Realignment separability criterion assisted with filtration for detecting continuous-variable entanglement

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We introduce a weak form of the realignment separability criterion which is particularly suited to detect continuous-variable entanglement and is physically implementable (it requires linear optics transformations and homodyne detection). Moreover, we define a family of states, called Schmidt-symmetric states, for which the weak realignment criterion reduces to the original formulation of the realignment criterion, making it even more valuable as it is easily computable especially in higher dimensions. Then, we focus in particular on Gaussian states and introduce a filtration procedure based on noiseless amplification or attenuation, which enhances the entanglement detection sensitivity. In some specific examples, it does even better than the original realignment criterion.

I. INTRODUCTION

When it comes to mixed states, determining whether a state is entangled or not is provably a hard decision problem [1,2]. Still, it has long been and it remains an active research topic because entanglement is a key resource for quantum information processing. Both for discrete- and continuous-variable systems, various separability criteria—conditions that must be satisfied by any separable state—have been derived. Probably the best known criterion is the Peres-Horodecki criterion [3,4], also called the positive partial transpose (PPT) criterion. Introduced for discrete-variable systems, it states that if a quantum state is separable, then its partial transpose must remain physical (i.e., positive semidefinite). This PPT condition is, in general, only a necessary condition for separability. It becomes sufficient only for systems of dimensions 2×2 and 2×3 [4]. The PPT criterion was generalized to continuous variables (i.e., infinite-dimensional systems) by Duan *et al.* [5] and Simon [6]. Interestingly, it is necessary and sufficient for all $1 \times n$ Gaussian states [7] and $n \times m$ bisymmetric Gaussian states [8]. In all other cases, when a state is entangled but its partial transpose remains positive semidefinite, we call it a bound entangled state [1,9]. These are entangled states from which no pure entangled state can be distilled through local (quantum) operations and classical communications (LOCC).

Many other separability criteria have been developed over years (see, e.g., [10–14], and consult [1] for an older, but still relevant, review). Among them, we focus in the present paper on the realignment criterion [15,16]. This criterion is unrelated to the PPT criterion and thereby enables the detection of some bound entangled states in both discrete-variable [16]

and continuous-variable cases [17]. Unfortunately, the realignment criterion happens to be generally hard to compute, especially for continuous-variable systems. To our knowledge, it has only been computed for Gaussian states by Zhang *et al.* [17] and yet, the difficulty increases with the number of modes.

In this paper, we introduce a weaker form of the realignment criterion which is much simpler to compute and comes with a physical implementation in terms of linear optics and homodyne detection; hence it is especially suited to detect continuous-variable entanglement. It is, in general, less sensitive to entanglement than the original realignment criterion and cannot detect bound-entangled states, but it happens to be equivalent to the original realignment criterion for the class of Schmidt-symmetric states. Furthermore, we show that by supplementing this criterion with a filtration method, it is possible to greatly improve it and sometimes even surpass the original realignment criterion while keeping the simplicity of computation.

In Sec. II, we review the definition of the realignment criterion, focusing especially on the realignment map *R*. We link different formulations of this criterion and list its main properties. In Sec. III, we introduce the weak realignment criterion and show that for a class of states that we call Schmidt symmetric, both the weak and original realignment criteria are equivalent (while the former is much easier to compute than the latter). In Sec. IV, we apply the weak realignment criterion to continuous-variable states and give special attention to Gaussian states. The idea is to compare to the work of Zhang *et al.* [17], which relied on the original formulation of the criterion. We notice that several entangled states remain undetected by the weak realignment criterion and, unfortunately,

the latter cannot detect bound entanglement. As a solution, we introduce in Sec. V a filtration procedure that enables a better entanglement detection by bringing the state closer to a Schmidt-symmetric state, hence increasing the sensitivity of the entanglement witness. In Sec. VI, we provide some specific examples for 1×1 and 2×2 Gaussian states. In some cases, the filtration procedure supplementing the weak realignment criterion enables a better entanglement detection than the original realignment criterion. Finally, we give our conclusions in Sec. VII.

II. REALIGNMENT CRITERION AND REALIGNMENT MAP

It is well known that any bipartite pure state $|\psi\rangle_{AB}$ can be decomposed according to the Schmidt decomposition $|\psi\rangle_{AB} = \sum_i \lambda_i |i_A\rangle |i_B\rangle$, where $|i_A\rangle$ and $|i_B\rangle$ form orthonormal bases of subsystems A and B, and the λ_i's are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$ known as the Schmidt coefficients [18]. The number of nonzero coefficients is called the Schmidt rank and denoted as r. A pure state is entangled if and only if $r > 1$. Interestingly, the entanglement classes under LOCC transformations are uniquely determined by the Schmidt rank [19].

An analogous Schmidt decomposition can also be defined for mixed states [20]. Let ρ be a mixed quantum state of a bipartite system AB; then it can be written in its operator Schmidt decomposition as

$$\rho = \sum_{i=1}^{r} \lambda_i A_i \otimes B_i, \tag{1}$$

with the Schmidt coefficients λ_i being some non-negative real numbers, the Schmidt rank r satisfying $1 \leqslant r \leqslant \min\{\dim A, \dim B\}$, and with $\{A_i\}$ and $\{B_i\}$ forming orthonormal bases[1] of the operator spaces for subsystems A and B with respect to the Hilbert-Schmidt inner product, i.e., $\mathrm{Tr}(A_i^\dagger A_j) = \mathrm{Tr}(B_i^\dagger B_j) = \delta_{ij}$. The Schmidt coefficients λ_i are unique for a bipartite state ρ and reveal some of its characteristic features. For example, the purity of ρ can be expressed as $\mathrm{Tr}\,\rho^2 = \sum_{i=1}^{r} \lambda_i^2$.

Similarly as for pure states, the operator Schmidt decomposition can be employed as an entanglement criterion for mixed bipartite states; this is called the computable cross-norm criterion and is defined as follows.

Theorem 1 (Computable cross-norm criterion [15]). Let ρ be a state with the operator Schmidt decomposition $\rho = \sum_{i=1}^{r} \lambda_i A_i \otimes B_i$. If ρ is separable, then $\sum_{i=1}^{r} \lambda_i \leqslant 1$. Conversely, if $\sum_{i=1}^{r} \lambda_i > 1$, then ρ is entangled.

The proof is given in Appendix A for completeness.

There exists an alternative formulation of the computable cross-norm criterion which, as we will see, turns out to be more convenient when considering continuous-variable states. This reformulation is done by defining a linear map R called

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[1]If the operator is Hermitian (such as ρ), then the operators A_i and B_i can be chosen Hermitian too. But the Schmidt decomposition is not unique and there exist other possible Schmidt decompositions of an Hermitian operator with non-Hermitian operators A_i and B_i.

a *realignment map*, whose action on the tensor product of matrices $A = \sum_{ij} a_{ij} |i\rangle\langle j|$ and $B = \sum_{kl} b_{kl} |k\rangle\langle l|$ is

$$R(A \otimes B) = \sum_{ijkl} a_{ij} b_{kl} |i\rangle|j\rangle\langle k|\langle l|. \tag{2}$$

Since any bipartite state ρ can be decomposed into $A \otimes B$ products according to Eq. (1), one can easily express its realignment $R(\rho)$ based on definition (2). Thus, the realignment map simply interchanges the bra vector $\langle j|$ of the first subsystem with the ket vector $|k\rangle$ of the second subsystem. Note that the map R is basis dependent; namely, it depends on the basis in which the matrix elements a_{ij} and b_{kl} are expressed. When applying R to continuous-variable states in Secs. IV, V, and VI, we will always assume that $|i\rangle$, $|j\rangle$, $|k\rangle$, and $|l\rangle$ are Fock states, so that Eq. (2) must be understood in the Fock basis.

Using the state-operator correspondence implied by the Choi-Jamiołkowski isomorphism [21,22], we can identify matrices with vectors living in the tensor-product ket space, namely, $|A\rangle = \sum_{ij} a_{ij} |i\rangle|j\rangle$ and $|B\rangle = \sum_{kl} b_{kl} |k\rangle|l\rangle$. Their corresponding dual vectors are noted $\langle A| = \sum_{ij} a_{ij}^* \langle i|\langle j|$ and $\langle B| = \sum_{kl} b_{kl}^* \langle k|\langle l|$, living in the tensor-product bra space. Hence, the above map can be reexpressed as

$$R(A \otimes B) = |A\rangle\langle B^*|, \tag{3}$$

where complex conjugation is also applied in the preferred basis. Using the fact that[2]

$$|A\rangle = \sum_{ij} a_{ij} |i\rangle|j\rangle = (A \otimes \mathbb{1})|\Omega\rangle$$

and

$$\langle B^*| = \sum_{ij} b_{ij} \langle i|\langle j| = \langle \Omega|(B^T \otimes \mathbb{1}), \tag{4}$$

where $|\Omega\rangle = \sum_i |i\rangle|i\rangle$ is the (unnormalized[3]) maximally entangled state and $\mathbb{1} = \sum_i |i\rangle\langle i|$ is the identity matrix, one can also rewrite the realignment map as

$$R(A \otimes B) = (A \otimes \mathbb{1})|\Omega\rangle\langle\Omega|(B^T \otimes \mathbb{1})$$
$$= (A \otimes \mathbb{1})|\Omega\rangle\langle\Omega|(\mathbb{1} \otimes B), \tag{5}$$

which will happen to be useful when considering the optical realization of the separability criterion.

It is obvious that $R(R(\rho)) = \rho$, so that definition (3) can also be restated as

$$R(|A\rangle\langle B|) = A \otimes B^*. \tag{6}$$

Note the special cases

$$R(\mathbb{1} \otimes \mathbb{1}) = |\Omega\rangle\langle\Omega|,$$
$$R(|\Omega\rangle\langle\Omega|) = \mathbb{1} \otimes \mathbb{1}, \tag{7}$$

which are trivial consequences of $|\mathbb{1}\rangle = |\Omega\rangle$ and $\hat{\Omega} = \mathbb{1}$.

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[2]$(A \otimes \mathbb{1})|\Omega\rangle = \sum_{ij} a_{ij}(|i\rangle\langle j| \otimes \mathbb{1}) \sum_k |k\rangle|k\rangle = \sum_{ij} a_{ij}|i\rangle|j\rangle$.

[3]This definition of $|\Omega\rangle$ remains useful even for continuous-variable (infinite-dimensional) systems, where it can be interpreted as a (unnormalized) two-mode squeezed vacuum state with infinite squeezing. The definition of R given by Eq. (5) remains thus valid with $|\Omega\rangle = \sum_{i=0}^{\infty} |ii\rangle$, where $|i\rangle$ stand for Fock states.

It will also be useful in the following to define the *dual realignment map* $R^\dagger$, which is such that $\text{Tr}(\rho_1 R(\rho_2)) = \text{Tr}(R^\dagger(\rho_1) \rho_2)$. Definitions (3) and (5) translate into

$$
\begin{aligned}
R^\dagger(A \otimes B) &= |B^T\rangle\langle A^\dagger|, \\
&= (B^T \otimes \mathbb{1})|\Omega\rangle\langle\Omega|(A \otimes \mathbb{1}), \\
&= (\mathbb{1} \otimes B)|\Omega\rangle\langle\Omega|(A \otimes \mathbb{1}).
\end{aligned} \tag{8}
$$

Coming back to the question of separability, let us now state the following theorem.

Theorem 2 (Realignment criterion [16]). If the bipartite state ρ is separable, then $\| R(\rho) \|_{\text{tr}} \leqslant 1$. Conversely, if $\| R(\rho) \|_{\text{tr}} > 1$, then ρ is entangled.

Proof. From Eq. (3), the realignment of a product state is given by

$$
R(\rho_A \otimes \rho_B) = |\rho_A\rangle\langle\rho_B^*|, \tag{9}
$$

and therefore

$$
\| R(\rho_A \otimes \rho_B) \|_{\text{tr}} = \text{Tr}\sqrt{|\rho_A\rangle\langle\rho_B^*|\rho_B^*\rangle\langle\rho_A|} \leqslant 1, \tag{10}
$$

where $\| \mathcal{O} \|_{\text{tr}} = \text{Tr}(\sqrt{\mathcal{O}\mathcal{O}^\dagger})$ denotes the trace norm[4] of an operator $\mathcal{O}$ and the inequality is found using the Hilbert-Schmidt inner product, $\langle A|B\rangle = \text{Tr}(A^\dagger B)$. The convexity of the trace norm implies that $\| R(\rho) \|_{\text{tr}} \leqslant 1$, for any separable state $\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B$, with $p_i \geqslant 0$ and $\sum_i p_i = 1$. ∎

Theorem 2 is called the realignment criterion as the detection of entanglement exploits the map R. But it is interesting to note that $\| R(\rho) \|_{\text{tr}}$ coincides with the sum of the Schmidt coefficients of ρ, so the realignment criterion is actually equivalent to Theorem 1 [23,24]. Indeed, let ρ be a state with the operator Schmidt decomposition $\rho = \sum_i^r \lambda_i A_i \otimes B_i$. Then, according to Eq. (3),

$$
R(\rho) = \sum_i \lambda_i R(A_i \otimes B_i) = \sum_i \lambda_i |A_i\rangle\langle B_i^*| \tag{11}
$$

and

$$
\begin{aligned}
\| R(\rho) \|_{\text{tr}} &= \text{Tr}\left[\sqrt{\sum_{i,j} \lambda_i \lambda_j |A_i\rangle\langle B_i^*|B_j^*\rangle\langle A_j|}\right] \\
&= \text{Tr}\left[\sqrt{\sum_i \lambda_i^2 |A_i\rangle\langle A_i|}\right] \\
&= \text{Tr}\left[\sum_i |\lambda_i||A_i\rangle\langle A_i|\right] = \sum_i \lambda_i
\end{aligned} \tag{12}
$$

since $\langle A_i|A_j\rangle = \langle B_i|B_j\rangle = \delta_{ij}$. Theorem 2 is thus equivalent to Theorem 1.

As a trivial example of Theorem 2, let us consider two d-dimensional systems (with $d \geqslant 2$). The maximally mixed state $\rho = \mathbb{1} \otimes \mathbb{1}/d^2$ is mapped to $R(\rho) = |\Omega\rangle\langle\Omega|/d^2$ [see Eq. (7)], so its trace norm is $\| R(\rho) \|_{\text{tr}} = 1/d < 1$ as expected since ρ is separable. Conversely, according to Eq. (7),

the maximally entangled state $\rho = |\Omega\rangle\langle\Omega|/d$ is mapped to $R(\rho) = \mathbb{1} \otimes \mathbb{1}/d$, so that $\| R(\rho) \|_{\text{tr}} = d > 1$ and the entanglement of ρ is well detected in this case.

Finally, it is worth adding that, by inspection, definition (2) of the realignment map can be decomposed as

$$
R(A \otimes B) = ((A \otimes B^T) F)^{T_2}, \tag{13}
$$

where $(\cdot)^{T_2}$ denotes a partial transposition on the second subsystem (B), and $F = \sum_{i,j} |ij\rangle\langle ji| = |\Omega\rangle\langle\Omega|^{T_2}$ is the exchange operator [25]. From this, we obtain the following:

Remark 1. For any state ρ, the realignment map can be defined as

$$
R(\rho) = (\rho^{T_2} F)^{T_2} = (\rho F)^{T_2} F. \tag{14}
$$

In other words, the map R boils down to the concatenation of partial transposition on subsystem B, then applying the exchange operator F to the right, followed by partial transposition on subsystem B again. Conversely, the roles of F and $(\cdot)^{T_2}$ can be exchanged. This alternative definition of R allows us to express the trace norm as

$$
\| R(\rho) \|_{\text{tr}} = \| (\rho F)^{T_2} F \|_{\text{tr}} = \| (\rho F)^{T_2} \|_{\text{tr}}, \tag{15}
$$

where the last equality comes from the fact that, for any operator A, we have

$$
\begin{aligned}
\| AF \|_{\text{tr}} &= \text{Tr}\sqrt{AF(AF)^\dagger} = \text{Tr}\sqrt{AFF^\dagger A^\dagger} \\
&= \text{Tr}\sqrt{AA^\dagger} = \| A \|_{\text{tr}}
\end{aligned} \tag{16}
$$

since $FF^\dagger = FF = \mathbb{1}$. From Eq. (15), it becomes obvious that for the special case of states ρ_s belonging to the symmetric subspace, i.e., states satisfying $F\rho_s = \rho_s F = \rho_s$, the realignment criterion coincides with the PPT criterion [26]. Indeed, $\| R(\rho_s) \|_{\text{tr}} = \| \rho_s^{T_2} \|_{\text{tr}}$ and $\| \rho_s^{T_2} \|_{\text{tr}} = \sum_i |\lambda_i'| > 1$ implies that at least one eigenvalue λ_i' of the partial-transposed state $\rho_s^{T_2}$ is negative, since $\text{Tr}(\rho_s) = \text{Tr}(\rho_s^{T_2}) = \sum_i \lambda_i' = 1$ (which is the PPT criterion). Beyond the case of states in the symmetric subspace, however, the realignment and PPT criteria are generally incomparable criteria.

Of course, the dual realignment map $R^\dagger$ can also be defined similarly as in Eq. (14), namely,

$$
R^\dagger(\rho) = (F\rho^{T_2})^{T_2} = F(F\rho)^{T_2}. \tag{17}
$$

The difference with the (primal) realignment map R is that the exchange operator F is applied to the left. To be complete, let us mention that maps R and $R^\dagger$ can also be defined using partial transposition on the first subsystem denoted as $(\cdot)^{T_1}$, namely,

$$
\begin{aligned}
R(\rho) &= (F\rho^{T_1})^{T_1} = F(F\rho)^{T_1}, \\
R^\dagger(\rho) &= (\rho^{T_1} F)^{T_1} = (\rho F)^{T_1} F.
\end{aligned} \tag{18}
$$

III. WEAK REALIGNMENT CRITERION

Let us now introduce the weak realignment criterion, which is in general not as strong as the original realignment criterion but has the advantage of being easily computable and physically implementable using standard optical components. The weak realignment criterion applies to all states but our main focus in this paper will be its application to continuous-variable states (see Sec. IV). We will in particular

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[4]The trace norm of $\mathcal{O}$ is equivalent to the sum of the singular values of $\mathcal{O}$, which are given by the square roots of the eigenvalues of $\mathcal{O}\mathcal{O}^\dagger$. For an Hermitian operator, the trace norm is simply equal to the sum of the absolute values of the eigenvalues.

provide some explicit calculations in the case of $n \times n$ mode Gaussian states, in which case it boils down to computing a simple quantity that only depends on the covariance matrix of the state. As expected, however, the easiness of computation comes with the price of a lower entanglement detection sensitivity than the one of the original realignment criterion for Gaussian states as calculated in [17]. As a way to overcome this problem, we show below that, for a special type of states that we call *Schmidt symmetric*, the weak form and original form of the realignment criterion become equivalent. This suggests the use of a filtration procedure for augmenting the detection sensitivity. As explored in Sec. V, we may "symmetrize" the state by locally applying a noiseless amplifier or attenuator (it does not affect the separability of the state, so we may apply the weak realignment criterion on the filtered state).

A. Formulation

It is well known that the trace norm of an operator is greater than or equal to its trace (and we have equality if and only if the operator is positive semidefinite). Using Eq. (14), we have that for any state

$$\| R(\rho) \|_{\text{tr}} \geqslant \text{Tr} R(\rho) = \text{Tr}(\rho^{T_2} F) = \text{Tr}(\rho F^{T_2})$$
$$= \text{Tr}(\rho |\Omega\rangle\langle\Omega|) = \langle\Omega|\rho|\Omega\rangle, \qquad (19)$$

where we have used the invariance of the trace under partial transposition $(\cdot)^{T_2}$ (first equality), the identity $\text{Tr}(A B^{T_2}) = \text{Tr}(A^{T_2} B)$ for any bipartite operators A and B, and the definition of F. Note that this result can also be obtained by noticing that

$$\text{Tr}(R(\rho) \mathbb{1} \otimes \mathbb{1}) = \text{Tr}(\rho R^{\dagger}(\mathbb{1} \otimes \mathbb{1})) = \text{Tr}(\rho |\Omega\rangle\langle\Omega|). \quad (20)$$

We can thus state the following theorem:

Theorem 3 (Weak realignment criterion). For any bipartite state ρ, the trace norm of the realigned state can be lower bounded as

$$\| R(\rho) \|_{\text{tr}} \geqslant \text{Tr} R(\rho) = \langle\Omega|\rho|\Omega\rangle. \qquad (21)$$

Hence, if ρ is separable, then $\langle\Omega|\rho|\Omega\rangle \leqslant 1$. Conversely, if $\langle\Omega|\rho|\Omega\rangle > 1$, then ρ is entangled.

In other words, the weak realignment criterion amounts to computing the fidelity of state ρ with respect to $|\Omega\rangle$. It is immediate that its entanglement detection capability can only be lower than that of the original realignment criterion, Theorem 2. Furthermore, if we deal with bound-entangled states, the weak realignment criterion cannot detect entanglement. Indeed, we can link the weak realignment criterion with the PPT criterion by expressing

$$\| R(\rho)^{T_2} \|_{\text{tr}} = \| \rho^{T_2} F \|_{\text{tr}} = \| \rho^{T_2} \|_{\text{tr}}, \qquad (22)$$

where we have used Eqs. (14) and (16), combined with the inequality

$$\| R(\rho)^{T_2} \|_{\text{tr}} \geqslant \text{Tr}(R(\rho)^{T_2}) = \text{Tr} R(\rho). \qquad (23)$$

Thus,

$$\| \rho^{T_2} \|_{\text{tr}} \geqslant \text{Tr} R(\rho), \qquad (24)$$

and we deduce that the weak realignment criterion is weaker than the PPT criterion. If a state is bound entangled, we

have $\| \rho^{T_2} \|_{\text{tr}} = 1$ which then implies that $\text{Tr} R(\rho) \leqslant 1$, so its entanglement cannot be detected with the weak realignment criterion.

It is instructive to apply the weak realignment criterion on each component of the operator Schmidt decomposition of ρ. Using Eq. (5), we have

$$\text{Tr} R(A \otimes B) = \langle\Omega|A \otimes B|\Omega\rangle = \text{Tr}(AB^T), \qquad (25)$$

which implies that if $\rho = \sum_i \lambda_i A_i \otimes B_i$, then

$$\text{Tr} R(\rho) = \sum_i \lambda_i \text{Tr}(A_i B_i^T) = \sum_i \lambda_i \langle B_i^*|A_i\rangle. \qquad (26)$$

Remembering that $\| R(\rho) \|_{\text{tr}} = \sum_i \lambda_i$, it appears that we must have $B_i = A_i^*$ in order to reach a situation where $\text{Tr} R(\rho) = \| R(\rho) \|_{\text{tr}}$. This is analyzed now.

B. Schmidt-symmetric states

We now show that for Schmidt-symmetric states, the weak and original forms of the realignment criterion become equivalent (while the weak form is much simpler to compute). Let us define Schmidt-symmetric states ρ_{sch} as the states that admit an operator Schmidt decomposition with $B_i = A_i^*, \forall i$, namely,

$$\rho_{\text{sch}} = \sum_i \lambda_i A_i \otimes A_i^*. \qquad (27)$$

These states satisfy $F \rho_{\text{sch}} F = \rho_{\text{sch}}^*$ since applying F on both sides is equivalent to exchanging the two subsystems and since the Schmidt coefficients are real. Note that the converse is not true as there exist states ρ that satisfy $F \rho F = \rho^*$ but are not Schmidt symmetric, for example, the state $\rho = \sum_i \lambda_i A_i \otimes (-A_i^*)$. For any state ρ that satisfies $F \rho F = \rho^*$, it is easy to see that $R(\rho)$ is Hermitian since[5]

$$R(\rho)^{\dagger} = ((\rho F)^{T_2} F)^{\dagger} = F(\rho^* F)^{T_1}$$
$$= F(F\rho)^{T_1} = R(\rho), \qquad (28)$$

where we have used Eqs. (14) and (18). Thus, $R(\rho_{\text{sch}})$ is necessarily an Hermitian operator.

Actually, using the definition (3) of the realignment map R, it appears that $R(\rho_{\text{sch}}) = \sum_i \lambda_i |A_i\rangle\langle A_i|$ is positive semidefinite, so that $\| R(\rho_{\text{sch}}) \|_{\text{tr}} = \text{Tr} R(\rho_{\text{sch}})$. Conversely, if the latter equality is satisfied for a state ρ, it means that $R(\rho)$ is positive semidefinite so it can be written as $R(\rho) = \sum_i \lambda_i |A_i\rangle\langle A_i|$, which is nothing else but the realignment of a Schmidt-symmetric state. We have thus proven the following theorem:

Theorem 4 (Schmidt-symmetric states). A bipartite state ρ is Schmidt symmetric (i.e., it admits the operator Schmidt decomposition $\rho = \sum_i \lambda_i A_i \otimes A_i^*$) if and only if

$$\| R(\rho) \|_{\text{tr}} = \text{Tr} R(\rho). \qquad (29)$$

This entails the coincidence between the weak form of the realignment criterion derived in Theorem 3 and the original realignment criterion of Theorem 2 in the special case of Schmidt-symmetric states.

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[5]Be aware that $R(\rho)^{\dagger}$ is distinct from the dual map $R^{\dagger}(\rho)$.

Incidentally, we note that the (necessary) condition $F\rho F = \rho^*$ for a state to be Schmidt symmetric resembles the (necessary and sufficient) condition $F\rho F = \rho$ for a state to be symmetric under the exchange of the two systems. For this reason, when building a filtration procedure in order to bring the initial state closer to a Schmidt-symmetric state (see Sec. V), we will "symmetrize" the state. More precisely, we will exploit the fact that the condition $F\rho F = \rho^*$ implies that $\mathrm{Tr}_1\,\rho = \mathrm{Tr}_2\,\rho^*$. In other words, Schmidt-symmetric states are such that the reduced states of both subsystems are complex conjugate of each other, namely, $\rho_{sch,2} = \rho_{sch,1}^*$, which is also a simple consequence of

$$\rho_{sch,1} = \mathrm{Tr}_2(\rho_{\mathrm{sch}}) = \sum_i \lambda_i A_i \,\mathrm{Tr}A_i^*,$$

$$\rho_{sch,2} = \mathrm{Tr}_1(\rho_{\mathrm{sch}}) = \sum_i \lambda_i A_i^* \,\mathrm{Tr}A_i. \tag{30}$$

Hence, they have the same eigenspectrum since their eigenvalues are real, and in particular the same purity (but the converse is not true),

$$\mathrm{Tr}\big(\rho_{sch,1}^2\big) = \mathrm{Tr}\big(\rho_{sch,2}^2\big). \tag{31}$$

The filtration procedure that we apply in Sec. V follows Eq. (31) in the sense that we will "symmetrize" the initial state so that the two subsystems reach the same purity.

IV. WEAK REALIGNMENT CRITERION FOR CONTINUOUS-VARIABLE STATES

A. Preliminaries and symplectic formalism

Now, we turn to the application of the weak realignment criterion for continuous-variable states (i.e., living in an infinite-dimensional Fock space). We start by briefly introducing the symplectic formalism employed for continuous-variable states. More details can be found, for example, in [27,28].

A continuous-variable system is represented by N modes, each of them associated with a Hilbert space spanned by the Fock basis and having its own mode operators a_i and $a_i^\dagger$ which verify the commutation relation $[a_i, a_i^\dagger] = 1$. We define the quadrature vector $\mathbf{r} = (x_1, p_1, x_2, p_2, \ldots, x_N, p_N)$, where

$$x_i = \frac{1}{\sqrt{2}}(a_i + a_i^\dagger), \quad p_i = -\frac{1}{\sqrt{2}}(a_i - a_i^\dagger) \quad \forall i = 1, \ldots, N. \tag{32}$$

Each quantum state ρ can be described by a quasiprobability distribution function, the Wigner function

$$W(\mathbf{x}, \mathbf{p}) = \frac{1}{(2\pi)^N} \int d\mathbf{y}\, e^{-i\mathbf{p}\cdot\mathbf{y}} \langle \mathbf{x} + \mathbf{y}/2|\rho|\mathbf{x} - \mathbf{y}/2\rangle, \tag{33}$$

which is normalized to one.

The first-order moments constitute the displacement vector, defined as $\langle \mathbf{r} \rangle = \mathrm{Tr}(\mathbf{r}\rho)$, while the second moments make up the covariance matrix γ whose elements are given by

$$\gamma_{ij} = \tfrac{1}{2}\langle\{r_i, r_j\}\rangle - \langle r_i\rangle\langle r_j\rangle, \tag{34}$$

where $\{\cdot, \cdot\}$ represents the anticommutator.

A *Gaussian* state is fully characterized by its displacement vector and covariance matrix and its Wigner function has a Gaussian shape. Some relevant examples of Gaussian states are the following:

(i) The coherent state $|\alpha\rangle$ is a displaced vacuum state (where $\alpha = 0$), meaning that the covariance matrix is the one of the vacuum $\gamma_{|\alpha\rangle} = \gamma_{|0\rangle} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ but the first moment depends on the value of α.

(ii) The squeezed state $|r\rangle$: the uncertainty of one quadrature is minimized by squeezing it according to the squeezing parameter r; the covariance matrix is given by $\gamma_{|r\rangle} = \frac{1}{2}\begin{pmatrix} e^{-2r} & 0 \\ 0 & e^{2r} \end{pmatrix}$.

(iii) The thermal state ρ_{th} is a mixed state where the uncertainties of each quadratures are equal, but not minimals; the covariance matrix is given by $\gamma_{th} = \frac{1}{2}\begin{pmatrix} 2\langle n\rangle + 1 & 0 \\ 0 & 2\langle n\rangle + 1 \end{pmatrix}$, where $\langle n\rangle$ is the mean photon number.

(iv) The two-mode squeezed vacuum state $|\mathrm{TMSV}\rangle$ is a two-mode state with covariance matrix

$$\gamma_{\mathrm{TMSV}} = \frac{1}{2}\begin{pmatrix} \cosh 2r & 0 & \sinh 2r & 0 \\ 0 & \cosh 2r & 0 & -\sinh 2r \\ \sinh 2r & 0 & \cosh 2r & 0 \\ 0 & -\sinh 2r & 0 & \cosh 2r \end{pmatrix}. \tag{35}$$

If the squeezing r tends to infinity, one recovers the well-known Einstein-Podolsky-Rosen (EPR) state.

A Gaussian unitary transformation is a unitary transformation that preserves the Gaussian character of a quantum state. In terms of quadrature operators, a Gaussian unitary transformation is described by the map

$$\mathbf{r} \to \mathcal{S}\mathbf{r} + \mathbf{d}, \tag{36}$$

where $\mathbf{d}$ is a real vector of dimension $2N$ and $\mathcal{S}$ is a real $2N \times 2N$ matrix which is symplectic.

B. Examples of realigned states

It is instructive first to check the action of the realignment map R on some of the well-known states of quantum optics:

(a) Fock states[6]:
$R(|n_1\rangle\langle n_2| \otimes |n_3\rangle\langle n_4|) = |n_1\rangle\langle n_3| \otimes |n_2\rangle\langle n_4|$.

(b) Position states[7]:
$R(|x_1\rangle\langle x_2| \otimes |x_3\rangle\langle x_4|) = |x_1\rangle\langle x_3| \otimes |x_2\rangle\langle x_4|$.

(c) Momentum states[8]:
$R(|p_1\rangle\langle p_2| \otimes |p_3\rangle\langle p_4|) = |p_1\rangle\langle -p_3| \otimes |-p_2\rangle\langle p_4|$.

(d) Coherent states:
$R(|\alpha\rangle\langle\beta| \otimes |\gamma\rangle\langle\delta|) = |\alpha\rangle\langle\gamma^*| \otimes |\beta^*\rangle\langle\delta|$. In particular, $R(|\alpha\rangle\langle\alpha| \otimes |\alpha^*\rangle\langle\alpha^*|) = |\alpha\rangle\langle\alpha| \otimes |\alpha^*\rangle\langle\alpha^*|$, so that a pair of phase-conjugate coherent states is invariant under R.

(e) Two-mode squeezed vacuum state:
Defining $|\mathrm{TMSV}\rangle = (1 - \tau^2)^{1/2} \sum_i \tau^i |i\rangle|i\rangle$ with $0 \leqslant \tau < 1$ characterizing the squeezing, we obtain $R(|\mathrm{TMSV}\rangle\langle\mathrm{TMSV}|) = \frac{1+\tau}{1-\tau}\rho_{th} \otimes \rho_{th}$ where $\rho_{th} = (1 - \tau)$

---

[6]Remember that the Fock basis $\{|n\rangle\}$ is used as the preferred basis with respect to which the realignment map R is defined.

[7]This can be proven using Eq. (5) and expressing $|\Omega\rangle = \sum_n |n, n\rangle$ in the position basis, namely, $|\Omega\rangle = \int dx\, |x, x\rangle$.

[8]This can be proven using Eq. (5) and expressing $|\Omega\rangle = \sum_n |n, n\rangle$ in the momentum basis: $|\Omega\rangle = \int dp\, |p, -p\rangle$.
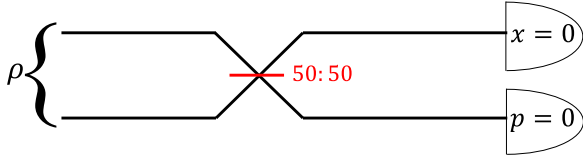
FIG. 1. Weak realignment criterion for a (two-mode) state ρ. The trace of the realigned state $R(\rho)$ is obtained by computing the probability of measuring $x_1 = p_2 = 0$ on the output state after processing ρ through a 50:50 beam splitter [see Eq. (38)].

$\sum_i \tau^i |i\rangle\langle i|$ is a thermal state. Entanglement is detected in this case since $\| R(|\text{TMSV}\rangle\langle\text{TMSV}|) \|_{\text{tr}} = \frac{1+\tau}{1-\tau} > 1$ as soon as $\tau > 0$.

(f) Tensor product of thermal states: $R(\rho_{th} \otimes \rho_{th}) = \frac{1-\tau}{1+\tau} |\text{TMSV}\rangle\langle\text{TMSV}|$ so that we have $\| R(\rho_{th} \otimes \rho_{th}) \|_{\text{tr}} = \frac{1-\tau}{1+\tau} \leqslant 1$, as expected for a separable state.

C. Expression of Tr(R) for arbitrary states

Let us now show how the weak form of the realignment criterion provides us with an implementable entanglement witness for all continuous-variable states. According to Eq. (21), in order to access $\text{Tr}\, R(\rho)$ we need to project state ρ onto $|\Omega\rangle$, which can be thought of as an unnormalized infinitely entangled two-mode vacuum squeezed state. As proven in Appendix B, the latter can be reexpressed as

$$|\Omega\rangle = \sqrt{\pi}\, U_{BS}^\dagger |0\rangle_{x_1} |0\rangle_{p_2}, \qquad (37)$$

that is, it can formally be obtained by applying (the reverse of) a 50:50 beam splitter Gaussian unitary U_{BS} on an input state of the product form $|0\rangle_{x_1} |0\rangle_{p_1}$, where $U_{BS} |z\rangle_{x_1} |z'\rangle_{x_2} = |(z-z')/\sqrt{2}\rangle_{x_1} |(z+z')/\sqrt{2}\rangle_{x_2}$ in the position eigenbasis and $|0\rangle_{x_1}$ ($|0\rangle_{p_2}$) is the position (momentum) eigenstate with zero eigenvalue. Therefore,

$$\text{Tr}\, R(\rho) = \pi \langle 0|_{x_1} \langle 0|_{p_2} U_{BS}\, \rho\, U_{BS}^\dagger |0\rangle_{x_1} |0\rangle_{p_2}. \qquad (38)$$

Hence, implementing the weak realignment criterion amounts to expressing the probability of projecting the state $\rho' = U_{BS}\, \rho\, U_{BS}^\dagger$ onto $|0\rangle_{x_1} |0\rangle_{p_2}$ where ρ' is the state obtained at the output of a 50:50 beam splitter (see Fig. 1 for the two-mode case). This yields an experimental way of constructing an entanglement witness using standard optical components since entanglement is detected simply by applying a Gaussian measurement on the state [27,29].

Furthermore, this entanglement witness can be generalized to $n \times n$ modes with quadrature components $\mathbf{x}_A = (x_1, \ldots, x_n)$, $\mathbf{p}_B = (p_{n+1}, \ldots, p_{2n})$. We have[9]

$$|\Omega_{n\times n}\rangle = \pi^{n/2} U_{BS}^\dagger |0\rangle_{\mathbf{x}_A} |0\rangle_{\mathbf{p}_B}, \qquad (39)$$

with the shorthand notation $|0\rangle_{\mathbf{x}_A} \equiv |0, \ldots, 0\rangle_{\mathbf{x}_A}$ and $|0\rangle_{\mathbf{p}_B} \equiv |0, \ldots, 0\rangle_{\mathbf{p}_B}$; hence

$$\text{Tr}\, R(\rho) = \pi^n \langle 0|_{\mathbf{x}_A} \langle 0|_{\mathbf{p}_B} \rho' |0\rangle_{\mathbf{x}_A} |0\rangle_{\mathbf{p}_B}. \qquad (40)$$

---

[9]To be more precise, if the n first modes belong to Alice and the n last modes belong to Bob, we apply n 50:50 beam splitters between Alice's ith mode and Bob's ith mode, for $i = 1, \ldots, n$.
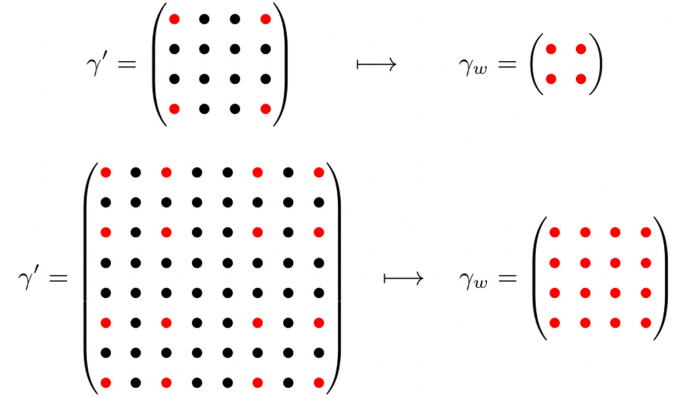

FIG. 2. Construction of the restricted covariance matrix γ_w from the covariance matrix γ', when $n = 1$ and $n = 2$. The red bullets correspond to the entries of γ' that are copied in γ_w while the black bullets are the entries that are dropped.

D. Expression of Tr(R) for Gaussian states

If the initial state ρ is an $n \times n$ Gaussian state, the state $\rho' = U_{BS}\, \rho\, U_{BS}^\dagger$ will be Gaussian too (since the beam splitter is a Gaussian unitary). Its Wigner function is thus given by

$$W_{\rho'}(\mathbf{r}) = \frac{1}{(2\pi)^{2n}\sqrt{\det \gamma'}} e^{-\frac{1}{2}\mathbf{r}(\gamma')^{-1}\mathbf{r}^T}, \qquad (41)$$

where $\mathbf{r} = (x_1, p_1, x_2, p_2, \ldots, x_{2n}, p_{2n})$ and γ' is the covariance matrix of ρ' obtained as

$$\gamma' = \mathcal{S}\gamma\mathcal{S}^{\mathcal{T}} \qquad \text{with} \qquad \mathcal{S} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1}_{2n} & -\mathbb{1}_{2n} \\ \mathbb{1}_{2n} & \mathbb{1}_{2n} \end{pmatrix} \qquad (42)$$

being the symplectic matrix representing the beam-splitting transformation and γ being the covariance matrix of ρ. The probability of projecting ρ' onto $|0\rangle_{\mathbf{x}_A} |0\rangle_{\mathbf{p}_B}$ as of Eq. (40) is thus easy to compute. Indeed, the probability distribution of measuring $\mathbf{x}_A$ on the n modes of the first system and $\mathbf{p}_B$ on the n modes of the second system is given by[10]

$$P(\mathbf{x}_A, \mathbf{p}_B) = \frac{1}{(2\pi)^n \sqrt{\det \gamma_w}} e^{-\frac{1}{2}(\mathbf{x}_A, \mathbf{p}_B)\gamma_w^{-1}(\mathbf{x}_A, \mathbf{p}_B)^T}, \qquad (43)$$

where γ_w ("w" is for witness) is the restricted covariance matrix obtained by removing the lines and columns of the unmeasured quadratures of γ' (see Fig. 2 for examples with $n = 1$ and 2). Thus $\langle\Omega|\rho|\Omega\rangle = \pi^n P(\mathbf{0}, \mathbf{0}) = \frac{1}{2^n \sqrt{\det \gamma_w}}$. In Appendix C, we show how $P(0, 0)$ can also be computed directly in a two-mode case ($n = 1$).

We are now ready to state the following theorem:

Theorem 5 (Weak realignment criterion for Gaussian states). For any $n \times n$ Gaussian state ρ^G, the trace norm of the realigned state can be lower bounded as

$$\| R(\rho^G) \|_{\text{tr}} \geqslant \text{Tr}\, R(\rho^G) = \frac{1}{2^n \sqrt{\det \gamma_w}}. \qquad (44)$$

---

[10]The probability distribution is Gaussian since we are dealing with Gaussian states.

Hence, if ρ is separable, then $\frac{1}{2^n\sqrt{\det\gamma_w}} \leqslant 1$. Conversely,

$$\text{if } \frac{1}{2^n\sqrt{\det\gamma_w}} > 1, \text{ then } \rho^G \text{ is entangled.} \qquad (45)$$

Incidentally, we note that condition (45) is equivalent to det $\gamma_w < 1/4^n$ and can thus be viewed as checking the nonphysicality of γ_w via the violation of the Schrödinger-Robertson uncertainty relation. This is in some sense similar to the PPT criterion, which is based on checking the nonphysicality of the partially transposed state.

Consider the special case of a two-mode Gaussian state ($n = 2$). Its covariance matrix can always be transformed into the normal form [5]

$$\gamma^G = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & d \\ c & 0 & b & 0 \\ 0 & d & 0 & b \end{pmatrix} \qquad (46)$$

by applying local Gaussian unitary operations,[11] which are combinations of squeezing transformations and rotations and hence do not influence the separability of the state. Applying Theorem 5, we get

$$\text{Tr}\, R(\rho^G) = \frac{1}{\sqrt{(a+b-2c)(a+b+2d)}} \qquad (47)$$

and the weak realignment criterion reads

$$\frac{1}{\sqrt{(a+b-2c)(a+b+2d)}} > 1 \Rightarrow \rho^G \text{ is entangled.}$$

In comparison, it was shown in [17] that for a covariance matrix in the normal form, Eq. (46), the trace norm of the realigned state is given by

$$\| R(\rho^G) \|_{\text{tr}} = \frac{1}{2\sqrt{(\sqrt{ab}-|c|)(\sqrt{ab}-|d|)}}. \qquad (48)$$

Comparing Eqs. (47) and (48) illustrates the fact that the weak realignment criterion is generally weaker than the original form of the realignment criterion [there exist states such that $\| R(\rho^G) \|_{\text{tr}} > 1$ while $\text{Tr}\, R(\rho^G) \leqslant 1$].

As already mentioned, both criteria become equivalent if the state is in a Schmidt-symmetric form. In this case, for a general $n \times n$ Gaussian state ρ^G described by the covariance matrix

$$\gamma^G = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \qquad (49)$$

it implies that both reduced covariance matrices must be identical, namely, $A = B$. Indeed, ρ^G being Schmidt symmetric implies that $F\rho^G F = (\rho^G)^*$. Exchanging Alice and Bob's systems yields a Gaussian state $F\rho^G F$ of covariance matrix $\begin{pmatrix} B & C^T \\ C & A \end{pmatrix}$, while $(\rho^G)^* = (\rho^G)^T$ is a Gaussian state that admits

the covariance matrix $\begin{pmatrix} A & C^T \\ C & B \end{pmatrix}$. Identifying these two covariance matrices, we conclude that any Schmidt-symmetric Gaussian state must have a covariance matrix of the form

$$\gamma^G_{\text{sch}} = \begin{pmatrix} A & C \\ C^T & A \end{pmatrix}. \qquad (50)$$

In particular, both reduced covariance matrices have the same determinant, i.e., det $A =$ det B, which is expected since we know from Eq. (31) that the two reduced states have the same purity, $\text{Tr}((\rho_1^G)^2) = \frac{1}{2^n\sqrt{\det A}}$ and $\text{Tr}((\rho_2^G)^2) = \frac{1}{2^n\sqrt{\det B}}$. In Sec. V, we apply a filtration procedure that brings the Gaussian state closer to a Schmidt-symmetric Gaussian state, which will have the effect of bringing the covariance matrix (49) closer to the form (50). More precisely, we will consider a filtration that equalizes the determinants of the reduced covariance matrices (hence, the two subsystems reach the same purity). We say that a covariance matrix of the form (49) has been *symmetrized* when det $A =$ det B.

Note that the covariance matrix in form (50) is a necessary but not sufficient condition for a Gaussian state to be Schmidt symmetric. A necessary and sufficient condition must imply additional constraints on matrix C. Let us show this for a two-mode Gaussian state with covariance matrix in the normal form

$$\gamma = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & d \\ c & 0 & a & 0 \\ 0 & d & 0 & a \end{pmatrix}, \qquad (51)$$

which is a special case of Eq. (50). Using Eq. (47), we obtain

$$\text{Tr}(R(\rho)) = \frac{1}{2\sqrt{(a-c)(a+d)}}, \qquad (52)$$

while Eq. (48) implies that

$$\| R(\rho) \|_{\text{tr}} = \frac{1}{2\sqrt{(a-|c|)(a-|d|)}}. \qquad (53)$$

Both formulas are thus equivalent only if $c \geqslant 0$ and $d \leqslant 0$, which gives the additional constraint on C. Thus, the necessary and sufficient condition for a two-mode state with covariance matrix in normal form (46) to be Schmidt symmetric is that $a = b$, $c \geqslant 0$, and $d \leqslant 0$.

This last point can be illustrated by considering $|\Omega\rangle = \sum_n |n\rangle|n\rangle = \int dx\, |x, x\rangle = \int dp\, |p, -p\rangle$, which can be viewed (up to normalization) as the limit of a two-mode squeezed vacuum state with infinite squeezing. It has $c > 0$ and $d < 0$ since the x's are correlated and p's are anticorrelated. It admits an operator Schmidt decomposition $|\Omega\rangle\langle\Omega| = \sum_{n,m} |n\rangle\langle m| \otimes |n\rangle\langle m|$ with all Schmidt coefficients being equal to one and the associated operators $A_{n,m} = B_{n,m} = |n\rangle\langle m|$; hence it is Schmidt symmetric since it satisfies $B_{n,m} = A_{n,m}^*$. Now, let us apply a phase shift of π on one of the modes, yielding $|\Omega'\rangle = \sum_n (-1)^n |n\rangle|n\rangle = \int dx\, |x, -x\rangle = \int dp\, |p, p\rangle$. Here, we have $c < 0$ and $d > 0$ since the x's are anticorrelated and p's are correlated, so it should not be Schmidt symmetric. Accordingly, it can be checked that $|\Omega'\rangle\langle\Omega'|$ does not admit an operator Schmidt decomposition with $B_{n,m} = A_{n,m}^*$. We may decompose it as $|\Omega'\rangle\langle\Omega'| = \sum_{n,m} A_{n,m} \otimes B_{n,m}$ where all

---

[11]The covariance matrix of the TMSV state, Eq. (35), is an example of the normal form.

Schmidt coefficients are again equal to one and, for example, $A_{n,m} = B_{n,m} = i^{n+m}|n\rangle\langle m|$ or $A_{n,m} = (-1)^{n+m}B_{n,m} = |n\rangle\langle m|$, but in all cases $B_{n,m} \neq A_{n,m}^*$. This is an example of an (unnormalized) state verifying $F\rho F = \rho^*$ but that is not Schmidt symmetric. Since $|\Omega\rangle$ and $|\Omega'\rangle$ share the same Schmidt coefficients, the trace norm of their realignments coincide and are equal to the trace of the realignment of $|\Omega\rangle$ only (in contrast, the trace of the realignment of $|\Omega'\rangle$ vanishes).

The link between $|\Omega\rangle$ and $|\Omega'\rangle$ suggests that a suitable local phase shift operation performed on one of the modes of a state can be useful to make the state closer to being Schmidt symmetric, and hence to enhance the detection capability of the weak realignment criterion (an example of this feature is shown in Sec. VI). Applying a local phase shift operation is, however, not always sufficient to make the state exactly Schmidt symmetric, as can be seen by considering a covariance matrix of the form (46), where we impose that $c \geqslant 0$ and $d \leqslant 0$. Indeed, as soon as $a \neq b$, one can verify that $\mathrm{Tr}(R(\rho)) < \| R(\rho) \|_{\mathrm{tr}}$ as a consequence of the well-known inequality between arithmetic and geometric means, $\sqrt{ab} \leqslant (a+b)/2$, which is saturated if and only if $a = b$.

V. IMPROVEMENT OF THE WEAK REALIGNMENT CRITERION VIA FILTRATION

According to Theorem 5, the trace norm of the realigned state $\| R(\rho) \|_{\mathrm{tr}}$ is greater than (or equal to) $\mathrm{Tr}\, R(\rho)$, which for $n \times n$ Gaussian states is a quantity that solely depends on the determinant of the restricted covariance matrix γ_w. For this reason, while it is easier to compute (especially in higher dimension), the weak realignment criterion has generally a lower entanglement detection performance than the realignment criterion (as applied in [17]). This suggests the possibility of improving the criterion by transforming the state via a suitable (invertible) operation prior to applying the criterion.

Since the trace norm and trace of the realigned state are equivalent for a Schmidt-symmetric state, the natural idea is to find a procedure that ideally transforms the initial state into a Schmidt-symmetric state without of course creating or destroying entanglement. We focus here on $n \times n$ Gaussian states and exploit the fact that any Schmidt-symmetric Gaussian state admits a covariance matrix of the form (50); in particular its reduced determinants are equal. Even if this is not a sufficient condition for a state to be Schmidt symmetric, we choose to symmetrize the initial state by equalizing the reduced determinants of its covariance matrix in order to reach a state that is closer to (ideally equal to) a Schmidt-symmetric state. We then apply Theorem 5 on the resulting symmetrized state in order to get an enhanced entanglement detection performance.

An $n \times n$ Gaussian state ρ is fully characterized by its displacement vector **d** and covariance matrix γ defined in Eq. (49). Since first-order moments are irrelevant as far as entanglement detection is concerned, we can restrict to states with **d** = 0 with no loss of generality. To symmetrize the state, we will exploit a filtering operation in the Fock basis as follows. Suppose that the first subsystem has a smaller noise variance or more precisely that $\det A < \det B$ in Eq. (49), meaning that the purity of the first subsystem is larger than

that of the second subsystem (the opposite case is treated below). We process each mode of the first subsystem through a (trace-decreasing) noiseless amplification map [30–32], that is,

$$\rho_{AB} \to \tilde{\rho}_{AB} = c\,(t^{\hat{n}/2} \otimes \mathbb{1})\rho_{AB}(t^{\hat{n}/2} \otimes \mathbb{1}), \qquad (54)$$

where c is a constant, $\hat{n}$ is the total photon number in the modes of the first subsystem, and $t > 1$ is the transmittance or gain ($\sqrt{t}$ is the corresponding amplitude gain). It can be checked that this map effects an increase of the noise variance of the first subsystem (it increases $\det A$). Note that if the input state ρ_{AB} is Gaussian, then the output state $\tilde{\rho}_{AB}$ remains Gaussian [33]. Crucially, this map does not change the separability of the state (the amount of entanglement might change, but no entanglement can be created from scratch or fully destroyed). Therefore, $\tilde{\rho}_{AB}$ should be closer to a Schmidt-symmetric state and is a good candidate for applying Theorem 5.

To find the covariance matrix of the output state $\tilde{\rho}_{AB}$, we follow the evolution of the Husimi function defined as

$$Q(\boldsymbol{\alpha}) = \frac{1}{\pi^n}\langle\boldsymbol{\alpha}|\rho|\boldsymbol{\alpha}\rangle, \qquad (55)$$

where $|\boldsymbol{\alpha}\rangle$ is a vector of coherent states. For an $n \times n$ Gaussian state ρ_{AB}, the Husimi function is given by

$$Q(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{1}{\pi^{2n}\sqrt{\det\left(\gamma + \frac{1}{2}\right)}}e^{-\frac{1}{2}r^T\Gamma r}, \qquad (56)$$

where α is associated to the first system and β to the second, $\Gamma = (\gamma + \mathbb{1}/2)^{-1}$, and

$$\boldsymbol{r} = \sqrt{2}(\mathrm{Re}(\alpha_1), \mathrm{Im}(\alpha_1), \dots, \mathrm{Re}(\alpha_n), \mathrm{Im}(\alpha_n),$$
$$\mathrm{Re}(\beta_1), \mathrm{Im}(\beta_1), \dots, \mathrm{Re}(\beta_n), \mathrm{Im}(\beta_n))^T \qquad (57)$$

with $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ representing the real and imaginary parts. The noiseless amplification map enhances the amplitude of a coherent state as $|\alpha\rangle \to e^{(t-1)|\alpha|^2/2}|\sqrt{t}\,\alpha\rangle$. Therefore, the Husimi function of the output state $\tilde{\rho}_{AB}$ is equal to (see [34] for more details)

$$\tilde{Q}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \propto \frac{1}{\pi^{2n}}\langle\boldsymbol{\alpha}, \boldsymbol{\beta}|(t^{\hat{n}/2} \otimes \mathbb{1})\rho_{AB}(t^{\hat{n}/2} \otimes \mathbb{1})|\boldsymbol{\alpha}, \boldsymbol{\beta}\rangle$$
$$= e^{(t-1)(|\alpha_1|^2 + \cdots + |\alpha_n|^2)}Q(\sqrt{t}\,\boldsymbol{\alpha}, \boldsymbol{\beta}). \qquad (58)$$

Since the output state $\tilde{\rho}_{AB}$ is a Gaussian state, its Husimi function is still of the form (56) with an output covariance matrix $\tilde{\gamma}$ (and corresponding $\tilde{\Gamma}$). Comparing the exponent of both expressions, we find that

$$\tilde{\Gamma} = M\Gamma M - (M^2 - \mathbb{1}),$$
$$\tilde{\gamma} = \left[M\left(\gamma + \frac{1}{2}\right)^{-1}M - (M^2 - \mathbb{1})\right]^{-1} - \frac{1}{2}, \qquad (59)$$

where

$$M = \begin{pmatrix} \sqrt{t}\,\mathbb{1}_{2n\times 2n} & 0 \\ 0 & \mathbb{1}_{2n\times 2n} \end{pmatrix}. \qquad (60)$$

The last point before applying the weak realignment criterion on $\tilde{\rho}_{AB}$ is to find a suitable value for the transmittance t (note that t must be greater than 1). A simple ansatz is to choose t so that the filtered state $\tilde{\rho}_{AB}$ is a symmetrized Gaussian state; that
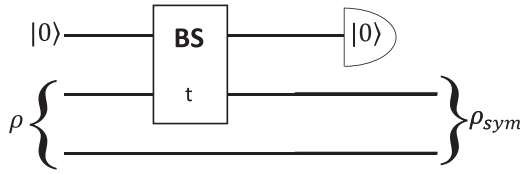
FIG. 3. Circuit implementing the filtration on a two-mode Gaussian state ρ: a noiseless attenuation map is applied on the first mode of ρ (BS represents a beam splitter of transmittance t). The resulting state ρ_{sym} is symmetrized if the value of t is well chosen.
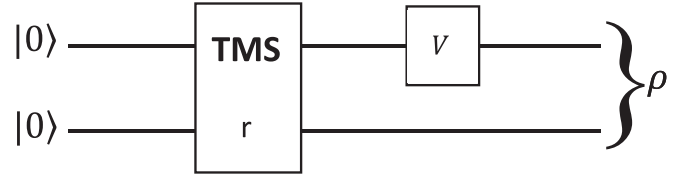

FIG. 4. Example of a Gaussian state created from a two-mode vacuum state processed through a two-mode squeezer (with squeezing parameter $r > 0$) and a Gaussian additive-noise channel acting on the first mode (with noise variance V).

is, the noise variances of both subsystems are equal ($\det A = \det B$).

Now, if the first subsystem has a larger noise variance (namely, $\det A > \det B$), we can simply exchange the roles of A and B and apply the noiseless amplification map on the modes of the second subsystem. Alternatively, we may consider another filtering operation in the Fock basis by processing each mode of the first subsystem through a (trace-decreasing) noiseless attenuation map [33,35]. Formally, it is defined exactly as the noiseless amplification map in Eq. (54) but with a transmittance $t < 1$, so it leads to very similar calculations. Physically, the noiseless attenuation map has the advantage to admit an exact physical implementation (unlike the noiseless amplification map), which provides us with another method to compute the output covariance matrix $\tilde{\gamma}$ (and corresponding $\tilde{\Gamma}$). Indeed, processing the state of a mode through a noiseless attenuation map is equivalent to processing it through a beam splitter of transmittance t (with vacuum on an ancillary mode) and then postselecting the output conditionally on the vacuum on the ancillary mode (see Fig. 3). We give the details of this alternative calculation for the two-mode case in Appendix D.

We note that the enhancement of the weak realignment criterion obtained via prior filtration can be viewed as the consequence of using $\mathrm{Tr} R(\rho) = \mathrm{Tr}(\rho \, |\Omega\rangle\langle\Omega|)$ but with a better witness operator than $|\Omega\rangle\langle\Omega|$. Let us define the filtration map as $\rho \rightarrow \Lambda_F(\rho)$. For example, consider the noiseless attenuation map $\Lambda_F(\rho) \propto (t^{\hat{n}/2} \otimes \mathbb{1})\rho(t^{\hat{n}/2} \otimes \mathbb{1})$ applied on a 1×1 state (with $t < 1$). The trace of the realigned state after filtration can be expressed as

$$
\begin{aligned}
\mathrm{Tr}(R(\Lambda_F(\rho))) &= \mathrm{Tr}(\Lambda_F(\rho) \, |\Omega\rangle\langle\Omega|) \\
&= \mathrm{Tr}(\rho \, \Lambda_F^\dagger(|\Omega\rangle\langle\Omega|)),
\end{aligned} \quad (61)
$$

where $\Lambda_F^\dagger$ stands for the dual filtration map. In this example, we note that $\Lambda_F^\dagger = \Lambda_F$ and $\Lambda_F(|\Omega\rangle\langle\Omega|)$ is proportional to the projector onto a two-mode squeezed vacuum state. In other words, the enhancement in this example is obtained by computing the fidelity of ρ with respect to $\sum_n t^{n/2}|n\rangle|n\rangle$ instead of $|\Omega\rangle = \sum_n |n\rangle|n\rangle$.

In the next section, we apply this filtration procedure on several examples of Gaussian states in order to show how the weak realignment criterion assisted with filtration can indeed improve entanglement detection.

VI. APPLICATIONS

A. Two-mode squeezed vacuum state with Gaussian additive noise

We first illustrate how the filtration procedure enables a better entanglement detection on two-mode entangled Gaussian states. In particular, we show that for specific examples, computing the trace of the realigned state (after filtration) is equivalent to computing its trace norm. Let us consider a two-mode squeezed vacuum state whose first mode is processed through a Gaussian additive-noise channel as shown in Fig. 4. We denote V the variance of this added noise. It is known that the entanglement of the two-mode squeezed state decreases when we increase the noise variance, until $V = 1$ at which point it becomes separable (if $V \geqslant 1$, the channel is entanglement breaking [36]). Our Gaussian state has a covariance matrix

$$
\gamma = \begin{pmatrix} V + \frac{\cosh 2r}{2} & 0 & \frac{\sinh 2r}{2} & 0 \\ 0 & V + \frac{\cosh 2r}{2} & 0 & -\frac{\sinh 2r}{2} \\ \frac{\sinh 2r}{2} & 0 & \frac{\cosh 2r}{2} & 0 \\ 0 & -\frac{\sinh 2r}{2} & 0 & \frac{\cosh 2r}{2} \end{pmatrix}, \quad (62)
$$

where $r > 0$ is the squeezing parameter. By applying Eq. (44) where

$$
\gamma_w = \begin{pmatrix} \frac{1}{2}(V + e^{-2r}) & 0 \\ 0 & \frac{1}{2}(V + e^{-2r}) \end{pmatrix}, \quad (63)
$$

we find $\mathrm{Tr}\, R(\rho) = 1/(V + e^{-2r})$. According to Theorem 5, entanglement is thus detected if $V < 1 - e^{-2r}$. Clearly, for a finite squeezing parameter r, there exist entangled states with $1 - e^{-2r} < V < 1$ which are not detected. As a result, the weak realignment criterion does not always detect entanglement in this example (it becomes perfect at the limit of infinite squeezing, $r \rightarrow \infty$).

In comparison, it was shown in [17] that for a matrix in the normal form [Eq. (46)] the trace norm of the realignment is given by Eq. (48). In our example, we obtain

$$
\| R(\rho) \|_{\mathrm{tr}} = \frac{1}{\sqrt{(\cosh 2r + 2V)\cosh 2r - \sinh 2r}} \quad (64)
$$

so that entanglement is detected if $V < \tanh 2r$. Here again, the realignment criterion leaves some entangled states undetected (but it is more sensitive than the weak realignment criterion since $1 - e^{-2r} < \tanh 2r, \forall r > 0$).

Let us now symmetrize the state with the filtration procedure introduced in Sec. V; that is, we process the first

mode (which has a larger noise variance) through a noise-less attenuation map. By inspection, we find that the optimal

transmittance is $t = \tanh^2 r$ and the resulting symmetrized Gaussian state ρ^{sym} admits the covariance matrix[12]

$$\gamma^{sym} = \frac{1}{8(V + \cosh 2r)}\begin{pmatrix} (4V\cosh 2r + \cosh 4r + 3)\,\mathbb{1} & (8\sinh^2 r\cosh^2 r)\,\sigma_z \\ (8\sinh^2 r\cosh^2 r)\,\sigma_z & (4V\cosh 2r + \cosh 4r + 3)\,\mathbb{1} \end{pmatrix}, \tag{65}$$

where $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We now apply Eq. (44) to ρ^{sym}, where

$$\gamma_w^{sym} = \frac{1}{2}\begin{pmatrix} \frac{V\cosh 2r + 1}{V + \cosh 2r} & 0 \\ 0 & \frac{V\cosh 2r + 1}{V + \cosh 2r} \end{pmatrix}, \tag{66}$$

which gives

$$\mathrm{Tr}\,R(\rho^{sym}) = \frac{V + \cosh 2r}{1 + V\cosh 2r}. \tag{67}$$

Thus, entanglement is detected if $\mathrm{Tr}R(\rho^{sym}) > 1$ which is equivalent to $V < 1$, for all r. Hence, all entangled states of the form (62) are now detected. Note that $\mathrm{Tr}\,R(\rho^{sym}) = \| R(\rho^{sym}) \|_{\mathrm{tr}}$ here according to Theorem 4. Indeed, the co-variance matrix (65) is in the form (51) with $c > 0$ and $d < 0$, so we have reached a Schmidt-symmetric state.

As a consequence, we have confirmed that the entan-glement detection for Gaussian states is improved if one symmetrizes the state before applying the weak realign-ment criterion. In particular, in this specific example, the weak realignment criterion is as strong as the original re-alignment criterion with symmetrization since $\mathrm{Tr}\,R(\rho^{sym}) = \| R(\rho^{sym}) \|_{\mathrm{tr}}$ and even stronger than the realignment criterion without symmetrization based on $\| R(\rho) \|_{\mathrm{tr}}$. Moreover, ap-plying the symmetrization procedure and computing the trace of the realigned state (via the determinant of the restricted covariance matrix) are much easier than computing the trace norm of the realigned state (as developed in [17]). In Fig. 5 (upper panel), we illustrate the fact that the trace and trace norm of the realigned state can be increased by the filtration procedure (the value without filtration is found when $t = 1$). We notice that, although the optimal value of the transmittance $t = \tanh^2(r)$ allows for the detection of entanglement, there are actually many other values of t that allow for such a detection too. Moreover, it seems that the symmetrized state $t = \tanh^2(r)$ is not necessarily the best way of filtering the state of this example as it does not give the highest possible value of the trace of $R(\rho)$.

As a second example, let us start with the same two-mode squeezed vacuum state but add noise on the second mode

instead of the first. The covariance matrix reads

$$\gamma = \begin{pmatrix} \frac{\cosh 2r}{2} & 0 & \frac{\sinh 2r}{2} & 0 \\ 0 & \frac{\cosh 2r}{2} & 0 & -\frac{\sinh 2r}{2} \\ \frac{\sinh 2r}{2} & 0 & V + \frac{\cosh 2r}{2} & 0 \\ 0 & -\frac{\sinh 2r}{2} & 0 & V + \frac{\cosh 2r}{2} \end{pmatrix}. \tag{68}$$

As before $\mathrm{Tr}\,R(\rho) = 1/(V + e^{-2r})$ so the weak realignment criterion alone does not detect all entangled states. To improve
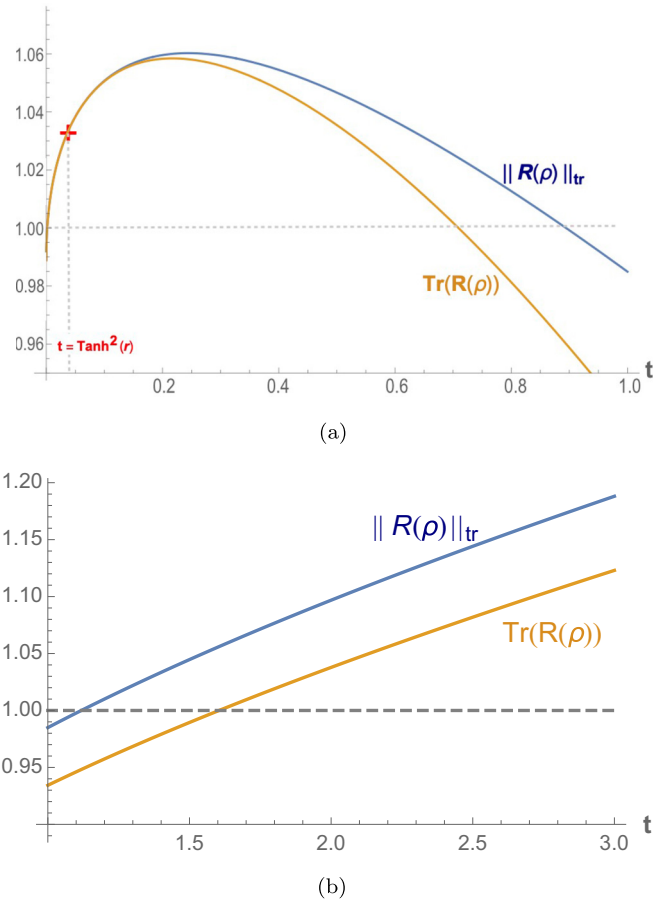

(a)

(b)

FIG. 5. Comparison of the trace $\mathrm{Tr}\,R(\rho)$ and trace norm $\| R(\rho) \|_{\mathrm{tr}}$ of the realigned state for a two-mode squeezed vac-uum state processed through a Gaussian additive-noise channel (we choose $r = 0.2$ and $V = 0.4$). (a) The noise is added on the first mode and filtering consists in processing this mode via a noiseless attenu-ation map. The red cross shows the point at $t = \tanh^2(r)$ where the covariance matrix has been symmetrized and $\mathrm{Tr}\,R(\rho) = \| R(\rho) \|_{\mathrm{tr}}$. (b) The noise is added on the second mode and filtering works by processing the first mode via a noiseless amplification map. The val-ues of the trace and the trace norm coincide when $t = \frac{1}{\tanh^2 r} \approx 25.7$.

_____

[12]Note that even if $V = 0$ (i.e., the state already has a symmetric covariance matrix) we may still process one of its modes through the noiseless attenuation map. It simply yields another (symmetric) two-mode squeezed vacuum state with lower entanglement.

on this, we could of course apply the noiseless attenuation map on the second mode, which would give the exact same results. Alternatively, we may explore another filtration procedure, which consists in applying the noiseless amplification map on the first mode. As can be seen in Fig. 5 (lower panel), the values of the trace and trace norm increase with t, and if t is chosen big enough, we detect entanglement. In order to symmetrize the covariance matrix, we would need a noiseless amplifier of transmittance $t = \frac{1}{\tanh^2 r}$, which is a limiting case that would yield a two-mode squeezed vacuum state with infinite squeezing. For this optimal value of t, we have $\mathrm{Tr}\, R(\rho_{\mathrm{sym}}) = \frac{1}{V}$ and thus entanglement is always detected when $V < 1$. Hence, here again, the weak realignment criterion assisted with filtration allows us to detect all entangled states.

B. Random two-mode Gaussian states

Let us consider two other examples of random two-mode Gaussian states with their covariance matrices written in the normal form:

$$
\gamma_1 = \begin{pmatrix} 1.46 & 0 & 0.83 & 0 \\ 0 & 1.46 & 0 & -0.23 \\ 0.83 & 0 & 0.80 & 0 \\ 0 & -0.23 & 0 & 0.80 \end{pmatrix},
$$

$$
\gamma_2 = \begin{pmatrix} 1.29 & 0 & -0.76 & 0 \\ 0 & 1.29 & 0 & 0.44 \\ -0.76 & 0 & 0.83 & 0 \\ 0 & 0.44 & 0 & 0.83 \end{pmatrix}. \tag{69}
$$

These states are not PPT so they are entangled. These examples are interesting because in both cases $\| R(\rho) \|_{\mathrm{tr}} > 1$ but $\mathrm{Tr}\, R(\rho) < 1$, so entanglement is detected by the realignment criterion but not by its weak formulation. We thus need to apply the filtration procedure in order to enhance the detection with the weak realignment criterion. In Fig. 6, we show the evolution of $\mathrm{Tr}\, R(\rho)$ as a function of the transmittance t of the noiseless attenuation map applied on the first mode ($t = 1$ corresponds to the initial value when no filtration is applied). The red cross indicates the exact point when the covariance matrix has been symmetrized. In the first example [see Fig. 6(a)], the filtration procedure works well and many values of t allow us to detect entanglement. In particular, the entanglement is detected at the optimal value of t (note that the trace and trace norm do not exactly coincide there, which witnesses the fact that the symmetrized state is not exactly a Schmidt-symmetric state). In the second example [see Fig. 6(b)], however, filtration alone is not sufficient and entanglement is never detected by the weak realignment criterion. Even if filtration is performed by applying a noiseless amplifier map on the second mode, we observe the same results. Nevertheless, entanglement can still be detected if we apply a local rotation (a π phase shift on one of the two modes which has the effect to flip the sign of the c and d elements in the normal form of the covariance matrix) prior to the filtration procedure, which makes the covariance matrix look similar to the first example. This is shown by the dashed green curve on Fig. 6(b). Furthermore, by applying
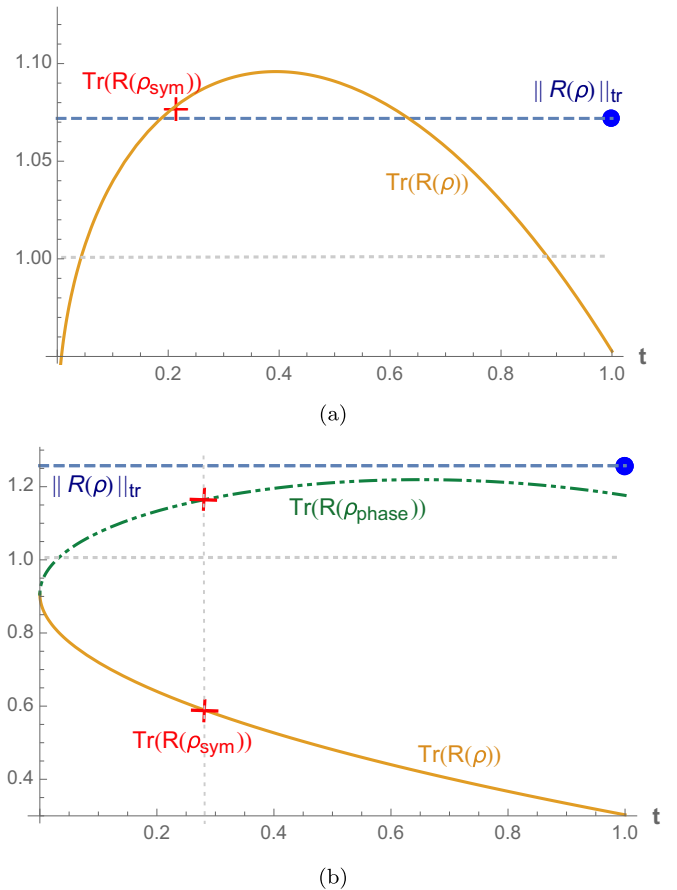

(a)

(b)

FIG. 6. Evolution of the trace $\mathrm{Tr}\, R(\rho)$ of the realigned state as a function of t for two-mode Gaussian states with covariance matrices (a) γ_1 and (b) γ_2 after filtering (noiseless attenuation on the first mode). Both examples are entangled states. The blue dashed line represents $\| R(\rho) \|_{\mathrm{tr}}$ before the filtration and the red cross shows the value of $\mathrm{Tr}\, R(\rho_{\mathrm{sym}})$ when the covariance matrix has been symmetrized. The green dashed curve shows the evolution of the trace $\mathrm{Tr}\, R(\rho_{\mathrm{phase}})$, where ρ_{phase} is the state obtained by applying a π phase shift prior to the filtration.

an appropriate local squeezing on the second mode of the state after the noiseless attenuator on the first mode, we may always reach a Schmidt-symmetric state [provided c and d have opposite signs in the covariance matrix (46) of the initial state; otherwise the state is anyway separable]. This indicates that applying a suitable local phase shift followed by a suitable noiseless attenuator (or amplifier) and finally a suitable local squeezer yields a filtration procedure that always allows the detection of entanglement for a two-mode Gaussian state.

Note that we cannot plot the evolution of $\| R(\rho) \|_{\mathrm{tr}}$ as a function of t in Fig. 6 (in contrast with Fig. 5) since the covariance matrix after filtration is not anymore in the form (46). The blue dashed line represents its initial value before the filtration is applied.

C. Examples of 2 × 2 NPT Gaussian states

Let us now move on to examples of 2×2 Gaussian states (in which case the PPT criterion is not any more necessary and

sufficient). We extend the example of Sec. VI A by considering that Alice and Bob share two instances of a two-mode squeezed vacuum state with added noise. The covariance matrix is thus given by

$$
\gamma_{EPR} = \begin{pmatrix} \left(V + \frac{\cosh 2r}{2}\right) \mathbb{1} & 0 & \frac{\sinh 2r}{2}\sigma_z & 0 \\ 0 & \left(V + \frac{\cosh 2r}{2}\right) \mathbb{1} & 0 & \frac{\sinh 2r}{2}\sigma_z \\ \frac{\sinh 2r}{2}\sigma_z & 0 & \frac{\cosh 2r}{2}\mathbb{1} & 0 \\ 0 & \frac{\sinh 2r}{2}\sigma_z & 0 & \frac{\cosh 2r}{2}\mathbb{1} \end{pmatrix}.
\tag{70}
$$

This state is always detected by the PPT separability criterion. We can also add some rotations on Bob's modes in order to get another state whose covariance matrix is given by $\gamma'_{EPR} = R(\theta, \tau)\gamma_{EPR}R^T(\theta, \tau)$ with

$$
R(\theta, \tau) = \begin{pmatrix} \mathbb{1}_{4\times4} & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & \mathbb{1}_{2\times2} \end{pmatrix} \begin{pmatrix} \mathbb{1}_{4\times4} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\tau} & 0 & -\sqrt{1-\tau} & 0 \\ 0 & 0 & \sqrt{\tau} & 0 & -\sqrt{1-\tau} \\ 0 & \sqrt{1-\tau} & 0 & \sqrt{\tau} & 0 \\ 0 & 0 & \sqrt{1-\tau} & 0 & \sqrt{\tau} \end{pmatrix}.
\tag{71}
$$

This rotated state is always entangled and detected by the PPT criterion. The entanglement detection effected by the weak realignment criterion is, however, depending on the values of θ and τ as follows:

(a) If $\theta = 0$ and $\tau = 1$ we have $\gamma'_{EPR} = \gamma_{EPR}$ and the calculations are exactly the same as in Sec. VI A (but everything is squared because we now have two states). It means in particular that $\mathrm{Tr}\, R(\rho_{EPR}) = \frac{1}{(e^{-2r}+V)^2}$ is not always greater than 1, but if we applied a suitable filtration with $t = \tanh^2(r)$, entanglement becomes always detected.

(b) If $\theta = \pi$ and regardless of the value of τ, we have $\mathrm{Tr}\, R(\rho'_{EPR}) = \frac{1}{1+V^2+2V\cosh 2r}$ which is always smaller than 1. Entanglement is thus never detected. Note that in this particular case, the filtration does not improve the value of $\mathrm{Tr}\, R(\rho')$ even if we try to add a rotation before the filtration. The key point is that this state does not have EPR-like correlations.

(c) If $\theta = 0$ and regardless of the value of τ, we have $\mathrm{Tr}\, R(\rho'_{EPR}) = \frac{1}{(\cosh 2r - \sqrt{\tau}\sinh 2r + V)^2}$. In some cases, entanglement is detected without any filtration. In some other cases,

entanglement is not straightforwardly detected, but the filtration helps in the detection. For example, if $r = 1$, $V = 0.8$, and $\tau = 0.9$, then $\mathrm{Tr}\, R(\rho'_{EPR}) \approx 0.8 < 1$ and entanglement is not detected as such. However, if we apply the filtration procedure, we see in Fig. 7 (upper panel) that there are many values of t that enable entanglement detection.

VII. CONCLUSIONS

We have introduced a weak formulation of the realignment criterion based on the trace of the realigned state $R(\rho)$, which has the advantage of being much easier to compute than the original formulation of the realignment criterion, especially in higher dimensions. It has a simple physical implementation as computing $\mathrm{Tr}\, R(\rho)$ is equivalent to measuring the $|\Omega\rangle$ component of state ρ via linear optics and homodyne measurements. Moreover, for states in the Schmidt-symmetric form, both realignment criteria—the weak and the original formulations—are equivalent. We focused especially on Gaussian states and showed that applying a suitable filtration procedure prior to applying the weak realignment criterion often allows for a better entanglement detection. In particular, we have explored a filtration based on noiseless amplification or attenuation, which is an invertible operation that transforms the state into a symmetrized form such that the entanglement detection is enhanced (this procedure may even surpass the original realignment criterion while it is simpler). We have provided examples of the application of this procedure for various 1×1 and 2×2 Gaussian states. These examples illustrate the power of the method (it can be made to detect all entangled 1×1 Gaussian states), even though we have found cases where it leaves the entanglement of 2×2 states undetected.

A question that we leave open in this work is whether the weak realignment criterion assisted with suitable prior filtration can be stronger than the PPT criterion to the degree that it can detect bound entangled states. The weak realignment criterion is weaker than both the original realignment and PPT criteria, which are two incomparable criteria (except for states in the symmetric subspace, where they coincide). Hence, as
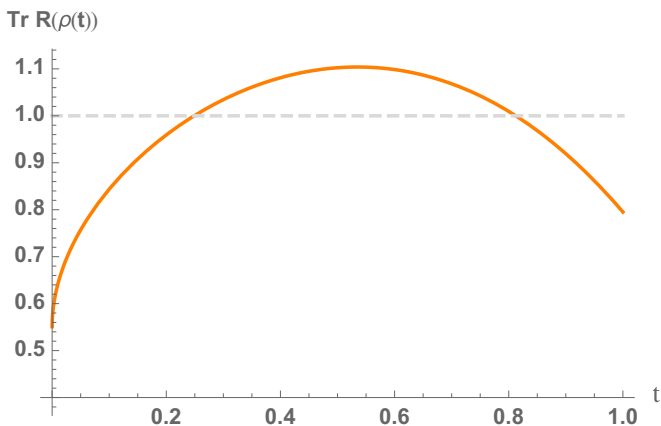
Tr R(ρ(t))

FIG. 7. Evolution of the trace $\mathrm{Tr}\, R(\rho'_{EPR})$ of the realigned state after filtration as a function of t for $r = 1$, $V = 0.8$, $\tau = 0.9$, and $\theta = 0$. The entanglement is detected in the interval of t values such that the trace exceeds 1.

such, it cannot detect bound entanglement. We have not been able to find instances where adding filtration allowed us to detect bound entangled states, although it should in principle be possible to bring the state close enough to a Schmidt-symmetric state so that bound entanglement is detected. Note, however, that this can only be the case if the original realignment criterion detects the entanglement of the state, that is, if the Schmidt-symmetric state that is approached via filtration is not within the symmetric subspace (otherwise, the weak realignment criterion tends to the realignment criterion which itself coincides with the PPT criterion, so no bound entanglement can be detected). This puts severe constraints on where to seek the detection of bound entanglement using the weak realignment criterion assisted with filtration.

As a future work, it would also be interesting to explore other possible filtration procedures in order to further improve the entanglement detection in higher dimensions. Alternatively, another interesting goal would be to find a physically implementable protocol for the original realignment criterion and not its weak form [that is, for evaluating the trace norm of $R(\rho)$ instead of its trace with optical components].

APPENDIX A: PROOF OF THEOREM 1

We give here the proof of Theorem 1. Let us construct an entanglement witness $\mathcal{W}$, that is, an observable with a positive expectation value on all separable states. Let us define $\mathcal{W} = \mathbb{1} - \sum_i^r A_i \otimes B_i$ and let us check that $\mathrm{Tr}(\rho_{\mathrm{sep}}\mathcal{W}) \geqslant 0$ where $\rho_{\mathrm{sep}} = (|a\rangle \otimes |b\rangle)(\langle a| \otimes \langle b|)$ is a separable (product) state. First we remark that

$$
\begin{aligned}
\mathrm{Tr}(\rho_{\mathrm{sep}}\mathcal{W}) &= (\langle a| \otimes \langle b|)\mathcal{W}(|a\rangle \otimes |b\rangle) \\
&= 1 - \sum_i^r \langle a|A_i|a\rangle \langle b|B_i|b\rangle \\
&\geqslant 1 - \sqrt{\sum_i^r |\langle a|A_i|a\rangle|^2} \sqrt{\sum_i^r |\langle b|B_i|b\rangle|^2}, \quad \text{(A1)}
\end{aligned}
$$

where we used the Cauchy-Schwarz inequality in the last step. Now, since the $\{A_i\}$ form a basis, we can write $|a\rangle\langle a| = \sum_j \alpha_j A_j$ where $\alpha_j = \langle a|A_i|a\rangle$, and similarly for $|b\rangle\langle b|$. This allows us to write

$$
\begin{aligned}
1 &= \|\,|a\rangle\langle a|\,\|^2 = \mathrm{Tr}(|a\rangle\langle a|(|a\rangle\langle a|)^{\dagger}) = \mathrm{Tr}\left(\sum_{ij} \alpha_i A_i \alpha_j^* A_j^{\dagger}\right) \\
&= \sum_{ij} \alpha_i \alpha_j^* \mathrm{Tr}(A_i A_j^{\dagger}) = \sum_i |\alpha_i|^2 = \sum_i |\langle a|A_i|a\rangle|^2 \quad \text{(A2)}
\end{aligned}
$$

and similarly $\sum_i |\langle b|B_i|b\rangle|^2 = 1$ so that $\mathrm{Tr}(\rho_{\mathrm{sep}}\mathcal{W}) \geqslant 0$. Thus, $\mathcal{W}$ is indeed an entanglement witness as any separable state is expressed as a convex mixture of states of the form ρ_{sep}. Let us now check under which condition it allows for entanglement detection. In other words, what is the condition to have $\mathrm{Tr}(\rho\mathcal{W}) < 0$? Consider a state ρ written in its operator Schmidt decomposition. Then,

$$
\begin{aligned}
\mathrm{Tr}(\rho\mathcal{W}) &= \mathbb{1} - \mathrm{Tr}\left(\sum_{ij}^r \lambda_i A_i A_j \otimes B_i B_j\right) \\
&= \mathbb{1} - \sum_{ij}^r \lambda_i \mathrm{Tr}(A_i A_j)\mathrm{Tr}(B_i B_j) \\
&= 1 - \sum_i^r \lambda_i. \quad \text{(A3)}
\end{aligned}
$$

Entanglement is thus detected when $\sum_i^r \lambda_i > 1$ which completes the proof.

APPENDIX B: FORMULATION OF THE $|\Omega\rangle$ STATE

We prove here that the state $|\Omega\rangle$ can be reexpressed as $\sqrt{\pi}\, U_{BS}^{\dagger}|0\rangle_{x_1}|0\rangle_{p_2}$ where U_{BS} is the unitary of a 50:50 beam splitter. By definition, it is expressed in the Fock basis as $|\Omega\rangle = \sum_n |n\rangle|n\rangle$. Thus, if $|x\rangle$ and $|y\rangle$ are position states, we have

$$
\begin{aligned}
\langle x|\langle y|\Omega\rangle &= \sum_n \langle x|n\rangle \langle y|n\rangle \\
&= \sum_n \langle x|n\rangle \langle n|y\rangle \\
&= \langle x|y\rangle = \delta(x - y), \quad \text{(B1)}
\end{aligned}
$$

so that $|\Omega\rangle$ can be written in the position basis as

$$
|\Omega\rangle = \int dx\, dy\, \delta(x - y)|x\rangle|y\rangle = \int dx\, |x\rangle|x\rangle. \quad \text{(B2)}
$$

Since the action of the 50:50 beam splitter unitary U_{BS} on the position eigenstates is defined as

$$
U_{BS}|x\rangle|y\rangle = \left|\frac{x - y}{\sqrt{2}}\right\rangle\left|\frac{x + y}{\sqrt{2}}\right\rangle, \quad \text{(B3)}
$$

we have

$$
\begin{aligned}
\langle x|\langle y|U_{BS}^{\dagger}|0\rangle_{x_1}|0\rangle_{p_2} &= \left\langle\frac{x - y}{\sqrt{2}}\right|\left\langle\frac{x + y}{\sqrt{2}}\right||0\rangle_{x_1}|0\rangle_{p_2} \\
&= \left\langle\frac{x - y}{\sqrt{2}}\Big|x = 0\right\rangle\left\langle\frac{x + y}{\sqrt{2}}\Big|p = 0\right\rangle \\
&= \delta\left(\frac{x - y}{\sqrt{2}}\right)\frac{1}{\sqrt{2\pi}} \\
&= \delta(x - y)/\sqrt{\pi}, \quad \text{(B4)}
\end{aligned}
$$

where we have used the fact that $\langle x|y\rangle = \delta(x - y)$ and $\langle x|p\rangle = \frac{1}{\sqrt{2\pi}}e^{ipx}$. Comparing with Eq. (B1), this completes the proof that $|\Omega\rangle = \sqrt{\pi}\, U_{BS}^{\dagger}|0\rangle_{x_1}|0\rangle_{p_2}$.

APPENDIX C: COMPUTATION OF EQ. (43)

We show how to directly compute Eq. (43) for a two-mode state. We have to compute the following integral where we set $x_1 = p_2 = 0$:

$$\langle x = 0|\langle p = 0|\rho'|x = 0\rangle|p = 0\rangle$$

$$= \int dx_2 dp_1 W_{\rho'}(0, p_1, x_2, 0)$$

$$= \frac{1}{(2\pi)^2 \sqrt{\det \gamma'}} \int dx_2 dp_1 \, e^{-\frac{1}{2}(0 \; p_1 \; x_2 \; 0)(\gamma')^{-1}(0 \; p_1 \; x_2 \; 0)^T}$$

$$= \frac{1}{(2\pi)^2 \sqrt{\det \gamma'}} \int dx_2 dp_1 \, e^{-\frac{1}{2}(x_2 \quad p_1)\Gamma\binom{x_2}{p_1}}$$

$$= \frac{1}{(2\pi)^2 \sqrt{\det \gamma'}} \frac{2\pi}{\sqrt{\det \Gamma}}$$

$$= \frac{1}{2\pi \sqrt{\det \gamma'}} \sqrt{\frac{\det \gamma'}{\det \gamma_w}} = \frac{1}{2\pi \sqrt{\det \gamma_w}}, \tag{C1}$$

where Γ is a 2×2 matrix with elements given by $\Gamma_{1,1} = (\gamma')^{-1}_{3,3}$, $\Gamma_{2,2} = (\gamma')^{-1}_{2,2}$, and $\Gamma_{1,2} = \Gamma_{2,1} = (\gamma')^{-1}_{2,3}$.

APPENDIX D: PHYSICAL INTERPRETATION OF THE SYMMETRIZATION PROCEDURE FOR A GAUSSIAN STATE

We present here an alternative way of computing the covariance matrix of the output state of a noiseless attenuation channel. To do so, we use the fact that the attenuation channel can be represented by a beam splitter followed by a postselection on the vacuum (see Fig. 3).

To filter the state, we process the modes of the subsystem with the higher variance (that is the higher value of the determinant of the reduced covariance matrix A or B) through a noiseless attenuation channel and then postselect the output conditionally to measuring the vacuum on the ancillary modes. Since it is a Gaussian channel, the output remains Gaussian. Processing the state through this channel will have for effect to lower the variance of the mode that traveled through the channel. The output state is the symmetrized Gaussian state ρ^{sym}. We chose the attenuator factor (that is the transmittance t of the beam splitter) so that the variance of both modes of ρ^{sym} are equal; that is, $\det A = \det B$. In terms of covariance matrix, the procedure is as follows.

Let us have an $n \times n$ Gaussian state ρ with covariance matrix (49) and let us assume $\det A \geqslant \det B$ with no loss of generality. We then add n vacuum state to the system. The new covariance matrix thus reads

$$\gamma_{|0\rangle^{\oplus n}+\rho} = \begin{pmatrix} \frac{1}{2}\mathbb{1}_{2n} & 0 \\ 0 & \gamma \end{pmatrix}. \tag{D1}$$

We now apply the transformation $\mathcal{S} \oplus \mathbb{1}$ where $\mathcal{S}$ is the beam splitter transformation,

$$\mathcal{S} \oplus \mathbb{1} = \begin{pmatrix} \sqrt{t}\mathbb{1}_{2n} & -\sqrt{1-t}\mathbb{1}_{2n} & 0 \\ \sqrt{1-t}\mathbb{1}_{2n} & \sqrt{t}\mathbb{1}_{2n} & 0 \\ 0 & 0 & \mathbb{1}_{2n} \end{pmatrix}, \tag{D2}$$

to the covariance matrix $\gamma_{|0\rangle^{\oplus n}+\rho}$:

$$\gamma_{\mathcal{S}\oplus\mathbb{1}} = \mathcal{S} \oplus \mathbb{1} \, \gamma_{|0\rangle^{\oplus n}+\rho} \, \mathcal{S}^\dagger \oplus \mathbb{1} = \begin{pmatrix} \mathcal{A} & \mathcal{C}^T \\ \mathcal{C} & \mathcal{B} \end{pmatrix}. \tag{D3}$$

Finally, we reduce the covariance matrix conditionally to measuring the vacuum on the first mode; that is [37,38],

$$\gamma^{sym} = \mathcal{B} - \mathcal{C}\left(\mathcal{A} + \frac{1}{2}\mathbb{1}\right)^{-1}\mathcal{C}^T = \begin{pmatrix} \mathcal{A}' & \mathcal{C}'^T \\ \mathcal{C}' & \mathcal{B}' \end{pmatrix}. \tag{D4}$$

At this stage we obtained a new covariance matrix which depends on t. If the filtration procedure is such that we want to symmetrize the covariance matrix, we need to make sure that the determinant of the covariance matrices of both subsystems are equal ($\det \mathcal{A}' = \det \mathcal{B}'$).

Explicit calculation for the two-mode case

Let us do the explicit calculations to obtain the symmetrized covariance matrix of a two-mode Gaussian state initially expressed in its normal form [5],

$$\gamma_\rho = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & d \\ c & 0 & b & 0 \\ 0 & d & 0 & b \end{pmatrix}. \tag{D5}$$

Any covariance matrix of a two-mode state can be transformed into this form by applying local linear unitary operations which are combinations of squeezing transformations and rotations. These operations do not influence the separability of the state, and are thus always allowed when studying entanglement. Note that we assume $a \geqslant b$ with no loss of generality. We first add the vacuum state to the system. The new covariance matrix reads

$$\gamma_{|0\rangle+\rho} = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & c & 0 \\ 0 & 0 & 0 & a & 0 & d \\ 0 & 0 & c & 0 & b & 0 \\ 0 & 0 & 0 & d & 0 & b \end{pmatrix}. \tag{D6}$$

We then apply the transformation $\mathcal{S} \oplus \mathbb{1}$ to the covariance matrix $\gamma_{|0\rangle+\rho}$ to obtain

$$\gamma_{\mathcal{S}\oplus\mathbb{1}} = \mathcal{S} \oplus \mathbb{1} \, \gamma_{|0\rangle+\rho} \, \mathcal{S}^\dagger \oplus \mathbb{1} = \begin{pmatrix} \mathcal{A} & \mathcal{C}^T \\ \mathcal{C} & \mathcal{B} \end{pmatrix} \tag{D7}$$

with

$$
\mathcal{A} = \begin{pmatrix} -ta + a + \frac{t}{2} & 0 \\ 0 & -ta + a + \frac{t}{2} \end{pmatrix}, \quad
\mathcal{B} = \begin{pmatrix} \left(a - \frac{1}{2}\right)t + \frac{1}{2} & 0 & c\sqrt{t} & 0 \\ 0 & \left(a - \frac{1}{2}\right)t + \frac{1}{2} & 0 & d\sqrt{t} \\ c\sqrt{t} & 0 & b & 0 \\ 0 & d\sqrt{t} & 0 & b \end{pmatrix},
$$

$$
\mathcal{C} = \begin{pmatrix} \frac{1}{2}(1 - 2a)\sqrt{-(t-1)t} & 0 \\ 0 & \frac{1}{2}(1 - 2a)\sqrt{-(t-1)t} \\ -c\sqrt{1 - t} & 0 \\ 0 & -d\sqrt{1 - t} \end{pmatrix}. \tag{D8}
$$

Finally, we reduce the covariance matrix conditionally to measuring the vacuum on the first mode; that is,

$$
\gamma^{sym} = \mathcal{B} - \mathcal{C}\left(\mathcal{A} + \frac{1}{2}\mathbb{1}\right)^{-1}\mathcal{C}^T = \begin{pmatrix} \frac{t - 2a(t+1) - 1}{4a(t-1) - 2(t+1)} & 0 & \frac{2c\sqrt{t}}{-2a(t-1) + t + 1} & 0 \\ 0 & \frac{t - 2a(t+1) - 1}{4a(t-1) - 2(t+1)} & 0 & \frac{2d\sqrt{t}}{-2a(t-1) + t + 1} \\ \frac{2c\sqrt{t}}{-2a(t-1) + t + 1} & 0 & \frac{2(t-1)c^2}{-2a(t-1) + t + 1} + b & 0 \\ 0 & \frac{2d\sqrt{t}}{-2a(t-1) + t + 1} & 0 & \frac{2(t-1)d^2}{-2a(t-1) + t + 1} + b \end{pmatrix}. \tag{D9}
$$

The covariance matrix will be symmetrized providing that the determinant of the covariance matrices of both subsystems are equal (det A = det B), meaning

$$
\left(\frac{t - 2a(t + 1) - 1}{4a(t - 1) - 2(t + 1)}\right)^2 = \left(\frac{2(t - 1)c^2}{-2a(t - 1) + t + 1} + b\right)\left(\frac{2(t - 1)d^2}{-2a(t - 1) + t + 1} + b\right). \tag{D10}
$$

Solving this equation for t gives the transmissivity of the beam splitter necessary to obtain a Gaussian state with a symmetric covariance matrix.

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