcal{P}_C^{\perp}(\cdot) ={}_{A_O-A}\ \cdot$, which is shorter and correct. But the former notation is preferred because it provides more intuition on the geometry behind the condition. The point is that the map ${}_{1-A_O+A}\cdot$ is the projector onto the linear subspace of 1-combs on $\mathcal{H}^A$ and can be noted as such $\mathcal{P}_C(\cdot) :={}_{1-A_O+A}\ \cdot$ whereas $\mathcal{P}_C^{\perp}$ is in fact the projector onto its orthogonal subspace $\mathcal{P}_C^{\perp}(\cdot) = 1-\mathcal{P}_C ={}_{1-(1-A_O+A)}$. Therefore the two definitions are strictly equivalent, but we prefer to say that *an operator is a 1-comb if and only if it belongs to the subspace of 1-combs and thus its projection onto this subspace equals the operator itself* $\iff \mathcal{P}_C(M) = M$ rather than *an operator is a 1-comb if its projection onto the orthogonal complement to the subspace of 1-combs is 0* $\iff \mathcal{P}_C^{\perp}(M) = 0$.

C.2.2 For n-comb

The procedure that was followed last section to express the 1-comb condition as a projector relation together with positivity and a normalisation constraints can naturally be generalised for the combs with more teeth. It is insightful to first look into what happen when we reformulate the deterministic 2-comb condition before doing it for the n-comb. Again, suppose we have a Hermitian operator $M^{(2)} := M$ on some Hilbert space $\mathcal{H}^{A_I^{(1)}\otimes A_O^{(1)}\otimes A_I^{(2)}\otimes A_O^{(2)}}$ and we want it to be a valid deterministic 2-comb with causal order $A^{(1)} \preceq A^{(2)}$. It thus has to be a PSD operator which is normalised, and whose subset of definition have been restricted to the linear subset of deterministic 2-combs with the appropriate causal order. If one states explicitly the conditions (3.1a) we get

$$M \geq 0 \tag{C.27a}$$

$$\mathrm{Tr}_{A_O^{(2)}} M = \mathbb{1}^{A_I^{(2)}} \otimes M^{(1)} \tag{C.27b}$$

$$\mathrm{Tr}_{A_0^{(1)}} M^{(1)} = \mathbb{1}^{A_I^{(1)}} \tag{C.27c}$$

with $M^{(1)} \in \mathcal{L}\left(\mathcal{H}^{A_I^{(1)}} \otimes \mathcal{H}^{A_O^{(1)}}\right)$. Defining the Hilbert-Schmidt expansion of the operator M as

$$M = \sum_{i,j,k,l} m_{ijkl} \sigma_i^{A_I^{(1)}} \sigma_j^{A_O^{(1)}} \sigma_k^{A_I^{(2)}} \sigma_l^{A_O^{(2)}}$$

some similar computation to the one did above yields to the result (3.4) that the most general M that satisfy (C.27) have the following expression

$$M = \frac{\mathbb{1}^{A^{(2)}}}{d_{A_O^{(2)}}} \otimes \left(\frac{\mathbb{1}^{A^{(1)}}}{d_{A_O^{(1)}}} + \sum_{\substack{i\geq 0\\ j>0}} m_{ij00}\ \sigma_i^{A_I^{(1)}} \sigma_j^{A_O^{(1)}} \right) + \sum_{\substack{i\geq 0\\ j\geq 0}} \sum_{\substack{k\geq 0\\ l>0}} m_{ijkl}\ \sigma_i^{A_I^{(1)}} \sigma_j^{A_O^{(1)}} \sigma_k^{A_I^{(2)}} \sigma_l^{A_O^{(2)}} \tag{C.28}$$

with coefficients $m_{ijkl} \in \mathbb{R}, \forall i,j,k,l$. The equations have been reorganised compared to (3.4) so it will be easier to find the projector onto it. But first, the **normalisation conditions** : applying $\frac{\mathbb{1}^{A_O^{(1)}}}{d_{A_O^{(1)}}}\otimes$ then $\frac{\mathbb{1}^{A_I^{(1)}}}{d_{A_I^{(1)}}}\otimes$ to the left of both sides of equation (C.27c) we get the conditions of 1-combs

$${}_{A_O^{(1)}}M^{(1)} = {}_{A^{(1)}}M^{(1)}$$

and

$${}_{A^{(1)}}M^{(1)} = \frac{\mathbb{1}^{A_I^{(1)}}}{d_{A_O^{(1)}}} \quad .$$

Then applying $\frac{\mathbb{1}^{A_O^{(2)}}}{d_{A_O^{(2)}}}\otimes$ on both sides of (C.27b) yields

$${}_{A_O^{(2)}}M = \frac{\mathbb{1}^{A_I^{(2)}}}{d_{A_O^{(2)}}} \otimes M^{(1)}$$

which is again equal as applying both $\frac{\mathbb{1}^{A_O^{(2)}}}{d_{A_O^{(2)}}}\otimes$ and $\frac{\mathbb{1}^{A_I^{(2)}}}{d_{A_I^{(2)}}}\otimes$ so that

$${}_{A_O^{(2)}}M = {}_{A^{(2)}}M$$

the trace condition is reached by applying both ${}_{A_I^{(1)}}(\cdot)$ and ${}_{A_O^{(1)}}(\cdot)$ to ${}_{A^{(2)}}M$, which gives

$$
\begin{aligned}
{}_{A^{(1)}A^{(2)}}M &= \frac{\mathbb{1}^{A_I^{(2)}}}{d_{A_O^{(2)}}} \otimes {}_{A^{(1)}}M^{(1)} \\
&= \frac{\mathbb{1}^{A_I^{(2)}}}{d_{A_O^{(2)}}} \otimes \frac{\mathbb{1}^{A^{(1)}}}{d_{A_O^{(1)}}}
\end{aligned}
$$

which is equivalent to

$$
{}_{A^{(1)}A^{(2)}}M = \frac{\mathbb{1}^{A^{(1)}A_I^{(2)}}}{d_{d_{A_O^{(1)}}A_O^{(2)}}} \iff \operatorname{Tr}\{M\} = d_{A_I^{(1)}} d_{A_I^{(2)}} \tag{C.29}
$$

We now have a normalisation (trace) condition, a positivity condition and already 2 constraints on the projector, namely that ${}_{A_O^{(2)}}M = {}_{A^{(2)}}M$ and ${}_{A_O^{(1)}A^{(2)}}M = {}_{A^{(1)}A^{(2)}}M$ which follows from 1-comb conditions.

Now, to build the projector one can notice that the reorganisation in (C.28) have been made to make the subspace appear more clearly. The lefthand side of the sum,

$$
\frac{\mathbb{1}^{A^{(2)}}}{d_{A^{(2)}}} \otimes \left(\frac{\mathbb{1}^{A}}{d_{A_O}} + \sum_{\substack{i \geq 0 \\ j > 0}} m_{ij00}\, \sigma_i^{A_I} \sigma_j^{A_O} \right)
$$

is nothing less than a 1-comb in tensor product with the unit matrix over subsystem $A^{(2)}$, so we can infer that this part of the space is

$$
\left\{ \operatorname{Span}\left\{ \mathbb{1}^{A^{(2)}} \right\} \otimes \mathcal{L}\left(\mathcal{H}^{A_I^{(1)}} \right) \otimes \operatorname{TL}\left(\mathcal{H}^{A_O^{(1)}} \right) \right\}
$$

where 'Span' means "the subspace spanned by". The projector to it is then $\mathcal{P}^{A^{(2)}} \otimes \mathcal{P}_C^{A^{(1)}} := {}_{A^{(2)} \times \left(1 - A_O^{(1)} + A^{(1)}\right)}(\cdot)$.

For the other part, it is again the orthogonal complement of

$$
\left\{ \mathcal{L}\left(\mathcal{H}^{A^{(1)}} \otimes \mathcal{H}^{A_I^{(2)}} \right) \otimes \operatorname{Span}\{ \mathbb{1}^{A_O^{(2)}} \} \right\}
$$

accessible through the projector $\left(\mathcal{I}^{A^{(1)}A_I^{(2)}} \otimes \mathcal{P}^{A_O^{(2)}} \right) := {}_{A_O^{(2)}}(\cdot)$, so the rightmost term of the sum belongs to the linear subspace

$$
\left\{ \mathcal{L}\left(\mathcal{H}^{A^{(1)}} \otimes \mathcal{H}^{A_I^{(2)}} \right) \otimes \operatorname{TL}\{ \mathcal{H}^{A_O^{(2)}} \} \right\}
$$

which is accessible through the map ${}_{\left(1 - A_O^{(2)}\right)}(\cdot)$. Hence the total subspace is

$$
\mathcal{L}_C^{(2)} \equiv \left\{ \left[\operatorname{Span}\{ \mathbb{1}^{A^{(2)}} \} \otimes \mathcal{L}_C^{(1)} \right] \oplus \left[\mathcal{L}\left(\mathcal{H}^{A^{(1)}} \otimes \mathcal{H}^{A_I^{(2)}} \right) \otimes \operatorname{Span}\{ \mathbb{1}^{A_O^{(2)}} \} \right] \right\}
$$

And the corresponding projector is

$$\mathcal{P}_C^{(2)} \equiv {}_{\left(1-A_O^{(2)}\right)} \cdot + \mathcal{P}_C^{(1)} \left({}_{A^{(2)}}(\cdot)\right) = {}_{\left(1-A_O^{(2)}\right)+\left(1-A_O^{(1)}+A^{(1)}\right)A^{(2)}}(\cdot) \tag{C.30}$$

Therefore the deterministic 2-comb conditions of validity are reduced to the following set of conditions

$$M \geq 0 \tag{C.31a}$$

$$_{A^{(1)}A^{(2)}}M = \frac{\mathbb{1}^{A^{(1)}A^{(2)}}}{d_{A_O^{(1)}} d_{A_O^{(2)}}} \tag{C.31b}$$

$$\mathcal{P}_C^{(2)}(M) \equiv \mathcal{P}_C^{A^{(1)} \preceq A^{(2)}}(M) = M \tag{C.31c}$$

These conditions are once again one for the positivity of each part of the operator that lives in a different subspace, one for the normalisation of the operator (note that this one can be chosen arbitrarily, but we stick to the one adopted in [40]) and finally one that requires that the operator belongs to the proper subspace, that is M is a 1-comb for $A^{(2)}$ and $_{A^{(2)}}M$ for $A^{(1)}$. Remark that the definition is recursive : we followed the same procedure to obtain the deterministic 2 comb conditions from the original theorem 3 than we did for the 1-comb.

The generalisation for n parties is now straightforward. First the positivity condition does not need to be changed according to the number of parties, as it applies on the n-comb and the depolarising operator is a CP map, so it is enough to preserve the PSD character of all the smaller combs contained inside the n-comb, as already observed in [40]. For the normalisation the same argument applies, but based on the TP characteristic, to say that the trace of a deterministic n-comb will always be the product of the dimension of its n input spaces.

For the projective conditions, we have seen that the 2-comb condition is built from the 1-comb conditions. We can subsequently infer a recursive relation (5.6) for this projector with the theorem 6 in the main text. Here we show the proof of this theorem

Proof. Since it is a recurrence relation, we will prove it by induction. Let there be a projector onto the subspace of quantum combs between N ordered parties $\mathcal{P}_C^{(N)}$, suppose that the projector onto the subspace of $N+1$ parties, with the $(N+1)$-th party being the last tooth. Suppose an PSD operator M defined on $\mathcal{L}\left(\mathcal{H}^{\mathfrak{N}} \otimes \mathcal{H}^{A_I^{(N+1)}} \otimes \mathcal{H}^{A_O^{(N+1)}}\right)$, then its projection on the subspace of (N+1)-combs is

$$\mathcal{P}_C^{N+1}(M) = {}_{\left(1-A_O^{(N+1)}\right)}(M) + {}_{A^{(N+1)}}\left(\mathcal{P}_C^{N}(M)\right) \tag{C.32}$$

Using Chiribella *et al.*'s deterministic quantum comb conditions expressed as depolarising superoperators, for $\mathcal{P}_C^{N+1}(M)$ to be a valid (N+1) deterministic comb it must obey

$$_{A_O^{(N+1)}}\mathcal{P}_C^{N+1}(M) = {}_{A^{(N+1)}}\mathcal{P}_C^{N+1}(M) \tag{C.33}$$

using the idempotency property (C.6) of the map, we have

$$\begin{aligned}
{}_{A_O^{(N+1)}}\mathcal{P}_C^{N+1}(M) &= {}_{A_O^{(N+1)}\left(1-A_O^{(N+1)}\right)}(M) + {}_{A_O^{(N+1)}A^{(N+1)}}\left(\mathcal{P}_C^N(M)\right) \\
&= {}_{\left(A_O^{(N+1)}-A_O^{(N+1)}\right)}(M) + {}_{A^{(N+1)}}\left(\mathcal{P}_C^N(M)\right) \\
&= {}_{A^{(N+1)}}\left(\mathcal{P}_C^N(M)\right) \\
{}_{A^{(N+1)}}\mathcal{P}_C^{N+1}(M) &= {}_{A^{(N+1)}\left(1-A_O^{(N+1)}\right)}(M) + {}_{A^{(N+1)}A^{(N+1)}}\left(\mathcal{P}_C^N(M)\right) \\
&= {}_{\left(A^{(N+1)}-A^{(N+1)}\right)}(M) + {}_{A^{(N+1)}}\left(\mathcal{P}_C^N(M)\right) \\
&= {}_{A^{(N+1)}}\left(\mathcal{P}_C^N(M)\right)
\end{aligned}$$

Which proves the recursion for N+1. Now suppose there is a non-zero element K that verify (C.33) but is not within the subspace where $\mathcal{P}_C^{N+1}$ projects to. Such an element verify ${}_{\left(1-A_O^{(N+1)}+A^{(N+1)}\right)}K = K$, which is trivially equal to $\mathcal{P}_C^{N+1}(K) = K$ when $N = 0$. For $N \geq 0$, if K doesn't belong to the set of combs, then $\mathcal{P}_C^{N+1}(K) = 0$, so

$$\begin{aligned}
\mathcal{P}_C^{N+1}(K) &= {}_{\left(1-A_O^{(N+1)}\right)}(K) + {}_{A^{(N+1)}}\left(\mathcal{P}_C^N(K)\right) \\
&= {}_{\left(1-A_O^{(N+1)}\right)}\left({}_{\left(1-A_O^{(N+1)}+A^{(N+1)}\right)}K\right) + {}_{A^{(N+1)}}\left(\mathcal{P}_C^N\left({}_{\left(1-A_O^{(N+1)}+A^{(N+1)}\right)}K\right)\right) \\
&= {}_{\left(1-A_O^{(N+1)}\right)}(K) + {}_{A^{(N+1)}}\left(\mathcal{P}_C^N(K)\right)
\end{aligned}$$

which is only equal to zero if K is itself equal to zero, this shows that the proposed projector encompass all eligible elements and thus concludes the proof. □

Altogether, the three conditions for n parties are gathered in the main text to obtain the equations (5.8) of definition 14.

C.3 PM projective conditions

C.3.1 Example : derivation of PM projective conditions with two parties.

Consider a matrix between two parties acting once, here called A and B. The validity conditions for this process matrix are

$$\begin{aligned}
&W \geq 0 \\
&\mathrm{Tr}\left[W\left(M^A \otimes M^B\right)\right] = 1 \\
&M^A\ ,\ M^B \geq 0 \\
&\mathrm{Tr}_{A_O}\left[M^A\right] = \mathbb{1}^{A_I} \\
&\mathrm{Tr}_{B_O}\left[M^B\right] = \mathbb{1}^{A_I^{(2)}}
\end{aligned}$$

First, notice that because they are combs, M^A and M^B can be substituted by ${}_{(1-A_O+A)}M^A$ and ${}_{(1-B_O+B)}M^B$ without changing anything (C.26c). Then using property (5.3) of the trace map, one can rewrite the tensor product with the projectors factored out

$$_{(1-A_O+A)}M^A \otimes {}_{(1-B_O+B)}M^B = {}_{(1-A_O+A)(1-B_O+B)}\left(M^A \otimes M^B\right)$$

which can be plugged into the trace condition

$$\mathrm{Tr}\left[W\left(M^A \otimes M^B\right)\right] = \mathrm{Tr}\left[W\ {}_{(1-A_O+A)(1-B_O+B)}\left(M^A \otimes M^B\right)\right] = 1$$

Two conditions will be extracted from this equation, the first one can be found by expanding the product of projectors as

$$\mathrm{Tr}\left[W\ {}_{(1-A_O)(1-B_O)+(1-A_O)B+A(1-B_O)}\left(M^A \otimes M^B\right)\right] + \mathrm{Tr}\left[W\ {}_{AB}\left(M^A \otimes M^B\right)\right] = 1$$

The rightmost trace is actually applied to some operators that are projected onto the real number space because the projector is acting in all the subsystems $A_I A_O A_I^{(2)} B_O$ constituting the space $\mathcal{L}\left(\mathcal{H}^{A_I} \otimes \mathcal{H}^{A_O} \otimes \mathcal{H}^{A_I^{(2)}} \otimes \mathcal{H}^{B_O}\right)$ we are working on. Therefore this term is equivalent to

$$\begin{aligned}
\mathrm{Tr}\left[W\ {}_{AB}\left(M^A \otimes M^B\right)\right] &= \mathrm{Tr}\left[W\frac{\mathbb{1}^{AB}}{d_{AB}} \otimes \mathrm{Tr}_{AB}\left[\left(M^A \otimes M^B\right)\right]\right] \\
&= \frac{\mathrm{Tr}\{W\}}{d_A d_B}\mathrm{Tr}_{AB}\left[\left(M^A \otimes M^B\right)\right] \\
&= \frac{\mathrm{Tr}\{W\}}{d_A d_B}\mathrm{Tr}_A\left[M^A\right]\mathrm{Tr}_B\left[M^B\right]
\end{aligned}$$

and we already know, according to (C.26b) that the trace of 1 combs must be equal to their input space dimension so

$$\begin{aligned}
\mathrm{Tr}\left[W\ {}_{AB}\left(M^A \otimes M^B\right)\right] &= \frac{\mathrm{Tr}\{W\}}{d_{A_I} d_{A_I^{(2)}} d_{A_O} d_{B_O}} d_{A_I} d_{A_I^{(2)}} \\
&= \frac{\mathrm{Tr}\{W\}}{d_{A_O} d_{B_O}} \qquad \text{(C.34)}
\end{aligned}$$

Now recall that we want this condition to hold regardless of the 1-comb being input[4]. Therefore both Alice and Bob could chose to do nothing which is translated as the matrix $\left(M^A \otimes M^B\right) = \frac{\mathbb{1}^{AB}}{d_{A_O} d_{B_O}}$. In this case the leftmost part of the trace is zero :

$$\mathrm{Tr}\left[W\ {}_{(1-A_O)(1-B_O)+(1-A_O)B+A(1-B_O)}\left(M^A \otimes M^B\right)\right] = 0 \qquad \text{(C.35)}$$

because

$$_{(1-A_O)(1-B_O)+(1-A_O)B+A(1-B_O)}\mathbb{1} = 0$$

Accordingly ,

$$\mathrm{Tr}\left[W\ {}_{AB}\left(M^A \otimes M^B\right)\right] = 1$$

[4]This sentence can be ambiguous, here we are talking about the kind of 1-comb that could be used to represent the operations performed by each party in their local laboratory, not the commentary some anthropomorphous 1-comb could make about this condition. Yes, this footnote is a bad pun.

which yields the first condition,

$$\operatorname{Tr}\left[W_{\ AB}\left(M^{A}\otimes M^{B}\right)\right]=\frac{\operatorname{Tr}\{W\}}{d_{A_O}d_{B_O}}=1$$

$$\operatorname{Tr}\{W\}=d_{A_O}d_{B_O}$$

which will be referred to as *the normalisation constraint*.

Going back to the general case, we are left with (C.35), that should be verified even when the operations are not trivial. Here's the trick : remark that this equation is an inner product with some projector applied on its right side. But a projector is self-dual (C.14), subsequently it can be passed to the left side without changing the equality

$$\begin{aligned}\operatorname{Tr}\left[W_{\ (1-A_O)(1-B_O)+(1-A_O)B+A(1-B_O)}\left(M^{A}\otimes M^{B}\right)\right]= \\ \operatorname{Tr}\left[{}_{(1-A_O)(1-B_O)+(1-A_O)B+A(1-B_O)}W\left(M^{A}\otimes M^{B}\right)\right] \\ =1\end{aligned}$$

In abstract notation the last two equations would have been written

$$\begin{aligned}&\left(W\Big|_{(1-A_O)(1-B_O)+(1-A_O)B+A(1-B_O)}\left(M^{A}\otimes M^{B}\right)\right)\\ &=\\ &\left({}_{(1-A_O)(1-B_O)+(1-A_O)B+A(1-B_O)}W\Big|\left(M^{A}\otimes M^{B}\right)\right)\end{aligned}$$

Since both W and $\left(M^{A}\otimes M^{B}\right)$ are positive semi-definite operators, and that the combs are in general non-zero matrices, the condition

$$ {}_{(1-A_O)(1-B_O)+(1-A_O)B+A(1-B_O)}W=0 \tag{C.36}$$

must hold all the time, and this is the second condition : *the projective constraints*.

To summarise, we started from the OCB conditions [1] and we modified it so it looked like the more recent definitions [12] :

$$W\geq 0 \tag{C.37a}$$

$$\operatorname{Tr}W=d_{A_O}d_{B_O} \tag{C.37b}$$

$$ {}_{(A_O+B_O-A_OB_O)-(1-A_O)B-A(1-B_O)}W=W \tag{C.37c}$$

And note that the last condition are in fact 3 linearly independent conditions because this is actually a sum of orthogonal projectors

$$ {}_{(1-A_O-B_O+A_OB_O)+(1-A_O)B+A(1-B_O)}W=0$$

$$ {}_{(1-A=_O-B_O+A_OB_O)}(\cdot)\perp {}_{(1-A_O)A_I^{(2)}B_O}(\cdot)\perp {}_{A_IA_O(1-B_O)}(\cdot)$$

from which we retrieve the 2-partite process matrix projective conditions as usually formulated

$$
\begin{aligned}
{}_{(A_O+B_O-A_OB_O)}W &= W \\
{}_{A_OB}W &=_B W \\
{}_{AB_O}W &=_A W
\end{aligned}
$$

C.3.2 Proof of theorem 7

Proof. This equation (5.14) is in fact a rephrasing of

$$
\mathcal{P}_V^{(n+1)} = 1 - \left({}_{\left(1-A_O^{(n+1)}+A_{IO}^{(n+1)}\right)}\mathcal{P}_{V^\perp}^{(n)} + {}_{(1-A_O^{(n+1)})\prod_{i=1}^{n} A_{IO}^{(i)}}\cdot \right)
$$

which is equivalent to saying that

$$
\mathcal{P}_{V^\perp}^{(n+1)} = {}_{\left(1-A_O^{(n+1)}+A_{IO}^{(n+1)}\right)}\mathcal{P}_{V^\perp}^{(n)} + {}_{(1-A_O^{(n+1)})\prod_{i=1}^{n} A_{IO}^{(i)}}\cdot \tag{C.38}
$$

the proof of this equation is by induction. For N=1, we have that

$$
\mathcal{P}_{V^\perp}^{(1)} = {}_{\left(1-A_O^{(1)}\right)}\cdot
$$

which is correct, suppose that

$$
\mathcal{P}_{V^\perp}^{(n)} = {}_{\prod_{i\in\mathcal{X}}\left(A_O^{(i)}+A_I^{(i)}A_O^{(i)}\right)\prod_{j\in\mathcal{N}\setminus\mathcal{X}} A_I^{(j)}A_O^{(j)}}\cdot
$$

then

$$
\begin{aligned}
\mathcal{P}_{V^\perp}^{(n+1)} &= {}_{\left(1-A_O^{(n+1)}+A_{IO}^{(n+1)}\right)}\mathcal{P}_{V^\perp}^{n} + {}_{(1-A_O^{(n+1)})\prod_{i=1}^{n} A_{IO}^{(i)}}(\cdot) \\
&= {}_{\left(1-A_O^{(n+1)}+A_{IO}^{(n+1)}\right)\left(\prod_{i\in\mathcal{X}}\left(1-A_O^{(i)}+A_I^{(i)}A_O^{(i)}\right)\prod_{j\in\mathcal{N}\setminus\mathcal{X}} A_I^{(j)}A_O^{(j)}\right)}(\cdot) + {}_{(1-A_O^{(n+1)})\prod_{i=1}^{n} A_{IO}^{(i)}}(\cdot) \\
&= {}_{\left(1-A_O^{(n+1)}\right)\left(\prod_{i\in\mathcal{X}}\left(1-A_O^{(i)}\right)\prod_{j\in\mathcal{N}\setminus\mathcal{X}} A_{IO}^{(j)}\right)+\left(A_{IO}^{(n+1)}\right)\left(\prod_{i\in\mathcal{X}}\left(1-A_O^{(i)}\right)\prod_{j\in\mathcal{N}\setminus\mathcal{X}} A_{IO}^{(j)}\right)}(\cdot) \\
&\quad + {}_{(1-A_O^{(n+1)})\prod_{i\in\mathcal{N}} A_{IO}^{(i)}}(\cdot)
\end{aligned}
$$

define $\mathcal{N}' = \mathcal{N} \cup \{n+1\}$, the newly associated ensemble of subset is

$$
\mathcal{X}' = \{\{\mathcal{X} + \emptyset\}, \{\mathcal{X} \cup \{n+1\}\}, \{n+1\}\}
$$

and finally

$$
\begin{aligned}
\mathcal{P}_{V^\perp}^{(n+1)} &= {}_{\left(\prod_{i\in\mathcal{X}\cup\{n+1\}}\left(1-A_O^{(i)}\right)\prod_{j\in\mathcal{N}\cup\{n+1\}\setminus\mathcal{X}\cup\{n+1\}} A_{IO}^{(j)}\right)+\left(\prod_{i\in\mathcal{X}\cup\emptyset}\left(1-A_O^{(i)}\right)\prod_{j\in\mathcal{N}\cup\{n+1\}\setminus\mathcal{X}} A_{IO}^{(j)}\right)}(\cdot) \\
&\quad + {}_{(1-A_O^{(n+1)})\prod_{i\in\mathcal{N}\cup\{n+1\}\setminus\{n+1\}} A_{IO}^{(i)}}(\cdot) \\
&= {}_{\prod_{i\in\mathcal{X}'}\left(A_O^{(i)}+A_I^{(i)}A_O^{(i)}\right)\prod_{j\in\mathcal{N}'\setminus\mathcal{X}'} A_I^{(j)}A_O^{(j)}}(\cdot)
\end{aligned}
$$

,which is the sought definition, this concludes the proof. □

Appendix D

Appendix to chapter 6

D.1 1-partite MPM as a special case of PM

In this section we explore how the one party MPM is linked to both comb and PM formalisms.

D.1.1 Quantum combs in process matrix

First, we show that MPM can stem from PM formalism under certain conditions. What the present section is arguing for is that representing the operations and side channel in an MPM by a n-comb is equivalent to a particular causally ordered PM. This PM have an inside channel that can be factored from the rest : this is the side-channel. What is left of the PM when the channel is factored out is the MPM. Of course, the expectation that the most general way of representing side operation was a deterministic quantum combs can be made without such an argument, but here we show it explicitly : the quantum comb characteristic of the ensemble formed by the side channels and individual operations can naturally be deduced from causally ordered PM. Moreover, there is an equivalence between link product and the generalised Born's rule. In this section we will prove these claims for a small example (a 2-partite PM).

Consider A 2-partite PM $\tilde{W}$, suppose that it has a defined causal order of $\tilde{A} \preceq \tilde{B}$ and admits a decomposition like the one presented in figure D.1. That is, starting from equation (3.15),

$$\tilde{W} = \frac{1}{d_{\tilde{A}_O} d_{\tilde{B}_O}} \left(\mathbb{1}^{\tilde{A}\tilde{B}} + \sigma^{\tilde{A} \preceq \tilde{B}} \right) \tag{D.1}$$

where we have neglected the non-signalling terms as they are not relevant in this development, (D.1) admits a decomposition like

$$\tilde{W} = \frac{1}{d_{A'_O}} \left(\mathbb{1} + \sigma^{A'_O \preceq B'_I} \right) \otimes \frac{1}{d_{A_O} d_{B_O}} \left(\mathbb{1}^{\tilde{A}\tilde{B}} + \sigma^{A \preceq B} \right) := V \otimes W \tag{D.2}$$

Where the prime subsystems refer to those associated with the side channel while those without it represent those of the MPM. The overall subsystems are represented with a tilde. Such decomposition can always be made, although it will sometimes be a trivial decomposition (V is a 1-dimensionnal unitary operator 1) when the PM can not be written in a product state. A quick computation shows that V, which can be written as

$$V = \frac{1}{d_{A'_O}} \left(\mathbb{1}^{A'_O B'_I} + \sigma^{A'_O \preceq B'_I} \right)$$

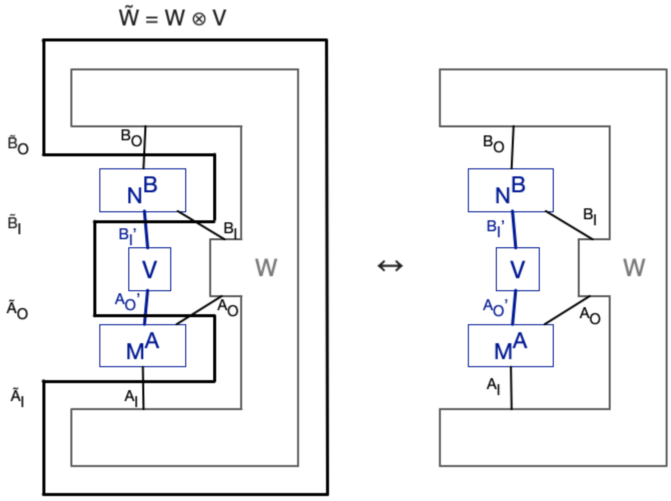


FIGURE D.1: Equivalence between particular 2-partite PM and 2-partite MPM, the parts drawn in blue are quantum comb while those in black are (multi-round) process matrix.

have the same form as a 1-comb from A'_O to B'_I, (3.3) :

$$V = \frac{1}{d_{B'_I}} \left(\frac{d_{B'_I}}{d_{A'_O}} \mathbb{1}^{A'_O B'_I} + \sum_{j>0} v_{0j}\, \sigma_j^{B'_I} + \sum_{i>0} \sum_{j>0} v_{ij}\, \sigma_i^{A'_O} \sigma_j^{B_I} \right), \quad v_{ij} \in \mathbb{R} \forall i, j; V \geq 0$$

and obey the deterministic 1-comb conditions with the exception of a different normalisation (notice the factor in front of the unit matrix term). As for W it has kept a 2-partite PM formulation :

$$\tilde{W} = \frac{1}{d_{A_O} d_{B_O}} \left(\mathbb{1}^{AB} + \sigma^{A \preceq B} \right)$$

and we can proceed to an identification of the terms like

$$\begin{aligned} \tilde{W} &= V^{A'_O B'_I} \otimes W^{AB} \\ &= \frac{1}{d_{A_O} d_{A'_o} d_{B_O}} \left(\mathbb{1}^{A A'_O B B'_I} + \mathbb{1}^{A'_O B'_I} \otimes \sigma^{A \preceq B} + \sigma^{A'_O \preceq B'_I} \otimes \mathbb{1}^{AB} + \sigma^{A'_O \preceq B'_I} \otimes \sigma^{A \preceq B} \right) \\ \sigma^{\tilde{A} \preceq \tilde{B}} &= \mathbb{1}^{A'_O B'_I} \otimes \sigma^{A \preceq B} + \sigma^{A'_O \preceq B'_I} \otimes \mathbb{1}^{AB} + \sigma^{A'_O \preceq B'_I} \otimes \sigma^{A \preceq B} \end{aligned}$$

Under this form, one can prove through the Hilbert-Schmidt inner product that it is equivalent to a 2-partite MPM taking in a 2 comb as

Lemma 1 (1-MPM as a special case of causally ordered 2-PM)**.** *A process matrix with defined global causal order have an inside channel that can be factorised through a tensor product, like for example equation* (D.2)*, and can then be shown as being equivalent to a one-partite MPM with the same causal order as*

$$\mathrm{Tr}\left\{ \tilde{W} \cdot \left(M^{\tilde{A}} \otimes N^{\tilde{B}} \right) \right\} = \mathrm{Tr}\left\{ C^T \cdot W \right\} \tag{6.3}$$

where $C \in \mathcal{L}\left(\mathcal{H}^{A_I \otimes A_O \otimes B_I \otimes B_O}\right)$ is a deterministic 2-comb formed by the link product (according to theorem 4) of M, V and N

$$C \equiv N \underset{B'_I}{*} V \underset{A'_O}{*} M \tag{D.3}$$

with the addition under texts below the link product symbols to point out over which space they are taken. Moreover the equation (6.3) *shows that the Hilbert-Schmidt inner product* (2.4) *is equivalent* *to the link product in this formulation*

$$\left(\tilde{W}\,\middle|\,M^{\tilde{A}} \otimes N^{\tilde{B}}\right) = W * C \tag{6.4}$$

Proof. We start by rearranging the terms on the left-hand side of the equation using (F.5d), (F.7), (F.5a) and (F.8)

$$\begin{aligned}
\mathrm{Tr}\left\{\tilde{W} \cdot \left(M^{\tilde{A}} \otimes N^{\tilde{B}}\right)\right\} &= \mathrm{Tr}\left\{(V \otimes W) \cdot \left(M^{\tilde{A}} \otimes N^{\tilde{B}}\right)\right\} \\
&= \mathrm{Tr}_{AB}\left[\mathrm{Tr}_{A'_O B'_I}\left[(V \otimes W) \cdot \left(M^{\tilde{A}} \otimes N^{\tilde{B}}\right)\right]\right] \\
&= \mathrm{Tr}_{AB}\left[\mathrm{Tr}_{A'_O B'_I}\left[\left(\left(\mathbb{1}^{A'_O B'_I} \otimes W\right) \cdot \left(V \otimes \mathbb{1}^{AB}\right)\right) \cdot \left(M^{\tilde{A}} \otimes N^{\tilde{B}}\right)\right]\right] \\
&= \mathrm{Tr}_{AB}\left[\mathrm{Tr}_{A'_O B'_I}\left[\left(\mathbb{1}^{A'_O B'_I} \otimes W\right) \cdot \left(\left(V \otimes \mathbb{1}^{AB}\right) \cdot \left(M^{\tilde{A}} \otimes N^{\tilde{B}}\right)\right)\right]\right]
\end{aligned}$$

$$\mathrm{Tr}\left\{\tilde{W} \cdot \left(M^{\tilde{A}} \otimes N^{\tilde{B}}\right)\right\} = \mathrm{Tr}_{AB}\left[W \cdot \mathrm{Tr}_{A'_O B'_I}\left[\left(\left(V^{A'_O B'_I} \otimes \mathbb{1}^{AB}\right) \cdot \left(M^{\tilde{A}} \otimes N^{\tilde{B}}\right)\right)\right]\right] \tag{D.4}$$

Now focusing on the inner part of the trace in the right-hand side of equation (D.4)

$$\mathrm{Tr}_{A'_O B'_I}\left[\left(\left(V \otimes \mathbb{1}^{AB}\right) \cdot \left(M^{\tilde{A}} \otimes N^{\tilde{B}}\right)\right)\right]$$

we can use the definition of the link product (3.5) to show that

$$\begin{aligned}
N \underset{B'_I}{*} V \underset{A'_O}{*} M &= (N \otimes M) \underset{A'_O B'_I}{*} (V \otimes \mathbb{1}) \\
(N \otimes M) \underset{A'_O B'_I}{*} (V \otimes \mathbb{1}) &\equiv \mathrm{Tr}_{A'_O B'_I}\left[\left(N^{BB'_I} \otimes M^{AA'_O}\right) \cdot \left(V^{A'_O B'_I} \otimes \mathbb{1}^{AB}\right)^{T_{A'_O B'_I}}\right]^{T_{AB}} \\
&= \mathrm{Tr}_{A'_O B'_I}\left[\left(V^{A'_O B'_I} \otimes \mathbb{1}^{AB}\right)^{T_{A'_O B'_I}} \cdot \left(N^{T_{BB'_I}} \otimes M^{T_{AA'_O}}\right)\right]
\end{aligned}$$

so comparing the two this yields

$$\mathrm{Tr}_{A'_O B'_I}\left[\left(\left(V \otimes \mathbb{1}^{AB}\right) \cdot \left(M^{\tilde{A}} \otimes N^{\tilde{B}}\right)\right)\right] = N^T \underset{B'_I}{*} V^T \underset{A'_O}{*} M^T \equiv C^T \tag{D.5}$$

And we end back to the equation (D.3), which when we plug it back into the right member of equation (D.4) gives

$$\mathrm{Tr}\left\{\tilde{W} \cdot \left(M^{\tilde{A}} \otimes N^{\tilde{B}}\right)\right\} = \mathrm{Tr}_{AB}\left[W \cdot C^T\right]$$

which is equation (6.3). □

The generalisation of this lemma is straightforward : as long as the PM is totally causally ordered and can be factored as $W \otimes V^{(1)} \otimes V^{(2)} \otimes \ldots$, the above proof mechanism can be used recursively, since the link product is associative (see section 3.1.3) and the resulting object is a deterministic n-comb. The thing is that there is no restriction on the form of the $V^{(i)}$'s so one can always take trivial matrices in 1-dimensional space like $V^{(i)} \in \mathbb{C}^1 \to V^{(i)} = 1$. Nonetheless, note that when the 1-dimensional unit matrix is used as side-channel between two combs, the link product trivially reduces to the tensor product, this will be the starting point for the generalisation to several parties. See the main text.

D.1.2 MPM as a constrained process matrix

We can also do the argument of last section the other way around, what happens when one decide to plug a quantum comb into a valid process matrix ? The mathematics remain usable in that case, but are they still leading to something meaningful ?

Let W be a 2-partite process matrix between A and B, suppose that there is means to impose that A is in the causal past of B, for example they're linked together by an ancilla that allows communication from A to B. Subsequently, one can represent their operations, which are made local by the presence of the ancilla, as a deterministic 2-comb of form

$$M^{A\preceq B} = {}_{1-B_O+(1-A_O+A)B}M^{A\preceq B}$$

that is semi-positive defined and is normalised as usual (C.27). For W, we haven't assumed anything but the fact that is a valid process matrix (C.37).

To see what precisely happens when the probability of the process to be deterministically realised is computed, i.e. when the inner product $\mathrm{Tr}\{W\ M^{A\preceq B}\}$ is calculated, one will need to express both elements into an Hilbert-Schmidt basis like it was done in chapter 3, equations (3.4) and (3.15). Using the properties of the Generalised Gell-Mann Basis (A.36), the trace to be computed will be greatly simplified as only the basis elements of the form $\sigma_0^{A_I} \otimes \sigma_0^{A_O} \otimes \sigma_0^{B_I} \otimes \sigma_0^{B_O} \equiv \mathbb{1}^{AB}$ will not be traceless. This yields the following result

$$\begin{aligned}\left(W\middle|M^{A\preceq B}\right) &= \mathrm{Tr}\left[WM^{A\preceq B}\right]\\ &= \frac{1}{d_{A_I}d_{A_O}d_{B_I}d_{B_O}}\left(\mathrm{Tr}\left\{\mathbb{1}^{AB}\right\} + \mathrm{Tr}\left[\sum_{\substack{i,l>0\\k\geq 0}}\sum_{\substack{r,s,t\geq 0\\u>0}} w_{i0kl}m^{(2)}_{rstu}\ \delta_{i,r}\delta_{0,s}\delta_{k,t}\delta_{l,u}\ \mathbb{1}^{AB}\right]\right)\\ &= 1 + \sum_{\substack{i,l>0\\k\geq 0}} w_{i0kl}m^{(2)}_{i0kl}\end{aligned}$$

There is a quirk ! The probability of this deterministic process to happen is no longer guaranteed to be 1, which is not physical : the process happening no matter what the parties do has no longer 100% probability to be observed.

This problem is in fact a special case of a much bigger problem : there is no linear way to parallel compose process matrices. If we indeed interpret what we were trying to do using lemma 1 it show us that we were composing $\tilde{W} = W \otimes V$, where V is the side channel between the 2 teeth of the comb, and happen to be either a (badly normalised, see the comment last section) 1-comb between A's output to B's input systems. But V can also be interpreted as a (badly normalised) 2-partite PM between A and B, with trivial system for Bob output and Alice input and only non trivial terms of the form $\sigma^{A\preceq B}$ (see equation (3.15)). The difference with what we did last section is that W haven't had its causal structure fixed to match with the one of the comb *a priori*. In a recent paper [91], Jia and Sakharwade showed that a composition of arbitrary 2-partite process matrices through tensor product as we are trying to do : $\tilde{W} = W \otimes V$ don't always lead to valid process matrix. They obtained the same kind of troublesome term, $\sum_{\substack{i,l>0\\k\geq 0}} w_{i0kl}m^{(2)}_{i0kl}$, that is a signalling term in the opposite direction of the causal structure of the one of the comb. This kind of term allows one to form a global causal loop between the 2 parties, $A \preceq B \preceq A$, which is forbidden in well-defined process matrices, hence the possibility of it messing with the unit normalisation of probabilities. It must be noted that the issue was raised in a context of developing a Shannon information theory for the resource of indefinite causal structure, which needs such a rule of parallel composition of PM.

The authors proposed several solutions to work around this problem. The first one is that if we stay in the 'orthodox' point of view about process matrices, the process is the description of the environment of an entire family of parties, thus there shouldn't exist a need to add parties and thus to parallel compose [1, 48]. Otherwise one have to restrict the local operations of parties so that the whole process remains valid. This however is not compatible with the object we are looking for as this kind of restriction sometimes impose that the causal order of the comb should be the other way around, and as noted in the paper this is not compatible with the composition of comb through the link product. A last solution would be to go for a broader framework that is less restrictive about the normalisation of the matrices and operations, like [7].

Moreover, subsequent work of Guérin and collaborators [92] proved a no-go theorem stating that there is no linear composition rule for processes. Their proposed way to go around the problem is *single-shot* information theory, which does not need to parallel compose the resource. This is not helpful in our case but they also proposed two other solutions. The first one would also be to extend the formalism further, as they guessed that there might be a non-linear way of doing so, using post-selection for example. The other one, based on the observation that the two way signalling cannot lead to an interpretation where the parties are at fixed space-time point, would be to restrict the two-way signalling terms in the composition[1].

It is the latter that we will consider as we do want for the successive operations of a party to be well localised in the global causal order between them. Therefore of all the proposed solutions the one compatible with the MPM framework we are trying to establish is

To be valid, a multi-round process matrix must only have terms that are compatible with the defined causal ordering existing between the local operations of its parties.

This leads to the conclusion that in such a case where A can send information to B through a side channel, the validity conditions of one of the object must be modified. As this whole situation arose from a restriction on the comb, changing the conditions on it would only lead us back to the general case. It's thus the constraints on the process matrix that must be modified accordingly just like argued by Guérin *et al.*, which bring us to the definition of the MPM as being this "constrained process matrix". Keep in mind that it is something else than a real process matrix as the object do not encode *all* the possibles correlations in the process because of the allowed side-channels.

To conclude the discussion, the chosen way of fixing the ill-defined probabilities obtained when plugging a comb in a PM is to impose that the supplementary term in the probabilities vanish

$$\sum_{i>0}\sum_{k\geq 0}\sum_{l>0} w_{i0kl}m^{(2)}_{i0kl} = 0 \qquad ,$$

Setting the terms in the comb part to zero is the trivial solution since it means that the comb no longer can use its inside channel and we are back to a 2-partite PM, so we won't consider it. Imposing that the coefficients always sum up to zero is also not going to be considered because of the arguments that we just presented above, *e.g.* it would require some non-linear way of composing combs with MPM to be consistent. Moreover, proper choice of coefficient that add up to zero can lead to partial two-way signalling, we want to reject this possibility of forming a fine-tuned (*i.e.* with an *ad hoc* choice of PM coefficients for each comb) causal loops, according to the discussions one can find in [48, 54]. Then,

[1] In this regard, the side-channels act a bit like in Castro-Ruiz *et al.*'s representation of dynamics of PM [73].

because the process matrix should be compatible with any kind of comb plugged in, the only set of coefficient $\{w_{i0kl}\}$ that will always lead to the sum being 0 regardless of the coefficients $\{m^{(2)}\}$ is $w_{i0kl} = 0 \forall i, k, l$ which correspond to enforcing that its causal order match the one of the comb, as the terms we are settings to zero are actually $\sigma^{B \preceq A}$. This leads to the way the MPM is defined in the main text.

D.2 The MPM for one party

D.2.1 Example : one party acting twice, the 2-partite-1-party MPM

We saw in last subsection that the validity conditions for a PM taking in a 2-comb with causal structure $A^{(1)} \preceq A^{(2)}$ are the same than those of a 2-partite PM (3.15) but with the term $\sigma^{A^{(2)} \preceq A^{(1)}}$ being zero. We could have obtained these conditions if the object considered to derive validity conditions of the PM had directly been the most general 2-comb with this causal structure. Indeed, the projective conditions for such and comb are $M = {}_{1-A_O^{(2)}+\left(1-A_O^{(1)}+A^{(1)}\right)A^{(2)}}M$. Using the procedure it gives :

$$
\begin{aligned}
1 = &\left(W \middle|_{1-A_O^{(2)}+\left(1-A_O^{(1)}+A^{(1)}\right)A^{(2)}} M\right) \\
&\left(W \middle|_{1-A_O^{(2)}+\left(1-A_O^{(1)}\right)A^{(2)}} M\right) + \left(W \middle|_{A^{(1)}A^{(2)}} M\right)
\end{aligned}
$$

Using (C.37b) to make the right term in the second line disappear, and then the duality of the projector (C.14), the two following lines are derived

$$
\begin{aligned}
0 &= \left(W \middle|_{1-A_O^{(2)}+\left(1-A_O^{(1)}\right)A^{(2)}} M\right) \\
&= \left({}_{1-A_O^{(2)}+\left(1-A_O^{(1)}\right)A^{(2)}} W \middle| M\right)
\end{aligned}
$$

and we end up with these projective constraints

$$
{}_{1-A_O^{(2)}+\left(1-A_O^{(1)}\right)A^{(2)}}W = 0 \tag{D.6}
$$

and thus the following matrix

$$
W = {}_{A_O^{(2)}-\left(1-A_O^{(1)}\right)A^{(2)}}W \tag{D.7}
$$

As expected, this matrix is compatible with the projective constraints on general 2-partite PM (the positivity and normalisation conditions are still the same) that were derived at equation (C.36):

$$
\begin{aligned}
&{}_{\left(1-A_O^{(1)}\right)\left(1-A_O^{(2)}\right)}\left[{}_{A_O^{(2)}-\left(1-A_O^{(1)}\right)A^{(2)}}W\right] = 0 \\
&{}_{\left(1-A_O^{(1)}\right)A^{(2)}}\left[{}_{A_O^{(2)}-\left(1-A_O^{(1)}\right)A^{(2)}}W\right] = {}_{\left(1-A_O^{(1)}\right)A^{(2)}-\left(1-A_O^{(1)}\right)A^{(2)}}W = 0 \\
&{}_{A^{(1)}\left(1-A_O^{(2)}\right)}\left[{}_{A_O^{(2)}-\left(1-A_O^{(1)}\right)A^{(2)}}W\right] = 0
\end{aligned}
$$

So the PM is a valid general PM, but that have inherited more restrictive constraints on the subspace it must live on. This conclusion can be drawn by comparing conditions (C.36) with (D.6).

We can prove that this is effectively the sought matrix and nothing less by going into the Hilbert-Schmidt expansion. Let $W = \sum_{ijkl} w_{ijkl}\ \sigma_i^{A_I^{(1)}} \sigma_j^{A_O^{(1)}} \sigma_k^{A_I^{(2)}} \sigma_l^{A_O^{(2)}}$, then enforcing condition (D.7) gives

$$
\begin{aligned}
{}_{A_O^{(2)}-\left(1-A_O^{(1)}\right)A^{(2)}}W &= W \\
{}_{A_O^{(2)}}W &= \sum_{ijk} w_{ijk0}\ \sigma_i^{A_I^{(1)}} \sigma_j^{A_O^{(1)}} \sigma_k^{A_I^{(2)}} \\
{}_{A^{(2)}}W &= \sum_{ij} w_{ij00}\ \sigma_i^{A_I^{(1)}} \sigma_j^{A_O^{(1)}} \\
{}_{A_O^{(1)}A^{(2)}}W &= \sum_{i} w_{i000}\ \sigma_i^{A_I^{(1)}} \\
{}_{\left(1-A_O^{(1)}\right)A^{(2)}}W &= \sum_{i}\sum_{j>0} w_{ij00}\ \sigma_i^{A_I^{(1)}} \sigma_j^{A_O^{(1)}} \\
{}_{A_O^{(2)}-\left(1-A_O^{(1)}\right)A^{(2)}}W &= \sum_{i} w_{i000}\ \sigma_i^{A_I^{(1)}} + \sum_{ij}\sum_{k>0} w_{ijk0}\ \sigma_i^{A_I^{(1)}} \sigma_j^{A_O^{(1)}} \sigma_k^{A_I^{(2)}}
\end{aligned}
$$

If we split it and impose normalisation constraints so it looks like (3.15) :

$$
\begin{aligned}
W := W^{A^{(1)}\preceq A^{(2)}} &= \frac{1}{d_{A_I^{(1)}} d_{A_O^{(2)}}} \left(\mathbb{1} + \sigma^{A^{(1)}\preceq A^{(2)}} + \sigma^{A^{(1)}\not\preceq\not\succeq A^{(2)}}\right) \qquad \text{(D.8)} \\
\sigma^{A^{(1)}\preceq A^{(2)}} &:= \sum_{\substack{j>0\\k>0}} w_{0jk0}\ \sigma_j^{A_O^{(1)}} \sigma_k^{A_I^{(2)}} + \sum_{\substack{i>0\\j>0\\k>0}} w_{ijk0}\ \sigma_i^{A_I^{(1)}} \sigma_j^{A_O^{(1)}} \sigma_k^{A_I^{(2)}} \\
\sigma^{A^{(1)}\not\preceq\not\succeq A^{(2)}} &:= \sum_{i>0} w_{i000}\ \sigma_i^{A_I^{(1)}} + \sum_{k>0} w_{i000}\ \sigma_k^{A_I^{(2)}} + \sum_{j>0} w_{i0k0}\ \sigma_i^{A_I^{(1)}} \sigma_k^{A_I^{(2)}}
\end{aligned}
$$

we indeed retrieve the actual conditions (3.15) minus the term with the wrong causal structure $\sigma^{A^{(2)}\preceq A^{(1)}}$.

D.2.2 Recursive Formulation of MPM projector

The recursive definition of the projector $\mathcal{P}_M^{(n)}$ onto the subspace of N-partite MPM for one party[2] can be deduced from equation (5.6), using the procedure of chapter 5. Let W be a one party MPM compatible with a deterministic N-comb M, then

$$
\begin{aligned}
\left(W \middle| \mathcal{P}_C^{(n)}(M)\right) &= 1 \\
\left(\left(\mathcal{P}_C^{(n)}(W) - {}_{\prod_{i=1}^n A_{IO}^{(i)}}(W)\right) |M\right) &= 0 \\
\mathcal{P}_C^{(n)}(W) - {}_{\prod_{i=1}^n A_{IO}^{(i)}}(W) &= 0
\end{aligned}
$$

[2] Or equivalently a PM with totally ordered causal structure, or a special $N+1$-deterministic comb, thanks to theorem 8.

hence

$$\mathcal{P}_M^{(n)} = \left({}_1(\cdot) - \mathcal{P}_C^{(n)} + {}_{\prod_{i=1}^n A_{IO}^{(i)}}(\cdot) \right) \tag{6.7}$$

The recursive definition of this projector is then obtained from eq. (5.6)

$$\begin{aligned}
\mathcal{P}_M^{(n+1)} &= \left({}_1(\cdot) - \mathcal{P}_C^{(n+1)} + {}_{\prod_{i=1}^{n+1} A_{IO}^{(i)}}(\cdot) \right) \\
&= \left({}_1(\cdot) - {}_{\left(1-A_O^{(n+1)}\right)}(\cdot) - {}_{A_{IO}^{(n+1)}}\mathcal{P}_C^{(n)} + {}_{\prod_{i=1}^{n+1} A_{IO}^{(i)}}(\cdot) \right) \\
&= \left({}_1(\cdot) - {}_{\left(1-A_O^{(n+1)}\right)}(\cdot) - {}_{A_{IO}^{(n+1)}}((\cdot) - (\cdot)) - {}_{A_{IO}^{(n+1)}}\mathcal{P}_C^{(n)} + {}_{A_{IO}^{(n+1)}\prod_{i=1}^{n} A_{IO}^{(i)}}(\cdot) \right) \\
&= \left({}_1(\cdot) - {}_{\left(1-A_O^{(n+1)}+A_{IO}^{(n+1)}\right)}(\cdot) + {}_{A_{IO}^{(n+1)}}\left({}_1(\cdot) - \mathcal{P}_C^{(n)} + {}_{\prod_{i=1}^{n} A_{IO}^{(i)}}(\cdot) \right) \right)
\end{aligned}$$

Finally,

$$\mathcal{P}_M^{(n+1)} = {}_{\left(A_O^{(n+1)}-A_{IO}^{(n+1)}\right)}(\cdot) + {}_{A_{IO}^{(n+1)}}\mathcal{P}_M^{(n)} \tag{6.8b}$$

D.2.3 One party MPM are causally ordered N-partite process matrix which are N+1 quantum combs

Starting back from the above example, the 2-partite MPM with only one party, it is possible to show that this is actually a 3-comb. This is an obvious realisation if we acknowledge the fact that the quantum comb *is* the most general object between parties that are causally ordered and thus must be equivalent to the one party MPM, but here we show it explicitly using the formalism we have developed. This will help to understand better what is happening when we generalise the situation to more parties. The only tricky part here will be (again) a change of notation convention.

To prove this is a 3-partite comb, first we rename the teeth. It is done as in figure D.2, the process matrix is formally extended by 2 trivial systems, both of dimension 1. Then the systems are labelled with numbers from bottom to top, so the wire added at the bottom of the process matrix will be associated with some Hilbert space $\mathcal{H}^{A^0}$ such that $d_A^0 = 1$, then $A_I^{(1)}$ is renamed A^2, with $d_{A_I^{(1)}} \equiv d_A^2$, ...etc until the added top wire that will correspond to Hilbert space $\mathcal{H}^{A^5}$ with $d_{A^5} = 1$. We thus have formed the extension

$$\tilde{W} = 1^{A^5} \otimes W \otimes 1^{A^0} \equiv W \qquad .$$

Conditions of validity were derived for this matrix above in the text, for the projective constraints this is equation (D.7) at section D.2.1. In the new notation this projector becomes

$$W = {}_{A^4-\left(1-A^2\right)A^3A^4}W \tag{D.9}$$

As for the positivity and the normalisation condition they're still the same

$$\begin{aligned}
W &\geq 0 \\
\operatorname{Tr} W &= d_{A_O^{(1)}} d_{A_O^{(2)}} \equiv d_{A^4} d_{A^2}
\end{aligned}$$

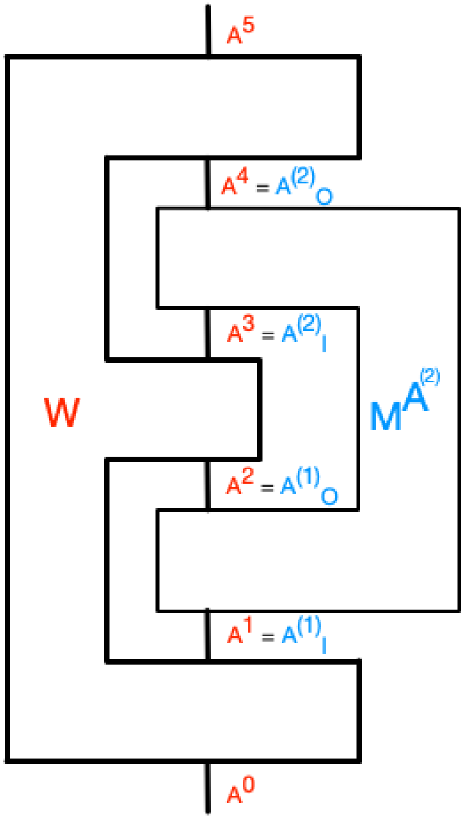


FIGURE D.2: 2-partite process compatible with a 2-partite quantum comb matrix as a 3-comb : the trick is to see the outputs [input] of the comb (written in blue) as inputs [outputs] of the process matrix. A notation that do not specify if the spaces are input or output is used (in red) to avoid confusion during the demonstration.

Now this is to be compared with the comb conditions for a 3-partite comb, using eq. (5.8) :

$$W \geq 0$$

$$_{A^5A^4A^3A^2A^1A^0}(W) = \frac{\mathbb{1}^{A^5A^4A^3A^2A^1A^0}}{d_{A^5}d_{A^3}d_{A^1}}$$

$$_{(1-A^5+(1-A^3+(1-A^1+A^0A^1)A^2A^3)A^4A^5)}W = W$$

This set of conditions is actually verified : notice that because the dimensions of A^5 and A^0 are both 1, the normalisation condition is verified straightforwardly :

$$\begin{aligned} _{A^5A^4A^3A^2A^1A^0}(W) &= \frac{\mathbb{1}^{A^5A^4A^3A^2A^1A^0}}{d_{A^5A^4A^3A^2A^1A^0}} \otimes \mathrm{Tr}_{A^5A^4A^3A^2A^1A^0}[W] \\ &= \frac{d_{A^4}d_{A^2}}{d_{A^5}d_{A^4}d_{A^3}d_{A^2}d_{A^1}d_{A^0}} \mathbb{1}^{A^5A^4A^3A^2A^1A^0} \\ &= \frac{\mathbb{1}^{A^5A^4A^3A^2A^1A^0}}{d_{A^5}d_{A^3}d_{A^1}} \end{aligned}$$

Also, their associated depolarising superoperator is trivially ${}_{A^0} = {}_{A^5} = 1$, so the projector to the subspace of valid 3-combs becomes

$$\begin{aligned} W &= {}_{\left(1-A^5+\left(1-A^3+\left(1-A^1+A^0A^1\right)A^2A^3\right)A^4A^5\right)}W \\ &= {}_{\left(1-1+\left(1-A^3+\left(1-A^1+1A^1\right)A^2A^3\right)A^41\right)}W \\ &= {}_{\left(1-A^3+A^2A^3\right)A^4}W \\ &= {}_{A^4-\left(1-A^2\right)A^3A^4}W \end{aligned}$$

which is indeed the projector to the subspace of valid 2-partite process matrices compatible with 2-combs like in equation (D.9).

Finally, the positivity conditions can be deduced by combining the condition $W \geq 0$ with the fact that the depolarising superoperator conserve the positive character of operators (C.10) (otherwise tracing out a subsystem could lead to negative probabilities, which means that ignoring the action of a subsystem of the PM could lead to a nonphysical situation).

Hence the causally ordered 2-partite process matrix, which is the one party MPM, is a special kind of 3-comb. This result is generalised in theorem 8 of the main text and proven now.

Proof of theorem 8

Proof. The first part of the theorem, that fully causally ordered PM are one party MPM is trivial and deducible from lemma 1 because the tensor product factoring is always possible for fully ordered PM. See the discussion below the lemma.
Now for the second part, suppose you have a n-partite MPM used by one party A whose output spaces are labelled by even number and input spaces by odd number. E.g. with A^1 being the input space of the first input (i.e. the one that cannot be signalled to) and A^{2n} being the space the her last output (i.e. the one that cannot signal to any other subpart). Extend it by a supplementary input space A^0 as well as an output space A^{2n+1}, both of dimension 1 and call the resulting object $\tilde{W} \equiv 1^{A^{(2n+1)}} \otimes W^{A^{2n}A^{(2n-1)}\ldots A^2A^1} \otimes 1^{A^0}$. It's obvious that both object are represented exactly by the same matrix and that it is at most a notation trick of the CJ picture. One can see that the process matrix positivity and normalisation conditions for PM W imply the counterpart positive and normalisation condition for comb on $\tilde{W}$. Indeed, if you treat the formerly input(output) spaces, i.e. the ones labelled with an odd(even) number, of your PM W as the input(output) space in the object $\tilde{W}$, you see that the process matrix condition from the odd indices to the even indices becomes comb condition from the even indices to the odd ones :

$$W \geq 0 \iff \tilde{W} \geq 0$$

$$\operatorname{Tr} W = \prod_{i=1}^{n} d_{A^{2i}} \iff {}_{A^{2n+1}A^{2n}\ldots A^1A^0}\tilde{W} = \frac{\mathbb{1}^{A^{2n+1}A^{2n}\ldots A^1A^0}}{\prod_{i=0}^{n} d_{A^{2i+1}}}$$

For the projective constraints, the projector $\mathcal{P}_V^{(n)}(W) \in \mathcal{L}\left(\bigotimes_{i=1}^{2n} \mathcal{H}^{A^i}\right)$ is indeed equivalent to the one onto the subspace of $n+1$ quantum comb with trivial input A^0 and output A^{2n+1}: $\mathcal{P}_C^{(n+1)}(\tilde{W}) \in \mathcal{L}\left(\left(\mathcal{L}\left(\bigotimes_{i=0}^{n} \mathcal{H}^{A^i}\right)\right) \rightarrow \left(\mathcal{L}\left(\bigotimes_{i=0}^{n} \mathcal{H}^{A^{i+1}}\right)\right)\right)$ because the first one can be extended to be applied on the trivially extended space of linear superoperators :

$$\mathcal{P}_V^{(n)} \in \mathcal{L}\left(\left(\mathcal{L}\left(\bigotimes_{i=1}^{2n}\mathcal{H}^{A^i}\right)\right) \to \left(\mathcal{L}\left(\bigotimes_{i=1}^{2n}\mathcal{H}^{A^i}\right)\right)\right)$$
$$\Longleftrightarrow$$
$$\left({}_1(\cdot)^{A^{2n+1}} \circ \mathcal{P}_V^{(n)} \circ {}_1(\cdot)^{A^0}\right) \in \mathcal{L}\left(\left(\mathcal{L}\left(\bigotimes_{i=0}^{2n+1}\mathcal{H}^{A^i}\right)\right) \to \left(\mathcal{L}\left(\bigotimes_{i=0}^{2n+1}\mathcal{H}^{A^i}\right)\right)\right)$$

and the second one admit a CJ representation onto that space as well.

$$\mathcal{P}_C^{(n+1)} \in \mathcal{L}\left(\left(\mathcal{L}\left(\bigotimes_{i=0}^{2n+1}\mathcal{H}^{A^i}\right)\right) \to \left(\mathcal{L}\left(\bigotimes_{i=0}^{2n+1}\mathcal{H}^{A^i}\right)\right)\right)$$

And we see that the extension of $\mathcal{P}_V$ is strictly equivalent to the comb conditions :

$$\left(\mathcal{I}^{A^{2n+1}} \circ \mathcal{P}_V^{(n)} \circ \mathcal{I}^{A^0}\right)(\tilde{W}) =$$
$$\left(1^{A^{2n+1}} \otimes {}_{A^{2n}-(1-A^{2n-2}+(1-A^{2n-4}+(...+(1-A^4+(1-A^2)A^3A^4)A^5A^6)...)A^{2n-3}A^{2n-2})A^{2n-1}A^{2n}}(W) \otimes 1^{A^0}\right) =$$
$${}_{A^{2n}-(1-A^{2n-2}+(1-A^{2n-4}+(...+(1-A^4+(1-A^2)A^3A^4)A^5A^6)...)A^{2n-3}A^{2n-2})A^{2n-1}A^{2n}}(1^{A^{2n+1}} \otimes W \otimes 1^{A^0}) =$$
$${}_{A^{2n}-(1-A^{2n-2}+(1-A^{2n-4}+(...+(1-A^4+(1-A^2)A^3A^4)A^5A^6)...)A^{2n-3}A^{2n-2})A^{2n-1}A^{2n}}(\tilde{W}) =$$
$${}_{[1-A^{2n+1}]+(A^{2n}-(1-A^{2n-2}+(1-A^{2n-4}+(...+(1-A^4+(1-A^2)A^3A^4)A^5A^6)...)A^{2n-3}A^{2n-2})A^{2n-1}A^{2n})[A^{2n+1}]}(\tilde{W}) =$$
$${}_{1-A^{2n+1}+(1-(1-A^{2n-2}+(1-A^{2n-4}+(...+(1-A^4+(1-A^2)A^3A^4)A^5A^6)...)A^{2n-3}A^{2n-2})A^{2n-1})A^{2n}A^{2n+1}}(\tilde{W}) =$$
$${}_{1-A^{2n+1}+(1-A^{2n-1}+(1-(1-A^{2n-4}+(...+(1-A^4+(1-A^2)A^3A^4)A^5A^6)...)A^{2n-3})A^{2n-2}A^{2n-1})A^{2n}A^{2n+1}}(\tilde{W}) =$$
$$\vdots$$
$${}_{1-A^{2n+1}+(1-A^{2n-1}+(1-A^{2n-3}+(...+(1-A^5+(1-A^3+A^2A^3)A^4A^5)...)A^{2n-4}A^{2n-3})A^{2n-2}A^{2n-1})A^{2n}A^{2n+1}}(\tilde{W}) =$$
$${}_{1-A^{2n+1}+(1-A^{2n-1}+(1-A^{2n-3}+(...+(1-A^5+(1-A^3+[1-A^1+A^1]A^2A^3)A^4A^5)...)A^{2n-4}A^{2n-3})A^{2n-2}A^{2n-1})A^{2n}A^{2n+1}}(\tilde{W}) =$$
$${}_{1-A^{2n+1}+(1-A^{2n-1}+(1-A^{2n-3}+(...+(1-A^5+(1-A^3+(1-A^1+[A^0]A^1)A^2A^3)A^4A^5)...)A^{2n-4}A^{2n-3})A^{2n-2}A^{2n-1})A^{2n}A^{2n+1}}(\tilde{W})$$
$$\equiv \mathcal{P}_C(\tilde{W})$$

Where, during the derivation, we used square brackets to indicate the elements that were added in a particular line using the fact that their presence do not modify the projector (like e.g. when using $1 = A^0$ to add it in a multiplication at the penultimate line, or adding $0 = 1 - A^7$ at the fourth). All the other things that were done from line to line were reshuffling of coefficients, except at the second line where the projector was extended onto the whole operator. This concludes the proof[3]. □

[3]Note that this proof can also be done recursively by noting that PM conditions will enforce the fact that ${}_{A^{2i}A^{2i-1}}(\cdot) = {}_{A^{2i}A^{2i-1}A^{2i-2}}(\cdot), \forall i > 0$ while the comb conditions, like defined in [40] are $\mathrm{Tr}_{A^{2i+1}}(\cdot) = \mathbb{1}^{A^{2i}} \otimes \mathrm{Tr}_{A^{2i+1}A^{2i}}(\cdot)$, $\forall i$. A short calculation will show that these 2 are equals, which is what is actually done formally in the proof.

D.3 MPM example : Two parties acting twice each

D.3.1 Derivation of generalised Born's rule and the MPM conditions of validity

The situation is represented in figure 6.1, we adopt the same convention as earlier this chapter namely that Alice and Bob's system are numbered by order, *e.g.* $B^{(1)} \preceq B^{(2)}$ to refer to Bob's subsystems. We will just change N_A to n and N_B to m in order to lighten the notation. The Hilbert spaces represented by a wire between a 1-comb and a side channel are noted with a bar like $\mathcal{H}^{\bar{B}_I^{(2)}}$ refers to the wire getting out of side-channel U into the 1-comb $N^{B^{(2)}}$, as for the Hilbert spaces between the MPM and the 1-combs they are noted with the same convention but without a bar. Finally a tilde means both bar and no bar *e.g.* $\mathcal{H}^{\tilde{B}_I^{(2)}} = \mathcal{H}^{B_I^{(2)}} \otimes \mathcal{H}^{\bar{B}_I^{(2)}}$. If we consider the full PM as

$$\tilde{W} = V \otimes W \otimes U$$

the generalised Born's rule is (3.11)

$$\left(\tilde{W} \middle| \left(M^{\tilde{A}^{(1)}} \otimes M^{\tilde{A}^{(2)}} \otimes N^{\tilde{B}^{(1)}} \otimes N^{\tilde{B}^{(2)}}\right)\right) = 1 \tag{D.10}$$

Now consider the following

$$\begin{aligned}
&\mathrm{Tr}\Big\{\tilde{W} \cdot \left(M^{\tilde{A}^{(1)}} \otimes M^{\tilde{A}^{(2)}} \otimes N^{\tilde{B}^{(1)}} \otimes N^{\tilde{B}^{(2)}}\right)\Big\} = 1 \\
&= \mathrm{Tr}\Big\{(V \otimes W \otimes U) \cdot \left(M^{\tilde{A}^{(1)}} \otimes M^{\tilde{A}^{(2)}} \otimes \mathbb{1}^{\tilde{B}^{(1)} \otimes \tilde{B}^{(2)}}\right) \cdot \left(\mathbb{1}^{\tilde{A}^{(1)} \otimes \tilde{A}^{(2)}} N^{\tilde{B}^{(1)}} \otimes N^{\tilde{B}^{(2)}}\right)\Big\} \\
&= \mathrm{Tr}_{AB}\left[W \cdot \mathrm{Tr}_{\bar{A}}\left[\mathrm{Tr}_{\bar{B}}\left[\left(V \otimes U \otimes \mathbb{1}^{AB}\right)\left(M^{\tilde{A}^{(1)}} \otimes M^{\tilde{A}^{(2)}} \otimes \mathbb{1}^{\tilde{B}^{(1)}\tilde{B}^{(2)}}\right) \cdot \left(\mathbb{1}^{\tilde{A}^{(1)}\tilde{A}^{(2)}} N^{\tilde{B}^{(1)}} \otimes N^{\tilde{B}^{(2)}}\right)\right]\right]\right] \\
&= \mathrm{Tr}_{A^{(1)}A^{(2)}B^{(1)}B^{(2)}}\left[W \cdot \mathrm{Tr}_{\bar{A}^{(1)}\bar{A}^{(2)}}\left[V \cdot \left(M^{\tilde{A}^{(1)}} \otimes M^{\tilde{A}^{(2)}}\right)\right] \otimes \mathrm{Tr}_{\bar{B}^{(1)}\bar{B}^{(2)}}\left[U \cdot \left(N^{\tilde{B}^{(1)}} \otimes N^{\tilde{B}^{(2)}}\right)\right]\right]
\end{aligned}$$

now using equation (D.5), stating that

$$\mathrm{Tr}_{\bar{A}^{(1)}\bar{A}^{(2)}}\left[V \cdot \left(M^{\tilde{A}^{(1)}} \otimes M^{\tilde{A}^{(2)}}\right)\right] = \left(M^{A^{(1)}} * V * M^{A^{(2)}}\right)^{T_A} = \left(M^A\right)^{T_A}$$

and

$$\mathrm{Tr}_{\bar{B}^{(1)}\bar{B}^{(2)}}\left[U \cdot \left(N^{\tilde{B}^{(1)}} \otimes N^{\tilde{B}^{(2)}}\right)\right] = \left(N^{B^{(1)}} * U * N^{B^{(2)}}\right)^{T_B} = \left(N^B\right)^{T_B}$$

we can finally reach the system of equations presented in the main text

$$\begin{gathered}
W * M^A * N^B = \left(\tilde{W} \middle| \left(M^{A^{(1)}} \otimes M^{A^{(2)}} \otimes N^{B^{(1)}} \otimes N^{B^{(2)}}\right)\right) \\
\mathrm{Tr}\Big\{W \cdot \left(M^A \otimes N^B\right)^T\Big\} = \mathrm{Tr}\Big\{\tilde{W} \cdot \left(M^{A^{(1)}} \otimes M^{A^{(2)}} \otimes N^{B^{(1)}} \otimes N^{B^{(2)}}\right)\Big\} \\
\tilde{W} = V \otimes W \otimes U \\
M^A = M^{A^{(1)}} * V * M^{A^{(2)}} \\
N^B = N^{B^{(1)}} * U * N^{B^{(2)}}
\end{gathered} \tag{6.11}$$

The positivity conditions follows accordingly, for the 2 others :

$$
\begin{aligned}
1 &= \mathrm{Tr}\Big\{W\cdot\left(M^A\otimes N^B\right)^T\Big\}\\
&= \mathrm{Tr}\Big\{W\cdot\left(\mathcal{P}_C^n\left\{M^A\right\}\right)^{T_A}\otimes\left(\mathcal{P}_C^m\left\{N^B\right\}\right)^{T_B}\Big\}\\
&= \mathrm{Tr}\Big\{W\cdot\mathcal{P}_C^n\mathcal{P}_C^m\left\{\left(M^A\right)^{T_A}\otimes\left(N^B\right)^{T_B}\right\}\Big\}\\
&= \mathrm{Tr}\Big\{W\cdot(\mathcal{P}_C^n\mathcal{P}_C^m-{}_{AB}(\cdot))\left\{\left(M^A\right)^{T_A}\otimes\left(N^B\right)^{T_B}\right\}\Big\}+\mathrm{Tr}\Big\{W\cdot{}_{AB}\left(\left(M^A\right)^{T_A}\otimes\left(N^B\right)^{T_B}\right)\Big\}
\end{aligned}
$$

with the rightmost term giving the normalisation condition

$$\mathrm{Tr}\{W\}=d_{A_O^{(1)}}d_{A_O^{(2)}}d_{B_O^{(1)}}d_{B_O^{(2)}}$$

and the left one giving the projective condition

$$
\begin{aligned}
0 &= \mathrm{Tr}\Big\{W\cdot(\mathcal{P}_C^n\mathcal{P}_C^m-{}_{AB}(\cdot))\left\{\left(M^A\right)^{T_A}\otimes\left(N^B\right)^{T_B}\right\}\Big\}\\
&= \mathrm{Tr}\Big\{(\mathcal{P}_C^n\mathcal{P}_C^m-{}_{AB}(\cdot))\left\{W\right\}\cdot\left(\left(M^A\right)^{T_A} otimes \left(N^B\right)^{T_B}\right)\Big\}\\
0 &= (\mathcal{P}_C^n\mathcal{P}_C^m-{}_{AB}(\cdot))\left\{W\right\}\\
W &= \left({}_1(\cdot)-\mathcal{P}_C^{A^{(1)}\preceq A^{(2)}}\mathcal{P}_C^{B^{(1)}\preceq B^{(2)}}+{}_{AB}(\cdot)\right)\left\{W\right\}
\end{aligned}
$$

here we have used the fact that the comb could be arbitrary to make the projected MPM equal to zero. We thus have found the projector onto the 4-partite, 2 parties MPM

$$\mathcal{P}_M^{(2)(2)}\equiv\left({}_1(\cdot)-\mathcal{P}_C^{A^{(1)}\preceq A^{(2)}}\mathcal{P}_C^{B^{(1)}\preceq B^{(2)}}+{}_{AB}(\cdot)\right) \tag{D.11}$$

In the next subsections we will explicit it and then provide some characterisation of it.

D.3.2 Explicit formulation of the projective conditions

For the 2 people acting twice each, it is easier to work with the orthogonal complement of (D.11), which is $\left(\mathcal{P}_C^{A^{(1)}\preceq A^{(2)}}\mathcal{P}_C^{B^{(1)}\preceq B^{(2)}}-{}_{AB}(\cdot)\right)$, so using (C.27) we have

$$
\begin{aligned}
&{}_{\left(1-A_O^{(2)}+\left(1-A_O^{(1)}\right)A^{(2)}\right)\left(1-B_O^{(2)}+\left(1-B_O^{(1)}\right)B^{(2)}\right)}W+\\
&\quad{}_{\left(1-A_O^{(2)}+\left(1-A_O^{(1)}\right)A^{(2)}\right)B^{(1)}B^{(2)}}W+\\
&\quad{}_{A^{(1)}A^{(2)}\left(1-B_O^{(2)}+\left(1-B_O^{(1)}\right)B^{(2)}\right)}W=0
\end{aligned}
\tag{D.12}
$$

where the parts that are obviously on orthogonal subspace[4] directly have been separated in order for the equation to be split on several lines. The two bottom lines are easy to interpret : those are the same matrices as (D.7), but with the other party that have been

[4]Lost reader can be wondering how is it so obvious that these elements are orthogonal. This is a fact that rely on the presence of terms that look like ${}_{((1-X_O)\times\ldots)+(X\times\ldots)}W$, where the symbol $\times$ is used to emphasise that it is a product of projectors. Recall that $X=X_IX_O$ and that, if you applied the projectors of both sides of the sum to an operator the result would look like $(1-X_O)X_IX_O\times\ldots=(X_O-X_O)X_I\times\ldots=0\times X_I\times\ldots=0$ and therefore be null, thus they project to orthogonal subspaces.

traced out ! So it's a process matrix with the actions of one of the parties being trivially ignored and that is compatible with a 2-comb between the actions of the other party.

The interpretation of the first line of eq. (D.12) is less trivial. In terms of direct projectors the conditions become

$$
\begin{aligned}
&{}_{A_0^{(2)}+B_O^{(2)}-A_O^{(2)}B_O^{(2)}}W \\
&- {}_{\left[1-\left(A_0^{(1)}+B_O^{(1)}-A_O^{(1)}B_O^{(1)}-\left(1-B_O^{(1)}\right)A^{(1)}-\left(1-A_O^{(1)}\right)B^{(1)}\right)\right]A^{(2)}B^{(2)}}W \\
&- {}_{\left[1-\left(A_0^{(1)}+B_O^{(2)}-A_O^{(1)}B_O^{(2)}-\left(1-B_O^{(2)}\right)A^{(1)}-\left(1-A_O^{(1)}\right)B^{(2)}\right)\right]A^{(2)}}W \\
&- {}_{\left[1-\left(A_0^{(2)}+B_O^{(1)}-A_O^{(2)}B_O^{(1)}-\left(1-B_O^{(1)}\right)A^{(2)}-\left(1-A_O^{(2)}\right)B^{(1)}\right)\right]B^{(2)}}W
\end{aligned}
\tag{D.13}
$$

because the projectors always contain a term of the form ${}_{\left(X_O^{(2)}+\left(1-X_O^{(1)}\right)X^{(2)}\right)\ldots}\cdot$ they are always simultaneously orthogonal to projectors that look like ${}_{\left(1-X_O^{(2)}\right)\ldots}\cdot$ as well as ${}_{\left(1-X_O^{(1)}\right)X^{(2)}\ldots}\cdot$. Since the conditions of validity of general comb, eq. (5.9c) always have either one of the terms in the definition of the orthogonal projector (like equation (C.36) for the 2 partite case, but with 15 possible combination in the 4 partite case), the 3 process matrices are all valid. This term, ${}_{\left(X_O^{(2)}+\left(1-X_O^{(1)}\right)X^{(2)}\right)}\cdot$ is actually the projector onto the space of valid process matrix that are compatible with 2-comb on X.

The interpretation of the equation becomes clear : the first line of equation (D.13) states that the process matrix must be an operator in which either $A^{(2)}$ or $B^{(2)}$ have trivial output system *i.e.* is last. Then, second line tells you to subtract to it the orthogonal complement of the projector onto the space of 2 PM between $A^{(1)}$ and $B^{(1)}$ when both $A^{(2)}$ and $B^{(2)}$ are trivial. This means that the remaining space when the second operation of both parties have been traced out is the space of 2PM between the 2 parties, as one would have expected. The last two lines are almost the same idea but when only one of the party have its second operation traced out : the remainder have the form of a 2 PM between her remaining tooth and the last tooth of the other party, while his first tooth can be anything since the comb makes it certain that this tooth will never be last.

D.4 Other MPM example and explicit characterisation

Here we introduce a smaller MPM than the example considered in the last section to support and give examples for a few points that are made in the main text and that don't require such a complex MPM. Subsequently, in this section we introduce the simplest non-trivial (*i.e.* that is not a PM) 2 parties MPM : where Alice acts twice and Bob once. Here we will just give it as it is without the development, as it's almost the same as in last section. These formulas will come in handy in the next two sections.

So let's W be a MPM between Alice, who acts twice, and Bob, acting once. Using definition 6.12, the projector is

$$
W = {}_{A_O^{(2)}+B_O-A_O^{(2)}B-\left(1-A_O^{(2)}\right)B-\left(1-A_O^{(1)}+A^{(1)}\right)A^{(2)}B}W
\tag{D.14}
$$

For the interpretation of the terms, see the above section. The expansion in a traceless basis, $W = \sum_{i,j,k,l,m,n} w_{ijklmn}\, \sigma_i^{A_I^{(1)}} \sigma_j^{A_O^{(1)}} \sigma_k^{A_I^{(2)}} \sigma_l^{A_O^{(2)}} \sigma_m^{B_I} \sigma_n^{B_O}$, reads

$$
\begin{aligned}
W &= \frac{1}{d_{A_O^{(1)}} d_{A_O^{(2)}} d_B} \left(\mathbb{1} + \sigma^{A^{(1)} \preceq B} + \sigma^{B \preceq A^{(1)}} + \sigma^{A^{(2)} \preceq B} + \sigma^{B \preceq A^{(2)}} + \sigma^{A^{(1)} \not\preceq\not\succeq A^{(2)} \not\preceq\not\succeq B} \right) \\
\sigma^{A^{(1)} \preceq B} &= \sum_{\substack{j>0 \\ m>0}} \left(\sum_i w_{ij00m0} \sigma_i^{A_I^{(1)}} \sigma_j^{A_O^{(1)}} \sigma_m^{B_I} + \sum_{\substack{i \\ k>0 \cup l>0}} w_{ijklm0} \sigma_i^{A_I^{(1)}} \sigma_j^{A_O^{(1)}} \sigma_k^{A_I^{(2)}} \sigma_l^{A_O^{(2)}} \sigma_m^{B_I} \right) \\
\sigma^{B \preceq A^{(1)}} &= \sum_{\substack{n>0 \\ i>0}} \left(\sum_m w_{i000mn} \sigma_i^{A_I^{(1)}} \sigma_m^{B_I} \sigma_n^{B_O} + \sum_{\substack{m \\ k>0}} \sigma_i^{A_I^{(1)}} \sigma_k^{A_I^{(2)}} \sigma_m^{B_I} \sigma_n^{B_O} \right) \\
\sigma^{A^{(2)} \preceq B} &= \sum_{\substack{l>0 \\ m>0}} \left(\sum_k w_{00klm0} \sigma_k^{A_I^{(2)}} \sigma_l^{A_O^{(2)}} \sigma_m^{B_I} + \sum_{\substack{k \\ j>0 \cup i>0}} w_{ijklm0} \sigma_i^{A_I^{(1)}} \sigma_j^{A_O^{(1)}} \sigma_k^{A_I^{(2)}} \sigma_l^{A_O^{(2)}} \sigma_m^{B_I} \right) \\
\sigma^{B \preceq A^{(2)}} &= \sum_{\substack{n>0 \\ k>0}} \left(\sum_m w_{00k0mn} \sigma_k^{A_I^{(2)}} \sigma_m^{B_I} \sigma_n^{B_O} + \sum_{\substack{m \\ i>0 \cup j>0}} w_{ijk0mn}\, \sigma_i^{A_I^{(1)}} \sigma_j^{A_O^{(1)}} \sigma_k^{A_I^{(2)}} \sigma_m^{B_I} \sigma_n^{B_O} \right) \\
\sigma^{A^{(1)} \not\preceq\not\succeq A^{(2)} \not\preceq\not\succeq B} &= \sum_{i>0} w_{i00000} \sigma_i^{A_I^{(1)}} + \sum_{\substack{i \\ k>0}} w_{i0k000} \sigma_i^{A_I^{(1)}} \sigma_k^{A_I^{(2)}} + \sum_{\substack{i \\ m>0}} \sigma_i^{A_I^{(1)}} \sigma_m^{B_I} + \sum_{\substack{i \\ k>0 \\ m>0}} w_{i0k0m} \sigma_i^{A_I^{(1)}} \sigma_k^{A_I^{(2)}} \sigma_m^{B_I}
\end{aligned} \tag{D.15}
$$

D.4.1 This MPM can violate 3-causal inequality

Here the game is played with two parties : Alice and Bob, they both receive 2 random bits a_1, a_2 and b, b' as settings, Alice can act twice on the system and Bob once. Alice receive only one bit per operation she makes, so she doesn't know a_2 before she have made her first action. It is played as follow : a referee gives all the random bits to the parties but a_2, which is revealed only after Alice have acted the first time. The goal of the game depends on Bob's second bit b'. This extra information fixes the order of the game; it has 3 values 0, 1 and 2 and correspond to these 3 scenarios:

- $b' = 0$: Alice must guess Bob setting b without playing twice;
- $b' = 1$: Bob must guess Alice's first setting a_1 and she also must guess his;
- $b' = 2$: Bob must guess Alice's second setting.

Therefore, the guesses of each part are the outcomes. We will label them x, y, z for, respectively, $A^{(1)}$, $A^{(2)}$ and B, it is possible to express the probability of winning in a formal manner like

$$
\begin{aligned}
P_{\text{succ}} &= P(x = b, b' = 0 | y, z, a_1, a_2, b) + P(y = b, z = a_1, b' = 1 | x, a_1, a_2, b)\Big) \\
&\quad + P(z = a_2, b' = 2 | x, y, a_1, a_2, b
\end{aligned}
$$

which can be simplified using Bayes' rule and assuming that a' is uniformly distributed between its 3 possible values[5]

$$\begin{aligned} P_{\text{succ}} =& \frac{1}{3}\, P(x=b|y,z,a_1,a_2,b,b'=0) + \frac{1}{3}\, P(y=b, z=a_1|x,a_1,a_2,b,b'=1) + \\ & \frac{1}{3}\, P(z=a_2|x,y,a_1,a_2,b,b'=2) \end{aligned} \tag{D.16}$$

This is the probability that we'll try to maximise through this section.

Causal Case

The main limitation when the causal structure is separable is that every player can only send one bit to only one person that he had previously chosen, and only Bob knows who have to guess whose setting. In the classical case, as well as the quantum one with predefined causally separable structure (and even the one that involves a probabilistic mixture of causal orders), no strategy can do better than the following procedure[6] :

1. Alice and Bob assume one of the three possible causal structure, like the one in which B is first followed A.
2. The referee distributes the first settings.
3. They all pass on their setting to the next people in the causal structure.
4. If they guessed the correct order ($b' = 0$) they won with certainty, as they just have to output the setting they are given.
5. Else, if the order is different, in the first case ($b' = 1$) Alice can still know Bob setting with certainty as he's passing it to her, but Bob have to random guess Alice's a_1. In the other case ($b' = 2$), Bob have to random guess Alice's a_2.

Actually, no matter the causal order, there is always two scenarios in which one of the party have to do a random guess. With such a strategy, the probability of success is

$$P_{\text{succ}} = \frac{1}{3} \times 1 + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{2}{3} \quad , \tag{D.17}$$

and hence this is the bound on the causal inequality, remark that Alice could freely pass on her settings and outcome from her first operation to the second, since she acts twice in a row.

[5]If this wasn't the case the analysis can still be performed using some variables to represent this unknown probability but this is an unnecessary complication as in both the classical and quantum cases players must consider different strategies. In any case, fixing this probability doesn't really affect the point of showing an advantage of the non-causally separable game over the classical one.

[6]To see why is true consider the 3 possible causal ordering : Bob is before Alice's two operations, Bob is in between her two operations or Bob is after. In the 3 cases there will always be a scenario that will match what Bob's b' setting is asking for and two that won't. In the scenarios were the wrong causal structure have been chosen, there will always be a first party that won't be able to be signalled the setting of the other party. In the matching scenarios this first party outcome is ignored so it is not a problem, but in the others it is. Note also that when Alice is the first party, she can influence the causal structure of the 2 remaining actions to be taken, so if she will be before or after Bob for her second operation. But because it is Bob that know the causal structure that is required to win the game, she cannot use this information to influence the ordering between the parties and increase the winning probability

(Multi-round) Process Matrix implementation

If the player have access to any valid multi-round process matrix of their choice, in this scenario an MPM that will have the general form of (D.15), they can optimise their strategy by being linked to each other by a causally non-separable process matrix like

$$W = \frac{1}{8}\left(\mathbb{1}^{\otimes 6} + \frac{1}{\sqrt{5}}\left(\sigma_x\sigma_y\sigma_y\mathbb{1}\sigma_z\sigma_x + \mathbb{1}\sigma_y\sigma_y\mathbb{1}\sigma_y\sigma_y + \mathbb{1}\sigma_y\sigma_y\sigma_x\sigma_x\mathbb{1}\right)\right) \tag{D.18}$$

where $\mathbb{1}$ is the 2-by-2 unit matrix, $\sigma_x, \sigma_y, \sigma_z$, the Pauli matrices and where the tensor product as well as the superscripts to indicate on which space where the operators act are both omitted (*e.g.* : $\mathbb{1}^{A_I^{(1)}} \otimes \sigma_y^{A_O^{(1)}} \otimes \sigma_y^{A_I^{(2)}} \otimes \sigma_x^{A_O^{(2)}} \otimes \sigma_x^{B_I} \otimes \mathbb{1}^{B_O} \equiv \mathbb{1}\sigma_y\sigma_y\sigma_x\sigma_x\mathbb{1}$, I and O subscripts indicating respectively the input or output systems).

Alice's actions are the same regardless of what is happening : first she measures in X basis and sends in Y, and during her second operation she does the opposite (measure in Y, sends in X). Whilst Bob's adapting his action depending on what the value of b' is

- $b' = 0$: he measures in Z, and sends his input in X basis;
- $b' = 1$: he measures in Y, and sends in Y;
- $b' = 2$: he measures in X, and sends a trivial system.

Therefore, let ξ, χ, η be the matrix representation of the operations of respectively Alice first time, Alice second time and Bob. We have that $\xi = \xi(x, a_1), \chi = \chi(y, a_2), \eta = \eta(z, b, b')$, and, explicitly,

$$\begin{aligned}
\eta &= \delta_{0,b'}\eta_0 + \delta_{1,b'}\eta_1 + \delta_{2,b'}\eta_2 \quad ; \\
\eta_0 &= \frac{1}{4}\left(\mathbb{1} + (-1)^z\sigma_z\right) \otimes \left(\mathbb{1} + (-1)^{b+z}\sigma_x\right) \\
\eta_1 &= \frac{1}{4}\left(\mathbb{1} + (-1)^z\sigma_y\right) \otimes \left(\mathbb{1} + (-1)^{b}\sigma_y\right) \\
\eta_2 &= \frac{1}{2}\left(\mathbb{1} + (-1)^z\sigma_x\right) \otimes \rho \quad ,
\end{aligned} \tag{D.19}$$

where δ is the Kronecker delta function and ρ some trivial system. As for Alice we have

$$\xi = \frac{1}{4}\left(\mathbb{1} + (-1)^x\sigma_x\right) \otimes \left(\mathbb{1} + (-1)^{a_1}\sigma_y\right) \quad , \tag{D.20}$$

and

$$\chi = \frac{1}{4}\left(\mathbb{1} + (-1)^y\sigma_y\right) \otimes \left(\mathbb{1} + (-1)^{a_2}\sigma_x\right) \quad . \tag{D.21}$$

With such a strategy the probability of winning (D.16) is computed using the generalised

Born's rule ($p = \text{Tr}\left[W\left(\eta \otimes \xi \otimes \chi\right)\right]$) combined with the definition of conditional probabilities. In a first time we need to calculate these quantities :

$$\begin{aligned} P(x = b|y, z, a_1, a_2, b, b' = 0) &= \frac{1}{32} \sum_{y,z,a_1,a_2,b} P(x = b, y, z, a_1, a_2, b|b' = 0) \\ &= \frac{1}{32} \sum_{x,a,b,c} \text{Tr}\left[W\left(\eta(z, b, b' = 0) \otimes \xi(x = b, a_1) \otimes \chi(y, a_2)\right)\right] \end{aligned} \tag{D.22a}$$

$$\begin{aligned} P(y = b, z = a_1|x, a_1, a_2, b, b' = 1) &= \frac{1}{16} \sum_{x,a_1,a_2,b} P(x, y = b, z = a_1, a_1, a_2, b|b' = 1) \\ = \frac{1}{16} \sum_{x,a_1,a_2,b} &\text{Tr}\left[W\left(\eta(z = a_1, b, b' = 1) \otimes \xi(x, a_1) \otimes \chi(y = b, a_2)\right)\right] \end{aligned} \tag{D.22b}$$

$$\begin{aligned} P(z = a_2|x, y, a_1, a_2, b, b' = 2) &= \frac{1}{32} \sum_{x,y,a_1,a_2,b} P(z = a_2, x, y, a_1, a_2, b|b' = 2) \\ &= \frac{1}{32} \sum_{x,y,a_1,a_2,b} \text{Tr}\left[W\left(\eta(z = a_2, b, b' = 2) \otimes \xi(x, a_1) \otimes \chi(y, a_2)\right)\right] \end{aligned} . \tag{D.22c}$$

To do so, the following relations will greatly simplify the math :

$$\text{Tr}(\eta_0 \sigma_z \sigma_x) = (-1)^b \quad \text{Tr}(\eta_1 \sigma_z \sigma_x) = 0 \quad \text{Tr}(\eta_2 \sigma_z \sigma_x) = 0 \tag{D.23a}$$

$$\text{Tr}(\eta_0 \sigma_y \sigma_y) = 0 \quad \text{Tr}(\eta_1 \sigma_y \sigma_y) = (-1)^{z+b} \quad \text{Tr}(\eta_2 \sigma_y \sigma_y) = 0 \tag{D.23b}$$

$$\text{Tr}(\eta_0 X \mathbb{1}) = 0 \quad \text{Tr}(\eta_1 \sigma_x \mathbb{1}) = 0 \quad \text{Tr}(\eta_2 \sigma_x \mathbb{1}) = (-1)^x \tag{D.23c}$$

Indeed, property (F.9) combined with GGB properties (A.36b) will greatly reduce the number of terms that must be computed. In a same fashion, one can use

$$\text{Tr}(\xi \sigma_x \sigma_y) = (-1)^{x+a_1} \quad \text{Tr}(\xi \mathbb{1} \sigma_y) = (-1)^{a_1} \tag{D.24a}$$

$$\text{Tr}(\chi \sigma_y \mathbb{1}) = (-1)^y \quad \text{Tr}(\chi \sigma_y \sigma_x) = (-1)^{y+a_2} \tag{D.24b}$$

$$\text{Tr}(\eta \xi \chi \mathbb{1}) = 1 \tag{D.24c}$$

Putting everything together the probability function become

$$\begin{aligned} P(x, y, z, a_1, a_2, b|b') =& \frac{\delta_{b',0}}{3} \left(1 + \frac{1}{\sqrt{5}}(-1)^{b+x}\right) + \\ & \frac{\delta_{b',1}}{3} \left(1 + \frac{1}{\sqrt{5}}(-1)^{(a_1+z)}(-1)^{(b+y)}\right) + \frac{\delta_{b',2}}{3} \left(1 + \frac{1}{\sqrt{5}}(-1)^{z+a_2}\right) \end{aligned} \tag{D.25}$$

Using this reduced form the conditional probabilities that correspond to a winning situation can be extracted from (D.22). Explicitly

$$P(x = b|y, z, a_1, a_2, b, b' = 0) = \frac{1}{6}\left(1 + \frac{1}{\sqrt{5}}\right)$$
$$P(y = b, z = a_1|x, a_1, a_2, b, b' = 1) = \frac{1}{6}\left(1 + \frac{1}{\sqrt{5}}\right)$$
$$P(z = a_2|x, y, a_1, a_2, b, b' = 2) = \frac{1}{6}\left(1 + \frac{1}{\sqrt{5}}\right)$$

Finally, the overall probability of winning (D.16) is

$$P_{\text{succ}} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{5}}\right) = \frac{5+\sqrt{5}}{10} \approx 0.7236 \quad , \tag{D.26}$$

which is strictly bigger than the classical case (D.17). We therefore have proven that there exist 3-partite, 2 parties MPM that can beat a 3-causal inequality.

D.4.2 Activation by side-channel

If now we allow the side channel to be used, one can find an example of activation of causal non-separability like the one in [54], but using the side-channel instead of the shared entangled ancillas. Here we provide this example to motivate the extension of the definition of causal separability that also consider all the side channels, as it was presented in the main text, def. 17. Consider the following MPM of form (D.15) for which all the dimensions of the subsystems have been set to 2 :

$$W^{A^{(1)}A^{(2)}B} = \frac{1}{8}\left(\mathbb{1} + \frac{1}{\sqrt{2}}\left[\sigma_x^{A_I^{(1)}}\sigma_z^{A_I^{(2)}}\sigma_z^{A_O^{(2)}}\sigma_z^{B_I} + \sigma_z^{A_I^{(1)}}\sigma_z^{A_I^{(2)}}\sigma_z^{B_O}\right]\right) \tag{D.27}$$

this is almost the same matrix as the example in [54] but the isolated party is now getting an non-trivial input instead of output. Therefore, following the same arguments as in the paper[7] we can see that it is a valid process matrix for which $A^{(1)}$ is causally separable from the two other partites and it cannot be used to beat a causal inequality. It cannot however get activated by the 'teleportation' of some part of $A^{(1)}$ to $A^{(2)}$ through entangled ancillas because the part of the PM to be teleported is now an input and not an output. But by allowing a side channel in the system we can simply pass on the input of $A^{(1)}$ to her causal future. Consider the extension by side channel

$$W^{A^{(1)}A_{O'}^{(1)}A_{I'}^{(2)}A^{(2)}B} = W^{A^{(1)}A^{(2)}B} \otimes \frac{|\Phi^+\rangle\langle\Phi^+|^{A_{O'}^{(1)}A_{I'}^{(2)}}}{2} \tag{D.28}$$

Where $|\Phi^+\rangle\langle\Phi^+|^{A_{O'}^{(1)}A_{I'}^{(2)}}$ is the identity unitary channel. Now if $A^{(1)}$'s action is to simply forward her input through the channel, $M^{A^{(1)}} = |\Phi^+\rangle\langle\Phi^+|^{A_I^{(1)}A_{O'}^{(1)}} \otimes \frac{1}{2}\mathbb{1}^{A_O^{(1)}}$ the reduced 2 partite process matrix is

$$W^{A^{(2)}A_{I'}^{(2)}B} = \frac{1}{8}\left(\mathbb{1} + \frac{1}{\sqrt{2}}\left[\sigma_z^{A_I^{(2)}}\sigma_z^{A_O^{(2)}}\sigma_x^{A_{I'}^{(2)}}\sigma_z^{B_I} + \sigma_z^{A_{I'}^{(1)}}\sigma_z^{A_I^{(2)}}\sigma_z^{B_O}\right]\right) \tag{D.29}$$

[7] We invite the curious reader to consult section III.D. pp.22-23 of [54] for the details.

in which one can factorise the $\sigma_z^{A_I^{(2)}}$ terms to let appear a formulation that is equivalent to the epitomical OCB process matrix [1] between the remaining subsystems, a typical example of causally non-separable process matrix that can be used to violate causal inequalities. As argued in [70], a proper definition of causal separability should prevent such a case of activation, this is why we also consider all possible extension by the side channels in definition 17.

D.4.3 Elements on the correlations made available by the MPM

Here we provide an approach on the exploration of the possible new correlations made available with the MPM (D.15). To understand where the new correlations could be coming from, it can be useful to compare the expansion of a deterministic 2-comb (C.28) with the one of a tensor product of deterministic 1-combs (3.3) :

$$\begin{aligned} M^{A^{(1)}} \otimes M^{A^{(2)}} = \frac{1}{d_{A_O^{(1)}} d_{A_O^{(2)}}} \Bigg(& \mathbb{1} + \sum_{\substack{i=0 \\ j>0}} m_{ij}^{(1)} \sigma_i^{A_I^{(1)}} \sigma_j^{A_O^{(1)}} \\ & + \sum_{\substack{k=0 \\ l>0}} m_{kl}^{(2)} \sigma_k^{A_I^{(2)}} \sigma_l^{A_O^{(2)}} + \sum_{\substack{i=0 \\ j>0}} \sum_{\substack{k=0 \\ l>0}} m_{ij}^{(1)} m_{kl}^{(2)} \sigma_i^{A_I^{(1)}} \sigma_j^{A_O^{(1)}} \sigma_k^{A_I^{(2)}} \sigma_l^{A_O^{(2)}} \Bigg) \end{aligned} \tag{D.30}$$

$$M^{A^{(1)} \preceq A^{(2)}} = \frac{1}{d_{A_O^{(1)}} d_{A_O^{(2)}}} \Bigg(\mathbb{1} + \sum_{\substack{i=0 \\ j>0}} m_{ij00} \sigma_i^{A_I^{(1)}} \sigma_j^{A_O^{(1)}} + \sum_{\substack{i,j,k=0 \\ l>0}} m_{ijkl} \sigma_i^{A_I^{(1)}} \sigma_j^{A_O^{(1)}} \sigma_k^{A_I^{(2)}} \sigma_l^{A_O^{(2)}} \Bigg) \tag{D.31}$$

We see that there is two things the 2-comb can do that the product of 1-combs cannot. First there is the terms that have the form $m_{i00l}\ \sigma_i^{A_I^{(1)}} \sigma_l^{A_O^{(2)}}$ and $m_{i0kl}\ \sigma_i^{A_I^{(1)}} \sigma_k^{A_I^{(2)}} \sigma_l^{A_O^{(2)}}$ that are purely forbidden in a tensor product of combs. Such terms can be understand as follows[8] : the first operation consists on measuring the input system in a given basis then output a maximally mixed state conditionally on the result of measurement. This is problematic if the operation was a 1-comb as it does not preserve the trace. But when it is a 2-comb, it correspond to switching the measured system with a depolarised system, then applying a unitary to the input system inside the side-channel and finally output it in the second operation (that is both i and l non-zero, if k is also non-zero, the unitary can get conditioned by the measurement of the input system of the second operation). The second kind of terms that are not feasible with a tensor product of quantum combs, is basically when the 2-comb cannot be factored in a tensor product of 1-combs, this 'entanglement' simply account for the fact that there is information flowing through the side-channel and thus it makes no sense to consider the operations as individual. Note that whilst the m_{i00kl} terms alone can be obtained by CP trace non-increasing maps applied by both parties when taken as 1-combs, the terms that cannot be factorised as a tensor product of combs are obviously impossible to be simulated by the combs acting on their own, no matter the kind of map they choose to apply on their side.

However, this will be insufficient for the case considered here (3 slots, 2 partites) to show a new kind of non-causal correlations using the side-channel. The reason for it is simple : from Bob's point of view, when he applies his operations, the reduced MPM is nothing else than a comb, and from Alice's point of view, when she applies her first round, the

[8] To see it explicitly apply the reverse direction Choi-Jamiołkowski isomorphism (2.29).

reduces MPM is a 2 parties PM. Both cannot present new dynamics other than the advantage gained by the presence of the side channel. So in that case, the correlations will only benefit from the 'supplementary room' provided by the side channel, which is for example a new channel between Alice's operations if there was none *a priori* or the advantage provided solely by the size of the comb that simply increase the dimension of Alice's channel to herself, and thus allow her to keep more information in memory between her operations.

For that reason, the bigger MPM (*e.g.*(D.13) that was introduced last section) is now needed to carry on with the search of genuinely new dynamics when allowing side channels. Recall that what we want to see is that if there is some correlations that an MPM can obtain only when objects bigger than 1-comb are plugged into it, and that cannot be explained by the sole size of the side-channel. Since we already discarded activation from the possible causes and that we changed the definition of causal separability accordingly, we will assume that all the improvement in correlations added by a side-channel that activated the process is not relevant. This is why we consider coherent superposition of different side channel as the only thing that could bring *genuinely* new correlations in the main text.

Appendix E

Characterisation of the set of valid process matrices

In this last note we present the sketch of an auxiliary result to the main topic of this thesis. The projective formulation developed for the description of the multi-round process matrices can indeed serve other purposes than just deriving validity conditions for it. Here we use it to show that the convex set of valid process matrices is the convex hull of all the one-party MPM with the same number of slots but every possible combination of causal order. We will need to consider the permutations of the causal order between parties[1]. If we consider N parties, whose set is noted as $\mathcal{N} = \left\{A^{(1)}, A^{(2)}, ..., A^{(N)}\right\}$, there will be $N!$ different ways of forming complete causal order. The complete set of possible causal orders will be denoted by Π :

$$\Pi = \Big\{A^{(1)} \preceq A^{(2)} \preceq ... \preceq A^{(N-1)} \preceq A^{(N)}, A^{(2)} \preceq A^{(1)} \preceq ... \preceq A^{(N-1)} \preceq A^{(N)},$$
$$..., A^{(1)} \preceq A^{(2)} \preceq ... \preceq A^{(N)} \preceq A^{(N-1)}, ..., A^{(N)} \preceq A^{(N-1)} \preceq ... \preceq A^{(2)} \preceq A^{(1)}\Big\}$$

An arbitrary element of this set will be denoted by π. We will also need to consider arbitrary ordered subset $(k_1, k_2, ...k_n)$ of $\mathcal{N}$ with n elements (with $1 \leq n \leq N$, $k_i \neq k_j$ for $i \neq j$), let $\Pi_{(k_1,k_2,...k_n)}$ be the set of permutations of $\mathcal{N}$ in which element k_1 have been fixed in first position, k_2 in second position , ...etc, *i.e.* $\Pi_{(k_1,k_2,...k_n)} = \{\pi \in \Pi | \pi(1) = k_1, \dots, \pi(n) = k_n\}$.

The main theorem of this appendix relies on the recursive formulation of the projectors to the subspace of PM (5.14) and to the subspace of one partite MPM (6.8), which can be shown to be connected using the projector properties, see C.1.3. The theorems states that every projector to the subspace of process matrix is exactly equivalent to the projector obtained by taking the union of projectors onto subspace of fully causally ordered PM, *i.e.* 1 party MPM. This union of MPM can in turn be understood as a weighted sum of quantum combs through theorem 8, with the weights being any real numbers, possibly negative. An analogue to this last result was already proven in [10] using the hierarchy of superoperator formalism (*i.e.* the axiomatisation of quantum combs), but the authors did not make the link with process matrix formalism explicitly. Here we will derive the result using the PM framework and we will see in the upcoming sections that this can be used to better characterise the process matrix itself.

The proof is actually quite simple to understand : for an N-partite process, the projector onto the subspace of general process matrix is actually equivalent to the union of the $N!$ projectors that each projects onto the subspace of process matrix that have a completely

[1]Since this chapter is mainly talking about process matrix, party and partite will often be confused.

determined causal order (which we have seen to be 1 party MPM), and there is $N!$ projectors of this kind because of every causal order one can obtain by permuting the N parties together. Because of this every element that belong to the set of valid PM can be written as a weighted sum of elements in the subspaces of valid one party MPM with a given fixed causal order. By means of theorem 8, we can show that these elements are in fact combs. The result, as we will see, can be used to show that non-dynamical, causally separable PM are the ones that can be decomposed in a convex sum of combs with different causal orders. And that the other kind of causally separable PM, the dynamical PM, also present a particular manner of being decomposed.

The proof requires the following lemma in order to be more easily done. It is about composition of projectors onto the subspace of valid PM with fixed causal order.

Lemma 2. *Let $\mathcal{P}_X$ and $\mathcal{P}_Y$, two projectors of type $\mathcal{P}_M$, defined as in equation* (6.8)*, both acting on the same set of parties but with different causal orders. Their union, i.e. the projector onto the union of both of their subspaces is given by formula* (C.18)*. We have that the extension of the union of these projectors by an extra party acting last is the same as the union of the extension of both projector individually :*

$$\mathcal{P}_{X\cup Y}^{(\ldots\preceq n+1)} = \mathcal{P}_X^{(\ldots\preceq n+1)} + \mathcal{P}_Y^{(\ldots\preceq n+1)} - \mathcal{P}_X^{(\ldots\preceq n+1)}\mathcal{P}_Y^{(\ldots\preceq n+1)} \tag{E.1}$$

or, written in recursive formulation,

$$_{A_{IO}^{(n+1)}}\mathcal{P}_{X\cup Y} + {}_{\left(A_O^{(n+1)} - A_{IO}^{(n+1)}\right)}(\cdot) = \tag{E.2}$$

$$_{A_{IO}^{(n+1)}}\mathcal{P}_X + {}_{\left(A_O^{(n+1)} - A_{IO}^{(n+1)}\right)}(\cdot) + {}_{A_{IO}^{(n+1)}}\mathcal{P}_Y + {}_{\left(A_O^{(n+1)} - A_{IO}^{(n+1)}\right)}(\cdot) \tag{E.3}$$

$$+ {}_{A_{IO}^{(n+1)}}(\mathcal{P}_X\mathcal{P}_Y) + {}_{\left(A_O^{(n+1)} - A_{IO}^{(n+1)}\right)}(\cdot) \tag{E.4}$$

Proof. Starting from the left-hand side of (E.4):

$$\begin{aligned}
{}_{A_{IO}^{(n+1)}}\mathcal{P}_{X\cap Y} + {}_{\left(A_O^{(n+1)} - A_{IO}^{(n+1)}\right)}(\cdot) &= {}_{A_{IO}^{(n+1)}}\mathcal{P}_X + {}_{A_{IO}^{(n+1)}}\mathcal{P}_Y - {}_{A_{IO}^{(n+1)}}(\mathcal{P}_X\mathcal{P}_Y) + {}_{\left(A_O^{(n+1)} - A_{IO}^{(n+1)}\right)}(\cdot) \\
&= {}_{A_{IO}^{(n+1)}}\mathcal{P}_X + {}_{A_{IO}^{(n+1)}}\mathcal{P}_Y - {}_{A_{IO}^{(n+1)}}(\mathcal{P}_X\mathcal{P}_Y) + 2\,{}_{\left(A_O^{(n+1)} - A_{IO}^{(n+1)}\right)}(\cdot) - {}_{\left(A_O^{(n+1)} - A_{IO}^{(n+1)}\right)}(\cdot) \\
&= \mathcal{P}_X^{(\ldots\preceq n+1)} + \mathcal{P}_Y^{(\ldots\preceq n+1)} - (\mathcal{P}_X\mathcal{P}_Y)^{(\ldots\preceq n+1)} \\
&= \mathcal{P}_X^{(\ldots\preceq n+1)} + \mathcal{P}_Y^{(\ldots\preceq n+1)} - \mathcal{P}_X^{(\ldots\preceq n+1)}\mathcal{P}_Y^{(\ldots\preceq n+1)}
\end{aligned}$$

where we have defined

$$(\mathcal{P}_X\mathcal{P}_Y)^{(\ldots\preceq n+1)} = {}_{A_{IO}^{(n+1)}}(\mathcal{P}_X\mathcal{P}_Y) + {}_{\left(A_O^{(n+1)} - A_{IO}^{(n+1)}\right)}(\cdot)$$

to get from line 2 to line 3, because we have the identity

$$\begin{aligned}
\mathcal{P}_X^{(\ldots\preceq n+1)}\mathcal{P}_Y^{(\ldots\preceq n+1)} &= \left({}_{A_{IO}^{(n+1)}}\mathcal{P}_X + {}_{\left(A_O^{(n+1)} - A_{IO}^{(n+1)}\right)}(\cdot)\right)\left({}_{A_{IO}^{(n+1)}}\mathcal{P}_Y + {}_{\left(A_O^{(n+1)} - A_{IO}^{(n+1)}\right)}(\cdot)\right) \\
&= {}_{A_{IO}^{(n+1)}}(\mathcal{P}_X\mathcal{P}_Y) + {}_{\left(A_O^{(n+1)} - A_{IO}^{(n+1)}\right)}{}_{A_{IO}^{(n+1)}}(\mathcal{P}_X) \\
&\quad + {}_{\left(A_O^{(n+1)} - A_{IO}^{(n+1)}\right)}{}_{A_{IO}^{(n+1)}}(\mathcal{P}_Y) + {}_{\left(A_O^{(n+1)} - A_{IO}^{(n+1)}\right)}{}_{\left(A_O^{(n+1)} - A_{IO}^{(n+1)}\right)}(\cdot) \\
&= {}_{A_{IO}^{(n+1)}}(\mathcal{P}_X\mathcal{P}_Y) + 0 + 0 + {}_{\left(A_O^{(n+1)} - A_{IO}^{(n+1)}\right)}(\cdot) \\
&= (\mathcal{P}_X\mathcal{P}_Y)^{(\ldots\preceq n+1)}
\end{aligned}$$

that was used to link lines 3 and 4. □

With this lemma, we can formulate the theorem :

Theorem 11. *For a set of N parties labelled $\mathcal{N} = \{A^{(1)}, ..., A^{(n)}\}$, the projector onto the space of process matrices between all these parties ,$\mathcal{P}_V^{(n)}$, given by equation* (5.14)*, is equal to the union of all N*! *projectors onto a subspace of the space of valid PM that admit a complete causal order.*

$$\mathcal{P}_V^{(n)} = \bigcup_{\pi_i \in \Pi} \mathcal{P}_M^{\pi_i\left(A^{(1)} \preceq ... \preceq A^{(n)}\right)} \tag{E.5}$$

Proof. By recursion, for $N = 1$, it's true, as formula (5.14) and (6.8) are equivalent in this case :

$$\mathcal{P}_V^{(1)} = {}_{A_O^{(1)}}(\cdot) = \mathcal{P}_M^{(1)}$$

Suppose this is true for the case $N = n$,

$$\mathcal{P}_V^{(n)} = \bigcup_{\pi_i \in \Pi_{\mathcal{N}}} \mathcal{P}_M^{\pi_i\left(A^{(1)} \preceq ... \preceq A^{(n)}\right)}$$

Where we have put a subscript to the group of permutation to highlight to which ensemble it is affiliated. Then, for $N = n + 1$, it must also be true. Let $\mathcal{N}'$ be the new set of parties, i.e. $\mathcal{N}' = \mathcal{N} \cup \{A^{(n+1)}\}$, the following relation must hold

$$\mathcal{P}_V^{(n+1)} = \bigcup_{\pi_i \in \Pi_{\mathcal{N}'}} \mathcal{P}_M^{\pi_i\left(A^{(1)} \preceq ... \preceq A^{(n)} \preceq A^{(n+1)}\right)}$$

Remark that the new union of projectors on the right-hand side is actually the union of all $n + 1$ possible sets of n elements in $\mathcal{N}'$ extended by the only element left in $\mathcal{N}'$ acting last :

$$\bigcup_{\pi_i \in \Pi_{\mathcal{N}'}} \mathcal{P}_M^{\pi_i\left(A^{(1)} \preceq ... \preceq A^{(n)} \preceq A^{(n+1)}\right)} = \bigcup_{i=1}^{n+1} \left(\bigcup_{\pi_j \in \Pi_{\mathcal{N}' \setminus \{A^{(i)}\}}} \mathcal{P}_M^{\pi_j\left(A^{(1)} \preceq ... \preceq A^{(i-1)} \preceq A^{(i+1)} \preceq ... \preceq A^{(n+1)}\right) \preceq A^{(i)}} \right)$$

Using (6.8) and lemma 2, this is equivalent to

$$\begin{aligned}
&\bigcup_{i=1}^{n+1} \left(\bigcup_{\pi_j \in \Pi_{\mathcal{N}' \setminus \{A^{(i)}\}}} \mathcal{P}_M^{\pi_j\left(A^{(1)} \preceq ... \preceq A^{(i-1)} \preceq A^{(i+1)} \preceq ... \preceq A^{(n+1)}\right) \preceq A^{(i)}} \right) \\
&= \bigcup_{i=1}^{n+1} \left(\bigcup_{\pi_j \in \Pi_{\mathcal{N}' \setminus \{A^{(i)}\}}} \left({}_{A_{IO}^{(i)}}\mathcal{P}_M^{\pi_j\left(A^{(1)} \preceq ... \preceq A^{(i-1)} \preceq A^{(i+1)} \preceq ... \preceq A^{(n+1)}\right)} + {}_{\left(A_O^{(i)} - A_{IO}^{(i)}\right)}(\cdot) \right) \right) \\
&= \bigcup_{i=1}^{n+1} \left({}_{A_{IO}^{(i)}} \left(\bigcup_{\pi_j \in \Pi_{\mathcal{N}' \setminus \{A^{(i)}\}}} \mathcal{P}_M^{\pi_j\left(A^{(1)} \preceq ... \preceq A^{(i-1)} \preceq A^{(i+1)} \preceq ... \preceq A^{(n+1)}\right)} \right) + {}_{\left(A_O^{(i)} - A_{IO}^{(i)}\right)}(\cdot) \right)
\end{aligned}$$

We can split this equation to make $\mathcal{P}_V^{(n)}$ appears

$$
\begin{aligned}
&\bigcup_{i=1}^{n+1}\left({}_{A_{IO}^{(i)}}\left(\bigcup_{\pi_j\in\Pi_{\mathcal{N}'\backslash\{A^{(i)}\}}}\mathcal{P}_M^{\pi_j\left(A^{(1)}\preceq\ldots\preceq A^{(i-1)}\preceq A^{(i+1)}\preceq\ldots\preceq A^{(n+1)}\right)}\right)+{}_{\left(A_O^{(i)}-A_{IO}^{(i)}\right)}(\cdot)\right)\\
&=\bigcup_{i=1}^{n}\left({}_{A_{IO}^{(i)}}\left(\bigcup_{\pi_j\in\Pi_{\mathcal{N}'\backslash\{A^{(i)}\}}}\mathcal{P}_M^{\pi_j\left(A^{(1)}\preceq\ldots\preceq A^{(i-1)}\preceq A^{(i+1)}\preceq\ldots\preceq A^{(n+1)}\right)}\right)+{}_{\left(A_O^{(i)}-A_{IO}^{(i)}\right)}(\cdot)\right)\\
&\quad\cup\left({}_{A_{IO}^{(n+1)}}\left(\bigcup_{\pi_i\in\Pi_{\mathcal{N}}}\mathcal{P}_M^{\pi_i\left(A^{(1)}\preceq\ldots\preceq A^{(n)}\right)}\right)+{}_{\left(A_O^{(n+1)}-A_{IO}^{(n+1)}\right)}(\cdot)\right)\\
&=\bigcup_{i=1}^{n}\left({}_{A_{IO}^{(i)}}\left(\bigcup_{\pi_j\in\Pi_{\mathcal{N}'\backslash\{A^{(i)}\}}}\mathcal{P}_M^{\pi_j\left(A^{(1)}\preceq\ldots\preceq A^{(i-1)}\preceq A^{(i+1)}\preceq\ldots\preceq A^{(n+1)}\right)}\right)+{}_{\left(A_O^{(i)}-A_{IO}^{(i)}\right)}(\cdot)\right)\\
&\quad\cup\left({}_{A_{IO}^{(n+1)}}\left(\mathcal{P}_V^{(n)}\right)+{}_{\left(A_O^{(n+1)}-A_{IO}^{(n+1)}\right)}(\cdot)\right)
\end{aligned}
$$

As for the other terms, notice they are also defined as a projector of n-partite valid process matrix but on other subsets of parties. It is possible to continue the decomposition like

$$
\begin{aligned}
&\bigcup_{i=1}^{n}\left({}_{A_{IO}^{(i)}}\left(\bigcup_{\pi_j\in\Pi_{\mathcal{N}'\backslash\{A^{(i)}\}}}\mathcal{P}_M^{\pi_j\left(A^{(1)}\preceq\ldots\preceq A^{(i-1)}\preceq A^{(i+1)}\preceq\ldots\preceq A^{(n+1)}\right)}\right)+{}_{\left(A_O^{(i)}-A_{IO}^{(i)}\right)}(\cdot)\right)\\
&\quad\cup\left({}_{A_{IO}^{(n+1)}}\left(\mathcal{P}_V^{(n)}\right)+{}_{\left(A_O^{(n+1)}-A_{IO}^{(n+1)}\right)}(\cdot)\right)\\
&=\bigcup_{i=1}^{n}\left(\bigcup_{\pi_j\in\Pi_{\mathcal{N}'\backslash\{A^{(i)}\}}}\mathcal{P}_M^{\pi_j\left(A^{(1)}\preceq\ldots\preceq A^{(i-1)}\preceq A^{(i+1)}\preceq\ldots\preceq A^{(n+1)}\right)\preceq A^{(i)}}\right)\\
&\quad+\left({}_{A_{IO}^{(n+1)}}\left(\mathcal{P}_V^{(n)}\right)+{}_{\left(A_O^{(n+1)}-A_{IO}^{(n+1)}\right)}(\cdot)\right)\\
&\quad-\bigcup_{i=1}^{n}\left(\bigcup_{\pi_j\in\Pi_{\mathcal{N}'\backslash\{A^{(i)}\}}}\mathcal{P}_M^{\pi_j\left(A^{(1)}\preceq\ldots\preceq A^{(i-1)}\preceq A^{(i+1)}\preceq\ldots\preceq A^{(n+1)}\right)\preceq A^{(i)}}\right)\left({}_{A_{IO}^{(n+1)}}\left(\mathcal{P}_V^{(n)}\right)+{}_{\left(A_O^{(n+1)}-A_{IO}^{(n+1)}\right)}(\cdot)\right)
\end{aligned}
$$

where formula (C.18) was used to go to the next line. The negative term can be simplify by a heuristic argument[2] : since it is the intersection of the union of all the subspaces that have a causal order but that do not ends by the party $A^{(n+1)}$ acting last with the union of the projectors onto the totally ordered PM that do end with $A^{(n+1)}$, the only possible term that they could possibly have in common is when $A^{(n+1)}$ is the only one party acting. This

[2]The heuristic approach is preferred here because the explicit computation, while more rigorous, would add an unnecessarily extra length to the proof.

is translated as the following projector :

$$\bigcup_{i=1}^{n}\left(\bigcup_{\pi_j\in\Pi_{\mathcal{N}'\setminus\{A^{(i)}\}}}\mathcal{P}_M^{\pi_j\left(A^{(1)}\preceq\ldots\preceq A^{(i-1)}\preceq A^{(i+1)}\preceq\ldots\preceq A^{(n+1)}\right)\preceq A^{(i)}}\right)$$
$$\left({}_{A_{IO}^{(n+1)}}\left(\mathcal{P}_V^{(n)}\right)+{}_{\left(A_O^{(n+1)}-A_{IO}^{(n+1)}\right)}(\cdot)\right)$$
$$={}_{\left(1-A_O^{(n+1)}\prod_{i=1}^{n}A_{IO}^{(i)}\right)}(\cdot)$$

As the left-hand term in the sum, the one that is the union of everything that do not end by $A^{(n+1)}$, since we postulated that the N partite objects that compose them are actually the union of all valid process matrix with this kind of causal order, it can be rewritten as the union of all the N partite valid PM $\mathcal{P}_V$ extended by the party $A^{(n+1)}$ but without it being last, i.e. extended by ${}_{\left(1-A_O^{(n+1)}\right)}\cdot$, since it's the dual of the extension (6.8), hence

$$\bigcup_{i=1}^{n}\left(\bigcup_{\pi_j\in\Pi_{\mathcal{N}'\setminus\{A^{(i)}\}}}\mathcal{P}_M^{\pi_j\left(A^{(1)}\preceq\ldots\preceq A^{(i-1)}\preceq A^{(i+1)}\preceq\ldots\preceq A^{(n+1)}\right)\preceq A^{(i)}}\right)$$
$$={}_{\left(1-A_O^{(n+1)}\right)}\left(\mathcal{P}_V^{(n)}\right)$$

Putting everything together,

$$\mathcal{P}_V^{(n+1)}={}_{\left(1-A_O^{(n+1)}\right)}\left(\mathcal{P}_V^{(n)}\right)+\left({}_{A_{IO}^{(n+1)}}\left(\mathcal{P}_V^{(n)}\right)+{}_{\left(A_O^{(n+1)}-A_{IO}^{(n+1)}\right)}(\cdot)\right)-{}_{\left(1-A_O^{(n+1)}\right)\prod_{i=1}^{n}A_{IO}^{(i)}}(\cdot)$$

Finally this can be reorganised as

$$\mathcal{P}_V^{(n+1)}={}_{\left(1-A_O^{(n+1)}+A_{IO}^{(n+1)}\right)}\left(\mathcal{P}_V^{(n)}\right)+{}_{\left(A_O^{(n+1)}-A_{IO}^{(n+1)}\right)}(\cdot)-{}_{\left(1-A_O^{(n+1)}\right)\prod_{i=1}^{n}A_{IO}^{(i)}}(\cdot)\tag{E.6}$$

which is indeed the recursive definition of $\mathcal{P}_V$, according to equation (5.14). This concludes the proof. □

This theorem imply that we can always decompose a valid process matrix as a sum of terms that have a completely defined causal order. Although it may looks trivial, we will now provide elements that shows that it can perhaps be used to differentiate causally separable PM from those who are not.

We have seen that the intersection of the subset of PM with the positive cone is a convex cone. So is the intersection of MPM with the positive cone. As theorem 11 states that the projector onto the PM subspace is the union of all the projectors to the subspaces of all the combs with the causal order the PM can have, the PM subspace is the convex hull formed by this union, and moreover this imply that the union of all the combs with different causal orders is itself a convex cone. Since in that context the normalisation conditions are the same for both combs and PM, the valid PM and Combs all lie in the hypersurface of $\frac{1}{\prod_X d_{X_I}}\mathbb{1}$.

Appendix F

Linear Algebra : Matrix and Kronecker Product Cheat Sheet

Here will be given a few properties of tensor product and matrix product that will be useful in the mathematical developpement.

The more advanced properties of linear algebra used in the thesis, if not available in the books about quantum theory, were found in [60, 105–109].

F.1 Kronecker Product properties

Let A, B, C, D be matrices:

$$\dim(A \otimes B) = \dim(A) \times \dim(A) \tag{F.1}$$

$$\text{Eigenval}(A \otimes B) = \text{Eigenval}(A) \otimes \text{Eigenval}(B) \tag{F.2}$$

where 'Eigenval' designates the set of eigenvalues.

$$A, B \geq 0 \Rightarrow (A \otimes B) \geq 0 \tag{F.3}$$

$$(A \otimes B)^T = A^T \otimes B^T \tag{F.4a}$$

$$(A \otimes B)^* = A^* \otimes B^* \tag{F.4b}$$

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger \tag{F.4c}$$

$$(A \otimes B) \otimes C = A \otimes (B \otimes C) \tag{F.5a}$$

$$(A + B) \otimes C = A \otimes C + B \otimes C \tag{F.5b}$$

$$A \otimes (B + C) = A \otimes B + A \otimes C \tag{F.5c}$$

$$(A \otimes B)(C \otimes D) = AC \otimes BD \tag{F.5d}$$

F.2 Trace and Partial Trace

Let $M^{AB} = \sum_{i,j} v_{ij}\, \sigma_i^A \sigma_j^B$; $P^{AB} = R^A \otimes T^B$; V^A, V^B, W^A, W^B

$$\mathrm{Tr}\{M\} = \mathrm{Tr}\left\{M^T\right\} \tag{F.6a}$$

$$\mathrm{Tr}_A\left[M^T\right] = \left[\mathrm{Tr}_A\left[M\right]\right]^T \tag{F.6b}$$

$$\mathrm{Tr}_B\left[M^T\right] = \left[\mathrm{Tr}_B\left[M\right]\right]^T \tag{F.6c}$$

$$\mathrm{Tr}\{P\} = \mathrm{Tr}\left\{R^A \otimes T^B\right\} = \mathrm{Tr}_A\left[\mathrm{Tr}_B\left[R^A \otimes T^B\right]\right] \mathrm{Tr}_A\left[R^A\right] \times \mathrm{Tr}_B\left[T^B\right] \tag{F.7a}$$

$$\mathrm{Tr}_A\left[P\right] = \mathrm{Tr}_A\left[R^A \otimes T^B\right] = \mathrm{Tr}_A\left[R\right] \otimes T^B \tag{F.7b}$$

$$\mathrm{Tr}_A\left[\left(\mathbb{1}^A \otimes V^B\right) \cdot P^{AB} \cdot \left(\mathbb{1}^A \otimes W^B\right)\right] = V^B \cdot \mathrm{Tr}_A\left[P^{AB}\right] \cdot W^B \tag{F.8}$$

$$\mathrm{Tr}\left(\left(A \otimes B\right)\left(C \otimes D\right)\right) = \mathrm{Tr}\left(AC\right) \times \mathrm{Tr}\left(BD\right) \tag{F.9}$$

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