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Additive uncertainty relations in quantum optics

Mémoire présenté en vue de l'obtention du diplôme
d'Ingénieur Civil physicien à finalité spécialisée

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Année académique
2019 - 2020

Werner Heisenberg and Erwin Schrödinger are traveling in a car and get pulled over by a police o cer. The o cer asks, “Do you know how fast you were going?”

Heisenberg replies “I do not, but I do know where I am !”

Full of suspicion, the o cer then asks to check their trunk. He looks inside and, horrified, asks “Do you know that there is a dead cat in your trunk ?”

To which Schrödinger replies, “Well, now I do !”

Abstract

The Heisenberg uncertainty principle is a fundamental pillar of quantum mechanics. It expresses the impossibility for the uncertainties on both the position and momentum of a particle to be vanishingly small, a feature that originates from the non-commutativity of the corresponding observables. Uncertainty relations are traditionally expressed as a non-trivial lower bound on the product of variances. Recently, however, alternative uncertainty relations have appeared in the literature which are based on the sum of variances. The interest of such additive uncertainty relations resides in the fact that the lower bound on the sum of variances remains non-trivial even when considering eigenstates of any one of the observables. We focus on the additive uncertainty relation due to Maccone and Pati, which holds for any pair of observables. We investigate the specific case where the observables are single-mode or two-mode quadratures of light, which are essential quantities in quantum optics. We find an additive uncertainty relation for the single-mode x and p quadrature operators and prove that it is saturated for all single-mode Gaussian states as well as for all displaced and squeezed single-mode Fock states, a situation which is in stark contrast with the usual Schrödinger-Robertson uncertainty relation. Interestingly, the relation is even saturated for Schrödinger cat states. For the two-mode case, we consider so-called EPR observables and show that the corresponding additive uncertainty relation is saturated for displaced two-mode squeezed vacuum states, displaced two-mode squeezed Fock states, and even for products of cat states. The fact that this additive uncertainty relation is saturated for such a large set of non-Gaussian states makes it a good candidate for building a new separability criterion that would detect a wide class of non-Gaussian entangled states, which is a notably hard problem. As a final touch, we consider the extension of additive uncertainty relations to more than two observables and establish a connection between the works of Song-Qiao, Kechrimparis-Weigert and Maccone-Pati, which in turn helps us prove a tighter additive uncertainty relation for N variables.

Résumé

Le principe d'incertitude d'Heisenberg est l'un des piliers fondamentaux de la mécanique quantique. Il exprime qu'il est impossible que les incertitudes sur la position et l'impulsion d'une particule soient simultanément arbitrairement petites, ce qui constitue une conséquence directe de la non-commutativité de ces deux observables. Traditionnellement, les relations d'incertitude sont exprimées comme une borne inférieure non-triviale sur le produit des variances. Toutefois, récemment, des relations d'incertitude alternatives ont vu le jour dans la littérature, basées, cette fois-ci, sur des sommes de variances. L'intérêt de telles relations d'incertitude additives réside dans le fait que la borne inférieure sur la somme des variances reste non-triviale, et ce, même lorsque l'état considéré est un état propre de l'un des deux observables. Nous nous concentrons sur la relation d'incertitude de Maccone et Pati, qui est valable pour n'importe quelle paire d'observables. Nous étudions le cas particulier où les observables considérés sont les quadratures de la lumière à un et deux modes, qui sont des grandeurs physiques essentielles en optique quantique. Nous trouvons une relation d'incertitude pour les opérateurs de quadrature x et p et nous prouvons qu'elle est saturée par tous les états gaussiens ainsi que pour les états de Fock à un mode comprimés et déplacés, une situation en contraste direct avec la traditionnelle relation d'incertitude de Robertson-Schrödinger. De façon intéressante, les états chats de Schödinger saturent également la relation de Maccone et Pati. Pour le cas à deux modes, nous considérons des observables de type EPR et nous montrons que la relation d'incertitude additive correspondante est saturée pour tous les états du vide à deux modes comprimés et déplacés, pour les états de Fock à deux modes comprimés et déplacés et même pour les produits d'états chat. Le fait que cette relation d'incertitude additive soit saturée par un ensemble aussi large d'états non-gaussiens fait d'elle une excellente candidate pour le développement d'un nouveau critère de séparabilité qui pourrait détecter une large gamme d'états intriqués non-gaussiens, ce qui constitue un problème particulièrement complexe. Enfin, nous considérons l'extension des relations d'incertitude au cas à plus de 2 variables et établissons un lien entre les travaux de Song-Qiao, Kechrimparis-Weigert et Maccone-Pati, ce qui nous permet de prouver une relation d'incertitude additive à N-variables plus stricte.

Remerciements

Tout d'abord, j'aimerais remercier le professeur Cerf pour m'avoir consacré son temps et pour m'avoir permis de réaliser ce mémoire. Du début jusqu'à la fin, il m'a guidé avec bienveillance, conseillé, rassuré et encouragé. Il m'a permis de me dépasser et je lui en suis très reconnaissant.

Je remercie aussi Célia, la meilleure superviseure dont un étudiant puisse rêver. Désolé de t'avoir embêtée avec mes questions et puis désolé aussi de ne pas t'avoir toujours tenu au courant pendant que je m'isolais pour travailler intensivement ! Tu m'as super bien encadré et je suis très content d'avoir pu travailler sur ce mémoire avec toi.

Je remercie mes confrères thugs : Casimir, Denis, Philippe et Timothée. Merci les gars pour votre soutien, j'ai senti vos encouragements à chaque ligne que j'écrivais ! Je remercie particulièrement Timothée qui m'a souvent relu, m'a apporté des commentaires pertinents et m'a ébloui avec ses "galaxy brain" times.

Je remercie Ardit l'intellectuel, qui m'a toujours incité à me dépasser.

Je remercie également Greg et Kevin pour leur soutien, ainsi que Kate qui a eu la patience de relire mon mémoire.

Je tiens aussi à remercier ma famille pour m'avoir soutenu et encouragé durant toutes mes années d'études, surtout durant les moments les plus difficiles. Je remercie particulièrement *gjyshi* qui m'a quotidiennement demandé si j'avançais bien dans mon mémoire. *Shpresoj se do të të pëlqej*.

Enfin, je souhaite remercier Phuong, qui a toujours été là, et qui m'a particulièrement soutenu durant la dernière ligne droite avant la remise. Du fond du coeur, merci pour tes efforts et tes encouragements.

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Introduction

When speaking about quantum mechanics, the most striking feature that comes to mind is arguably its inherent probabilistic nature. Indeed, in contrast with classical mechanics, a particle is not described by a point in phase space anymore, but instead, by a complex wave function $\psi(x)$. The probability to find the particle at a point x_0 will then be given, using Born's rule, with a probability density $|\psi(x_0)|^2$. This statistical interpretation of quantum mechanics naturally implies the notion of uncertainty, since it is impossible to know the position of the particle with certainty before actually making a measurement. This notion of uncertainty took a step further when, in 1927, Heisenberg [1] formulated the famous *Heisenberg uncertainty principle*

$$\sigma_x^2 \sigma_p^2 \geq \frac{\hbar}{4}$$

where σ_x^2 and σ_p^2 are, respectively, the variance on the position and momentum and where $\hbar$ is the reduced Planck constant. This inequality expresses that it is impossible for a system to be in a state where both the uncertainties σ_x^2 and σ_p^2 are simultaneously small. To put it in other words, although it is possible for the uncertainties on the position and momentum to be arbitrarily small, when measured individually, it is impossible for the product of these uncertainties to be small, when measured simultaneously. Later, in 1929, Robertson generalized the Heisenberg uncertainty relation and showed that for any pair of observables $\hat{A}$ and $\hat{B}$ [2] :

$$\sigma_A^2 \sigma_B^2 \geq \frac{1}{4} |\langle \psi | [\hat{A}, \hat{B}] | \psi \rangle|^2$$

where $[\hat{A}, \hat{B}]$ denotes the commutator. Let us note that in contrast with the lower bound of Heisenberg's inequality, the lower bound in the Robertson uncertainty relation depends on the state ψ, which can sometimes be problematic. Indeed, if the state ψ is an eigenstate of any of the two observables, the lower bound becomes trivial and therefore, the uncertainty relation loses interest.

To address this issue, many recent uncertainty relations based on the sum of variances have emerged in the literature [3, 4, 5, 6, 7]. These **additive** uncertainty relations present the advantage of guaranteeing the lower bound to be nontrivial whenever the state considered is an eigenstate of one of the observables. The study of such additive uncertainty relations is the main topic of this thesis.

As elegant and puzzling as they are, quantum uncertainty relations are not only studied because of their fundamental nature. Recently, the study of quantum uncertainty has been motivated by the rise of quantum information theory. More specifically, quantum key distribution is readily available in the market and its security relies on the uncertainty principle [8]. We will not go into much details here, but as a consequence of the uncertainty principle, the amount of eavesdropping noise as detected by Alice and Bob is complementary to the amount of noise impairing Eve's tapped version of the conversation. Another application of interest for the uncertainty principle is the determination of separability criteria, which allow us to distinguish between an entangled and a separable state. In a few words, the entanglement of a state can be detected with the so-called Positive Partial Transpose (PPT) criterion. This criterion states that if a state is separable, then its density matrix remains positive (and thus, physical) after a partial transposition[1]. For the case of continuous variables, the physicality of the partial transpose is precisely verified using uncertainty relations. The tighter the relation, the better the separability criterion, as it facilitates the detection of entanglement.

Initially, the uncertainty principle was expressed using variances. In 1948, however, Shannon came up with another way to quantify the uncertainty linked to the informational content of a random message, namely Shannon entropy [9]. Recently, entropic uncertainty relations have seen quite a success, as it has been shown that they directly imply the Heisenberg uncertainty principle [10] and as they can take account of non-classical correlations between the measured system and its environment [8].

This thesis is divided in two parts : in the first part, we present the theoretical background that will be of use in this work as well as the state of the art regarding uncertainty relations while in the second part, we present the results obtained in the course of this work. The first chapter covers the basic notions of quantum optics such as the mode operators, the number states, the Gaussian states and unitaries and the cat states. In Chapter 2, we give a brief review about Shannon's information theory. The third chapter combines the former two and is about uncertainty relations, whether they are variance-based or entropy-based. In the second part of this work, we thoroughly analyze an additive uncertainty relation due to Maccone and Pati [3] and investigate its application to the pair of canonically conjugate variables x and p. We apply the derived inequality to various common states in quantum optics and attempt at explicitly maximizing the lower bound of the inequality so that it becomes saturated for the set of all the states we introduced in the first part of this thesis. We also attempt to formulate an additive entropic uncertainty relation, although this route turns out to be less promising than expected. Lastly, we study a three-variable additive uncertainty relation and prove a tighter N-variables additive uncertainty relation. In the conclusion of this thesis, we discuss the obtained results and comment on the prospects.

[1] Let us note that the PPT criterion is a necessary but usually not sufficient condition for the separability of a state.

Part I

Theoretical Background

Chapter 1

Basics of quantum optics

In this chapter, we review all the basic notions in quantum mechanics that will be used through this thesis. We first start with a theoretical reminder of the density operator and the Wigner function, which are mathematical tools used to characterize quantum physical states. Then, we introduce the ladder operators and the quadratures of the electromagnetic field, as well as the Fock states. These states play an essential role in quantum optics, which is a branch of physics that uses quantum mechanics to describe light as particles, named photons, rather than considering light as a classical electromagnetic field. In the following section, we address Gaussian states and the Gaussian unitaries that are necessary to produce them. Finally, we also present the Schrödinger cat states.

Across this chapter, we will mainly refer to [11, 12] for the equations related to the density operator and the Wigner function, to [13] for the section about Fock states and to [14, 15, 16] for the definitions of Gaussian states and unitaries. Note that for simplicity, we will use the convention $\hbar = 1$ in the rest of this chapter.

1.1 Density operator

In quantum mechanics, the physical state of a system is described by a vector in a Hilbert space $\mathcal{H}$ and is noted with a ket $|\psi\rangle$. This representation is perfectly sufficient when considering *pure states*, which are states that can be completely characterized by a single ket $|\psi\rangle$. However, in practice, the system is often in a statistical mixture of pure states. For instance, we know that a photon from a natural source of light can have any polarization state, each with an equal probability [11, p. 303]. By representing a physical state with a density matrix, which can be viewed as the generalization of a state vector, it becomes possible to take account of such *mixed states*. The density matrix associated to a system is defined as :

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_i p_i \hat{\rho}_i \tag{1.1}$$

where $\hat{\rho}_i$ is the projector on the pure state $|\psi_i\rangle$. This notation highlights the fact that the density operator is a combination of pure density operators $\hat{\rho}_i$, weighted by a probability p_i. Of course, the probabilities must sum up to 1 :

$$\sum_i p_i = 1 \tag{1.2}$$

It is important to keep in mind that a statistical mixture is very different from a superposition of states. In the case of a superposition, e.g. $|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle$, the system is simultaneously in two states and it only collapses to either $|\psi_1\rangle$ or $|\psi_2\rangle$ after a measurement. This is in total contrast with a mixed state which is a statistical distribution of well defined states.

The density operator possesses three well-known properties :

1. $\hat{\rho} = \hat{\rho}^\dagger$, implying thus that the eigenvalues of the density operator are real.
2. $\hat{\rho}$ is a positive operator and its eigenvalues are therefore positive.
3. $\mathrm{Tr}(\hat{\rho}) = 1$, meaning that the sum of the eigenvalues of $\hat{\rho}$ is equal to 1. Here, $\mathrm{Tr}(\hat{A})$ designates the trace of the operator $\hat{A}$ and is defined as follows :

$$\mathrm{Tr}(\hat{A}) = \sum_i \langle i|\hat{A}|i\rangle \tag{1.3}$$

where $\{|i\rangle\}$ is an arbitrary orthonormal basis of the Hilbert space.

1.2 Wigner function

In classical mechanics, it is possible to specify, with an arbitrary precision, both the position x and the momentum p of a particle at the same time. This particle is then represented in phase space by a single point (x, p). Such a deterministic representation is impossible for a quantum particle, due to the probabilistic nature of quantum mechanics. It becomes therefore crucial to introduce a function that allows to convey information about position and momentum while also satisfying the general rules of quantum mechanics, which impose a certain limit to the precision of simultaneous measurements [12, p. 735]. This function is the *Wigner quasi-probability distribution*, proposed by Eugene Wigner in 1932 [17].

The aim of this section is to exhibit how we can associate each density operator $\hat{\rho}$ to a Wigner quasi-probability distribution. By definition :

$$W(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi\hbar)^3}\int_{-\infty}^{\infty} d^3y\; e^{-i\mathbf{p}.\mathbf{y}/\hbar}\left\langle \mathbf{x}+\frac{\mathbf{y}}{2}\right|\hat{\rho}\left|\mathbf{x}-\frac{\mathbf{y}}{2}\right\rangle \tag{1.4}$$

where $\mathbf{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ and $\mathbf{p} = (\hat{p}_1, \hat{p}_2, \hat{p}_3)$ are 3-dimensional canonically conjugate[1] variables that span the 6-dimensional phase space.

The Wigner function is a real function and, just like any other probability distribution, it is normalized to one. This mathematically translates into :

$$\int_{-\infty}^{\infty} d\mathbf{x}\, d\mathbf{p}\; W(\mathbf{x},\mathbf{p}) = 1 \tag{1.5}$$

However, because $W(\mathbf{x},\mathbf{p})$ can sometimes take negative values, it is called *quasi*-probability distribution[2]. Nevertheless, it has been shown by R.L. Hudson in [18] that $W(\mathbf{x},\mathbf{p})$ is always positive if and only if we consider Gaussian states. In the particular case where the particle is described by a pure state $|\psi\rangle$, we have :

$$W(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi\hbar)^3}\int_{-\infty}^{\infty} d^3y\; e^{-i\mathbf{p}.\mathbf{y}/\hbar}\,\psi^*\!\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi\!\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right) \tag{1.6}$$

where $\psi(\mathbf{x}) = \langle \mathbf{x}|\psi\rangle$ is the wave function of the particle in the position basis.

[1] By *canonically conjugate* variables, we mean that the variables are linked by a Fourier transform.

[2] To be more accurate, since we are dealing with continuous variables, it is a quasi-probability *density* function. From here on out, we will simply refer to it as the Wigner distribution, having in mind that it is a misuse of language.

Self-evidently, equations (1.4) and (1.6) can also be derived in the momentum representation :

$$W(x,p) = \frac{1}{(2\pi\hbar)^3}\int_{-\infty}^{\infty} d^3q \; e^{-iq.x/\hbar} \left\langle p + \frac{q}{2} \right| \hat{\rho} \left| p - \frac{q}{2} \right\rangle \tag{1.7}$$

and

$$W(x,p) = \frac{1}{(2\pi\hbar)^3}\int_{-\infty}^{\infty} d^3q \; e^{-iq.x/\hbar} \; \phi^*(p + \frac{q}{2}) \; \phi(p - \frac{q}{2}) \tag{1.8}$$

where $\phi(p) = \langle p|\psi\rangle$ is the wave function of the particle in the momentum basis.

If the Wigner function is integrated over x or p, one gets the (classical) probability distributions for p or x. These marginals of the Wigner distribution are positive and normalized to one.

$$\int_{-\infty}^{\infty} dx \; W(x,p) = W(p) \tag{1.9}$$

$$\int_{-\infty}^{\infty} dp \; W(x,p) = W(x) \tag{1.10}$$

1.3 Mode and quadrature operators

1.3.1 Fock states

A first way of describing mode operators, is by defining them as operators that act, in a particular manner, on *Fock states*. Fock states, also known as *number states*, are quantum states which are characterized by a fixed number of (quasi-)particles. They are noted $|n\rangle$ and they correspond to the eigenstates of the quantum harmonic oscillator :

$$\hat{H}|n\rangle = E_n|n\rangle = \hbar\left(n + \frac{1}{2}\right)\omega|n\rangle \tag{1.11}$$

Let us note that the Hamiltonian of the harmonic oscillator can be written in terms of the *number operator* or the *quadratures* of the electromagnetic field, that will be defined later in this section :

$$\hat{H} = \hat{N} + \frac{1}{2} = \frac{1}{2}(\hat{x}^2 + \hat{p}^2) \tag{1.12}$$

Since the eigenstates of the Hamiltonian form an orthonormal basis, we have :

$$\langle n'|n\rangle = \delta_{n'n} \tag{1.13}$$

Moreover, the Fock state basis is complete, or mathematically speaking :

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{1} \tag{1.14}$$

where $\mathbb{1}$ is the identity operator.

We now define the mode operators, namely the creation operator $\hat{a}^\dagger$ and the annihilation operator $\hat{a}$. Their action on the Fock states are given by :

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \quad \text{and} \quad \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \tag{1.15}$$

From these definitions, $\hat{a}^\dagger$ can be interpreted as the creation operator which creates one particle by raising the system from the state $|n\rangle$ to $|n+1\rangle$ while $\hat{a}$ is the annihilation operator which deletes one particle and lowers the system from the state $|n\rangle$ to $|n-1\rangle$.

More generally, we can consider an n-mode Fock state, which can be viewed as a Fock state with n spatial degrees of freedom :

$$|n_1, n_2, ..., n_i, ..., n_n\rangle \tag{1.16}$$

where n_i is the number of particles that occupy the i^{th} mode. Consequently, the mode operators become :

$$\begin{aligned} \hat{a}_i^\dagger|n_1, n_2, ..., n_i, ..., n_n\rangle &= \sqrt{n_i+1}\,|n_1, n_2, ..., n_i+1, ..., n_n\rangle \\ \hat{a}_i|n_1, n_2, ..., n_i, ..., n_n\rangle &= \sqrt{n_i}\,|n_1, n_2, ..., n_i-1, ..., n_n\rangle \end{aligned} \tag{1.17}$$

and they obey the following commutation relations :

$$\begin{aligned} [\hat{a}_i, \hat{a}_j] &= 0 \\ [\hat{a}_i^\dagger, \hat{a}_j^\dagger] &= 0 \\ [\hat{a}_i, \hat{a}_j^\dagger] &= \delta_{ij} \end{aligned} \tag{1.18}$$

These relations indicate that creating or annihilating a particle in the i^{th} then in the j^{th} mode is equivalent to creating or annihilating a particle in the j^{th} then in the i^{th}. However, creating then annihilating a particle in the same mode is not the same as annihilating then creating : the order of the operators is important.

Another way to define the Fock states is to define them as the eigenstates of the number operator, which is defined as :

$$\hat{N}_i = \hat{a}_i^\dagger\hat{a}_i \tag{1.19}$$

Using equations (1.17) and (1.19), it is easy to see that the number operator provides the number of particles in the i^{th} mode :

$$\begin{aligned} \hat{N}_i|n_1, n_2, ..., n_i, ..., n_n\rangle &= \hat{a}_i^\dagger\hat{a}_i|n_1, n_2, ..., n_i, ..., n_n\rangle \\ &= \hat{a}_i^\dagger\sqrt{n_i}\,|n_1, n_2, ..., n_i-1, ..., n_n\rangle \\ &= \sqrt{n_i-1+1}\sqrt{n_i}\,|n_1, n_2, ..., n_i, ..., n_n\rangle \\ &= n_i|n_1, n_2, ..., n_i, ..., n_n\rangle \end{aligned} \tag{1.20}$$

1.3.2 Quantization of the electromagnetic field

The mode operators we defined in the above subsection play an important role in the quantization of the electromagnetic field. To understand why, we use Maxwell equations to first classically describe the electromagnetic field, and we will then make the transition from classical to quantum optics. Without going too much into details, we define the potential vector $A(r,t)$, from which we can determine the magnetic field $B(r,t)$ and the electric field $E(r,t)$ [19] :

$$B(r,t) = \nabla \times A(r,t) \quad \text{and} \quad E(r,t) = -\frac{\partial A(r,t)}{\partial t} \tag{1.21}$$

with the Coulomb gauge condition :

$$\nabla \cdot \mathbf{A}(\mathbf{r},t) = 0 \tag{1.22}$$

From the Maxwell equations, we can derive :

$$\nabla^2 \mathbf{A}(\mathbf{r},t) = \frac{1}{c^2}\frac{\partial^2 \mathbf{A}(\mathbf{r},t)}{\partial t^2} \tag{1.23}$$

By solving this equation, and using (1.21), we can get the expression of the electric field :

$$\mathbf{E}(\mathbf{r},t) = \sum_{k}\sum_{\lambda=1,2} \frac{1}{\sqrt{2}}\mathcal{E}_k \mathbf{e}_{k\lambda}\left\{A_{k\lambda}\, e^{-i(\omega_k t+\mathbf{k}.\mathbf{r})} + A^*_{k\lambda}\, e^{i(\omega_k t-\mathbf{k}.\mathbf{r})}\right\} \tag{1.24}$$

where k is the index of the mode, λ the polarization, $\mathbf{e}_{k\lambda}$ the polarization vector, ω_k is the angular frequency of the k-mode, $\mathcal{E}_k$ is a constant containing all the dimensional components and $A_{k\lambda}$ and $A^*_{k\lambda}$ are the complex field amplitudes. Roughly speaking, the transition from the classical electromagnetic field to quantum mechanics is accomplished by replacing the complex amplitudes by the mode operators.

$$\begin{aligned} A_{k\lambda} &\rightarrow \hat{a}_{k\lambda} \\ A^*_{k\lambda} &\rightarrow \hat{a}^\dagger_{k\lambda} \end{aligned} \tag{1.25}$$

For simplicity, we assume that the light is monochromatic, that it is polarized along the x-axis and that the wave's direction of propagation is along the z-axis. This allows us to drop the indices k and λ. The electric field then becomes :

$$E_x(z,t) = \frac{1}{\sqrt{2}}\mathcal{E}\,\mathbf{1}_x\left\{\hat{a}\, e^{-i\omega t+kz} + \hat{a}^\dagger e^{i\omega t-kz}\right\} \tag{1.26}$$

We now define the dimensionless position $\hat{x}$ and momentum $\hat{p}$ operators :

$$\hat{x} = \frac{1}{\sqrt{2}}(\hat{a}+\hat{a}^\dagger) \tag{1.27}$$

$$\hat{p} = \frac{i}{\sqrt{2}}(\hat{a}-\hat{a}^\dagger) \tag{1.28}$$

which correspond to the position and momentum of a harmonic oscillator of mass m and angular frequency ω. By expressing the mode operators in terms of the position and momentum operator, eq. (1.26) becomes :

$$E_x(z,t) = \mathcal{E}\,\mathbf{1}_x\left\{\hat{x}\cos(-i\omega t+kz) + \hat{p}\sin(i\omega t-kz)\right\} \tag{1.29}$$

Therefore, we see that $\hat{x}$ and $\hat{p}$ are (respectively) the amplitudes of the co-sinusoidal and sinusoidal components of the electric field. Because there is a phase shift of $\pi/2$ between these two components, the operators $\hat{x}$ and $\hat{p}$ are called the *quadratures* of the electromagnetic field. From (1.18) we can see that they satisfy the canonical commutation relation :

$$[\hat{x},\hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i \tag{1.30}$$

Unlike the ladder and the number operators, these quadratures have continuous spectra. Their eigenvectors are defined as :

$$\hat{x}|x\rangle = x|x\rangle \quad \text{and} \quad \hat{p}|p\rangle = p|p\rangle \tag{1.31}$$

where x, p are real numbers and $\{|x\rangle\}, \{|p\rangle\}$ are two bases that are linked by a Fourier transform :

$$|x\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixp} |p\rangle \, dp \quad \text{and} \quad |p\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixp} |x\rangle \, dx \tag{1.32}$$

Just like Fock states, $\{|x\rangle\}, \{|p\rangle\}$ form complete and orthogonal bases :

$$\int_{-\infty}^{\infty} |x\rangle\langle x| \, dx = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} |p\rangle\langle p| \, dp = 1 \tag{1.33}$$

$$\langle x'|x\rangle = \delta_{x'x} \quad \text{and} \quad \langle p'|p\rangle = \delta_{p'p} \tag{1.34}$$

Let us end this section by giving the Wigner function $W_n(x,p)$ of Fock states $|n\rangle$, which depends on the Laguerre polynomials $L_n(x)$ [11] :

$$W_n(x,p) = \frac{(-1)^n}{\pi} e^{-(x^2+p^2)} L_n(2(x^2+p^2)) \tag{1.35}$$

As illustrated in Fig. 1.1, the Wigner function can take negative values, which was already mentioned in section 1.2. The wave function $\psi_n(x)$ of Fock states is given by [20]:

$$\psi_n(x) = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} H_n(x) e^{-x^2/2} \tag{1.36}$$

where $H_n(x)$ are the Hermite polynomials.

(a) Vacuum state $|0\rangle$ (b) State $|1\rangle$ (c) State $|3\rangle$

Figure 1.1: Wigner functions of different Fock states. It should be noted that these states are centered on the origin.

1.4 One-mode Gaussian states and unitaries

Gaussian states are continuous-variable states that are described with a Gaussian Wigner function. The most relevant quantities that characterize these states are their statistical moments [14], and more specifically, the first and second moments. The first moment is called the *displacement vector* or more simply, the *mean value*:

$$\bar{\mathbf{r}} \equiv \langle \hat{\mathbf{r}} \rangle \equiv \mathrm{Tr}(\hat{\mathbf{r}}\hat{\rho}) \tag{1.37}$$

where $\hat{\mathbf{r}} = (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, \ldots, \hat{x}_n, \hat{p}_n)^T$ is the quadratures vector, while the second moment is called the *covariance matrix*, whose elements are defined by :

$$\gamma_{ij} \equiv \frac{1}{2}\langle \{\Delta\hat{r}_i, \Delta\hat{r}_j\} \rangle \tag{1.38}$$

where $\Delta\hat{r}_i = \hat{r}_i - \langle\hat{r}_i\rangle$ and $\{\hat{A},\hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ stands for the anti-commutator. For a single-mode state, the covariance matrix is a 2×2 real and symmetric matrix that can be written as :

$$\gamma = \begin{pmatrix} \sigma_x^2 & \sigma_{xp} \\ \sigma_{px} & \sigma_p^2 \end{pmatrix} \tag{1.39}$$

where σ_x^2 and σ_p^2 are the position and momentum variances and $\sigma_{xp} = \sigma_{px}$ is the covariance. Note that depending on the reference considered, the notation of the variance of the random variable x (for instance) is written as σ_x^2 or Δx^2.

As stated previously, the Wigner function of an n-mode Gaussian state is a Gaussian distribution and is given by :

$$W_G(r) = \frac{1}{(2\pi)^n\sqrt{\det\gamma}}\, e^{-\frac{1}{2}(r-\langle r\rangle)^T\gamma^{-1}(r-\langle r\rangle)} \tag{1.40}$$

This expression clearly shows that the displacement vector and the covariance matrix are indeed the only information needed in order to completely describe a Gaussian state.

The most basic example of a Gaussian state is the one with 0 photon, i.e. *the vacuum state*, whose Wigner function is plotted in Fig. 1.1a. Its mean value and covariance are equal to 0 while the position and momentum variances are both equal to 1/2. The uncertainties on the x and p quadratures can be seen in Fig. 1.2.

Figure 1.2: Projection of the vacuum state Wigner function in phase space.

The thermal state is very similar to the vacuum state, in the sense that both quadratures have the same uncertainties, only this time, the variances are greater than 1/2 (we also have, just like for the vacuum state, $\langle\hat{r}\rangle = \sigma_{xp} = \sigma_{px} = 0$).

In the following subsections, we will study Gaussian unitaries that will allow us to create other types of Gaussian states, starting from the vacuum state. Gaussian unitaries are, in particular, unitary transformations $U^{-1} = U^{\dagger}$ that map a Gaussian state onto another Gaussian state :

Figure 1.3: The coherent state $|\alpha\rangle$ (blue) is created by applying the displacement operator on the vacuum state $|0\rangle$ (red-dashed).

$$\hat{\rho} \rightarrow \hat{U}\hat{\rho}\hat{U}^{\dagger} \tag{1.41}$$

In terms of the quadrature operators, a Gaussian unitary is more simply described by an affine map [14]:

$$\hat{\mathbf{r}} \rightarrow S\hat{\mathbf{r}} + \mathbf{d} \tag{1.42}$$

where $\mathbf{d}$ is a real vector of dimension $2n$ and S a $2n \times 2n$ symplectic matrix. By definition, a symplectic matrix is a real matrix such as :

$$S\,\Omega\,S^{T} = \Omega \tag{1.43}$$

with Ω known as the *symplectic form* :

$$\Omega = \bigoplus_{k=1}^{n} \omega \; ; \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.44}$$

where $\bigoplus$ denotes the matrix direct sum.

Let us mention that every symplectic matrix has a determinant equal to 1.

1.4.1 Coherent states and displacement operator

Let us start by introducing the displacement operator:

$$\hat{D}(\alpha) \equiv e^{\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}} \tag{1.45}$$

where $\alpha = (x + ip)/\sqrt{2}$ is the complex amplitude. A coherent state $|\alpha\rangle$ can simply be seen as a displaced vacuum state, such as shown in Fig. 1.3 :

$$|\alpha\rangle \equiv \hat{D}(\alpha)|0\rangle \tag{1.46}$$

Under the action of $\hat{D}(\alpha)$, the creation and annihilation operators become :

$$\begin{aligned}\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) &= \hat{a} + \alpha\\ \hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha) &= \hat{a}^\dagger + \alpha^*\end{aligned} \tag{1.47}$$

while the quadratures transform as :

$$\begin{aligned}\hat{D}^\dagger(\alpha)\hat{x}\hat{D}(\alpha) &= \hat{x} + \mathrm{Re}(\alpha)\\ \hat{D}^\dagger(\alpha)\hat{p}\hat{D}(\alpha) &= \hat{p} + \mathrm{Im}(\alpha)\end{aligned} \tag{1.48}$$

where $\mathrm{Re}(\alpha)$ and $\mathrm{Im}(\alpha)$ respectively stand for the real and imaginary part of α. It is clear from these equations that the displacement operator translates the vacuum state in phase space. Or in other words, it modifies the mean value vector while leaving the covariance matrix intact :

$$\langle\hat{r}\rangle = \sqrt{2}\begin{pmatrix}\mathrm{Re}(\alpha)\\ \mathrm{Im}(\alpha)\end{pmatrix} \quad \text{and} \quad \gamma = \gamma_{vac} = \frac{1}{2}\mathbb{1} \tag{1.49}$$

Coherent states can also be defined as a linear superposition of Fock states [21, p.191] :

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{1/2}}|n\rangle \tag{1.50}$$

These states are normalized ($\langle\alpha|\alpha\rangle = 1$) but are never orthogonal as, for two different complex number α and β, we have :

$$\langle\beta|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2+\beta^*\alpha} \tag{1.51}$$

Let us point out that the lack of orthogonality between coherent states is a result from the fact that they form an overcomplete set of states [21, p. 191].The coherent states can also be viewed as eigenstates of the annihilation operator :

$$\begin{aligned}\hat{a}|\alpha\rangle &= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{1/2}}\sqrt{n}|n-1\rangle\\ &= \alpha e^{-|\alpha|^2/2}\sum_{n=1}^{\infty}\frac{\alpha^{n-1}}{(n-1)!^{1/2}}|n-1\rangle\\ &= \alpha e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{1/2}}\sqrt{n}|n-1\rangle\\ &= \alpha|\alpha\rangle\end{aligned} \tag{1.52}$$

Of course, the following conjugate relation holds :

$$\begin{aligned}(\hat{a}|\alpha\rangle)^\dagger &= (\alpha|\alpha\rangle)^\dagger\\ \Leftrightarrow \langle\alpha|\hat{a}^\dagger &= \langle\alpha|\alpha^*\end{aligned} \tag{1.53}$$

Figure 1.4: The vacuum circle (red-dashed) has been squeezed along the x-axis into an ellipse (blue).

1.4.2 Squeezed states and squeezing operator

The vacuum and coherent states mentioned previously are both examples of states that saturate the Heisenberg uncertainty relation[3] :

$$\Delta_x^2 \Delta_p^2 \geq \frac{1}{4} \tag{1.54}$$

Moreover, the uncertainties of their quadratures are both equal to 1/2. However, the uncertainty relation can also be saturated for states with different uncertainties on their quadratures. This can be achieved using the *squeezing operator*:

$$\hat{S}(z) \equiv e^{\frac{1}{2}(z^* \hat{a}^2 - z \hat{a}^{\dagger 2})} \tag{1.55}$$

where $z = re^{i\theta}$ and $r \in \mathbb{R}^+$ is the squeezing parameter, while θ is the squeezing angle. In all generality, a *squeezed state* is written :

$$|\alpha; z\rangle \equiv \hat{D}(\alpha)\hat{S}(z)|0\rangle \tag{1.56}$$

Note that the $\hat{D}(\alpha)$ and $\hat{S}(z)$ do not commute, meaning that $\hat{D}(\alpha)\hat{S}(z) \neq \hat{S}(z)\hat{D}(\alpha)$. However, as calculated in [22], we have :

$$\hat{D}(\alpha)\hat{S}(z) = \hat{S}(z)\hat{D}(\beta) \quad ; \quad \beta = \alpha\cosh r - \alpha^* e^{i\theta}\sinh r \tag{1.57}$$

which implies that the order of these operators in the definition of squeezed states is only a convention, as it is possible to generate all the possible squeezed states, regardless

[3] More details in Chapter 3.

of whether we first apply the displacement operator and then the squeezing operator or inversely. When there is no displacement, we can write the squeezed vacuum state as :

$$|0;z\rangle = \hat{S}(z)\,|0\rangle = \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} e^{in\phi} (-\tanh r)^n\, |2n\rangle \tag{1.58}$$

As shown in Fig. 1.4, the *squeezed state* is an ellipse which has the same area as the uncertainty circle of the vacuum in phase space. This is because symplectic transformations conserve the area of a state in phase space.

Under the action of $\hat{S}(z)$, the ladder operator transform as :

$$\begin{aligned} \hat{S}^\dagger(z)\hat{a}\hat{S}(z) &= \hat{a}\cosh r - \hat{a}^\dagger e^{i\phi}\sinh r \\ \hat{S}^\dagger(z)\hat{a}^\dagger\hat{S}(z) &= \hat{a}^\dagger\cosh r - \hat{a}e^{-i\phi}\sinh r \end{aligned} \tag{1.59}$$

while the $\hat{x}$ and $\hat{p}$ quadratures transform as :

$$\begin{aligned} \hat{S}^\dagger(z)\hat{x}\hat{S}(z) &= (\cosh r - \cos\phi\,\sinh r)\,\hat{x} - (\sin\phi\,\sinh r)\,\hat{p} \\ \hat{S}^\dagger(z)\hat{p}\hat{S}(z) &= -(\sin\phi\,\sinh r)\,\hat{x} + (\cosh r + \cos\phi\,\sinh r)\,\hat{p} \end{aligned} \tag{1.60}$$

Fixing $\phi = 0$ and therefore $z = r$, the quadratures become :

$$\begin{aligned} \hat{S}^\dagger(z)\hat{x}\hat{S}(z) &= e^{-r}\,\hat{x} \\ \hat{S}^\dagger(z)\hat{p}\hat{S}(z) &= e^{r}\,\hat{p} \end{aligned} \tag{1.61}$$

which clearly highlights that the squeezing operator squeezes one quadrature and dilates the other one in phase space. Consequently, the symplectic transformation associated to this operator is :

$$S_{squeeze} = \begin{pmatrix} e^{-r} & 0 \\ 0 & e^{r} \end{pmatrix} \tag{1.62}$$

and the covariance matrix transforms into :

$$\gamma_{squeeze} = S\,\gamma_{vac}S^T = \frac{1}{2}\begin{pmatrix} e^{-2r} & 0 \\ 0 & e^{2r} \end{pmatrix} \tag{1.63}$$

1.4.3 Phase rotation

The last single-mode Gaussian unitary we introduce is called the *phase-shift operator*, or the rotation operator :

$$\hat{R}(\theta) \equiv e^{-i\theta\hat{a}^\dagger\hat{a}} \tag{1.64}$$

This operator simply adds a phase to the mode operators :

$$\begin{aligned} \hat{R}^\dagger(\theta)\hat{a}\hat{R}(\theta) &= e^{-i\theta}\,\hat{a} \\ \hat{R}^\dagger(\theta)\hat{a}^\dagger\hat{R}(\theta) &= e^{i\theta}\,\hat{a}^\dagger \end{aligned} \tag{1.65}$$

or, in phase space, it rotates the quadratures by an angle θ:

$$\begin{aligned} \hat{R}^\dagger(\theta)\hat{x}\hat{R}(\theta) &= \hat{x}\cos\theta + \hat{p}\sin\theta \\ \hat{R}^\dagger(\theta)\hat{p}\hat{R}(\theta) &= -\hat{x}\sin\theta + \hat{p}\cos\theta \end{aligned} \tag{1.66}$$

Consequently, the associated symplectic matrix is :

$$S_{rot} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{1.67}$$

Now that this last unitary has been reviewed, we can define the notation :

$$|\alpha, \theta, z\rangle \equiv \hat{D}(\alpha)\hat{R}(\theta)\hat{S}(z)\,|0\rangle \tag{1.68}$$

which is the most general one-mode pure Gaussian state : a rotated and displaced squeezed state. Note that $\hat{R}(\theta)$ does not commute with $\hat{D}(\alpha)$ or $\hat{S}(z)$, but as already pointed out in the previous subsection, the order of the operators does not matter in order to generate any arbitrary pure Gaussian state.

1.5 Two-mode Gaussian states and unitaries

We now review the two-mode Gaussian states which are produced from two harmonic oscillators. Each oscillator can be described by its own set of ladder operators ($\hat{a}_1; \hat{a}_1^\dagger$ and $\hat{a}_2; \hat{a}_2^\dagger$), or equivalently, by its own dimensionless quadratures ($\hat{x}_1; \hat{p}_1$ and $\hat{x}_2; \hat{p}_2$). We first examine the two-mode vacuum state, which is denoted by the tensor product of two single-mode vacuum states $|0\rangle_1 \otimes |0\rangle_2$, or, more concisely, $|0; 0\rangle$. In the position basis, its wave function [23]

$$\psi_{00}(x_1; x_2) = \frac{1}{\sqrt{\pi}} e^{-x_1^2/2} e^{-x_2^2/2} \tag{1.69}$$

can be rewritten as :

$$\psi_{00}(x_1; x_2) = \frac{1}{\sqrt{\pi}} e^{-(x_1 - x_2)^2/4} e^{-(x_1 + x_2)^2/4} \tag{1.70}$$

Following this equation, we can define the observables $\hat{x}_\pm = (\hat{x}_1 \pm \hat{x}_2)/\sqrt{2}$, which have Gaussian distributions and which variances can be computed, by definition, as :

$$\Delta x_\pm^2 = \langle 0; 0|\hat{x}_\pm^2|0; 0\rangle - \langle 0; 0|\hat{x}_\pm|0; 0\rangle^2 = \frac{1}{2} \tag{1.71}$$

This result was to be expected because the double-vacuum state corresponds to a physical state where there is no photon in either mode, meaning that there cannot be any interaction (or correlation) between the two modes. Therefore, we could have also calculated the variances of $\hat{x}_\pm$ by knowing that $\Delta x_1^2 = \Delta x_2^2 = 1/2$ for a single mode vacuum state and by remembering that the variance of a sum of two *uncorrelated* variables A and B is given by $\mathrm{Var}(A + B) = \mathrm{Var}(A) + \mathrm{Var}(B)$, and that the variance of a variable A multiplied by a scalar a is given by $\mathrm{Var}(aA) = a^2\mathrm{Var}(A)$:

$$(\Delta x_\pm)^2 = \Delta\left(\frac{(x_1 \pm x_2)}{\sqrt{2}}\right)^2 = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2} \tag{1.72}$$

The same reasoning can be followed for the momentum quadratures $\hat{p}_\pm = (\hat{p}_1 \pm \hat{p}_2)/\sqrt{2}$

As we will see, things get a little bit trickier when we consider a two-mode squeezed vacuum state, but let us first define the two-mode displacement operator, which does not cause any correlation between the two modes.

1.5.1 Two-mode displaced states

The two-mode displacement operator is simply the tensor product of two single-mode displacement operators, each acting on its own corresponding mode [15, p. 54]. The two-mode displaced states are thus written :

$$\begin{aligned}\hat{D}_{TM}(\alpha_1,\alpha_2)|0\rangle_1|0\rangle_2 &\equiv (\hat{D}(\alpha_1)\otimes\hat{D}(\alpha_2))|0\rangle_1|0\rangle_2\\ &= \hat{D}(\alpha_1)|0\rangle_1\hat{D}(\alpha_2)|0\rangle_2\\ &= |\alpha_1\rangle|\alpha_2\rangle\end{aligned} \tag{1.73}$$

1.5.2 Two-mode squeezing operator

It is important to distinguish between the product of two single-mode squeezing operators, which physically represents a situation where two harmonic oscillators are separately squeezed, and a two-mode squeezing operator, where the two harmonic oscillators become correlated [16]. We introduce the two-mode squeezing operator :

$$\hat{S}_{TM} \equiv e^{\frac{1}{2}(z^*\hat{a}_1\hat{a}_2 - z\hat{a}_1^\dagger\hat{a}_2^\dagger)} \tag{1.74}$$

The action of $\hat{S}_{TM}$ on the ladder operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ is given by [16] :

$$\begin{aligned}\hat{S}^\dagger_{TM}(z)\hat{a}_1\hat{S}_{TM}(z) &= \hat{a}_1\cosh r + \hat{a}_2^\dagger e^{i\theta}\sinh r\\ \hat{S}^\dagger_{TM}(z)\hat{a}_1^\dagger\hat{S}_{TM}(z) &= \hat{a}_1^\dagger\cosh r + \hat{a}_2 e^{-i\theta}\sinh r\\ \hat{S}^\dagger_{TM}(z)\hat{a}_2\hat{S}_{TM}(z) &= \hat{a}_2\cosh r + \hat{a}_1^\dagger e^{i\theta}\sinh r\\ \hat{S}^\dagger_{TM}(z)\hat{a}_2^\dagger\hat{S}_{TM}(z) &= \hat{a}_2^\dagger\cosh r + \hat{a}_1 e^{-i\theta}\sinh r\end{aligned} \tag{1.75}$$

When applying the two-mode squeezing operator on the double-vacuum state, we obtain the two-mode squeezed vacuum state, also called the EPR state (for Einstein-Podolski-Rosen) [24]:

$$|EPR\rangle \equiv \hat{S}_{TM}(r)|0,0\rangle = \frac{1}{\cosh r}\sum_{n=0}^{\infty}(\tanh r)^n|n,n\rangle \tag{1.76}$$

The particularity of this EPR state is that it exhibits a *non-classical* correlation between the two modes. To understand what we mean by *non-classical*, we consider the extreme case of an infinite squeezing $r\to\infty$, and a squeezing angle $\theta=\pi$. Therefore, the wave functions of the two-mode squeezed state takes the form of :

$$\begin{aligned}\psi(x_1,x_2) &\propto \delta(x_1 - x_2)\\ \psi(p_1,p_2) &\propto \delta(p_1 + p_2)\end{aligned} \tag{1.77}$$

Let us assume that Alice and Bob, two fictional observers, are sharing the EPR state : the quadratures of the first mode are associated to Alice, while the quadratures of the second mode are associated to Bob. Suppose now that Alice and Bob are located far away from each other so that any form of communication is impossible (in the short time during which Alice and Bob operate). Without measurement, the position and momentum of Alice's half of the state are completely uncertain (the same goes for Bob's position and momentum). However, if Alice measures the position of her part of the state and gets x_1, then, according to eq. (1.77), Bob's position becomes precisely equal to that of Alice and he should thus also measure x_1. If Alice measures the momentum and gets p_1, then again, according to (1.77), Bob's momentum becomes precisely opposite to that of Alice and he obtains $-p_1$.

Figure 1.5: Action of a beam splitter on two different input beams.

The problem in this thought experiment resides in the fact that when Alice makes a measurement, Bob's half of the state *instantaneously* collapses to a state with a well defined position or momentum, even though Alice and Bob are far away from each other. This would imply that the information of the collapse of Alice's state had to travel faster than the speed of light from Alice to Bob, which is impossible according to the special theory of relativity. EPR proposed instead that the states of Alice and Bob had a well-defined position and momentum from the very beginning, or in other words, that the information was locally hidden in the two states so that when they were moved apart, no communication had to take place. This is known as the *local hidden-variable theory*.

If Alice and Bob's states had a predetermined position and momentum from the start, then the Heisenberg uncertainty relation, which states that the position and momentum cannot simultaneously have a precise value, would be violated. With this argument, EPR challenged the completeness of quantum mechanics.

In 1964, J.S. Bell proved with his inequalities that the local hidden-variable theory was incompatible with the statistical predictions of quantum mechanics [25], exposing thus that quantum states can exhibit correlations that cannot be reproduced or explained by classical physics. *Quantum entanglement* is an example of such correlation. The EPR state we considered previously is an *entangled state*, i.e. a state such that when the wave function of one part of the system collapses, then the state of the other part of the system is determined by the measurement on the first one.

In practice, the EPR state can be achieved by using a beam splitter, which is an optical device that splits an incoming beam into two parts : one that is reflected and another that is transmitted. Its action on two beams is schematized in Fig. 1.5 . The *beam splitter* operator is defined by :

$$B(\theta) = e^{\theta(\hat{a}_1 \hat{a}_2^{\dagger} - \hat{a}_1^{\dagger} \hat{a}_2)} \tag{1.78}$$

where θ determines the transmissivity of the beam splitter $\tau = \cos^2(\theta) \in [0;1]$. The beam splitter is said to be balanced (or 50:50) if $\tau = 1/2$.

An ideal beam splitter transforms the ladder operators as :

$$\begin{pmatrix} \hat{a}_1' \\ \hat{a}_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} \tag{1.79}$$

As shown in [26], the two-mode squeezed vacuum state produced by the operator in eq. (1.74) is equivalent to the two-mode state emerging from a 50:50 beam splitter with a single-mode vacuum state squeezed along x and a single-mode vacuum state squeezed along p at the input.

1.5.3 Two-mode rotation operation

The beam splitter is an example of a two-mode rotation operation. Another example is simply the tensor product of two single-mode rotation operators, each acting on its corresponding mode. The two-mode rotated states are thus written :

$$\begin{aligned} \hat{R}_{TM}(\theta_1;\theta_2)|\psi\rangle_1|\phi\rangle_2 &\equiv (\hat{R}(\theta_1)\otimes\hat{R}(\theta_2))|\psi\rangle_1|\phi\rangle_2 \\ &= \hat{R}(\theta_1)|\psi\rangle_1\hat{R}(\theta_2)|\phi\rangle_2 \end{aligned} \tag{1.80}$$

1.6 Cat states

We conclude this chapter by briefly defining quantum (Schrödinger) *cat states*. These states are a superposition of coherent states, taking the form of :

$$|cat\rangle_\pm \equiv \frac{1}{\sqrt{N_\pm}}(|\alpha\rangle \pm |-\alpha\rangle) \quad \text{with} \quad N_\pm = 2(1 \pm e^{-2|\alpha|^2}) \tag{1.81}$$

where $N_\pm$ is the normalization factor, $|cat\rangle_+$ is called an *even cat state* and $|cat\rangle_-$ is called an *odd cat state*. Their names come from the fact that, when expressed in terms of a sum of Fock states (see eq. (1.50)), they only contain (respectively) even or odd number states. As seen in Fig. 1.6, $|\alpha\rangle$ and $|-\alpha\rangle$ are diametrically opposed in phase space.

It can be proven that even and odd cat states are orthogonal :

$$\begin{aligned} {}_-\langle cat|cat\rangle_+ &= \frac{1}{N}(\langle\alpha| - \langle-\alpha|)(|\alpha\rangle + |-\alpha\rangle) \\ &= \frac{1}{N}(\langle\alpha|\alpha\rangle + \langle\alpha|-\alpha\rangle - \langle-\alpha|\alpha\rangle - \langle-\alpha|-\alpha\rangle) \\ &= \frac{1}{N}(1 + e^{-2|\alpha|^2} - e^{-2|\alpha|^2} - 1) = 0 \end{aligned} \tag{1.82}$$

where we used eq. (1.51). The cat states can be generalized to a superposition of N coherent states of the same amplitude $|\alpha|$ but evenly-distributed phases $2\pi n/N$ $(n = 0, 1, \ldots N-1)$. They can be represented by [27] :

$$|cat_N(\alpha)\rangle \equiv \frac{1}{\sqrt{M_N(\alpha)}}\sum_{n=0}^{N-1}|\alpha e^{i2\pi\frac{n}{N}}\rangle \tag{1.83}$$

where $M_N(\alpha)$ is the normalization constant. Such states are also refered to as *multi-headed cat states*.

Note that the constituent states of a Schrödinger cat state (SCS) can become macroscopically distinguishable in the limit of a large amplitude, and the SCS may become an important tool to study a lot of fundamental issues, e.g. the decoherence of macroscopic superposition states [27].

Figure 1.6: Even and odd cat states projected onto phase space.

1.7 Conclusion

In this chapter we have reviewed the basic notions of quantum mechanics that will be used in this thesis. We first started with a reminder about the density operator, which is a generalization of the state vector representation, followed by a definition of the Wigner function. Then, we introduced the number state formalism that allows us to characterize states by the number of particles (or photons) that they contain. The ladder operator and the quadratures of the electromagnetic field were also defined. Next, we studied different Gaussian states which were all generated by applying some unitary transformation to the single-mode or two-mode vacuum state. Finally, the cat states have been briefly defined as the superposition of two diametrically opposed coherent states.

Chapter 2

Basics of information theory

In 1948, Claude E. Shannon published an article called "The Mathematical Theory of Communication" [9] which has become the pillar of information theory. One key quantity of the whole theory is called the *Shannon entropy*. Interestingly, as we will see later, this classical physical quantity will also be useful to measure the uncertainty of a quantum random variable. Although we are mainly interested in continuous variables, we will first review the Shannon entropy of discrete variables, since information theory was first developed for discrete variables and then extended to continuous variables.

Information theory is such a wide field of study that the focus of this chapter will be on the definitions that are useful in the scope of this thesis.

2.1 Shannon entropy

Let us consider a discrete random variable X which can take the values $x \in \mathcal{X}$ with a probability $p(x)$. The Shannon entropy of this variable is given by :

$$H(X) \equiv H(p) = -\sum_{x \in \mathcal{X}} p(x)\log(p(x)) \qquad (2.1)$$

As stated by Shannon in his paper [9], the choice of a logarithmic base corresponds to the choice of a unit for measuring information. Here, the logarithm is in base 2, meaning that the entropy is expressed in binary digits, or *bits*. The entropy $H(X)$ can be interpreted as the number of bits needed, on average, to describe the outcome of a random variable. The most common example often given to illustrate this interpretation is the coin flipping. If we toss a coin, there are two possible outcomes, heads or tails, each with a probability of 1/2. The entropy of the distribution is then :

$$H(Coin\ flip) = -\sum_{x} p(x)\log(p(x)) = -\frac{1}{2}\log(\frac{1}{2}) - \frac{1}{2}\log(\frac{1}{2}) = 1 \qquad (2.2)$$

This tells us that to describe this variable, we only need one bit. For instance, we can associate the value 0 to the outcome "heads" and the value 1 to the outcome "tails".

To complete this definition, we must add the convention :

$$0\log(0) \equiv 0 \qquad (2.3)$$

Indeed, adding a term with a zero probability shouldn't change the entropy. This is justified when we look at the limit of the contribution of a vanishing probability :

Figure 2.1: The Shannon discrete entropy is always positive, since $0 \le p(x) \le 1$.

$$\lim_{x \to 0} x \log(x) = 0 \tag{2.4}$$

An important property of the Shannon entropy is that it always is positive, as shown in Fig 2.1. This property is however not verified anymore for continuous variables, as we will see in the next section.

Another interesting property of the Shannon entropy is that it is a concave function and that it respects thus the Jensen's inequality, given by [28]:

$$H\left(\sum_{k=1}^{n} \lambda_k p_k\right) \ge \sum_{k=1}^{n} \lambda_k H(p_k) \tag{2.5}$$

where $\lambda_k \in [0;1]$ and $\sum_{k=1}^{n} \lambda_n = 1$.

2.2 Shannon differential entropy

When examining the entropy of continuous distributions, we speak of differential entropy, which is very similar to the discrete case. It is defined as :

$$h(X) \equiv h(p) = -\int_{-\infty}^{\infty} dx\, p(x) \log(p(x)) \tag{2.6}$$

In the case of a probability distribution $p(x_1; x_2; \dots; x_n)$ of n continuous variables, we define the joint differential entropy of the vector $X = (X_1; \dots; X_n)$, which is written as :

$$h(X) = -\int_{-\infty}^{\infty} dx_1 \dots dx_n \; p(x_1; \dots; x_n) \log(x_1; \dots; x_n) \tag{2.7}$$

Although eq. (2.6) looks like we took eq. (2.1) and replaced the sum by an integral, the reality is not that simple. To be more precise, the relation between the differential entropy $h(X)$ and the Shannon entropy $H(X)$ is given by :

$$\lim_{\Delta \to 0} H(X^{\Delta}) + \log \Delta = h(X) \tag{2.8}$$

where X is the quantized entropy of X (more details in [29]). In other words, the differential entropy is the continuous version of the Shannon entropy, up to a constant. The difference between the discrete case and the continuous case is, in short, that the Shannon entropy measures the *absolute* randomness of a variable, whereas the differential entropy measures the *relative* randomness to an assumed standard, namely the coordinate system over which we perform the integration [9]. Due to this difference, the differential entropy can sometimes be negative. Let us, for example, take a random variable uniformly distributed on an interval from 0 to a, so that its density is 1/a from 0 to a and 0 elsewhere. Its differential entropy is then :

$$h(X) = \int_0^a \frac{1}{a}\log(\frac{1}{a})dx = \log a \tag{2.9}$$

which is negative for $a < 1$.

2.2.1 Entropy of Gaussian distributions

As detailed in Chapter 1, Gaussian states are states that are characterized by Gaussian functions. It feels only natural to now compute the differential entropy of Gaussian distributions, as they exhibit an interesting property regarding differential entropies.

Let us assume a Gaussian random variable :

$$X \quad p(x) = \frac{1}{\sqrt{2 \quad ^2}} e^{\frac{x^2}{2 \quad ^2}} \tag{2.10}$$

Then, calculating the differential entropy in nats, i.e. using the natural logarithm instead of the logarithm to the base 2, we obtain :

$$\begin{aligned} h(X) &= \int^{1} p(x)\ln(p(x))\,dx \\ &= \int_{1}^{1} p(x) \quad \frac{x^2}{2 \quad ^2} \quad \ln \sqrt{2 \quad ^2} \\ &= \frac{Var(p(x))}{2 \quad ^2} + \frac{1}{2}\ln 2 \quad ^2 \\ &= \frac{1}{2} + \frac{1}{2}\ln 2 \quad ^2 \\ &= \frac{1}{2}\ln e + \frac{1}{2}\ln 2 \quad ^2 \\ &= \frac{1}{2}\ln 2 \; e \quad ^2 \end{aligned} \tag{2.11}$$

Changing the base of the logarithm, we have :

$$h(X) = \frac{1}{2}\log 2 \; e \quad ^2 \tag{2.12}$$

It is possible to generalize this result to a normal distribution of n variables, which expression is given by:

$$p_N(x) = \frac{1}{\sqrt{(2 \quad)^n \det \quad }} e^{\frac{1}{2}(x \; h \; x i)^T \quad ^1 (x \; h \; x i)} \tag{2.13}$$

where is the covariance matrix defined in Chapter 1. In this case, the entropy is equal to :

$$h(p_N) = \frac{1}{2}\ln((2 \; e)^n \det \quad) \tag{2.14}$$

The striking feature of Gaussian distributions is that they maximize the differential entropy over all distributions with the same covariance, or mathematically :

$$h(p_N) \geq h(p) \quad \forall p(x) \tag{2.15}$$

For curious readers, the proof of this relation can be found in [29, p. 255-256].

2.3 Entropy powers

Sometimes, it is more convenient to work with the *entropy power* of a random variable, which is a derived quantity of the differential entropy $h(X)$. The entropy power of an n-variables distribution is written :

$$N(X) = \frac{1}{2\pi e} e^{\frac{2h(X)}{n}} \tag{2.16}$$

where $X = (X_1; X_2; \ldots; X_n)^T$ is a multivariate random variable.

In the particular case of a one-mode Gaussian distribution of variance σ_G^2, we have :

$$N_G = \frac{1}{2\pi e} e^{\ln(2\pi e \sigma_G^2)} = \sigma_G^2 \tag{2.17}$$

Combining this result with the fact that the differential entropy is maximized by Gaussian distributions, we have the following inequality :

$$\sigma_x^2 \geq N(x) \tag{2.18}$$

As we will see, entropy powers are very appropriate to write uncertainty relations. More specifically, variance-based quantum uncertainty relations can be derived from entropic uncertainty relations.

2.4 Conclusion

In this chapter, we briefly reviewed some definitions in information theory which help us measure uncertainty differently than by using variances, namely, using the Shannon entropy. We will see in this next chapter, that although Shannon information theory is a classical theory, it can still be applied for quantum uncertainty relations.

Chapter 3

Uncertainty Relations

In classical physics, when performing an experiment and then, repeating it under the same exact conditions, it is expected of the two outcomes to be exactly the same. This doctrine is called *determinism* [30, p. 5]. However, in quantum mechanics, there is an inherent probabilistic aspect that is well emphasized by the *uncertainty relations*. Indeed, it is, for instance, impossible to know with absolute precision the position and the momentum of a quantum particle *simultaneously*. This was first expressed by Heisenberg in 1927 and since then, many new uncertainty relations have been established. Some relations hold for any pair of arbitrary observables $\hat{A}$ and $\hat{B}$ [2, 3], some are extended to more than two variables [31, 32, 4, 5, 6, 7], and others are independent of the quantum physical state [32, 33]. More recently, entropic uncertainty relations have emerged as a tool to verify the security of quantum cryptographic protocols, such as quantum key distribution or two-party quantum cryptography [8].

The work presented in this thesis is based on the study of specific uncertainty relations and is appreciably inspired by the PhD thesis of A. Hertz [34]. In this chapter, we review some historical uncertainty relations that are based on the *product* of variances. Then, we give an overview of more "modern" uncertainty relations that are based on the *sum* of variances, some of which will be more thoroughly studied in the second part of this thesis. Finally, we close this chapter by giving a few well-known entropic uncertainty relations.

Note that for clarity, we often refer to uncertainty relations that are based on products as *multiplicative* uncertainty relations, as opposed to *additive* uncertainty relations, which are based on sums.

3.1 Uncertainty relations based on product of variances

3.1.1 Heisenberg uncertainty relation

In 1927, Heisenberg conducted a thought experiment, in which he intended to determine the position of an electron by illuminating it with a photon and then looking at it under the microscope [1]. Classical optics tell us that the electron position can only be resolved up to an uncertainty Δx, which depends on the wavelength λ of the incoming light and on the diameter of the lens D. Let us assume that we use a γ ray microscope that allows us to measure the position of the electron with most possible precision, i.e. by selecting a wide lens and a short wavelength [1]. This arbitrarily precise measurement cre-

[1] Note that in practice, it is impossible to observe the electron with an uncertainty strictly equal to 0, because the diameter of the lens is finite and the wavelength cannot be null. This is completely analog to the fact that one cannot create a quantum state where the position of a particle is perfectly known.

ates another problem on its own. Indeed, due to the Compton effect, the incident photon is scattered by the electron and as a result, the electron gains some momentum. However, because the wavelength of the photon was set small, the photon carried a more important momentum (following de Broglie's formula $p = h/\lambda$) and could potentially transmit more energy to the electron, increasing thus the uncertainty on the momentum. In conclusion, to know the position precisely, one must use short wavelengths, but doing so increases the uncertainty on the momentum. On the contrary, to know the momentum precisely, one must use greater wavelengths, at the expense of a larger uncertainty on the position. Heisenberg formulated that that there was a trade-off between the two measurements and expressed it as :

$$\Delta x \Delta p \gtrsim h \tag{3.1}$$

Soon after, Kennard mathematically formalized Heisenberg's results and demonstrated that [35]:

$$\sigma_x^2 \sigma_p^2 \geq \frac{\hbar^2}{4} \tag{3.2}$$

where σ_x^2 and σ_p^2 are the position and momentum variances of a quantum particle.

At this point, it is relevant to make the distinction between *uncertainty relations* and the *uncertainty principle*. As explained by Peres in [30], the uncertainty relation given by eq. (3.2) only reflects the intrinsic randomness of the outcomes of quantum tests, while the uncertainty principle refers to the disturbance in the measurements induced by the apparatus, such as described in the recent literature by Ozawa or Busch [36, 37]. In Heisenberg's initial paper, it is not evident whether the uncertainties originate from the physical state itself or from the disturbances induced by the measurements. However, eq. (3.2) is clearly an uncertainty *relation* : it highlights that it is impossible for a physical state to have both its position and momentum uncertainties simultaneously small. This intrinsic property of the physical state is a direct consequence of Fourier transform.

3.1.2 Robertson-Schrödinger uncertainty relation

In 1929, Robertson generalized the Heisenberg relation for two arbitrary observables $\hat{A}$ and $\hat{B}$ [2]:

$$\sigma_A^2 \sigma_B^2 \geq \frac{1}{4} |\langle\psi|[\hat{A},\hat{B}]|\psi\rangle|^2 \tag{3.3}$$

Of course, if we consider the operators $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}$, we get back to the Heisenberg relation, since $[\hat{x},\hat{p}] = i\hbar$. Note that for convenience, we fix $\hbar = 1$ throughout the rest of this thesis.

Relation (3.2) is invariant under (x,p)-displacement in phase space, meaning that if the state $|\psi\rangle$ is displaced in phase space with the use of an operator $D(\alpha)$, then the inequality remains the same. This is because variances are central moments and are thus invariant under translation, i.e. for any random variable A and constant a, we have $\mathrm{Var}(A + a) = \mathrm{Var}(A)$. Furthermore, if we apply the displacement operator on the right side of the inequality, we get :

$$\begin{aligned}\sigma_x^2\sigma_p^2 &\geq \frac{1}{4}|\langle\psi|\hat{D}^\dagger(\alpha)[\hat{x},\hat{p}]\hat{D}(\alpha)|\psi\rangle|^2\\ &\geq \frac{1}{4}|\langle\psi|\hat{D}^\dagger(\alpha)\hat{D}(\alpha)[\hat{x},\hat{p}]|\psi\rangle|^2\\ &\geq \frac{1}{4}|\langle\psi|[\hat{x},\hat{p}]|\psi\rangle|^2\end{aligned} \tag{3.4}$$

where we used the fact that the displacement operator is a unitary and that the commutator $[\hat{x},\hat{p}]$ is a scalar. This proves that the inequality is indeed invariant under (x,p)-translation.

Let us note that the inequality (3.2) is saturated by the vacuum state, coherent states and squeezed states, provided that they are squeezed along the x- or p-axis.

Robertson's uncertainty relation was later strengthened by Schrödinger who pointed out that one could add an anti-commutator term [38]:

$$\sigma_A^2\sigma_B^2 \geq \frac{1}{4}|\langle\{\hat{A},\hat{B}\}\rangle - 2\langle\hat{A}\rangle\langle\hat{B}\rangle|^2 + \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2 \tag{3.5}$$

Taking $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}$, the Robertson-Schrödinger uncertainty can be expressed as :

$$\det(\gamma) \geq \frac{1}{4} \tag{3.6}$$

where γ is the covariance matrix defined in Chapter 1. The uncertainty relation (3.6) is invariant under all Gaussian unitary transformations (displacement, squeezing and rotation) and is therefore saturated by all pure Gaussian states. This can be simply verified by taking the covariance matrix γ of any Gaussian state and then applying a symplectic transformation onto it. Similarly to eq. (1.63), γ transforms into $S\gamma S^T$. Then, remembering that the determinant of a product is equal to the product of the determinants, that the determinant of a matrix is equal to the determinant of its transpose and that the determinant of a symplectic matrix is equal to one, we get :

$$\det(S\gamma S^T) = \det(S)\det(\gamma)\det(S) = \det(\gamma) \tag{3.7}$$

which proves that $\det(\gamma)$ is invariant under symplectic transformation.

3.2 Uncertainty relations based on sum of variances

The Robertson-Schrödinger uncertainty relation exhibits the impossibility to precisely measure two non-commuting (or *incompatible*) observables. Indeed, if the two observables commute, it is easy to see that the lower bound of eq. (3.5) becomes null and that therefore, there is no limitation in our knowledge of the physical state. However, this lower bound can also become trivial when the physical state is an eigenstate of one of the two observables. For instance, if the physical state is an eigenstate $|\psi_A\rangle$ of the observable $\hat{A}$, such that $\hat{A}|\psi\rangle = a|\psi\rangle$, we'll have :

$$\sigma_A^2 = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2 = \langle\psi_A|\hat{A}^2|\psi_A\rangle - \langle\psi_A|\hat{A}|\psi_A\rangle^2 = a^2\langle\psi_A|\psi_A\rangle - (a\langle\psi_A|\psi_A\rangle)^2 = 0 \tag{3.8}$$

and therefore, the lower bound becomes trivial even if the other variance is non-zero : this is because the uncertainty relation is based on a *product* of variances. To solve this issue, many recent papers focused on uncertainty relations that are based on *sum* of variances, as they guarantee the lower bound to be greater than 0 even if the state is an eigenstate of one of observables.

3.2.1 Maccone and Pati uncertainty relations

The Maccone and Pati uncertainty relation [3] is at the core of this thesis, and will be later studied for the specific case of the $\hat{x}$ and $\hat{p}$ observables. This uncertainty relation is characterized by two inequalities, the first one being :

$$\Delta A^2 + \Delta B^2 \geq \pm i\langle[\hat{A},\hat{B}]\rangle + |\langle\psi|\hat{A} \pm i\hat{B}|\psi^{\perp}\rangle|^2 \tag{3.9}$$

where $|\psi^{\perp}\rangle$ designates an arbitrary state which is orthogonal to $|\psi\rangle$. The second inequality reads :

$$\Delta A^2 + \Delta B^2 \geq \frac{1}{2}|\langle\psi^{\perp}_{A+B}|\hat{A}+\hat{B}|\psi\rangle|^2 \tag{3.10}$$

where $|\psi^{\perp}_{A+B}\rangle \propto (\hat{A}+\hat{B} - \langle\hat{A}+\hat{B}\rangle)|\psi\rangle$ is a state orthogonal to $|\psi\rangle$. The notation $|\psi^{\perp}_{A+B}\rangle$ implies that the right-hand-side of the inequality is non-trivial unless $|\psi\rangle$ is an eigenstate of $\hat{A}+\hat{B}$. By combining eq. (3.9) and (3.10), we obtain the Maccone Pati uncertainty relation :

$$\Delta A^2 + \Delta B^2 \geq \max(L_1, L_2) \tag{3.11}$$

with $L_{1,2}$ being the right-hand-side of eq. (3.9) and (3.10) respectively.

Two different proofs were provided in the original paper. One given by Maccone and Pati themselves and the other given by an anonymous referee. Because this relation is at the root of this thesis, we present here the proof given by the anonymous referee, which is also the most elegant one, according to us but also according to the original authors.

To prove (3.9), we define the following operators :

$$\begin{aligned} \hat{C} &\equiv \hat{A} - \langle\hat{A}\rangle \\ \hat{D} &\equiv \hat{B} - \langle\hat{B}\rangle \end{aligned} \tag{3.12}$$

so that

$$\begin{aligned} \Delta A &= \|\hat{C}|\psi\rangle\| \\ \Delta B &= \|i\hat{D}|\psi\rangle\| \end{aligned} \tag{3.13}$$

where $\||\psi\rangle\| \equiv \sqrt{\langle\psi|\psi\rangle}$ designates the norm of $|\psi\rangle$ and where the imaginary unit i has been added for later convenience. We have thus :

$$\begin{aligned} \|(\hat{C} \mp i\hat{D})|\psi\rangle\|^2 &= \langle\psi|(\hat{C}^{\dagger} \pm i\hat{D}^{\dagger})(\hat{C} \mp i\hat{D})|\psi\rangle \\ &= \|\hat{C}|\psi\rangle\|^2 + \|\hat{D}|\psi\rangle\|^2 \mp i\langle\psi|(\hat{C}\hat{D} - \hat{D}\hat{C})|\psi\rangle \\ &= \|\hat{C}|\psi\rangle\|^2 + \|i\hat{D}|\psi\rangle\|^2 \mp i\langle\psi|(\hat{A}\hat{B} - \hat{B}\hat{A})|\psi\rangle \\ &= \|\hat{C}|\psi\rangle\|^2 + \|i\hat{D}|\psi\rangle\|^2 \mp i\langle\psi|[\hat{A},\hat{B}]|\psi\rangle \\ &= \Delta A^2 + \Delta B^2 \mp i\langle[\hat{A},\hat{B}]\rangle \end{aligned} \tag{3.14}$$

The left-hand-side can be lower bounded through the Cauchy-Schwarz inequality which states that for for any pair of non-null vectors $|\psi\rangle$ and $|\phi\rangle$ in a Hilbert space $\mathcal{H}$:

Figure 3.1: The parallelogram law states that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.

$$|\langle \ | \ \rangle|^2 \quad \langle \ | \ \rangle\langle \ | \ \rangle = \| \ \|^2 \| \ \|^2 \tag{3.15}$$

Therefore, using the Cauchy-Schwarz inequality, we can write :

$$\begin{aligned} \langle \ |(\hat{A} \quad i\hat{B}) \quad ^{?} \ ^{2} &= \langle \ |(\hat{A} \quad i\hat{B} \ \langle \hat{A} \quad i\hat{B}\rangle) \quad ^{?} \ ^{2} \\ &= \langle \ |(\hat{C} \quad i\hat{D}) \quad ^{?} \ ^{2} \\ & \quad (\hat{C} \quad i\hat{D})| \ \rangle \ ^{2} \end{aligned} \tag{3.16}$$

which is valid for all $^{?}$ orthogonal to $| \ \rangle$. By combining (3.14) and (3.16), we naturally obtain eq. (3.9).

To prove eq. (3.10), we use (3.12) and (3.13), with the di erence that this time, we do not add the imaginary unit i, i.e. $B = \hat{D}| \ \rangle$. We then apply the parallelogram law in Hilbert space (see Fig. 3.1) :

$$2 \ A^2 + 2 \ B^2 = (\hat{C} + \hat{D})| \ \rangle \ ^2 + (\hat{C} \quad \hat{D})| \ \rangle \ ^2 \tag{3.17}$$

Since $(\ A + B) = (\hat{C} + \hat{D})| \ \rangle$ and $(\ A \quad B) = (\hat{C} \quad \hat{D})| \ \rangle$, we can rewrite the previous equation as :

$$\begin{aligned} A^2 + \ B^2 &= \frac{1}{2}[(\ A + B)^2 + (\ A \quad B)^2] \\ & \frac{1}{2}(\ A + B)^2 \end{aligned} \tag{3.18}$$

which is equivalent to (3.10), since

$$
\begin{aligned}
(\Delta A + \Delta B)^2 &= \left\|(\hat{C} + \hat{D})|\psi\rangle\right\|^2 \\
&= \langle\psi|(\hat{C} + \hat{D})^\dagger(\hat{C} + \hat{D})|\psi\rangle \\
&= \langle\psi|(\hat{C} + \hat{D})^\dagger(\hat{C} + \hat{D})|\psi\rangle \langle\psi^\perp_{A+B}|\psi^\perp_{A+B}\rangle \\
&\geq \left|\langle\psi^\perp_{A+B}|\hat{C} + \hat{D}|\psi\rangle\right|^2 \\
&= \left|\langle\psi^\perp_{A+B}|(\hat{A} + \hat{B}) - (\langle\hat{A}\rangle + \langle\hat{B}\rangle)|\psi\rangle\right|^2 \\
&= \left|\langle\psi^\perp_{A+B}|\hat{A} + \hat{B}|\psi\rangle - (\langle\hat{A}\rangle + \langle\hat{B}\rangle)\langle\psi^\perp_{A+B}|\psi\rangle\right|^2 \\
&= \left|\langle\psi^\perp_{A+B}|\hat{A} + \hat{B}|\psi\rangle\right|^2
\end{aligned}
\tag{3.19}
$$

Note that at the fourth equality sign, we used the fact that $|\psi^\perp_{A+B}\rangle \propto (\hat{A}+\hat{B} - \langle\hat{A}+\hat{B}\rangle)|\psi\rangle$ in order to turn the Cauchy-Schwarz inequality into an equality.

3.2.2 Kechrimparis and Weigert uncertainty relations

Independently of Maccone and Pati, Kechrimparis and Weigert have recently worked on uncertainty relations for more than two variables that were based on sums and products of variances. In 2014, they proved that for $\hat{x}$, $\hat{p}$ and $\hat{r} = -\hat{x} - \hat{p}$, which are three pairwise canonically conjugate observables (i.e. $[\hat{p}, \hat{x}] = [\hat{x}, \hat{r}] = [\hat{r}, \hat{p}] = -i$), the lower bound on the product of variances is given by [31]:

$$\Delta x^2 \, \Delta p^2 \, \Delta r^2 \geq \left(\frac{\tau}{2}\right)^3 \tag{3.20}$$

where the number τ is the triple constant with value

$$\tau \equiv \sqrt{\frac{4}{3}} \tag{3.21}$$

In terms of the sum of variances, the uncertainty relation reads :

$$\Delta x^2 + \Delta p^2 + \Delta r^2 \geq \frac{3}{2}\tau = \sqrt{3} \tag{3.22}$$

This uncertainty relation is saturated by the generalized squeezed state $|\Xi_0\rangle$:

$$|\Xi_0\rangle \equiv \hat{S}_{\frac{i}{4}\ln 3}|0\rangle \tag{3.23}$$

which is generated by contracting a coherent state $|0\rangle$ by an amount of $\ln\sqrt[4]{3}$ at a squeezing angle of $3\pi/4$ (as an illustration, $|\Xi_0\rangle$ is represented in Fig. 3.2).

Later, in 2018 [32], they generalized this uncertainty relation for linear combinations of position and momentum operators, defined as :

$$\hat{r}_j = a_j \hat{p}_j + b_j \hat{x}_j \, ; \qquad a_j, b_j \in \mathbb{R}; \qquad j = 1, \ldots, N. \tag{3.24}$$

where at least two of the operators $\hat{r}_j$, $j = 1, \ldots, N$, should not commute (to exclude a trivial situation). Moreover, we note that by using a system of units in which both position and momentum have physical dimension $\sqrt{\hbar}$, the coefficients, a_j and b_j are dimensionless. As seen in Fig. 3.3, each observable $\hat{r}_j$ can be represented by a vector in a two-dimensional Euclidean space :

Figure 3.2: The state $|\psi_0\rangle$ (full line) is generated by squeezing the vacuum state (dashed). Figure reprinted from [31].

$$\mathbf{r}_j = \begin{pmatrix} a_j \\ b_j \end{pmatrix} \in \mathbb{R}^2; \qquad j = 1, \ldots, N. \tag{3.25}$$

We call $\|\hat{r}_j\| = \sqrt{a_j^2 + b_j^2}$ the "length" of the observable $\hat{r}_j$.

In the particular case of N observables ($N > 2$) arranged in a symmetric way, meaning that we assume that the tips of the vectors $\mathbf{r}_j \in \mathbb{R}^2$ are located on a circle of radius $R \in (0, \infty)$ and that they are distributed homogeneously, we have :

$$\hat{r}_j = (R\cos\theta_j)\hat{p} + (R\sin\theta_j)\hat{x}; \qquad \theta_j = \frac{2\pi(j-1)}{N}; \qquad j = 1, \ldots, N. \tag{3.26}$$

Remark that from a structural point of view, the value of the constant R is not important as it only rescales the all observables. We can fix it in such a way that any two adjacent observables form a canonical pair, i.e. :

$$[\hat{r}_j, \hat{r}_{j+1}] = i\hat{I}; \qquad \hat{r}_{N+1} \equiv \hat{r}_1; \qquad j = 1, \ldots, N \tag{3.27}$$

where $\hat{I}$ is the identity operator. These conditions are satisfied if the circumradius R of the polygon takes the value

$$R_N = \frac{1}{\sqrt{\sin\theta_N}}; \qquad \theta_N = \frac{2\pi}{N} \tag{3.28}$$

For the observables described by eq. (3.26), the N-variables uncertainty relation takes the form of :

$$\sum_{j=1}^{N} \Delta^2 r_j \geq \frac{N}{2\sin\theta_N} \quad \text{or} \quad \prod_{j=1}^{N} \Delta^2 r_j \geq \left(\frac{1}{2\sin\theta_N}\right)^N \tag{3.29}$$

Figure 3.3: A regular pentagon in the dimensionless "phase space" $\mathbb{R}^2$.

It is possible to absorb the factor $\sin\theta_N$ on the right-hand-side of these inequalities by considering the $\hat{r}_j$ operators with tips located on the unit circle (meaning $R_N = 1$), then, the uncertainty relations take particularly simple forms [32] :

$$\sum_{j=1}^{N} \Delta r_j^2 \geq \frac{N}{2} \quad \text{or} \quad \prod_{j=1}^{N} \Delta r_j^2 \geq \left(\frac{1}{2}\right)^N \tag{3.30}$$

These uncertainty relations are saturated for coherent states $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$, contrarily to the uncertainty relations (3.20) and (3.22) that were saturated for generalized squeezed states $|\alpha,\xi\rangle$. This difference can be explained by the fact that the observables involved in eq. (3.29) all have the same scaling and are evenly distributed.

It is interesting to note that unlike Maccone and Pati relations, the Kechrimparis and Weigert uncertainty relations do not depend on the measured state. However, they are restricted to quadrature observables

3.2.3 Song and Qiao uncertainty relation for 3 observables

Following the work of Maccone and Pati, Song and Qiao proposed an uncertainty relation based of the sum of variances of three observables $\hat{A}, \hat{B}, \hat{C}$:

$$\begin{aligned} \Delta A^2 + \Delta B^2 + \Delta C^2 \geq\; & \frac{1}{3} \left|\langle \psi^{\perp}_{ABC} | \hat{A} + \hat{B} + \hat{C} | \psi \rangle\right|^2 \\ & + \frac{\sqrt{3}}{3} \left| i\langle[\hat{A},\hat{B},\hat{C}]\rangle \right| + \frac{2}{3} \left|\langle \psi | \hat{A} + \hat{B} e^{\pm 2i\pi/3} + \hat{C} e^{\pm 4i\pi/3} | \psi^{\perp} \rangle\right|^2 \end{aligned} \tag{3.31}$$

where $|\psi^{\perp}\rangle$ is a state orthogonal to the state of the system $|\psi\rangle$, $|\psi^{\perp}_{ABC}\rangle \propto (\hat{A} + \hat{B} + \hat{C} - \langle \hat{A} + \hat{B} + \hat{C} \rangle)|\psi\rangle$ and $\langle[\hat{A},\hat{B},\hat{C}]\rangle \equiv \langle[\hat{A},\hat{B}]\rangle + \langle[\hat{B},\hat{C}]\rangle + \langle[\hat{C},\hat{A}]\rangle$. The sign in the last term of (3.31) is $+(-)$ when $i\langle[\hat{A},\hat{B},\hat{C}]\rangle$ is positive (negative).

Remark that applying the three pairwise canonical observables from Kechrimparis and Weigert's relation (3.22) to the inequality (3.31), one gets :

$$x^2 + \quad p^2 + \quad r^2 \quad \sqrt{3} + \frac{2}{3} h \;\; j\hat{x} + \hat{p}e^{2 i = 3} + \hat{r}e^{4 i = 3} \;\; ?^{\;2} \tag{3.32}$$

which is obviously stronger than the Kechrimparis and Weigert's relation since it contains an extra positive term to the right side of eq. (3.22).

3.3 Entropic uncertainty relations

Because entropies are natural quantities in (quantum) information sciences, entropic uncertainty relations have gained a lot of interest over the recent years. For instance, they can be used as a way to distinguish entangled states, for quantum key distribution or for two-party cryptography [8].

3.3.1 Hirschman uncertainty relation

In Chapter 2, we have seen that the Shannon entropies measure the randomness (or uncertainty) of a variable. In 1957, Hirschman assumed that it should be therefore possible to derive uncertainty relations that are based on Shannon entropies and conjectured that for any n-modal state :

$$h(x) + h(p) \quad n \ln(e \hbar) \tag{3.33}$$

where $x = (\hat{x}_1; :::\hat{x}_n)$ and $p = (\hat{p}_1; :::\hat{p}_n)$. This relation was formally proved in 1975 by Beckner [39] and independently, in the same year, by Biaªynicki-Birula and Mycielski [40].

At rst sight, eq. (3.33) may seem odd because we take the logarithm of a quantity with dimension $\hbar$ in the right-hand-side. However, since the de nition of the di erential entropy $h(\,)$ involves taking the logarithm of a quantity and since $h(x) + h(p) = h(x{:}p)$ (where $x.p$ represents an inner product), we notice that the problem cancels out since we have dimension $\hbar$ in both sides of the inequality. This motivates the use of the convention $\hbar = 1$ that we used earlier in this chapter, being aware that it is an abuse of notation.

An interesting feature of the entropic uncertainty relation (3.33) is that it implies the Heisenberg relation. Indeed, if we write (3.33) in terms of entropy powers, we get [41]:

$$\begin{aligned} N(x)N(p) &= \frac{1}{2e} e^{2h(x)} \frac{1}{2e} e^{2h(p)} \\ &= \left(\frac{1}{2e}\right)^2 e^{2[h(x)+h(p)]} \\ & \quad \left(\frac{1}{2e}\right)^2 e^{2\ln(e)} = \frac{1}{4} \end{aligned} \tag{3.34}$$

As detailed in Chapter 2, we have $N(x) \quad {}^2_x$, which highlights the fact that Gaussian distributions maximize the entropy for a xed variance. This allows us to write [41] :

$${}^2_x \, {}^2_p \quad N(x)N(p) \quad \frac{1}{4} \tag{3.35}$$

which clearly shows that the entropic uncertainty relation implies the Heisenberg relation (3.2).

3.3.2 Other entropic uncertainty relations

In 2009, Guanleiet al. formulated a similar entropic uncertainty relation to (3.33) but for two single-mode rotated quadratures [42] :

$$h(x_\theta) + h(x_\phi) \geq \ln(\pi e|\sin(\theta - \phi)|) \tag{3.36}$$

where $\hat{x}_\theta = \hat{x}\cos\theta + \hat{p}\sin\theta$ and $\hat{x}_\phi = \hat{x}\cos\phi + \hat{p}\sin\phi$. The n-modal generalization of the entropic uncertainty relation was derived in [43], where the lower bound is expressed in terms of the determinant of a matrix of commutators between the measured variables.

More recently, an improved version of the single-mode entropic uncertainty relation (3.33), which is saturated for all Gaussian states, was proposed by Hertz, Jabbour and Cerf in [41] :

$$h(x) + h(p) - \frac{1}{2}\ln\left(\frac{\sigma_x^2\sigma_p^2}{\det\gamma}\right) \geq \ln(\pi e) \tag{3.37}$$

This uncertainty relation was proved under two reasonable assumptions (for more details, see [34, p. 71]).

Morever, following a similar approach, Hertz and Cerf gave a proof (in [10]) of the entropic version of Kechrimparis and Weigert uncertainty relation that was initially conjectured in [32] :

$$\frac{2}{N}(h_1 + h_2 + \dots + h_N) \geq \ln(e\pi) \tag{3.38}$$

3.4 Conclusion

In this chapter, we have briefly outlined the history of uncertainty relations. In 1927, Heisenberg showcased that there was a trade-off between the measurement of the position of a particle and the measurement of its momentum. In this thesis, we focus on uncertainty relations, i.e. relations that highlight the intrinsic random aspect in quantum measurements. The Heisenberg relation was first mathematically formalized by Kennard and later generalized by Schrödinger and Robertson for two arbitrary observables. In the recent years, many new uncertainty relations based on sum of variances have emerged, as they present the advantages of having a non-trivial lower-bound when the quantum state measured is an eigenstate of one of the two observables. The Maccone and Pati relation will be thoroughly analyzed in the second part of this thesis, as well as the Kechrimparis-Weigert relation, and the Song-Qiao relation. In parallel, many entropic uncertainty relations were established, as they find many applications in quantum information sciences.

Part II

Results

Chapter 4

Additive uncertainty relation for single-mode quadratures

In this chapter, we thoroughly analyze the Maccone-Pati (M-P) uncertainty relation for the specific case where the arbitrary observables $\hat{A}$ and $\hat{B}$ are the position and momentum operators $\hat{x}$ and $\hat{p}$. We study the lower bound of the inequality for different single-mode states and determine the state $\psi^{\perp}$ that maximizes it. Then, we establish a connexion between this uncertainty relation and some other work that can be found in the literature, regarding nonclassicality criteria. Finally, we attempt to derive an entropic uncertainty relation based on the sum of entropy powers. Note that in this chapter and in the next ones, we use the convention $\hbar = 1$.

4.1 M-P relation for the quadratures of light

As we have seen in Chapter 3, the M-P relation is an additive uncertainty relation that holds for any pair of arbitrary observables $\hat{A}$ and $\hat{B}$. It is expressed as :

$$\Delta A^2 + \Delta B^2 \geq \max(L_1; L_2) \tag{4.1}$$

where

$$\begin{aligned} L_1 &= \pm i\langle\psi|[\hat{A};\hat{B}]|\psi\rangle + |\langle\psi|\hat{A} \pm i\hat{B}|\psi^{\perp}\rangle|^2 \\ L_2 &= \frac{1}{2}|\langle\psi^{\perp}_{A+B}|\hat{A}+\hat{B}|\psi\rangle|^2 \end{aligned} \tag{4.2}$$

In this chapter, and in the rest of this thesis, we focus on L_1, as it appears, from the original paper of Maccone and Pati, that selecting the orthogonal state $\psi^{\perp}$ which maximizes the quantity $|\langle\psi|\hat{A} \pm i\hat{B}|\psi^{\perp}\rangle|^2$ makes the inequality tight. By substituting $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}$, we obtain the following expression for L_1 :

$$\begin{aligned} L_1 &= \hbar + |\langle\psi|\hat{x} - i\hat{p}|\psi^{\perp}\rangle|^2 \\ &= \hbar + 2|\langle\psi|\hat{a}^{\dagger}|\psi^{\perp}\rangle|^2 \\ &= \hbar + 2|\langle\psi^{\perp}|\hat{a}|\psi\rangle|^2 \end{aligned} \tag{4.3}$$

where the sign - has been selected so that $-i\langle\psi|[\hat{x};\hat{p}]|\psi\rangle$ becomes a positive quantity and where we used eq. (1.27) and (1.28) to define the creation operator in terms of the quadratures of light. The Maccone and Pati uncertainty relation for the quadratures of light is thus given by (setting $\hbar = 1$):

$$\Delta x^2 + \Delta p^2 \geq 1 + 2 \max_{\{|\psi^\perp\rangle\}} |\langle\psi^\perp|\hat{a}|\psi\rangle|^2 \tag{4.4}$$

where $\max_{\{|\psi^\perp\rangle\}}$ indicates that we choose the state $\psi^\perp$ which maximizes $|\langle\psi^\perp|\hat{a}|\psi\rangle|^2$.

4.1.1 Invariances of the inequality

When faced with any uncertainty relation, an important question to ask is whether or not the relation is invariant under Gaussian unitaries. In this subsection, we show that the M-P uncertainty relation is indeed invariant under displacement and rotation. These results will be useful in the following sections, where we will compute the variances of the quadratures of light for different states, as well as the maximum value of the term $|\langle\psi^\perp|\hat{a}|\psi\rangle|^2$.

Invariance under displacement

We already know that the left member of the inequality is invariant under displacement, because variances are invariant under translation. We now need to check that this is also the case for the right member. This is mathematically achieved by applying the displacement operator on the states $|\psi\rangle$ and $|\psi^\perp\rangle$. Indeed, it is easily verified that applying the displacement operators on these two states does not change their orthogonality :

$$\langle\psi^\perp|\hat{D}^\dagger(\alpha)\hat{D}(\alpha)|\psi\rangle = \langle\psi^\perp|1|\psi\rangle = 0 \tag{4.5}$$

Consequently, the right member of eq. (4.4) becomes :

$$\begin{aligned} 1 + 2|\langle\psi^\perp|\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha)|\psi\rangle|^2 &= 1 + 2|\langle\psi^\perp|\hat{a} + \alpha|\psi\rangle|^2 \\ &= 1 + 2|\langle\psi^\perp|\hat{a}|\psi\rangle|^2 \end{aligned} \tag{4.6}$$

where we used eq. (1.47) to compute the action of the displacement operator on the annihilation operator. This proves that the M-P uncertainty relation is invariant under displacement.

Invariance under rotation

Similarly, by applying the phase-shift operator on the states $|\psi\rangle$ and $|\psi^\perp\rangle$ and by remembering that the action of the phase-shift operator on the mode operators is given by eq. (1.65) :

$$\begin{aligned} 1 + 2|\langle\psi^\perp|\hat{R}^\dagger(\phi)\hat{a}\hat{R}(\phi)|\psi\rangle|^2 &= 1 + 2|\langle\psi^\perp|\hat{a}e^{-i\phi}|\psi\rangle|^2 \\ &= 1 + 2|e^{-i\phi}|^2|\langle\psi^\perp|\hat{a}|\psi\rangle|^2 \\ &= 1 + 2|\langle\psi^\perp|\hat{a}|\psi\rangle|^2 \end{aligned} \tag{4.7}$$

which clearly shows that the right member is invariant under rotation. In parallel, we know that the left member is invariant under rotation because it corresponds to the trace of the covariance matrix and the trace of a matrix is invariant under any orthogonal transformation (and in particular, a rotation).

4.2 Single-mode Gaussian states

In this section, we prove that the M-P uncertainty relation is saturated for all single-mode Gaussian states.

The vacuum state

We start with the simplest type of Gaussian state : the vacuum state. After computing the right member of the M-P inequality, we obtain:

$$\begin{aligned} & \Delta x^2 + \Delta p^2 \geq 1 + 2|\langle\psi^{\perp}|\hat{a}|0\rangle|^2 \\ \Leftrightarrow \quad & \Delta x^2 + \Delta p^2 \geq 1 \end{aligned} \tag{4.8}$$

for any $\psi^{\perp}$ which respects $\langle\psi^{\perp}|0\rangle = 0$.

The variances Δx^2 and Δp^2 for the vacuum state are given by :

$$\begin{aligned} \Delta x^2 + \Delta p^2 &= \langle 0|\hat{x}^2 + \hat{p}^2|0\rangle - \langle 0|\hat{x}|0\rangle^2 - \langle 0|\hat{p}|0\rangle^2 \\ &= \langle 0|1 + 2\hat{a}^{\dagger}\hat{a}|0\rangle - \frac{1}{2}\langle 0|\hat{a} + \hat{a}^{\dagger}|0\rangle^2 + \frac{1}{2}\langle 0|\hat{a} - \hat{a}^{\dagger}|0\rangle^2 \\ &= 1 + 2\langle 0|\hat{N}|0\rangle \\ &= 1 \end{aligned} \tag{4.9}$$

The left and right members of the inequality are thus identical : this indicates that the vacuum state saturates the M-P uncertainty relation.

Coherent states

Since the M-P uncertainty relation is invariant under displacement, coherent states also saturate the inequality.

Squeezed states

We now focus on another class of Gaussian states : squeezed states. As seen in Chapter 1, a squeezed state can be written as :

$$|\psi\rangle = \hat{S}(z)|0\rangle \tag{4.10}$$

The most general orthogonal state to this squeezed state is the complex linear combination of squeezed Fock states given by:

$$|\psi^{\perp}\rangle = \sum_{n\geq 1}^{\infty} \alpha_n \hat{S}(z)|n\rangle \qquad \text{with} \qquad \sum_{n\geq 1}^{\infty} |\alpha_n|^2 = 1 \tag{4.11}$$

Indeed, it is quite straightforward to see that :

$$\langle\psi^{\perp}|\psi\rangle = \langle n|\sum_{n\geq 1}^{\infty} \alpha_n^* \hat{S}^{\dagger}(z)\hat{S}(z)|0\rangle = \sum_{n\geq 1}^{\infty} \alpha_n^* \langle n|0\rangle = 0 \tag{4.12}$$

The M-P uncertainty relation becomes thus :

$$\Delta x^2 + \Delta p^2 \geq 1 + 2 \max_{\{\alpha_n\}} \left| \sum_{n \geq 1} \alpha_n \langle n | \hat{S}(z)^\dagger \hat{a} \hat{S}(z) | 0 \rangle \right|^2 \tag{4.13}$$

The coefficients α_n must now be chosen in order to maximize the right member of the inequality. Conveniently, remembering the action of the squeezing operator on the annihilation operator, which is given by eq. (1.59), we determine that, in order to maximize the right member of the inequality, every coefficient α_n must be null, except for the coefficient α_1, which is linked to the state $|1\rangle$ and whose modulus $|\alpha_1|^2$ is equal to 1 for the normalization:

$$\begin{aligned}
\Delta x^2 + \Delta p^2 &\geq 1 + 2 \max_{\{\alpha_n\}} \left| \sum_{n \geq 1} \alpha_n \langle n | \hat{S}(z)^\dagger \hat{a} \hat{S}(z) | 0 \rangle \right|^2 \\
&\geq 1 + 2 \max_{\{\alpha_n\}} \left| \sum_{n \geq 1} \alpha_n \langle n | (\hat{a} \cosh r - \hat{a}^\dagger e^{i\phi} \sinh r) | 0 \rangle \right|^2 \\
&\geq 1 + 2 \max_{\{\alpha_n\}} \left| \alpha_1 (-e^{i\phi} \sinh r) \langle 1 | \hat{a}^\dagger | 0 \rangle \right|^2 \\
&\geq 1 + 2 \sinh^2 r \\
&\geq \cosh 2r
\end{aligned} \tag{4.14}$$

To determine the left member of the inequality, we can compute the trace of the covariance matrix for a Gaussian squeezed state, given by eq. (1.63). The sum of variances $\Delta x^2 + \Delta p^2$ is thus equal to :

$$\mathrm{Tr}(\gamma) = \mathrm{Tr} \begin{pmatrix} \Delta x^2 & \sigma_{xp} \\ \sigma_{px} & \Delta p^2 \end{pmatrix} = \mathrm{Tr} \begin{pmatrix} \frac{e^{-2r}}{2} & \sigma_{xp} \\ \sigma_{px} & \frac{e^{2r}}{2} \end{pmatrix} = \frac{e^{-2r} + e^{2r}}{2} = \cosh 2r \tag{4.15}$$

Subsequently, the M-P uncertainty relation is also saturated for squeezed states.

Generalized Gaussian states

Since the M-P uncertainty relation is saturated for squeezed states, and since it is also invariant under displacement and rotation, we conclude that the generalized Gaussian states $|\alpha; \phi; z\rangle$ saturate the uncertainty relation.

$$|\alpha; \phi; z\rangle \equiv \hat{D}(\alpha) \hat{R}(\phi) \hat{S}(z) |0\rangle \tag{4.16}$$

4.3 Single-mode Fock states

When considering an arbitrary number state $|n\rangle$, the most general orthogonal state is given by :

$$|\psi^\perp\rangle = \sum_{m \neq n} \beta_m |m\rangle \quad \text{with} \quad \sum_{m \neq n} |\beta_m|^2 = 1 \tag{4.17}$$

and the M-P uncertainty relation becomes :

$$
\begin{aligned}
\Delta x^2 + \Delta p^2 &\geq 1 + 2\max_{\{\alpha_m\}} \left| \sum_{m \neq n}^{\infty} \alpha_m \langle m|\hat{a}|n\rangle \right|^2 \\
&\geq 1 + 2\max_{\{\alpha_m\}} \left| \sum_{m \neq n}^{\infty} \alpha_m \langle m|\sqrt{n}\,|n-1\rangle \right|^2 \\
&\geq 1 + 2\left| \alpha_{n-1}\sqrt{n}\,\langle n-1|n-1\rangle \right|^2 \\
&\geq 1 + 2n
\end{aligned}
\tag{4.18}
$$

where $|\alpha_{n-1}|^2 = 1$ and all the other coefficients α_m vanished in order to maximize the right side of the inequality. Looking at the variances, we obtain :

$$
\begin{aligned}
\Delta x^2 + \Delta p^2 &= \langle n|\hat{x}^2 + \hat{p}^2|n\rangle - \langle n|\hat{x}|n\rangle^2 - \langle n|\hat{p}|n\rangle^2 \\
&= \langle n|1 + 2\hat{a}^\dagger\hat{a}|n\rangle - 0 \\
&= 1 + 2n
\end{aligned}
\tag{4.19}
$$

where we used the fact that Fock states are *centered* states, i.e. that the mean values $\langle n|\hat{x}|n\rangle$ and $\langle n|\hat{p}|n\rangle$ are null. Consequently, Fock states also saturate the M-P uncertainty relation.

Squeezed Fock states

We will handle squeezed Fock states similarly to what we have done for the squeezed coherent states. Instead of squeezing the vacuum, we squeeze an arbitrary number state $|n\rangle$. Consequently, we have :

$$
|\psi\rangle = \hat{S}(z)|n\rangle \quad \text{and} \quad |\psi^{\perp}\rangle = \sum_{m \neq n}^{\infty} \alpha_m \hat{S}(z)|m\rangle \quad \text{with} \quad \sum_{m \neq n}^{\infty} |\alpha_m|^2 = 1
\tag{4.20}
$$

Simplifying the right member of the M-P uncertainty relation gives us :

$$
\begin{aligned}
& 1 + 2\max_{\{\alpha_m\}} \left| \sum_{m \neq n}^{\infty} \alpha_m \langle m|\hat{S}(z)^\dagger \hat{a} \hat{S}(z)|n\rangle \right|^2 \\
&= 1 + 2\max_{\{\alpha_m\}} \left| \sum_{m \neq n}^{\infty} \alpha_m \langle m|\hat{a}\cosh r - \hat{a}^\dagger e^{i\theta}\sinh r|n\rangle \right|^2 \\
&= 1 + 2\max_{\{\alpha_m\}} \left| \sum_{m \neq n}^{\infty} \alpha_m \langle m|\sqrt{n}\cosh r|n-1\rangle - \langle m|\sqrt{n+1}\,e^{i\theta}\sinh r|n+1\rangle \right|^2 \\
&= 1 + 2\max_{\{\alpha_m\}} \left| \alpha_{n-1}\sqrt{n}\cosh r - \alpha_{n+1}\sqrt{n+1}\,e^{i\theta}\sinh r \right|^2
\end{aligned}
\tag{4.21}
$$

In order to find the state $|\psi^{\perp}\rangle$ that maximizes the lower bound of the uncertainty relation, we must determine the coefficients α_{n-1} and α_{n+1}. For this purpose, we develop the square modulus in eq. (4.21) and use the Lagrange multipliers method [44], with the constraint that the square moduli of ψ_{n-1} and ψ_{n+1} must sum up to one. For simplicity, we rewrite the complex coefficients in their exponential form and define the angle of dephasing :

$$\begin{aligned} \psi_{n-1} &\equiv x\, e^{i\phi_x} \\ \psi_{n+1} &\equiv y\, e^{i\phi_y} \\ \phi &\equiv \phi_x - \phi_y \end{aligned} \tag{4.22}$$

with $x; y \in \mathbb{R}^+$. The square modulus in eq.(4.21) becomes thus :

$$\begin{aligned} \left| \psi_{n-1}\sqrt{n}\cosh r - \psi_{n+1}\sqrt{n+1}\, e^{i\theta} \sinh r \right|^2 &= x^2 n \cosh^2 r + y^2 (n+1) \sinh^2 r \\ &- xy\, e^{i(\phi+\theta)} \cosh r \sinh r \sqrt{n^2+n} - xy\, e^{-i(\phi+\theta)} \cosh r \sinh r \sqrt{n^2+n} \end{aligned} \tag{4.23}$$

and the associated Lagrangian L is :

$$L = x^2 n \cosh^2 r + y^2 (n+1) \sinh^2 r - xy \sinh 2r \sqrt{n^2+n} \cos(\phi+\theta) - \lambda(x^2+y^2-1) \tag{4.24}$$

where we further simplified eq. (4.23) by factoring out the exponentials in order to reveal a cosine and where we used the hyperbolic identity $\sinh 2r = 2 \sinh r \cosh r$.

We now have to solve the following system of equations :

$$\begin{cases} \dfrac{\partial L}{\partial x} = 2xn \cosh^2 r - y \sinh 2r \sqrt{n^2+n} \cos(\phi+\theta) - 2x\lambda = 0 & (4.25) \\ \dfrac{\partial L}{\partial y} = 2y(n+1) \sinh^2 r - x \sinh 2r \sqrt{n^2+n} \cos(\phi+\theta) - 2y\lambda = 0 & (4.26) \\ \dfrac{\partial L}{\partial \phi} = xy \sinh 2r \sqrt{n^2+n} \sin(\phi+\theta) = 0 & (4.27) \\ \dfrac{\partial L}{\partial \lambda} = x^2 + y^2 - 1 = 0 & (4.28) \end{cases}$$

From eq.(4.27), we deduce three possible cases to consider :

1. $x = 0$

From eq. (4.28), we get $y = 1$. The reduced system takes the form of :

$$\sinh 2r \sqrt{n^2+n} \cos(\phi+\theta) = 0 \tag{4.29}$$

$$2(n+1) \sinh^2 r - 2\lambda = 0 \tag{4.30}$$

We thus get $\phi = (k+1)\frac{\pi}{2} - \theta$ (with $k \in \mathbb{Z}$). As a result, the right member of the M-P uncertainty relation is found to be equal to :

$$1 + 2\left| \psi_{n-1}\sqrt{n}\cosh r - \psi_{n+1}\sqrt{n+1}\, e^{i\theta} \sinh r \right|^2 = 1 + 2(n+1)\sinh^2 r \tag{4.31}$$

2. $y = 0$

From eq. (4.28), we get $x = 1$. The reduced system takes the form of :

$$2n \cosh^2 r - 2\lambda = 0 \tag{4.32}$$

$$\sinh 2r \sqrt{n^2+n} \cos(\phi+\theta) = 0 \tag{4.33}$$

We thus get $\phi = (k+1)\frac{\pi}{2} - \theta$ (with $k \in \mathbb{Z}$). As a result, the right member of the M-P uncertainty relation is found to be equal to :

$$1 + 2\left| \psi_{n-1}\sqrt{n}\cosh r - \psi_{n+1}\sqrt{n+1}\, e^{i\theta} \sinh r \right|^2 = 1 + 2n\cosh^2 r \tag{4.34}$$

3. $\theta = k\pi$ (with $k \in \mathbb{Z}$)

Note that in this case, the two coefficients are in phase opposition. The reduced system takes the form of :

$$\begin{cases} 2x\, n \cosh^2 r - y \sinh 2r \sqrt{n^2+n} - 2\lambda x = 0 & (4.35) \\ 2y(n+1)\sinh^2 r - x \sinh 2r \sqrt{n^2+n} - 2\lambda y = 0 & (4.36) \\ y = \sqrt{1-x^2} & (4.37) \end{cases}$$

A lengthy calculation gives us :

$$x = \frac{\sqrt{n}\cosh r}{\sqrt{n\cosh^2 r + (n+1)\sinh^2 r}} \tag{4.38}$$

$$y = \frac{\sqrt{n+1}\sinh r}{\sqrt{n\cosh^2 r + (n+1)\sinh^2 r}} \tag{4.39}$$

This results in the following lower bound :

$$\begin{aligned} 1 + 2\left(|\alpha_{n-1}|\sqrt{n}\cosh r - |\alpha_{n+1}|\sqrt{n+1}\, e^{i\theta}\sinh r\right)^2 &= 1 + 2(n\cosh^2 r + (n+1)\sinh^2 r) \\ &= (2n+1)\cosh 2r \end{aligned} \tag{4.40}$$

where we used the hyperbolic identities $\cosh^2 r + \sinh^2 r = \cosh 2r$ and $\cosh^2 r - \sinh^2 r = 1$ to further simplify the expression. The lower bound of the M-P uncertainty relation is thus given by eq. (4.40), as it is greater than the two sub-optimal lower bounds given by eq. (4.31) and (4.34).

We now compute the left member of the M-P uncertainty relation :

$$\begin{aligned} \Delta x^2 + \Delta p^2 &= \langle n|\hat{S}^\dagger(z)(\hat{x}^2+\hat{p}^2)\hat{S}(z)|n\rangle - \langle n|\hat{S}(z)^\dagger\hat{x}\hat{S}(z)|n\rangle^2 - \langle n|\hat{S}(z)^\dagger\hat{p}\hat{S}(z)|n\rangle^2 \\ &= 1 + 2\langle n|\hat{S}^\dagger(z)(\hat{a}^\dagger\hat{a})\hat{S}(z)|n\rangle \\ &= 1 + 2\langle n|\hat{S}^\dagger(z)\hat{a}^\dagger\hat{S}(z)\hat{S}^\dagger(z)\hat{a}\hat{S}(z)|n\rangle \\ &= 1 + 2\langle n|(\hat{a}^\dagger\cosh r - \hat{a}e^{-i\theta}\sinh r)(\hat{a}\cosh r - \hat{a}^\dagger e^{i\theta}\sinh r)|n\rangle \\ &= 1 + 2\langle n|\hat{a}^\dagger\hat{a}\cosh^2 r + \hat{a}\hat{a}^\dagger\sinh^2 r|n\rangle - 0 \\ &= 1 + 2\langle n|(\hat{a}^\dagger\hat{a}\cosh^2 r + (1+\hat{a}^\dagger\hat{a})\sinh^2 r)|n\rangle \\ &= 1 + 2(n\cosh^2 r + (n+1)\sinh^2 r) \\ &= (2n+1)\cosh 2r \end{aligned} \tag{4.41}$$

where the mean values of $\hat{x}$ and $\hat{p}$ vanished because the states are centered, where $\hat{S}^\dagger(z)\hat{S}(z) = \mathbb{1}$ because the squeezing operator is a unitary transformation and where we used the fact that $[\hat{a}, \hat{a}^\dagger] = 1$. Because expression (4.40) and (4.41) are equal, we conclude that squeezed Fock states saturate the M-P uncertainty relation.

Generalized Fock states

From the above results, we conclude that rotated and displaced squeezed Fock states, which we call *generalized Fock states* $|\alpha, \phi, z; n\rangle$, form a set of states that saturate the M-P uncertainty relation.

$$|\alpha, \phi, z; n\rangle \equiv \hat{D}(\alpha)\hat{R}(\phi)\hat{S}(z)|n\rangle \tag{4.42}$$

4.4 Single-mode cat states

In this section, we examine the M-P uncertainty relation for two specific single-mode cat states : the even and the odd cat states, previously defined in Chapter 1. Let us start our calculations with the odd cat state defined by :

$$|\psi\rangle = |cat\rangle_- = \frac{1}{\sqrt{N_-}}(|\alpha\rangle - |-\alpha\rangle) \quad \text{with} \quad N_- = 2(1 - e^{-2|\alpha|^2}) \tag{4.43}$$

Intuitively, we are tempted to choose the even cat state as the orthogonal state $\psi^\perp$ [1] :

$$\psi^\perp = |cat\rangle_+ = \frac{1}{\sqrt{N_+}}(|\alpha\rangle + |-\alpha\rangle) \quad \text{with} \quad N_+ = 2(1 + e^{-2|\alpha|^2}) \tag{4.44}$$

We will later see that this state maximizes in fact the lower bound of the inequality. By noticing that :

$$\hat{a}|cat\rangle_- = \frac{1}{\sqrt{N_-}}\alpha(|\alpha\rangle + |-\alpha\rangle) = \alpha\sqrt{\frac{N_+}{N_-}}|cat\rangle_+ \tag{4.45}$$

we can easily simplify the lower bound of the M-P relation :

$$\begin{aligned} \Delta x^2 + \Delta p^2 &\geq 1 + 2\left|{}_+\langle cat|\alpha\sqrt{\frac{N_+}{N_-}}|cat\rangle_+\right|^2 \\ &\geq 1 + 2\frac{N_+}{N_-}|\alpha|^2 \end{aligned} \tag{4.46}$$

This inequality is indeed saturated for odd cat states, since we have :

$$\begin{aligned} \Delta x^2 + \Delta p^2 &= {}_-\langle cat|1 + 2\hat{a}^\dagger\hat{a}|cat\rangle_- - 0 \\ &= 1 + 2\,{}_-\langle cat|\hat{a}^\dagger\hat{a}|cat\rangle_- \\ &= 1 + 2\,{}_+\langle cat|\alpha^*\sqrt{\frac{N_+}{N_-}}\sqrt{\frac{N_+}{N_-}}\alpha|cat\rangle_+ \\ &= 1 + 2\frac{N_+}{N_-}|\alpha|^2 \end{aligned} \tag{4.47}$$

where we used the fact that the mean value of $\hat{x}$ and $\hat{p}$ for cat states are null, as illustrated by Fig. 1.6.

Following the same reasoning for even cat states, we obtain the following uncertainty relation :

$$\Delta x^2 + \Delta p^2 \geq 1 + 2\frac{N_-}{N_+}|\alpha|^2 \tag{4.48}$$

which is also saturated, meaning that two-headed cat states form a set of states that saturate the M-P uncertainty relation.

[1] The proof of the orthogonality between the odd and the even cat state was already given by eq. (1.82).

4.5 M-P relation and nonclassicality

In the previous sections, we have analyzed the M-P uncertainty relation and showed that it was saturated for all single-mode Gaussian states as well as for all single-mode generalized Fock states and even for Schrödinger cat states. This was achieved by selecting an adequate state $|\psi^\perp\rangle$ that maximizes the lower bound of the inequality. Another way to look at this uncertainty relation is by expressing the variances of $\hat{x}$ and $\hat{p}$ in terms of the ladder operator and by defining the operator $\hat{\Lambda} = \hat{a}|\psi\rangle\langle\psi|\hat{a}^\dagger$:

$$
\begin{aligned}
& \Delta x^2 + \Delta p^2 \geq 1 + 2 \max_{\{|\psi^\perp\rangle\}} |\langle\psi^\perp|\hat{a}|\psi\rangle|^2 \\
\Leftrightarrow\ & \langle\psi|1+2\hat{a}^\dagger\hat{a}|\psi\rangle - \frac{1}{2}\langle\psi|\hat{a}+\hat{a}^\dagger|\psi\rangle^2 + \frac{1}{2}\langle\psi|\hat{a}-\hat{a}^\dagger|\psi\rangle^2 \geq 1 + 2 \max_{\{|\psi^\perp\rangle\}} |\langle\psi^\perp|\hat{a}|\psi\rangle|^2 \\
\Leftrightarrow\ & 2\langle\psi|\hat{a}^\dagger\hat{a}|\psi\rangle - 2\langle\psi|\hat{a}^\dagger|\psi\rangle\langle\psi|\hat{a}|\psi\rangle \geq 2 \max_{\{|\psi^\perp\rangle\}} |\langle\psi^\perp|\hat{a}|\psi\rangle|^2 \\
\Leftrightarrow\ & \langle\hat{a}^\dagger\hat{a}\rangle - \langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle \geq \max_{\{|\psi^\perp\rangle\}} \langle\psi^\perp|\hat{\Lambda}|\psi^\perp\rangle
\end{aligned}
\tag{4.49}
$$

In this notation, the left member of the inequality represents the total number of *chaotic* photons, i.e., the total number of photons minus the number of coherent photons. The right member, on the other hand, shows another approach at finding the maximal lower bound of the inequality : instead of determining the state $|\psi^\perp\rangle$ that maximizes the quantity $|\langle\psi^\perp|\hat{a}|\psi\rangle|^2$, one can focus on computing the maximal eigenvalue $\lambda \in \mathbb{R}^+$ of the restriction of the operator $\hat{\Lambda}$ to the subspace orthogonal to $|\psi\rangle$, which further simplifies the uncertainty relation into :

$$
\langle\hat{a}^\dagger\hat{a}\rangle - \langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle \geq \lambda \tag{4.50}
$$

Let us point out that the left member of the inequality corresponds to the second order determinant d_2, where the N-th order determinant d_N is defined by [45]:

$$
d_N = \begin{vmatrix}
1 & \langle\hat{a}\rangle & \langle\hat{a}^\dagger\rangle & \langle\hat{a}^2\rangle & \langle\hat{a}^\dagger\hat{a}\rangle & \dots \\
\langle\hat{a}^\dagger\rangle & \langle\hat{a}^\dagger\hat{a}\rangle & \langle\hat{a}^{\dagger 2}\rangle & \langle\hat{a}^\dagger\hat{a}^2\rangle & \langle\hat{a}^{\dagger 2}\hat{a}\rangle & \dots \\
\langle\hat{a}\rangle & \langle\hat{a}^2\rangle & \langle\hat{a}^\dagger\hat{a}\rangle & \langle\hat{a}^3\rangle & \langle\hat{a}^\dagger\hat{a}^2\rangle & \dots \\
\langle\hat{a}^{\dagger 2}\rangle & \langle\hat{a}^{\dagger 2}\hat{a}\rangle & \langle\hat{a}^{\dagger 3}\rangle & \langle\hat{a}^{\dagger 2}\hat{a}^2\rangle & \langle\hat{a}^{\dagger 3}\hat{a}\rangle & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{vmatrix} \tag{4.51}
$$

The d_N determinants play a central role in the determination of a nonclassicality criterion, as detailed by Schukin and Vogel in [45]. This criterion can be expressed by the following theorem : "A quantum state is nonclassical if and only if at least one of the determinants d_N is strictly negative."

Unfortunately, since d_2 is always non-negative, it yields no condition for nonclassicality, but we deemed that the connection between the M-P uncertainty relation and the nonclassicality criteria detailed in [45] or [46] was worth mentioning.

4.6 Towards an entropic uncertainty relation

As explained in Chapter 3, the entropic formulation of an uncertainty relation is more robust than the variance-based formulation, as the former implies the latter. Moreover, entropic uncertainty relations find many applications in quantum information science. This motivates our attempt at finding an entropic version of the additive uncertainty relation of Maccone and Pati. In this section, we first give an entropic uncertainty relation based on the sum of entropy powers. Then, we numerically check the validity of a variation of the Maccone and Pati uncertainty relation where we replace the variances by entropy powers.

4.6.1 Entropic formulation of the M-P uncertainty relation

Putting together an entropic formulation of the M-P uncertainty relation is a challenging task on its own, especially because of the second term in the right member of the inequality. In this subsection we attempt to derive an additive entropic uncertainty relation by checking, as a first step, if the following uncertainty relation holds :

$$N_x + N_p \overset{?}{\geq} 1 \tag{4.52}$$

where N_x, N_p are the entropy powers of x and p, defined by eq. (2.16). This inequality corresponds to the M-P uncertainty relation (4.4) where we discarded the square modulus in the right member and where we replaced the variances in the left member by the entropy powers. To better understand what motivated us to prove (4.52) we refer to Fig. 4.1. As illustrated, the Robertson-Schrödinger relation is stronger than the restricted M-P uncertainty relation[2] : if a state is in the red region, it means that it respects the Robertson-Schrödinger uncertainty relation, but also the M-P restricted uncertainty relation. As a consequence, if the Robertson-Schrödinger uncertainty relation holds if we replace the variances by entropy powers (as shown by (3.35)), then, relation (4.52) is also expected to hold.

Proving (4.52) is equivalent to proving :

$$\begin{aligned} &\frac{1}{2\pi e}\left(e^{2h(x)} + e^{2h(p)}\right) \overset{?}{\geq} 1 \\ \Leftrightarrow\quad &\frac{1}{2}e^{2h(x)} + \frac{1}{2}e^{2h(p)} \overset{?}{\geq} \pi e \end{aligned} \tag{4.53}$$

Taking the logarithm of both sides, we get :

$$\ln\left(\frac{1}{2}e^{2h(x)} + \frac{1}{2}e^{2h(p)}\right) \overset{?}{\geq} \ln(\pi e) \tag{4.54}$$

Using the property of the concavity of a logarithm given by eq. (2.5), we obtain :

$$\ln\left(\frac{1}{2}e^{2h(x)} + \frac{1}{2}e^{2h(p)}\right) \geq \frac{1}{2}\ln(e^{2h(x)}) + \frac{1}{2}\ln(e^{2h(p)}) = h(x) + h(p) \tag{4.55}$$

However, we know from eq. (3.33) that $h(x) + h(p) \geq \ln(\pi e)$. Thus, the entropic version of this restricted version of the M-P relation holds, but it is unfortunately not constraining as it results from the Hirschman uncertainty relation.

We now attempt to derive an entropic version of the *unrestricted* M-P uncertainty relation by conjecturing the following inequality for single-mode Fock states :

$$N_x + N_p \overset{?}{\geq} 1 + 2\max_{\{|\psi^\perp\rangle\}} |\langle\psi^\perp|\hat{a}|\psi\rangle|^2 = 1 + 2n \tag{4.56}$$

[2] By restricted M-P uncertainty relation, we mean $\sigma_x^2 + \sigma_p^2 \geq 1$.

Figure 4.1: Graphical comparison between the Robertson-Schrödinger uncertainty relation (red) and the restricted Maccone-Pati uncertainty relation (blue).

Note that the lower bound was calculated in (4.19). We numerically compute[3] the sum of the entropy powers N_x and N_p and then we compare it to the lower bound of the above inequality. Unfortunately, as shown by Fig. 4.2, the inequality is violated since we have $N_x + N_p$ $1 + 2n$. Let us stress that eq. (4.56) was a naive conjecture based on the correspondence with variance-based uncertainty relations and had no deep justi cation, but we still list it in this report in order to prevent the reader from losing time on making the calculations and to satisfy his/her curiosity.

4.7 Conclusion

In this chapter, we have derived an additive uncertainty relation for the quadratures of light $\hat{x}$ and $\hat{p}$ from the M-P uncertainty relation. We have proved that when considering these observables, the relation was invariant under displacement and rotation in phase space. We showed that the inequality was saturated for displaced and rotated squeezed Gaussian states but also for some non-Gaussian states, including displaced and rotated squeezed Fock states as well as for cat states. Then, we expressed the uncertainty relation in such a way that it exhibits a second order determinant of a particular matrix involved in the determination of a nonclassicality criterion. Finally, we attempted to establish an additive entropic uncertainty relation, but our e orts were proved to be rather unsuccessful.

[3] See Appendix A

Figure 4.2: The green line represents the minimal uncertainty for Fock states with respect to the mean number of photons $\langle n \rangle$, while the blue dots correspond to the sum of the entropy powers N_x and N_p for a Fock state $|n\rangle$.

Chapter 5

Additive uncertainty relation for two-mode quadratures

The aim of this chapter is to conduct the same analysis of the M-P uncertainty relation that was done in the previous chapter, only this time, we consider two-mode EPR-like operators. We hope that our calculations will serve as a stepping stone to the formulation of a new separability criterion, i.e., a criterion that will allow us to distinguish a separable state from an entangled state.

5.1 M-P relation for two-mode position and momentum operators

In order to get a two-mode additive uncertainty relation, we take the M-P uncertainty relation that was studied in Chapter 4 :

$$\Delta A^2 + \Delta B^2 \geq \pm i\langle[\hat{A},\hat{B}]\rangle + |\langle\psi|\hat{A} \pm i\hat{B}|\psi^{\perp}\rangle|^2 \tag{5.1}$$

and we substitute $\hat{A}$ and $\hat{B}$ by two-mode position and momentum operators. Following the notation of Duan *et al.* [47], we define the EPR-like operators :

$$\hat{u} = |a|\hat{x}_1 + \frac{1}{a}\hat{x}_2 \tag{5.2}$$

$$\hat{v} = |a|\hat{p}_1 - \frac{1}{a}\hat{p}_2 \tag{5.3}$$

where a is a non-zero real number. Let us point out that in the particular case where $a = 1$, these EPR-like operators become the two-mode position and momentum operators[1] (x_+ and p_-) that were defined in section 1.5.

We now proceed to simplify the right member of (5.1) by first computing the following commutator :

$$\begin{aligned}[\hat{u},\hat{v}] &= \left[|a|\hat{x}_1 + \frac{1}{a}\hat{x}_2, |a|\hat{p}_1 - \frac{1}{a}\hat{p}_2\right] \\ &= i|a|^2 - i\frac{1}{a^2} = i\left(a^2 - \frac{1}{a^2}\right)\end{aligned} \tag{5.4}$$

[1] Ignoring a multiplicative constant $\frac{1}{\sqrt{2}}$.

where we used the canonical commutation relation $[\hat{x}_i; \hat{p}_j] = i_{\ ij}$ and the following commutator properties :

$$\begin{aligned} [\hat{A}; \hat{B} + \hat{C}] &= [\hat{A}; \hat{B}] + [\hat{A}; \hat{C}] \\ [\ \hat{A}; \hat{B}] &= \ [\hat{A}; \hat{B}] \end{aligned} \tag{5.5}$$

where is a scalar. Consequently, since the sign in eq. (5.1) is selected so that the commutator becomes a positive quantity, we obtain 2 di erent uncertainty relations, depending on the value of a. For $|a| > 1$ and $|a| < 1$, we get (respectively) :

$$u^2 + \quad v^2 \quad a^2 \quad \frac{1}{a^2} + 2 \max_{\{|\ ^? \rangle\}} \quad {}^? \ |a|\hat{a}_1 + \frac{1}{a}\hat{a}_2^{\dagger} | \ \rangle^{2} \tag{5.6}$$

$$u^2 + \quad v^2 \quad \frac{1}{a^2} \quad a^2 + 2 \max_{\{|\ ^? \rangle\}} \quad {}^? \ |a|\hat{a}_1^{\dagger} + \frac{1}{a}\hat{a}_2 | \ \rangle^{2} \tag{5.7}$$

Note however, it can be shown that these two inequalities provide the same lower bounds. In the rest of this chapter, we will focus on the inequality (5.6).

5.1.1 Invariances of the inequality

In the previous chapter, we proved that the M-P uncertainty relation for the single-mode quadratures of light was invariant under displacement and rotation in phase-space. We now verify if it is still the case for the two-mode inequality. Similarly to what has been done in eq. (4.6) and using the two-mode displacement operator de ned by eq. (1.73), we nd for the right member of (5.6) :

$$\begin{aligned} &\frac{1}{a^2} \quad a^2 + \quad {}^? \ \hat{D}^{\dagger}(\ _2)\hat{D}^{\dagger}(\ _1) \ |a|\hat{a}_1 + \frac{1}{a}\hat{a}_2^{\dagger} \ \hat{D}(\ _1)\hat{D}(\ _2) | \ \rangle^{2} \\ =&\frac{1}{a^2} \quad a^2 + \quad {}^? \quad |a|\hat{a}_1 + |a| \ _1 + \frac{1}{a}\hat{a}_2^{\dagger} + \frac{1}{a} \ _2 \quad | \ \rangle^{2} \\ =&\frac{1}{a^2} \quad a^2 + \quad {}^? \ |a|\hat{a}_1 + \frac{1}{a}\hat{a}_2^{\dagger} | \ \rangle^{2} \end{aligned} \tag{5.8}$$

which proves that the uncertainty relation (5.6) is invariant under displacement in phase space, since variances are invariant under translation. Nevertheless, the relation is not invariant under rotation anymore, as shown by the below equation :

$$\begin{aligned} &\frac{1}{a^2} \quad a^2 + \quad {}^? \ \hat{R}^{\dagger}(\ _2)\hat{R}^{\dagger}(\ _1) \ |a|\hat{a}_1 + \frac{1}{a}\hat{a}_2^{\dagger} \ \hat{R}(\ _1)\hat{R}(\ _2) | \ \rangle^{2} \\ =&\frac{1}{a^2} \quad a^2 + \quad {}^? \quad e^{\ i \ _1}|a|\hat{a}_1 + e^{i \ _2}\frac{1}{a}\hat{a}_2^{\dagger} \quad | \ \rangle^{2} \end{aligned} \tag{5.9}$$

where we used the de nition of the two-mode rotation operator given by eq. (1.80). Note, however, that in the special case where $_1 = \ _2$, eq. (5.9) remains unchanged. The same applies for the left member of the inequality, implying thus that the relation is invariant under restricted rotation.

5.2 Two-mode Gaussian states

In this section, we compute the lower bound for the two-mode M-P uncertainty relation and show that the relation is saturated by displaced two-mode squeezed vacuum states.

Two-mode vacuum state

We start with the two-mode vacuum state, which is a tensor product of two single-mode vacuum states :

$$|0\rangle \equiv |0\rangle_1 \otimes |0\rangle_2 = |0\rangle_1|0\rangle_2 \tag{5.10}$$

where each subscript indicates the index of the mode. To compute the lower bound of the uncertainty relation, we need to simplify the following term :

$$\begin{aligned}\langle\psi^\perp|\left(|a|\hat{a}_1 + \frac{1}{a}\hat{a}_2^\dagger\right)|0;0\rangle &= 0 + \langle\psi^\perp|\frac{\hat{a}_2^\dagger}{a}|0;0\rangle \\ &= \frac{1}{a}\langle\psi^\perp|0;1\rangle\end{aligned} \tag{5.11}$$

From this expression, we conclude that the orthogonal state that maximizes the lower bound of the M-P inequality is $|\psi^\perp\rangle = |0;1\rangle$. By injecting this result in (5.6), we get :

$$\Delta u^2 + \Delta v^2 \geq \left|a^2 - \frac{1}{a^2}\right| + 2\frac{1}{|a|^2} = a^2 + \frac{1}{a^2} \tag{5.12}$$

Let us remark that this relation is very similar to the separability criterion established by Duan et al. in [47], which states that if a two-mode state is separable, then its EPR variance Δ_{EPR} obeys the following inequality :

$$\Delta_{EPR} \equiv \langle(\Delta u)^2\rangle + \langle(\Delta v)^2\rangle \geq a^2 + \frac{1}{a^2} \tag{5.13}$$

where $\langle(\Delta u)^2\rangle$ corresponds to the usual variance $\langle u^2\rangle - \langle u\rangle^2$. This sufficient[2] criterion for separability implies thus that if a state violates the inequality (5.13), for any value of a, it is entangled. The resemblance between (5.12) and (5.13) further motivates the work in this Chapter and we have high hopes that the M-P uncertainty relation can lead to a new, improved separability criterion. We want to stress however that, although the inequalities are similar, they do not express the same thing. The M-P inequality is a physicality criterion : if a state violates the inequality, it is not physical and can never be observed, whereas the separability criterion only gives us a bound that allows us to determine whether a state is entangled or not.

We now directly compute the variances of the EPR-like operators for the vacuum state:

$$\begin{aligned}&\Delta u^2 + \Delta v^2 \\ &= \langle 0;0|\hat{u}^2 + \hat{v}^2|0;0\rangle - \langle 0;0|\hat{u}|0;0\rangle^2 - \langle 0;0|\hat{v}|0;0\rangle^2 \\ &= \langle 0;0|\hat{u}^2 + \hat{v}^2|0;0\rangle - 0 - 0 \\ &= \langle 0;0|a^2\hat{x}_1^2 + \frac{1}{a^2}\hat{x}_2^2 + 2\,\mathrm{sign}(a)\hat{x}_1\hat{x}_2|0;0\rangle + \langle 0;0|a^2\hat{p}_1^2 + \frac{1}{a^2}\hat{p}_2^2 - 2\,\mathrm{sign}(a)\hat{p}_1\hat{p}_2|0;0\rangle \\ &= a^2 + \frac{1}{a^2} + 2a^2\langle 0;0|\hat{a}_1^\dagger\hat{a}_1|0;0\rangle + \frac{2}{a^2}\langle 0;0|\hat{a}_2^\dagger\hat{a}_2|0;0\rangle + 2\,\mathrm{sign}(a)\langle 0;0|\hat{a}_1\hat{a}_2 + \hat{a}_1^\dagger\hat{a}_2^\dagger|0;0\rangle \\ &= a^2 + \frac{1}{a^2} + 0\end{aligned} \tag{5.14}$$

where the mean values $\langle 0;0|\hat{u}|0;0\rangle$ and $\langle 0;0|\hat{v}|0;0\rangle$ vanished because the state is centered and where, again, we used (1.27) and (1.28) to express $\hat{x}_i$ and $\hat{p}_i$ in terms of the ladder operators. As we expected, the M-P uncertainty relation is also saturated by the two-mode vacuum state.

[2]For two-mode Gaussian states only.

Two-mode coherent states

Since the uncertainty relation is invariant under displacement in phase space, and since it is saturated for the two-mode vacuum state, it is also saturated for two-mode coherent states.

Two-mode squeezed states

We continue this section by computing the maximal lower bound of the M-P uncertainty relation for two-mode squeezed states. Those states are created by applying the two-mode squeezing operator de ned by eq. (1.74) on the two-mode vacuum state :

$$|\ \rangle = \hat{S}_{TM}(z)\,|0;0\rangle \tag{5.15}$$

Accordingly, the orthogonal state $^{\perp}$ is given by :

$$^{\perp} = \sum_{n_1;n_2\ 1}^{\infty} {}_{n_1n_2}\hat{S}_{TM}(z)\,|n_1;n_2\rangle \quad \text{with} \quad \sum_{n_1;n_2\ 1}^{\infty} |{}_{n_1n_2}|^2 = 1 \tag{5.16}$$

Substituting these two expressions in the right member of (5.6), we nd :

$$\begin{aligned} & a^2 \quad \frac{1}{a^2} + 2 \max_{\{|\ ^{\perp}\rangle\}} \quad ^{\perp} |\,|a|\hat{a}_1 + \frac{1}{a}\hat{a}_2^{\dagger}\,|\ \rangle^2 \\ &= a^2 \quad \frac{1}{a^2} \\ &+ 2 \max_{\{|\ ^{\perp}\rangle\}} \sum_{n_1;n_2\ 1}^{\infty} {}_{n_1n_2} \quad |a|\,\langle n_1;n_2|\,\hat{S}^{\dagger}_{TM}(z)\hat{a}_1\hat{S}_{TM}(z)\,|0;0\rangle + \frac{1}{a}\langle n_1;n_2|\,\hat{S}^{\dagger}_{TM}(z)\hat{a}_2^{\dagger}\hat{S}_{TM}(z)\,|0;0\rangle^2 \end{aligned} \tag{5.17}$$

To further simplify the expression, we remember the action of the two-mode squeezing operator on the ladder operator which was given by eq. (1.75) :

$$\hat{S}^{\dagger}_{TM}(z)\hat{a}_1\hat{S}_{TM}(z) = \hat{a}_1\cosh r + \hat{a}_2^{\dagger}e^{i}\ \sinh r \tag{5.18}$$

$$\hat{S}^{\dagger}_{TM}(z)\hat{a}_2^{\dagger}\hat{S}_{TM}(z) = \hat{a}_2^{\dagger}\cosh r + \hat{a}_1 e^{\ i}\ \sinh r \tag{5.19}$$

Eq. (5.17) therefore reduces to (with the optimal state $^{\perp} = \hat{S}^{\dagger}_{TM}(z)\,|0;1\rangle$) :

$$\begin{aligned} & a^2 \quad \frac{1}{a^2} + \quad \langle 0;1| \quad \frac{1}{a}\cosh r + |a|e^{i}\ \sinh r \quad |0;1\rangle^2 \\ &= a^2 \quad \frac{1}{a^2} + |a|^2\sinh^2 r + \frac{1}{a^2}\cosh^2 r + 2\,\mathrm{sign}(a)\cosh r\,\sinh r\,\cos \\ &= a^2 \quad \frac{1}{a^2} + 2 \quad a^2\sinh^2 r + \frac{1}{a^2}(\sinh^2 r + 1) + 2\ \mathrm{sign}(a)\cosh r\,\sinh r\,\cos \\ &= \quad a^2 + \frac{1}{a^2} \quad (1 + 2\sinh^2 r) + 2\ \mathrm{sign}(a)\cosh r\,\sinh r\,\cos \end{aligned} \tag{5.20}$$

Let us point out that in the particular case where $a = 1$, $=$ and where the squeezing is in nite ($r \to 1$), this lower bound tends towards zero, which was to be expected because in the limit of an in nite squeezing, the EPR state given by (5.15) represents a maximally entangled state : we observe a perfect correlation between the two position operators and a perfect anti-correlation between the two momentum operators.

Of course, we obtain the same result by directly computing the variances of $\hat{u}$ and $\hat{v}$:

$$
\begin{aligned}
\Delta^2 \hat{u} + \Delta^2 \hat{v} &= a^2 + \frac{1}{a^2} + 2a^2 \langle 0;0|\, \hat{S}^\dagger_{TM}(z)\hat{a}_1^\dagger \hat{S}_{TM}(z)\hat{S}^\dagger_{TM}(z)\hat{a}_1 \hat{S}_{TM}(z)\,|0;0\rangle \\
&\quad + \frac{2}{a^2} \langle 0;0|\, \hat{S}^\dagger_{TM}(z)\hat{a}_2^\dagger \hat{S}_{TM}(z)\hat{S}^\dagger_{TM}(z)\hat{a}_2 \hat{S}_{TM}(z)\,|0;0\rangle \\
&\quad + 2\,\mathrm{sign}(a) \langle 0;0|\, \hat{S}^\dagger_{TM}(z)\hat{a}_1 \hat{S}_{TM}(z)\hat{S}^\dagger_{TM}(z)\hat{a}_2 \hat{S}_{TM}(z)\,|0;0\rangle \\
&\quad + 2\,\mathrm{sign}(a) \langle 0;0|\, \hat{S}^\dagger_{TM}(z)\hat{a}_1^\dagger \hat{S}_{TM}(z)\hat{S}^\dagger_{TM}(z)\hat{a}_2^\dagger \hat{S}_{TM}(z)\,|0;0\rangle \\
&= a^2 + \frac{1}{a^2} + 2a^2 \langle 0;1|\sinh^2 r\,|0;1\rangle + \frac{2}{a^2}\langle 1;0|\sinh^2 r\,|1;0\rangle \\
&\quad + 2\,\mathrm{sign}(a)\cosh r \sinh r\, e^{i\varphi} + 2\,\mathrm{sign}(a)\cosh r \sinh r\, e^{-i\varphi} \\
&= \left(a^2 + \frac{1}{a^2}\right)(1 + 2\sinh^2 r) + 2\,\mathrm{sign}(a)\cosh r \sinh r \cos\varphi
\end{aligned}
\tag{5.21}
$$

which proves that the inequality is saturated for all two-mode squeezed vacuum states (and the EPR state).

5.3 Two-mode Fock states

We now take a look at the uncertainty relation while considering two-mode Fock states. Similarly to what has been done in the previous section, we define :

$$
\begin{aligned}
|\psi\rangle &= |n_1;n_2\rangle \\
|\psi^\perp\rangle &= \sum_{m_1;m_2 \neq n_1;n_2}^{\infty} c_{m_1 m_2} |m_1;m_2\rangle \quad \text{with} \quad \sum_{m_1;m_2 \neq n_1;n_2}^{\infty} |c_{m_1 m_2}|^2 = 1
\end{aligned}
\tag{5.22}
$$

The square modulus in the right member of (5.6) becomes thus :

$$
\begin{aligned}
&2 \max_{\{|\psi^\perp\rangle\}} \left| \langle \psi^\perp | \, |a|\hat{a}_1 + \frac{1}{a}\hat{a}_2^\dagger \, |\psi\rangle \right|^2 \\
&= 2 \max_{\{c_{m_1 m_2}\}} \left| \sum_{m_1;m_2 \neq n_1;n_2}^{\infty} c_{m_1 m_2} \langle m_1;m_2| \, |a|\hat{a}_1 + \frac{1}{a}\hat{a}_2^\dagger \, |n_1;n_2\rangle \right|^2 \\
&= 2 \max_{\{c_{m_1 m_2}\}} \left| c_{n_1-1;n_2} \langle n_1-1;n_2| \, |a|\sqrt{n_1}\, |n_1-1;n_2\rangle + \langle n_1;n_2+1| \, c_{n_1;n_2+1} \frac{1}{a} |n_1;n_2+1\rangle \right|^2 \\
&= 2 \max_{\{c_{m_1 m_2}\}} \left| c_{n_1-1;n_2} |a|\sqrt{n_1} + \frac{c_{n_1;n_2+1}}{a}\sqrt{n_2+1} \right|^2
\end{aligned}
\tag{5.23}
$$

By defining :

$$
\begin{aligned}
c_{n_1-1;n_2} &\equiv x\, e^{i\varphi_x} \\
c_{n_1;n_2+1} &\equiv y\, e^{i\varphi_y} \\
&\varphi_x \quad \varphi_y
\end{aligned}
\tag{5.24}
$$

with $x, y \in \mathbb{R}^+$, we can develop the square modulus and, just like we did in Chapter 4 for single-mode Fock states, use the Lagrange multipliers method to find the coefficients that maximize the lower bound of the inequality.

The Lagrangian $\mathcal{L}$ is given by :

$$\mathcal{L} = a^2x^2n_1 + \frac{y^2}{a^2}(n_2+1) + 2\,xy\sqrt{n_1(n_2+1)}\,\mathrm{sign}(a)\cos\varphi - \lambda(x^2+y^2-1) \tag{5.25}$$

and the system we have to solve is :

$$\begin{cases} \frac{\partial\mathcal{L}}{\partial x} = 2a^2xn_1 + 2y\sqrt{n_1(n_2+1)}\,\mathrm{sign}(a)\cos\varphi = 0 & (5.26)\\ \frac{\partial\mathcal{L}}{\partial y} = \frac{y^2}{a^2}(n_2+1) + 2\,x\sqrt{n_1(n_2+1)}\,\mathrm{sign}(a)\cos\varphi = 0 & (5.27)\\ \frac{\partial\mathcal{L}}{\partial\varphi} = -2xy\sqrt{n_1(n_2+1)}\,\mathrm{sign}(a)\sin\varphi = 0 & (5.28)\\ \frac{\partial\mathcal{L}}{\partial\lambda} = x^2+y^2-1 = 0 & (5.29) \end{cases}$$

Remark that, similarly to the calculations for the single-mode squeezed Fock states, we are faced with three different cases :

1. $x = 0\,;\, y = 1$
2. $x = 1\,;\, y = 0$
3. $\varphi = k\pi$ (with $k \in \mathbb{Z}$)

We find that (5.23) is maximized when we consider $\varphi = 0$ and the coefficients are therefore given by :

$$\begin{aligned} x &= \frac{a^2\sqrt{n_1}}{\sqrt{a^4n_1 + n_2 + 1}}\\ y &= \frac{\sqrt{n_2+1}}{\sqrt{a^4n_1 + n_2 + 1}} \end{aligned} \tag{5.30}$$

Consequently, the lower bound of the inequality becomes :

$$\begin{aligned} a^2\left(\frac{1}{a^2} + 2\max_{\{\lambda_{m_1m_2}\}}\left(\lambda_{n_1-1;n_2}|a|\sqrt{n_1} + \frac{\lambda_{n_1;n_2+1}}{a}\sqrt{n_2+1}\right)^2\right) &= a^2\left(\frac{1}{a^2} + 2(a^2n_1 + \frac{n_2}{a} + \frac{1}{a^2})\right)\\ &= a^2(1+2n_1) + \frac{1}{a^2}(1+2n_2) \end{aligned} \tag{5.31}$$

To see if the inequality is saturated, we directly compute the variances :

$$\begin{aligned} \Delta u^2 + \Delta v^2 = {} & a^2 + \frac{1}{a^2} + 2a^2\langle n_1;n_2|\hat{a}_1^\dagger\hat{a}_1|n_1;n_2\rangle + \frac{2}{a^2}\langle n_1;n_2|\hat{a}_2^\dagger\hat{a}_2|n_1;n_2\rangle\\ & + 2\,\mathrm{sign}(a)\langle n_1;n_2|\hat{a}_1\hat{a}_2 + \hat{a}_1^\dagger\hat{a}_2^\dagger|n_1;n_2\rangle\\ = {} & a^2 + \frac{1}{a^2} + 2a^2n_1 + \frac{2}{a^2}n_2 = a^2(1+2n_1) + \frac{1}{a^2}(1+2n_2) \end{aligned} \tag{5.32}$$

Therefore, we conclude that two-mode Fock states, as well as two-mode displaced Fock states (because the uncertainty relation is invariant under displacement) saturate the uncertainty relation.

Two-mode squeezed Fock states

Let us now examine the two-mode M-P uncertainty relation for two-mode squeezed Fock states. Those states are created after applying the two-mode squeezing operator to an arbitrary two-mode Fock state :

$$|\psi\rangle = \hat{S}_{TM}(z)\,|n_1;n_2\rangle \tag{5.33}$$

Accordingly, the orthogonal state $|\psi^\perp\rangle$ is given by :

$$|\psi^\perp\rangle = \sum_{m_1;m_2\neq n_1;n_2}^{\infty} c_{m_1m_2}\hat{S}_{TM}(z)\,|m_1;m_2\rangle \quad \text{with} \quad \sum_{m_1;m_2\neq n_1;n_2}^{\infty} |c_{m_1m_2}|^2 = 1 \tag{5.34}$$

By injecting these states in the right member of (5.6), we obtain :

$$\begin{aligned} &a^2\left(\frac{1}{a^2} + 2\max_{\{|\psi^\perp\rangle\}}\left|\langle\psi^\perp|\,|a|\hat{a}_1 + \frac{1}{a}\hat{a}_2^\dagger\,|\psi\rangle\right|^2\right)\\ &= a^2\left(\frac{1}{a^2} + 2\max_{\{|\psi^\perp\rangle\}}\left|\sum_{m_1;m_2\neq n_1;n_2}^{\infty} c^*_{m_1m_2}\langle m_1;m_2|\,\hat{S}^\dagger_{TM}(z)(|a|\hat{a}_1 + \frac{1}{a}\hat{a}_2^\dagger)\hat{S}_{TM}(z)\,|n_1;n_2\rangle\right|^2\right) \end{aligned} \tag{5.35}$$

Using (5.18) and (5.19), the expression breaks down to :

$$\begin{aligned} &a^2\Bigg(\frac{1}{a^2} + 2\max_{\{|\psi^\perp\rangle\}}\Bigg|\sum_{m_1;m_2\neq n_1;n_2}^{\infty} c^*_{m_1m_2}\Big(\langle m_1;m_2|\,\hat{a}_1\left(|a|\cosh r + \frac{e^{-i\theta}}{a}\sinh r\right)|n_1;n_2\rangle\\ &\qquad + \langle m_1;m_2|\,\hat{a}_2^\dagger\left(|a|e^{i\theta}\sinh r + \frac{1}{a}\cosh r\right)|n_1;n_2\rangle\Big)\Bigg|^2\Bigg)\\ &= a^2\Bigg(\frac{1}{a^2} + 2\max_{\{|\psi^\perp\rangle\}}\Bigg|c^*_{n_1-1;n_2}\sqrt{n_1}\left(|a|\cosh r + \frac{e^{-i\theta}}{a}\sinh r\right)\\ &\qquad + c^*_{n_1;n_2+1}\sqrt{n_2+1}\left(|a|e^{i\theta}\sinh r + \frac{1}{a}\cosh r\right)\Bigg|^2\Bigg) \end{aligned} \tag{5.36}$$

Although the expression is more complicated than previously, the methodology is the same as before : we use the Lagrange multipliers method in order to find the coefficients that maximize the lower bound of the M-P uncertainty relation. We spare the reader from the painful calculations and skip straight to the results. The M-P uncertainty relations for two-mode squeezed Fock states reads :

$$\begin{aligned} \Delta u^2 + \Delta v^2 \geq{}& a^2\left(1 + 2(n_1\cosh^2 r + n_2\sinh^2 r)\right) + \frac{1}{a^2}\left(1 + 2(n_2\cosh^2 r + n_1\sinh^2 r)\right)\\ &+ 4\,\mathrm{sign}(a)\,(n_1\sinh r\cosh r + n_2\sinh r\cosh r)\cos(\theta) \end{aligned} \tag{5.37}$$

Directly computing the variances gives us :

$$\begin{aligned}(\Delta u)^2 + (\Delta v)^2 = a^2 + \frac{1}{a^2} &+ 2a^2 \langle n_1; n_2| \hat{S}^{\dagger}_{TM}(z)\hat{a}_1^{\dagger}\hat{S}_{TM}(z)\hat{S}^{\dagger}_{TM}(z)\hat{a}_1\hat{S}_{TM}(z)|n_1; n_2\rangle \\ &+ \frac{2}{a^2} \langle n_1; n_2| \hat{S}^{\dagger}_{TM}(z)\hat{a}_2^{\dagger}\hat{S}_{TM}(z)\hat{S}^{\dagger}_{TM}(z)\hat{a}_2\hat{S}_{TM}(z)|n_1; n_2\rangle \\ &+ 2\,\mathrm{sign}(a) \langle n_1; n_2| \hat{S}^{\dagger}_{TM}(z)\hat{a}_1\hat{S}_{TM}(z)\hat{S}^{\dagger}_{TM}(z)\hat{a}_2\hat{S}_{TM}(z)|n_1; n_2\rangle \\ &+ 2\,\mathrm{sign}(a) \langle n_1; n_2| \hat{S}^{\dagger}_{TM}(z)\hat{a}_1^{\dagger}\hat{S}_{TM}(z)\hat{S}^{\dagger}_{TM}(z)\hat{a}_2^{\dagger}\hat{S}_{TM}(z)|n_1; n_2\rangle \\ = a^2 + \frac{1}{a^2} &+ 2a^2(n_1\cosh^2 r + n_2\sinh^2 r) + \frac{2}{a^2}(n_2\cosh^2 r + n_1\sinh^2 r) \\ &+ 4\,\mathrm{sign}(a)(n_1\sinh r\cosh r + n_2\sinh r\cosh r)\cos(\theta)\end{aligned} \tag{5.38}$$

which is equivalent to (5.37). Therefore, we have shown that two-mode squeezed Fock states saturate the uncertainty relation.

5.4 Two-mode cat states

We finish this chapter by considering the product of 2 single-mode cat states :

$$|cat_\alpha\rangle_\pm \equiv \frac{1}{\sqrt{N_\pm}}(|\alpha\rangle \pm |-\alpha\rangle) \quad \text{with} \quad N_\pm = 2(1 \pm e^{-2|\alpha|^2}) \tag{5.39}$$

$$|cat_\beta\rangle_\pm \equiv \frac{1}{\sqrt{N_\pm}}(|\beta\rangle \pm |-\beta\rangle) \quad \text{with} \quad N_\pm = 2(1 \pm e^{-2|\beta|^2}) \tag{5.40}$$

This leads to the following set of two-mode product cat states :

$$|\psi_1\rangle = |cat_\alpha\rangle_- |cat_\beta\rangle_- \tag{5.41}$$
$$|\psi_2\rangle = |cat_\alpha\rangle_+ |cat_\beta\rangle_+ \tag{5.42}$$
$$|\psi_3\rangle = |cat_\alpha\rangle_+ |cat_\beta\rangle_- \tag{5.43}$$
$$|\psi_4\rangle = |cat_\alpha\rangle_- |cat_\beta\rangle_+ \tag{5.44}$$

with $|cat\rangle_-$, $|cat\rangle_+$, N_-, N_+ having the same expression as in section 4.4. Remark that all these states are orthogonal. We analyze the lower bound of two mode M-P uncertainty relation for $|\psi\rangle = |\psi_1\rangle$:

$$\begin{aligned}&a^2 - \frac{1}{a^2} + 2 \max_{\{|\psi^\star\rangle\}} \left|\langle\psi^\star| |a|\hat{a}_1 + \frac{1}{a}\hat{a}_2^{\dagger}|\psi\rangle\right|^2 \\ &= a^2 - \frac{1}{a^2} + 2 \max_{\{|\psi^\star\rangle\}} \left|\langle\psi^\star| |a|\hat{a}_1 |cat_\alpha\rangle_- |cat_\beta\rangle_- + \langle\psi^\star| \frac{1}{a}\hat{a}_2^{\dagger} |cat_\alpha\rangle_- |cat_\beta\rangle_-\right|^2\end{aligned} \tag{5.45}$$

From this expression, and by keeping in mind (4.45), we deduce that the state $\psi^\star$ which optimizes the lower bound is given by the complex linear combination of :

$$\psi^\star = x|\psi_3\rangle + y|\psi_4\rangle \quad \text{with} \quad |x|^2 + |y|^2 = 1 \tag{5.46}$$

Again, we are faced with an optimization problem that we solve using the Lagrange multipliers method. Doing so gives us the following lower bound :

$$
\begin{aligned}
& a^2 \quad \frac{1}{a^2} + 2 \max_{\{| \; ? \rangle\}} \quad x \; |a| \sqrt{\frac{N_+}{N}} + \frac{y}{a} \sqrt{\frac{N}{N_+}} \quad ^2 \\
& = a^2 \quad 1 + 2| \quad |^2 \frac{N_+}{N} \quad + \frac{1}{a^2} \quad 1 + 2| \quad |^2 \frac{N}{N_+}
\end{aligned}
\tag{5.47}
$$

By directly computing the variances, we obtain the same result. We can repeat this reasoning for $| \; \rangle = | \; _2\rangle ; | \; _3\rangle$ or $| \; _4\rangle$ to conclude that the set of two-mode two-headed cat states saturate the M-P uncertainty relation.

5.5 Conclusion

In this chapter, we have studied the M-P uncertainty relation for two-mode EPR-like operators. We proved that, analogously to the one-mode case, the relation is invariant under displacement. However, the relation is not generally invariant under rotation anymore, except for the case where the two rotation angles are opposite. We showed that the inequality was saturated for the following set of product states : displaced and squeezed vacuum states, two-mode displaced and squeezed Fock states and for two-mode two-headed cat states. These results are very encouraging for the formulation of a separability criterion, but much work remains to be done in order to actually derive it.

Chapter 6

Additive uncertainty relation for more than two observables

In this chapter, we aim at making connections between three different uncertainty relations, namely the Maccone-Pati uncertainty relation [3], the Kechrimparis-Weigert uncertainty relation [32] and the Song-Qiao uncertainty relation [32]. Concretely, we show that substituting the 3 observables from the Song-Qiao inequality with the rotated quadratures from the Kechrimparis-Weigert uncertainty leads to an uncertainty relation that closely resemble the Maccone-Pati uncertainty relation that we derived in Chapter 4. Inspired by this result, we conjecture, then prove a tighter additive uncertainty relation for N rotated quadratures than the one proposed by Kechrimparis and Weigert in [32].

6.1 The Song-Qiao uncertainty relation

The Song-Qiao uncertainty relation is an additive uncertainty relation that holds for any three observables $\hat A, \hat B, \hat C$:

$$\begin{aligned} \Delta A^2 + \Delta B^2 + \Delta C^2 \geq\; & \frac{1}{3}\left|\langle\psi^\perp_{ABC}|\hat A + \hat B + \hat C|\psi\rangle\right|^2 \\ & + \frac{\sqrt{3}}{3}\left|i\langle[\hat A,\hat B,\hat C]\rangle\right| \pm \frac{2}{3}\left|\langle\psi|\hat A + \hat B e^{2\pi i/3} + \hat C e^{4\pi i/3}|\psi^\perp\rangle\right|^2 \end{aligned} \tag{6.1}$$

where $|\psi^\perp\rangle$ is a state orthogonal to the state of the system $|\psi\rangle$, $|\psi^\perp_{ABC}\rangle \propto (\hat A + \hat B + \hat C - \langle\hat A + \hat B + \hat C\rangle)|\psi\rangle$ and $\langle[\hat A,\hat B,\hat C]\rangle \equiv \langle[\hat A,\hat B]\rangle + \langle[\hat B,\hat C]\rangle + \langle[\hat C,\hat A]\rangle$. The sign in the last term of (3.31) is $+$ ($-$) when $i\langle[\hat A,\hat B,\hat C]\rangle$ is positive (negative).

We already pointed out in Chapter 3, that replacing $\hat A, \hat B, \hat C$ by the three operators $\hat x, \hat p$ and $\hat r = -\hat x - \hat p$ results in a stronger uncertainty relation that the one given by Kechrimparis and Weigert in 2014 [31]. In this section, we substitute the $\hat A, \hat B, \hat C$ operators by 3 equally distributed quadratures, as expressed in [32] :

$$\begin{aligned} \hat p_0 &\equiv \hat p \\ \hat p_{\frac{2\pi}{3}} &\equiv \cos\frac{2\pi}{3}\hat p + \sin\frac{2\pi}{3}\hat x = -\frac{1}{2}\hat p + \frac{\sqrt{3}}{2}\hat x \\ \hat p_{\frac{4\pi}{3}} &\equiv \cos\frac{4\pi}{3}\hat p + \sin\frac{4\pi}{3}\hat x = -\frac{1}{2}\hat p - \frac{\sqrt{3}}{2}\hat x \end{aligned} \tag{6.2}$$

We now compute the right member of the Song-Qiao inequality, by assessing each term one by one. For the first term, we have :

$$\frac{1}{3}\left|\langle\psi_{ABC}|\hat{A}+\hat{B}+\hat{C}|\psi\rangle\right|^2=\frac{1}{3}\left|\langle\psi_{ABC}|\left(\hat{p}-\frac{1}{2}\hat{p}+\frac{\sqrt{3}}{2}-\frac{1}{2}\hat{p}-\frac{\sqrt{3}}{2}\right)|\psi\rangle\right|^2=0 \quad (6.3)$$

For the second term, we have :

$$\begin{aligned}\frac{\sqrt{3}}{3}\, i\langle[\hat{A},\hat{B},\hat{C}]\rangle &= \frac{\sqrt{3}}{3}\, i\left(\langle[\hat{p}_0,\hat{p}_{\frac{2\pi}{3}}]\rangle+\langle[\hat{p}_{\frac{2\pi}{3}},\hat{p}_{\frac{4\pi}{3}}]\rangle+\langle[\hat{p}_{\frac{4\pi}{3}},\hat{p}_0]\rangle\right)\\ &= \frac{\sqrt{3}}{3}\, i\left(-\frac{i\sqrt{3}}{2}-\frac{i\sqrt{3}}{2}-\frac{i\sqrt{3}}{2}\right)\\ &= \frac{3}{2}\end{aligned} \quad (6.4)$$

where we calculated the commutators using the commutator properties from eq. (5.5) . Finally, for the third term, we have :

$$\begin{aligned}&\frac{2}{3}\left|\langle\psi|\hat{A}+\hat{B}e^{2i\pi/3}+\hat{C}e^{4i\pi/3}|\psi\rangle\right|^2\\ =&\frac{2}{3}\left|\langle\psi|\hat{p}+\left(-\frac{1}{2}\hat{p}+\frac{\sqrt{3}}{2}\hat{x}\right)e^{2i\pi/3}+\left(-\frac{1}{2}\hat{p}-\frac{\sqrt{3}}{2}\hat{x}\right)e^{4i\pi/3}|\psi\rangle\right|^2\\ =&\frac{2}{3}\left|\langle\psi|\hat{p}\left(1+\frac{e^{i\pi/3}}{2}+\frac{e^{-i\pi/3}}{2}\right)+\hat{x}\sqrt{3}\left(\frac{e^{i\pi/3}}{2}+\frac{e^{-i\pi/3}}{2}\right)|\psi\rangle\right|^2\\ =&\frac{2}{3}\left|\langle\psi|\frac{3}{2}\hat{p}+i\frac{3}{2}\hat{x}|\psi\rangle\right|^2\\ =&\frac{3}{2}\left|\langle\psi|(\hat{p}+i\hat{x})|\psi\rangle\right|^2\\ =&3|\langle\psi|\hat{a}|\psi\rangle|^2\end{aligned} \quad (6.5)$$

Reassembling each term, we get the following uncertainty relation for our three rotated quadratures :

$$\Delta^2 p_0+\Delta^2 p_{\frac{2\pi}{3}}+\Delta^2 p_{\frac{4\pi}{3}}\geq\frac{3}{2}\left(1+2|\langle\psi|\hat{a}|\psi\rangle|^2\right) \quad (6.6)$$

We notice that the right member of the inequality strongly resembles (up to a multiplicative factor) to the right member of the one-mode M-P uncertainty relation that was thoroughly studied in Chapter 4. Moreover, if we discard the term $2|\langle\psi|\hat{a}|\psi\rangle|^2$, we get back to the Kechrimparis and Weigert additive uncertainty relation for three variables (3.30). This result highlights the importance of the Maccone and Pati uncertainty relation, as it seems to improve the Kechrimparis and Weigert uncertainty relation[1].

6.2 Uncertainty relation for N variables

Looking back at (6.6), and comparing it to (3.29), we conjecture the following uncertainty relation :

$$\sum_{j=1}^{N}\Delta^2 r_j\geq\frac{N}{2}\left(1+2|\langle\psi|\hat{a}|\psi\rangle|^2\right) \quad (6.7)$$

[1]Indeed, as mentioned in Chapter 3, the Kechrimparis and Weigert relation is saturated for the displaced vacuum state only, while the M-P uncertainty relation is saturated for a larger set of states, as shown in Chapter 4 and 5

where $\hat{r}_j$ is the rotated quadrature defined by :

$$\hat{r}_j = \cos\theta_j\,\hat{p} + \sin\theta_j\,\hat{x}; \qquad \theta_j = \frac{2\pi(j-1)}{N}; \qquad j = 1;\ldots;N : \tag{6.8}$$

which is very similar to eq. (3.26), except for the fact that we dropped the scaling factor R for simplicity.

We start the proof of our conjecture with the following equality for the sum of the variances Δr_j^2, given by Kechrimparis and Weigert in [32] :

$$\sum_{j=1}^{N} \Delta r_j^2 = |a|^2\,\Delta p^2 + |b|^2\,\Delta q^2 + 2\,a.b\,C_{pq} \tag{6.9}$$

where a, b are vectors, and a.b denotes a scalar product:

$$|a| = \sqrt{\sum_{j=1}^{N}\cos^2\theta_j}\ ;\ |b| = \sqrt{\sum_{j=1}^{N}\sin^2\theta_j}\ ;\ a.b = \sum_{j=1}^{N}\cos\theta_j\sin\theta_j \tag{6.10}$$

and where C_{pq} is the covariance :

$$C_{pq} = \frac{1}{2}\left(\langle\psi|(\hat{p}\hat{q} + \hat{q}\hat{p})|\psi\rangle\right) - \langle\psi|\hat{p}|\psi\rangle\langle\psi|\hat{q}|\psi\rangle \tag{6.11}$$

Using the following trigonometric identities (see Appendix B) :

$$\begin{aligned}
\sum_{j=1}^{N}\cos\theta_j\sin\theta_j &= \sum_{j=1}^{N}\sin 2\theta_j = 0\\
\sum_{j=1}^{N}\cos^2\theta_j &= \sum_{j=1}^{N}\frac{1+\cos 2\theta_j}{2} = \frac{N}{2}\\
\sum_{j=1}^{N}\sin^2\theta_j &= \sum_{j=1}^{N}\frac{1+\sin 2\theta_j}{2} = \frac{N}{2}
\end{aligned} \tag{6.12}$$

Consequently, we have :

$$\sum_{j=1}^{N}\Delta r_j^2 = \frac{N}{2}(\Delta p^2 + \Delta q^2) \tag{6.13}$$

And keeping in mind the Maccone Pati uncertainty relation, we can write :

$$\sum_{j=1}^{N}\Delta r_j^2 = \frac{N}{2}(\Delta p^2 + \Delta q^2) \geq \frac{N}{2}(1 + 2|\langle\psi^{\perp}|\hat{a}|\psi\rangle|^2) \tag{6.14}$$

which proves our conjecture.

6.3 Conclusion

In this chapter, we have established that when considering three rotated quadrature operators, as defined by Kechrimparis and Weigert, the uncertainty relation they derived in [32] is equivalent to that of Song and Qiao [7]. Moreover, it corresponds to the Maccone-Pati uncertainty relation, up to a multiplicative factor. Based on this result, we conjectured and proved a tighter version of the Kechrimparis and Weigert uncertainty relation.

Chapter 7

Conclusion and prospects

The aim of this work was to investigate additive uncertainty relations for the quadratures of light $\hat{x}$ and $\hat{p}$. This study was motivated by the recent success of such uncertainty relations, as they guarantee the lower bound of the inequality to be nontrivial whenever the state under consideration is an eigenstate of any of the observables, which is an improvement over the traditional product-based uncertainty relations. The common root of most of the new additive uncertainty relations is the Maccone-Pati relation [3], hence our interest in this particular uncertainty relation.

The lower bound of the M-P inequality is not easily calculated because it depends on a particular state $|\psi^{\perp}\rangle$, which is orthogonal to the state $|\psi\rangle$ we consider. This can be considered as a drawback when compared to the original Heisenberg uncertainty relation, whose lower bound is given by a fixed value of $1/4$. However, it should be noted that this difficulty to compute the lower bound is a trade-off for a stronger uncertainty relation. As mentioned by Maccone and Pati [3], it is always possible to find a state $|\psi^{\perp}\rangle$ which maximizes the lower bound and thus, makes the inequality tight for the considered state $|\psi\rangle$. Nevertheless, determining the explicit form of this "optimal" $|\psi^{\perp}\rangle$ state proves to be a nontrivial problem.

We first developed the formalism of Maccone and Pati (M-P) for the single-mode quadratures of light and explicitly computed the lower bound for a large set of standard states in quantum optics. We showed that all single-mode Gaussian states, displaced and squeezed single-mode Fock states and Schrödinger cat states saturate the uncertainty relation. The interesting feature here is that the optimal orthogonal state can be easily expressed explicitly. Keeping in mind that entropic uncertainty relations are more robust than their variance-based counterparts, we attempted to derive an entropic formulation of the M-P uncertainty relation, based on the sum of entropy powers instead of sum of variances. Although very appealing, this attempt appeared to be rather unsuccessful.

Following these first results, we developed the M-P formalism for two-mode so-called EPR (for Einstein-Podolsky-Rosen) observables. We showed that the corresponding uncertainty relation is saturated for displaced two-mode squeezed vacuum states, displaced two-mode squeezed Fock states and even for products of cat states. Although we did not provide concrete results about a separability criterion in this master thesis, we anticipate that there could be a link between the two-mode M-P uncertainty relation and a separability criterion, in view of the fact that the Duan-Simon criterion [47][48] is also expressed as a lower bound on the sum of the EPR variances. Moreover, the M-P relation seems very promising since it is saturated (with an easy choice of the orthogonal state) for a large set of non-Gaussian states.

Lastly, we established a connection between the work of Kechrimparis-Weigert [32], Song-

Qiao [6] and Maccone-Pati. We showed that when considering 3 rotated quadratures, evenly distributed in phase space, the Kechrimparis-Weigert and Song-Qiao additive uncertainty relations are, in fact, equivalent. Moreover, they also correspond to the M-P uncertainty relation, multiplied by a factor of $3/2$. This made us conjecture and prove that for N quadratures even distributed in phase space, the Kechrimparis-Weigert and Song-Qiao uncertainty relations correspond to the Maccone-Pati uncertainty relation, multiplied by $N/2$.

With this thesis as a starting point, we hope to develop a new continuous-variables separability criterion, allowing us to detect the entanglement of a large set of non-Gaussian states. This would be a really interesting result since as of today, the necessary and sufficient criteria that allow us to detect an entangled state in an infinite-dimensional Hilbert space only exist for Gaussian states. In parallel, the formulation of an entropic, more robust, version of the M-P uncertainty relation seems to be a promising, yet very challenging task.

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Appendix A

Matlab code for the simulations

In this appendix, we present the Matlab code used to display the Wigner functions from Chapter 1 and to compute the entropy powers for Fock states from Chapter 3. Note that this code is sub-optimal but since it quickly disproved our postulated inequality, we did not bother to improve it after noticing that the script was not efficiently written.

```
%This script computes the sum of entropy power Nx and Np for Fock states

n=7;%Number of photons

%Wigner function of a Fock state
Wigner = @(x,p) laguerreL(n,2 (x.^2+p.^2)). exp( x.^2 p.^2) . (( 1)^n/pi);

%Displays the Wigner function of a Fock State
%fsurf(Wigner)

%Computes the Wigner marginal functions
WignerX = @(wx)arrayfun(@(x)integral(@(p)Wigner(x,p), inf,inf), wx);
WignerP = @(wp)arrayfun(@(p)integral(@(x)Wigner(x,p), inf,inf), wp);

hXargs = @(wx) log(WignerX(wx)). WignerX(wx);
hPargs = @(wp) log(WignerP(wp)). WignerX(wp);

%Computes the differential entropies
hX =   integral(@(wx) hXargs(wx), 5,5);
hP =   integral(@(wp) hXargs(wp), 5,5);

%Computes the entropy powers
Nx = (1/(2 pi exp(1))) exp(2 hX);
Np = (1/(2 pi exp(1))) exp(2 hP);
N = Nx+Np
```