Multiparticle quantum interference in Bogoliubov bosonic transformations

Michael G. Jabbour [1,2,3,*] and Nicolas J. Cerf [1,†]

[1] *Quantum Information and Communication, Ecole polytechnique de Bruxelles, CP 165/59, Université libre de Bruxelles, 1050 Brussels, Belgium*

[2] *Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, United Kingdom*

[3] *Department of Physics, Technical University of Denmark, 2800 Kongens Lyngby, Denmark*

(Received 10 March 2021; accepted 9 July 2021; published 22 October 2021)

We explore the multiparticle transition probabilities in Gaussian unitaries effected by a two-mode Bogoliubov bosonic transformation on the mode annihilation and creation operators. We show that the transition probabilities can be characterized by remarkably simple, yet unsuspected recurrence equations involving a linear combination of probabilities. The recurrence exhibits an interferometric suppression term—a *negative* probability—which generalizes the seminal Hong-Ou-Mandel effect to more than two indistinguishable photons impinging on a beam splitter of rational transmittance. Unexpectedly, interferences thus originate in this description from the cancellation of probabilities instead of amplitudes. Our framework, which builds on the generating function of the non-Gaussian matrix elements of Gaussian unitaries in Fock basis, is illustrated here for the most common passive and active linear coupling between two optical modes driven by a beam splitter or a parametric amplifier. Hence, it also allows us to predict unsuspected multiphoton interference effects in an optical amplifier of rational gain. In particular, we confirm the newly found two-photon interferometric suppression effect in an amplifier of gain 2 originating from timelike indistinguishability [Proc. Natl. Acad. Sci. **117**, 33107 (2020)]. Overall, going beyond standard two-mode optical components, we expect our method will prove valuable for describing general quantum circuits involving Bogoliubov bosonic transformations.

DOI: 10.1103/PhysRevResearch.3.043065

I. INTRODUCTION

Quantum interference is a cornerstone of quantum physics. While it challenges our understanding of the universe as for instance witnessed in Young's celebrated double slit experiment, it has various applications such as quantum computing [1], quantum cryptography [2], or superconducting quantum interference devices [3]. Quantum interference is notably a key to implementing future quantum technologies with photonic integrated devices, which has resulted in a vigorous research effort on harnessing multimode multiphoton interferences over the last decade, see, e.g., [4,5]. This is also significant in connection with the boson sampling paradigm [6], which builds on the computational hardness of simulating the coherent propagation of many identical bosons through a multimode linear interferometer and holds the promise of substantiating the advantage of quantum computers [7–10]. More generally, this has led to a revived interest for quantum interferometry going beyond the celebrated Hong-Ou-Mandel (HOM) effect [11], e.g., the generalized bunching effect in linear networks [12], the signatures of nonclassicality in a multimode interferometer [13], the observation of intrinsically three-photon interference [14,15], or the suppression laws in an eight-mode optical Fourier interferometer [16].

Formally, quantum interferences originate from adding up the amplitudes of (often a large number of) possible paths. Since amplitudes are complex, taking the square modulus of the resulting sum typically gives rise to constructive or destructive interferences. The HOM effect is a paradigmatic example of two-photon quantum interference: The probability of detecting two photons in coincidence at the output of a 50:50 beam splitter (one in each mode) vanishes when one photon impinges on each of the two input modes. The sum of the amplitudes of the two possible paths (both photons being either reflected or transmitted) vanishes, giving rise to destructive interference. In a nutshell, when only two paths of amplitudes α_1 and α_2 interfere, the resulting probability is $p = |\alpha_1 + \alpha_2|^2 = p_1 + p_2 + 2\sqrt{p_1 p_2}\cos\theta$, where $p_1 = |\alpha_1|^2$, $p_2 = |\alpha_2|^2$, and θ is the relative phase.

In this paper we explore multiparticle quantum interferences that emerge in Bogoliubov bosonic transformations. Bogoliubov transformations are ubiquitous in physics, appearing in various fields such as superconductivity, superfluidity, nuclear physics, and quantum field theory. They are also essential in understanding phenomena such as Hawking radiation [17,18] and the Unruh effect [19,20]. While our methods and results could be applied in various situations involving bosonic systems, we choose to illustrate them here by focusing on the quantum optics framework. Specifically, we investigate the optical Gaussian unitaries effected by Bogoliubov transformations in phase space, which closely model

*mgija@dtu.dk

†ncerf@ulb.ac.be

a great amount of operations performed in quantum optics experiments [21]. We start by examining the generic case of i and k photons impinging on the two input modes of a beam splitter, one of the simplest yet most essential operations in any optical setting. The probability $B_n^{(i,k)}$ of any output pattern is known to be expressible as a multiple summation involving four binomial coefficients [see Eq. (9)]. This complicated expression, owing to the many interfering paths, can of course be evaluated but cannot easily be exploited analytically. Here we derive an unexpectedly simple formula (Theorem 1) which governs these probabilities. Counterintuitively it involves a simple linear combination of probabilities (with no usual $\sqrt{p_1 p_2}$ terms) and the discrepancy with respect to the corresponding classical formula for distinguishable photons appears as a *negative* probability [22].

Our technique relies on calculating the generating function of the matrix elements of Gaussian unitaries in Fock basis, which can be expressed in a simple closed form with the Gaussian toolbox although it encapsulates complex non-Gaussian features such as the multiphoton transition probabilities $B_n^{(i,k)}$. It allows us to extend the HOM effect to many photons by predicting a simple negative contribution to the transition probability. More generally, our framework is suited to Gaussian unitaries describing the passive but also the active linear coupling between two bosonic modes. Hence, we predict a similar interference suppression term—a negative probability—in an optical amplifier, such as a nonlinear crystal pumped in the nondegenerate parametric amplification regime or a four-wave mixer (see Theorem 2). This corroborates and extends the recent finding of a two-boson interference effect in a gain-2 amplifier originating from timelike indistinguishability (bosons from the past and future cannot be distinguished) [23]. Active optical components are essential in continuous-variable quantum information processing [24,25] as they give access to invaluable resources and protocols, such as universal computing with Gaussian cluster states [26–28], optical multimode entanglement [29], Gaussian quantum steering [30], or Gaussian quantum cloning [31].

As a last result, we provide a further generalization of the HOM effect and two-boson active interference effect [23] by predicting a full interferometric suppression for any rational value of the transmittance (or gain) of a passive (or active) transformation provided specific photon numbers are chosen. Furthermore, we also briefly show that the asymptotic behavior of interferences with large photon numbers can easily be accessed based on generating functions. This illustrates the potential of our framework for describing multiparticle interferences in quantum circuits involving bosonic Bogoliubov transformations in phase space.

II. MODEL AND DERIVATIONS

A. Bosonic Gaussian unitaries

Bosonic modes are common carriers of continuous-variable quantum information [24,25]. A bosonic mode (e.g., a quantized mode of the electromagnetic field) is modeled by a quantum harmonic oscillator in an infinite-dimensional Fock space. It is associated with the usual pair of bosonic mode operators $\hat{a}$ and $\hat{a}^\dagger$, which must satisfy the commutation relation $[\hat{a}, \hat{a}^\dagger] = \mathbb{I}$. In this context, Bogoliubov transformations [32] (i.e., linear canonical transformations in $\hat{a}$ and $\hat{a}^\dagger$) are of particular interest as they correspond to Gaussian unitaries (i.e., quadratic Hamiltonians in $\hat{a}$ and $\hat{a}^\dagger$). They are especially valuable in the framework of quantum optics, where they conserve Gaussian-shaped Wigner functions in phase space and, most importantly, model ubiquitous transformations in experimental conditions and form the core of Gaussian quantum information [21]. They can be divided into passive and active transformations as effected by linear optical interferometry or parametric amplification, respectively. In this work, we illustrate our method for the most fundamental passive and active two-mode Gaussian unitaries, namely the beam splitter (BS) and two-mode squeezer (TMS). The BS unitary U_η^{BS} effects an energy-conserving linear coupling between two modes and acts in the Heisenberg picture as

$$\begin{aligned} U_\eta^{\mathrm{BS}\dagger}\,\hat{a}\,U_\eta^{\mathrm{BS}} &= \sqrt{\eta}\,\hat{a} + \sqrt{1-\eta}\,\hat{b}, \\ U_\eta^{\mathrm{BS}\dagger}\,\hat{b}\,U_\eta^{\mathrm{BS}} &= -\sqrt{1-\eta}\,\hat{a} + \sqrt{\eta}\,\hat{b}, \end{aligned} \tag{1}$$

where $\hat{a}$ and $\hat{b}$ are the mode operators, while η is the transmittance. Similarly, the TMS unitary U_λ^{TMS} models the generation of pairs of entangled photons by parametric amplification due to the pumping of a nonlinear crystal, and acts on mode operators as

$$\begin{aligned} U_\lambda^{\mathrm{TMS}\dagger}\,\hat{a}\,U_\lambda^{\mathrm{TMS}} &= \cosh(r)\,\hat{a} + \sinh(r)\,\hat{b}^\dagger, \\ U_\lambda^{\mathrm{TMS}\dagger}\,\hat{b}\,U_\lambda^{\mathrm{TMS}} &= \sinh(r)\,\hat{a}^\dagger + \cosh(r)\,\hat{b}. \end{aligned} \tag{2}$$

with $\lambda := \tanh^2(r)$ for a parametric gain $g := \cosh^2(r)$. The transformations characterized by Eqs. (1) and (2) happen to be useful in various contexts involving the evolution of bosonic systems. For instance, they can be exploited in black hole theory, where they describe the interaction of a Gaussian bosonic state with an already formed Schwarzchild black hole [33].

B. Generating functions

The generating function (GF) of a sequence $\{c_n\}_{n\in\mathbb{N}_0}$ is defined as

$$g(z) := \mathcal{T}_n[c_n](z) = \sum_{n=0}^{\infty} c_n z^n, \quad z \in \mathbb{C}. \tag{3}$$

It is a powerful tool as $g(z)$ encapsulates the entire sequence via $c_n = n!^{-1}(\partial^n g/\partial z^n)_{z=0}$. Here we exploit the properties of GFs in quantum optics when applied to the squared modulus of the matrix elements of Gaussian unitaries in Fock basis. Unlike the matrix elements in a coherent (Gaussian) basis, these happen to be quite difficult to handle because Fock states are non Gaussian, so it is helpful to characterize them via their GFs. Consider the four-dimensional sequence of transition probabilities $|\langle n, m| U |i, k\rangle|^2$ for some unitary U, where $|i\rangle$, $|k\rangle$, $|n\rangle$, and $|m\rangle$ denote Fock states ($i, k, n, m \in \mathbb{N}_0$). Its four-variate GF can be written as (see [34])

$$f(\mathbf{v}) = \frac{\mathrm{Tr}[(\tau_z \otimes \tau_w) U (\tau_x \otimes \tau_y) U^\dagger]}{(1-x)(1-y)(1-z)(1-w)}, \tag{4}$$

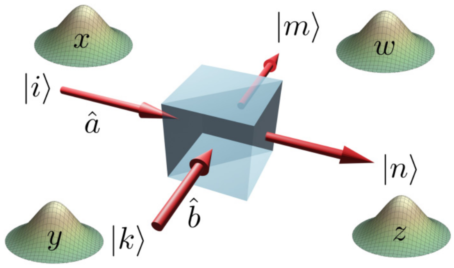

FIG. 1. Conventions in the definition of $f(x, y, z, w)$, which is the generating function of the transition probability for sending the Fock states $|i\rangle$ and $|k\rangle$, respectively, in modes $\hat{a}$ and $\hat{b}$ and, after processing through the unitary U, measuring the Fock states $|n\rangle$ and $|m\rangle$, respectively, in modes $\hat{a}$ and $\hat{b}$.

where we chose $\mathbf{v} := (x, y, z, w)$ such that $(x, y) \in [0, 1)^2$ and $(w, z) \in [0, 1]^2$, with the conventions shown in Fig. 1. Thus, $f(\mathbf{v})$ is proportional to the overlap between two Gaussian states, one of which being the product of two thermal states of the form $\tau_x := (1 - x)\sum_{n=0}^{\infty} x^n |n\rangle\langle n|$, while the other is the product of two thermal states processed through the unitary U. This makes $f(\mathbf{v})$ very easy to compute when U is Gaussian, regardless of the complexity of $|\langle n, m| U |i, k\rangle|^2$ itself, by exploiting the Gaussian formalism in phase space. Recalling that the overlap between two zero-mean Gaussian states ρ_1 and ρ_2 with covariance matrices V_1 and V_2 is given by $\mathrm{Tr}[\rho_1\rho_2] = 1/\sqrt{\det[(V_1 + V_2)/2]}$ [35], the GF of $|\langle n, m| U_\eta^{\mathrm{BS}} |i, k\rangle|^2$ can be expressed using standard tools of quantum optics as [34]

$$f_\eta^{\mathrm{BS}}(\mathbf{v}) = \frac{1}{1 - \eta(xz + yw) - \bar{\eta}(xw + yz) + xyzw}, \quad (5)$$

where $\bar{\eta} := 1 - \eta$, while the GF of $|\langle n, m| U_\lambda^{\mathrm{TMS}} |i, k\rangle|^2$ can be written as [34]

$$f_\lambda^{\mathrm{TMS}}(\mathbf{v}) = \frac{\bar{\lambda}}{1 - \lambda(xy + zw) - \bar{\lambda}(xz + yw) + xyzw}, \quad (6)$$

where $\bar{\lambda} := 1 - \lambda$. As a consistency check, we note that

$$f_\eta^{\mathrm{BS}}(\mathbf{0}) = |\langle 0, 0| U_\eta^{\mathrm{BS}} |0, 0\rangle|^2 = 1,$$
$$f_\lambda^{\mathrm{TMS}}(\mathbf{0}) = |\langle 0, 0| U_\lambda^{\mathrm{TMS}} |0, 0\rangle|^2 = \bar{\lambda}, \quad (7)$$

while normalization $\sum_{n,m=0}^{\infty} |\langle n, m| U |i, k\rangle|^2 = 1, \forall i, k,$ translates into

$$f^{\mathrm{BS/TMS}}(x, y, 1, 1) = (1 - x)^{-1}(1 - y)^{-1}. \quad (8)$$

Interestingly, energy conservation in U_η^{BS} manifests itself through $f_\eta^{\mathrm{BS}}(x, y, z, w) = f_\eta^{\mathrm{BS}}(tx, ty, z/t, w/t), \forall t$, while the conservation of the photon number difference in U_λ^{TMS} is reflected by $f_\lambda^{\mathrm{TMS}}(x, y, z, w) = f_\lambda^{\mathrm{TMS}}(tx, y/t, z/t, tw), \forall t$, see [34].

C. Multiphoton transition probabilities

We use Eq. (5) to derive a surprisingly simple recurrence equation for the multiphoton transition probabilities in a BS, denoted as $B_n^{(i,k)} := |\langle n, i + k - n| U_\eta^{\mathrm{BS}} |i, k\rangle|^2$, with $i, k, n \in \mathbb{N}_0$. Incidentally, note that a direct calculation yields [34]

$$B_n^{(i,k)} = \eta^k \bar{\eta}^i \sum_{m,j=\max(0,n-k)}^{\min(i,n)} (-1)^{m+j} \gamma_{n,m,j}^{(i,k)} \left(\frac{\eta}{\bar{\eta}}\right)^{m+j-n}, \quad (9)$$

where

$$\gamma_{n,m,j}^{(i,k)} := \binom{i}{m}\binom{k}{n-m}\binom{n}{j}\binom{i+k-n}{i-j}, \quad (10)$$

which is quite cumbersome to manipulate. Nevertheless, the following theorem provides an alternative.

Theorem 1. If $i = k = n = 0$, then $B_n^{(i,k)} = 1$, else,

$$B_n^{(i,k)} = \eta\, B_{n-1}^{(i-1,k)} + \eta\, B_n^{(i,k-1)} + \bar{\eta}\, B_n^{(i-1,k)} + \bar{\eta}\, B_{n-1}^{(i,k-1)} - B_{n-1}^{(i-1,k-1)}. \quad (11)$$

The definition of $B_n^{(i,k)}$ is extended here to all integers i, k, n, setting it to zero when either of them is negative.

Proof. We set $\mathbf{u} := (x, y, z)$, $\mathbf{j} := (i, k, n)$ and denote by $g_\eta^{\mathrm{BS}}(\mathbf{u})$ the three-variate GF of $B_n^{(i,k)}$ with the conventions of Fig. 1. Since $g_\eta^{\mathrm{BS}}(\mathbf{u}) = f_\eta^{\mathrm{BS}}(x, y, z, 1)$, Eq. (5) implies

$$[1 - \eta(xz + y) - \bar{\eta}(x + yz) + xyz]\, g_\eta^{\mathrm{BS}}(\mathbf{u}) = 1. \quad (12)$$

Using the shifting property of the GFs and the notation of Eq. (3), it can easily be shown that multiplying the GF by $\mathbf{u}_l$ for $l = 1, 2, 3$ corresponds to decreasing the index $\mathbf{j}_l$ of $B_n^{(i,k)}$ by one unit, so that for instance

$$\mathcal{T}_{\mathbf{j}}\big[B_{n-1}^{(i-1,k)}\big](\mathbf{u}) = xz\, \mathcal{T}_{\mathbf{j}}\big[B_n^{(i,k)}\big](\mathbf{u}) = xz\, g_\eta^{\mathrm{BS}}(\mathbf{u}). \quad (13)$$

In addition, we know that the three-variate GF of the product $\delta_{i,0}\,\delta_{k,0}\,\delta_{n,0}$ of three Kronecker deltas is 1. Using this, we see that Eq. (12) is equivalent to the relation

$$B_n^{(i,k)} - \eta\big[B_{n-1}^{(i-1,k)} + B_n^{(i,k-1)}\big] - \bar{\eta}\big[B_n^{(i-1,k)} + B_{n-1}^{(i,k-1)}\big] + B_{n-1}^{(i-1,k-1)} = \delta_{i,0}\,\delta_{k,0}\,\delta_{n,0},$$

which proves the theorem. ■

This recurrence can be nicely interpreted in the context of the HOM effect. As illustrated in Fig. 2, the first four terms of the right-hand side of Eq. (11) corroborate the classical intuition one may have about $B_n^{(i,k)}$: One should add the probabilities corresponding to the different scenarios in which the nth photon has not reached the BS yet, multiplied by the right probability (η or $\bar{\eta}$) depending on which path it takes. For example, $B_{n-1}^{(i-1,k)}$ must be multiplied by η since the extra photon must be injected on the input mode a and exit on the output mode a. Crucially, as a consequence of bosonic statistics, a fifth term appears in Eq. (11) with a minus sign that accounts for quantum interference and may be viewed as an interference suppression term. In the special case where $i = k = 1$ and $\eta = 1/2$, we recover the standard HOM effect,

$$B_1^{(1,1)} = \tfrac{1}{2}\, B_0^{(0,1)} + \tfrac{1}{2}\, B_1^{(1,0)} + \tfrac{1}{2}\, B_1^{(0,1)} + \tfrac{1}{2}\, B_0^{(1,0)} - B_0^{(0,0)}$$
$$= 4 \times \tfrac{1}{2} \times \tfrac{1}{2} - 1 = 0. \quad (14)$$

Let us stress that this is a very unconventional proof of the HOM effect as Eq. (14) does not involve a linear combination of amplitudes but of probabilities. The first two terms account for both photons being transmitted while the third and fourth terms correspond to both of them being reflected. The fifth

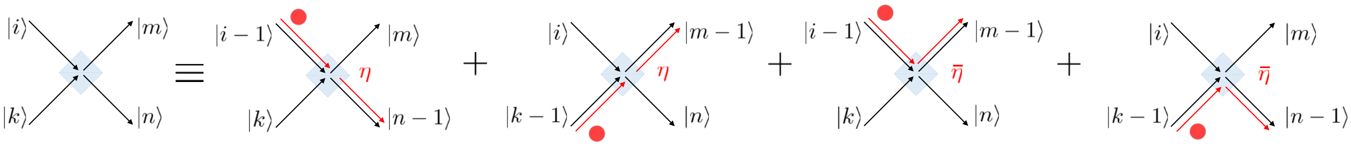

FIG. 2. Classical components of the recurrence formula (11) for the transition probability $B_n^{(i,k)}$ in a BS.

(negative) term has no classical counterpart. Note that if $k = 0$ and $i \geqslant 0$, the interference term disappears in Eq. (11) and one gets the recurrence $B_n^{(i,0)} = \eta\, B_{n-1}^{(i-1,0)} + \bar\eta\, B_n^{(i-1,0)}$, which had been derived in the context of majorization theory applied to bosonic transformations [36].

D. Distinguishable photons

It is instructive to give Eq. (11) further interpretation by juxtaposing it with its classical counterpart for distinguishable photons, which may for instance happen if the incident photons occupy different temporal modes. The classical probability of detecting n photons on output mode a when i and k photons impinge on input modes a and b is given by the convolution $p_{n|i,k} = \sum_{n'=0}^{n} p^A_{n'|i}\, p^B_{n-n'|k}$, where $p^A_{n|i}$ (or $p^B_{n|k}$) is the probability of getting n photons if i (or k) distinguishable photons enter mode a (or b), which itself follows a binomial distribution of parameter η (or $\bar\eta$), see [34] for details. Hence, the three-variate GF of $p_{n|i,k}$ is given by $g^{\rm cl}_\eta(\mathbf{u}) = g^A_\eta(x,z)\, g^B_\eta(y,z)$, where $g^A_\eta(x,z)$ and $g^B_\eta(y,z)$ are the two-variate GFs of $p^A_{n|i}$ and $p^B_{n|k}$. For instance, it is easy to show that $g^A_\eta(x,z) = 1/(1-\eta xz - \bar\eta x)$, so that $g^{\rm cl}_\eta(\mathbf{u})$ satisfies the relation

$$(1-\eta xz - \bar\eta x)\, g^{\rm cl}_\eta(\mathbf{u}) = 1(x)\, g^B_\eta(y,z), \tag{15}$$

where $1(x) \equiv 1$ is a constant function of x. Using again the shifting property of GFs, Eq. (15) implies the classical recurrence relation

$$p_{n|i,k} = \eta\, p_{n-1|i-1,k} + \bar\eta\, p_{n|i-1,k}, \quad i > 0, \tag{16}$$

where we have used the fact that $1(x)\, g^B_\eta(y,z)$ is the GF of $\delta_{i,0}\, p^B_{n|k}$ and can be ignored for $i > 0$. Interchanging $p^A_{n|i}$ and $p^B_{n|k}$, a similar reasoning yields

$$p_{n|i,k} = \eta\, p_{n|i,k-1} + \bar\eta\, p_{n-1|i,k-1}, \quad k > 0. \tag{17}$$

We notice here that Eq. (16) coincides with the first and third terms in Eq. (11), while Eq. (17) coincides with the second and fourth terms. If either $i = 0$ or $k = 0$ (i.e., no photon in one of the two input modes), then Eq. (11) reduces to the classical recurrence [for instance, Eq. (16) covers the case $k = 0$]. As advertised, the fifth (negative) term in Eq. (11) thus captures quantum interference (it appears as soon as $i, k > 0$) since it is absent from the classical formulas (16) and (17). Note also that removing this negative quantum term in Eq. (11) would then lead to twice the classical probability.

E. Active Gaussian transformations

An even more appealing application of our framework is to explore multiphoton interferences in an active transformation, such as a TMS. As proven in [23], a TMS may be viewed as a BS undergoing "partial time reversal," namely $\langle n, m|\, U^{\rm TMS}_\lambda\, |i, k\rangle = \sqrt{1-\lambda}\, \langle n, k|\, U^{\rm BS}_{1-\lambda}\, |i, m\rangle$. Indeed, indices k and m are interchanged, which may be interpreted as reverting the arrow of time of mode b [37]. Similarly, interchanging variables y and w, we see that the GFs are connected by $f^{\rm TMS}_\lambda(x,y,z,w) = (1-\lambda)\, f^{\rm BS}_{1-\lambda}(x,w,z,y)$, which is consistent with Eqs. (5) and (6). This allows us to write a recurrence for the transition probabilities $A_n^{(i,k)} := |\,\langle n, k-i+n|\, U^{\rm TMS}_\lambda\, |i,k\rangle\,|^2$ in a TMS (the definition of $A_n^{(i,k)}$ is extended to all integers i, k, n, setting it to zero when either of them is negative).

Theorem 2. If $i = k = n = 0$, then $A_n^{(i,k)} = \bar\lambda$, else,

$$A_n^{(i,k)} = \lambda\, A_n^{(i-1,k-1)} + \lambda\, A_{n-1}^{(i,k)} + \bar\lambda\, A_{n-1}^{(i-1,k)} + \bar\lambda\, A_n^{(i,k-1)} - A_{n-1}^{(i-1,k-1)}. \tag{18}$$

Proof. The relation can be easily proven by making use of Theorem 1, exploiting the fact that $A_n^{(i,k)} = \bar\lambda\, B_n^{(i,k-i+n)}$ with $\eta = \bar\lambda$ (or $\eta = 1/g$), see [34]. ■

Equation (18) is quite intriguing at first sight, as it is unclear how interferences take place in an active medium. However, as illustrated in Fig. 3, we may build an interpretation of Eq. (18) by considering all possible classical scenarios. The first term corresponds to the stimulated annihilation of an extra input photon pair, while the second term corresponds to the stimulated emission of an extra output photon pair (both occurring with probability $\propto\lambda$). The third and fourth terms correspond to an extra photon crossing the nonlinear medium without stimulating pair emission nor absorption (both with probability $\propto\bar\lambda$). Most importantly, the fifth (negative) term is again responsible for an unsuspected quantum interference effect, which has no classical counterpart. In the special case where $i = k = 1$ and $\lambda = 1/2$, we predict a complete extinction of the output state $|1\rangle\,|1\rangle$, which confirms a newly discovered two-photon interference effect in an amplifier of gain 2 [23] originating from *timelike* indistinguishability between the input and output photon pairs (exactly like the HOM effect can be viewed as a consequence of *spacelike* indistinguishability between two photons entering a BS of transmittance 1/2). Here again we find a surprising explanation of this effect based on the cancellation of probabilities (not amplitudes), namely

$$A_1^{(1,1)} = \tfrac{1}{2} A_1^{(0,0)} + \tfrac{1}{2} A_0^{(1,1)} + \tfrac{1}{2} A_0^{(0,1)} + \tfrac{1}{2} A_1^{(1,0)} - A_0^{(0,0)} = 4 \times \tfrac{1}{2} \times \tfrac{1}{4} - \tfrac{1}{2} = 0. \tag{19}$$

The first two terms account for events consisting of the stimulated annihilation of the input photon pair accompanied with the stimulated emission of a distinct output pair, while the third and fourth terms correspond to events where both photons cross the TMS. The fifth term is intrinsically quantum. Note that for $k = 0$, Eq. (18) reduces to the recurrence

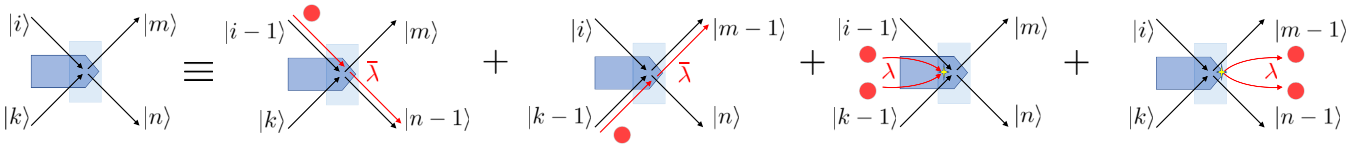

FIG. 3. Classical components of the recurrence formula (18) for the transition probabilities $A_n^{(i,k)}$ in a TMS.

$A_n^{(i,0)} = \lambda A_{n-1}^{(i,0)} + \bar{\lambda} A_{n-1}^{(i-1,0)}$ implying a majorization relation in a bosonic amplifier channel that was proven in [38].

F. Rational transmittance and gain

Coming back to passive BS transformations, it is easy to predict the existence of a HOM-like suppression effect for any rational value of the transmittance $\eta < 1$ provided some specific numbers of impinging photons are considered, namely

$$B_1^{(i,k)} = 0 \quad \text{if } \eta = k/(i+k), \tag{20}$$

as illustrated in Fig. 4. This can be understood as the result of amplitude cancellation between two scenarios, taking as a reference the situation where $i-1$ photons on mode a are reflected and $k-1$ photons on mode b are transmitted. The single photon observed on the output mode a may come from input mode a or b. Either the ith photon on mode a is transmitted (there are i possible choices) and all k photons on mode b are transmitted, which yields an amplitude $\propto i\,\eta$, or the kth photon on mode b is reflected (there are k possible choices) and all i photons on mode a are reflected, which yields an amplitude $\propto k\,\bar{\eta}$. Hence, we have $B_1^{(i,k)} \propto (i\,\eta - k\,\bar{\eta})^2$, which is consistent with Eq. (20). However, we provide a distinct interpretation in terms of probability cancellation as implied by Eq. (11), see [34]. For a quantum optical amplifier, we observe a similar effect for any rational value of the gain $g > 1$, namely

$$A_1^{(i,i+k-1)} = 0 \quad \text{if } \lambda = i/(i+k), \tag{21}$$

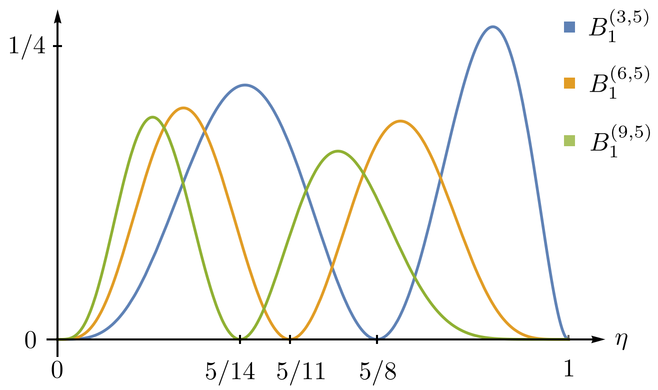

FIG. 4. Probability $B_1^{(i,k)}$ of observing a single photon on mode a at the output of a BS for $k = 5$ and for three different values $i = 3,\ 6,\ 9$ as a function of the transmittance η. We observe a HOM-like suppression effect for the corresponding rational values of $\eta = 5/8,\ 5/11,\ 5/14$.

corresponding to $g = 1 + i/k$ (see Fig. 5). This heretofore unknown interference effect can again be viewed as a consequence of probability cancellation in Eq. (18), see [34].

III. CONCLUSION AND OUTLOOK

Gaussian bosonic unitaries are readily described as affine transformations in phase space. Yet, addressing their action on Fock states typically leads to cumbersome calculations, which makes multiphotonic interferences in common Gaussian optical components hard to grasp. As a consequence, it is often an intractable task to prove fundamental entropy inequalities for Gaussian bosonic channels, while these are of major importance in optical quantum communication (see, e.g., the entropy photon-number inequality [39–41]). Here we have shown that the generating function of the matrix elements of a BS or TMS in Fock space can be expressed in a closed form, which, as a central consequence, yields simple recurrence equations for the multiphoton transition probabilities. In spite of the many interfering paths, Theorems 1 and 2 then provide a simple, intuitively appealing picture of multiphoton interference in passive and active bosonic circuits. It is amazing that such a simple account of quantum interferences in terms of probabilities (instead of amplitudes) in so well-studied optical components had yet remained unnoticed.

We have then predicted several multiphoton generalizations of the HOM effect in a BS of rational transmittance and have exploited the correspondence between a BS and TMS under partial time reversal [23] in order to reveal the existence of similar interferometric suppression effects in a quantum

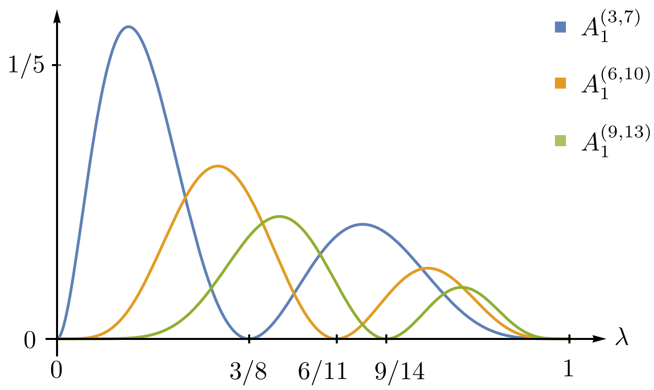

FIG. 5. Probability $A_1^{(i,i+k-1)}$ of observing a single photon on mode a at the output of a TMS for $k = 5$ and for three different values $i = 3,\ 6,\ 9$ as a function of the parameter λ. Note that k denotes here the number of output photons on mode b. We observe a suppression effect akin to the HOM effect for the corresponding rational values of $\lambda = 3/8,\ 6/11,\ 9/14$, or equivalently for values of the gain $g = 8/5,\ 11/5,\ 14/5$.

optical amplifier of rational gain. Interestingly, these predicted effects seem to escape the general framework for quantum suppression laws that has been derived in Refs. [42,43].

Let us stress that the generating function of transition probabilities can also be useful in studying other properties of Gaussian unitaries, for example their asymptotic behavior. Using Tauberian theorems, which state that if $g(z) \sim 1/(1-z)$ for $z \to 1$, then $\sum_{l=0}^{n} c_l \sim n$ for $n \to \infty$, it is indeed possible to approximate $B_n^{(i,i)}$ when $i \to \infty$ [34]. For $\eta = 1/2$, this exactly coincides with the asymptotic analysis of a BS with a large photon number in both input ports [44]. Note also that the generating function has recently been exploited in order to connect boson sampling with Fock-state inputs to boson sampling with thermal-state inputs [45], which is reminiscent of Eq. (4). Moreover, the technique developed here yields a powerful tool for characterizing certain non-Gaussian bosonic channels (those that are Gaussian dilatable), for example photon-added or photon-subtracted channels [46–48] as well as the linear coupling of a signal mode together with a passive environment [49].

Overall, beyond the results for a BS and TMS highlighted in this paper, we expect that our framework can be amenable to address any Bogoliubov transformation acting on an arbitrary number of modes. The special case of a multimode linear interferometer has already been considered in [50,51]. Although it does not seem to have implications for the complexity of simulating bosonic interferences, it provides a neat description of multimode multiphoton interference involving negative probabilities. Furthermore, we may anticipate other applications of this framework going beyond photonic systems. The same approach should indeed prove valuable for nonphotonic bosonic systems as well, since the transformations described by Eqs. (1) and (2) are not restricted to optical components but have quite a broad range of applications. In short, we have at hand a distinct approach to quantum multiparticle interferences in (passive and active) Bogoliubov transformations acting on any bosonic quantum systems.

ACKNOWLEDGMENTS

M.G.J. acknowledges support from the Wiener-Anspach Foundation and from the Carlsberg Foundation. This work was supported by the F.R.S.-FNRS under Project No. PDR T.0224.18 and by the EC under project ShoQC within the ERA-NET Cofund in Quantum Technologies (QuantERA) program.

[1] T. D. Ladd, F. Jelezko, R. Laflamme, Y. Nakamura, C. Monroe, and J. L. O'Brien, Quantum computers, Nature (London) **464**, 45 (2010).

[2] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Quantum cryptography, Rev. Mod. Phys. **74**, 145 (2002).

[3] D. Vasyukov, Y. Anahory, L. Embon, D. Halbertal, J. Cuppens, L. Neeman, A. Finkler, Y. Segev, Y. Myasoedov, M. L. Rappaport, M. E. Huber, and E. Zeldov, A scanning superconducting quantum interference device with single electron spin sensitivity, Nat. Nanotechnol. **8**, 639 (2013).

[4] A. Peruzzo, A. Laing, A. Politi, T. Rudolph, and J. L. O'Brien, Multimode quantum interference of photons in multiport integrated devices, Nat. Commun. **2**, 224 (2011).

[5] B. A. Bell, G. S. Thekkadath, R. Ge, X. Cai, and I. A. Walmsley, Testing multi-photon interference on a silicon chip, Opt. Express **27**, 35646 (2019).

[6] S. Aaronson and A. Arkhipov, The computational complexity of linear optics, in *Proceedings of the Forty-Third Annual ACM Symposium on Theory of Computing*, STOC '11 (Association for Computing Machinery, New York, 2011), p. 333.

[7] M. Tillmann, B. Dakić, R. Heilmann, S. Nolte, A. Szameit, and P. Walther, Experimental boson sampling, Nat. Photonics **7**, 540 (2013).

[8] A. Crespi, R. Osellame, R. Ramponi, D. J. Brod, E. F. Galvão, N. Spagnolo, C. Vitelli, E. Maiorino, P. Mataloni, and F. Sciarrino, Integrated multimode interferometers with arbitrary designs for photonic boson sampling, Nat. Photonics **7**, 545 (2013).

[9] M. A. Broome, A. Fedrizzi, S. Rahimi-Keshari, J. Dove, S. Aaronson, T. C. Ralph, and A. G. White, Photonic boson sampling in a tunable circuit, Science **339**, 794 (2013).

[10] J. Carolan, J. D. A. Meinecke, P. J. Shadbolt, N. J. Russell, N. Ismail, K. Wörhoff, T. Rudolph, M. G. Thompson, J. L. O'Brien, J. C. F. Matthews, and A. Laing, On the experimental verification of quantum complexity in linear optics, Nat. Photonics **8**, 621 (2014).

[11] C. K. Hong, Z. Y. Ou, and L. Mandel, Measurement of Subpicosecond Time Intervals between Two Photons by Interference, Phys. Rev. Lett. **59**, 2044 (1987).

[12] V. S. Shchesnovich, Universality of Generalized Bunching and Efficient Assessment of Boson Sampling, Phys. Rev. Lett. **116**, 123601 (2016).

[13] L. Rigovacca, C. Di Franco, B. J. Metcalf, I. A. Walmsley, and M. S. Kim, Nonclassicality Criteria in Multiport Interferometry, Phys. Rev. Lett. **117**, 213602 (2016).

[14] S. Agne, J. Jin, J. Z. Salvail, K. J. Resch, T. Kauten, E. Meyer-Scott, D. R. Hamel, G. Weihs, and T. Jennewein, Observation of Genuine Three-Photon Interference, Phys. Rev. Lett. **118**, 153602 (2017).

[15] A. J. Menssen, A. E. Jones, B. J. Metcalf, M. C. Tichy, S. Barz, W. S. Kolthammer, and I. A. Walmsley, Distinguishability and Many-Particle Interference, Phys. Rev. Lett. **118**, 153603 (2017).

[16] A. Crespi, R. Osellame, R. Ramponi, M. Bentivegna, F. Flamini, N. Spagnolo, N. Viggianiello, L. Innocenti, P. Mataloni, and F. Sciarrino, Suppression law of quantum states in a 3d photonic fast Fourier transform chip, Nat. Commun. **7**, 10469 (2016).

[17] S. W. Hawking, Particle creation by black holes, Commun. Math. Phys. **43**, 199 (1975).

[18] J. Traschen, An Introduction to Black Hole Evaporation, in *Mathematical Methods in Physics*, edited by A. A. Bytsenko and F. L. Williams (World Scientific, 2000), p. 180.

[19] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 1982).

[20] L. C. B. Crispino, A. Higuchi, and G. E. A. Matsas, The Unruh effect and its applications, Rev. Mod. Phys. **80**, 787 (2008).

[21] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Gaussian quantum information, Rev. Mod. Phys. **84**, 621 (2012).

[22] R. P. Feynman, Negative probability, in *Quantum Implications: Essays in Honour of David Bohm* (Routledge & Kegan Paul Ltd, London, 1987), pp. 235–248.

[23] N. J. Cerf and M. G. Jabbour, Two-boson quantum interference in time, Proc. Natl. Acad. Sci. **117**, 33107 (2020).

[24] S. L. Braunstein and P. van Loock, Quantum information with continuous variables, Rev. Mod. Phys. **77**, 513 (2005).

[25] N. J. Cerf, G. Leuchs, and E. S. Polzik, *Quantum Information with Continuous Variables of Atoms and Light* (Imperial College Press, London, 2007).

[26] P. van Loock, C. Weedbrook, and M. Gu, Building Gaussian cluster states by linear optics, Phys. Rev. A **76**, 032321 (2007).

[27] M. Gu, C. Weedbrook, N. C. Menicucci, T. C. Ralph, and P. van Loock, Quantum computing with continuous-variable clusters, Phys. Rev. A **79**, 062318 (2009).

[28] M. V. Larsen, X. Guo, C. R. Breum, J. S. Neergaard-Nielsen, and U. L. Andersen, Deterministic generation of a two-dimensional cluster state, Science **366**, 369 (2019).

[29] S. Gerke, J. Sperling, W. Vogel, Y. Cai, J. Roslund, N. Treps, and C. Fabre, Full Multipartite Entanglement of Frequency-Comb Gaussian States, Phys. Rev. Lett. **114**, 050501 (2015).

[30] I. Kogias, A. R. Lee, S. Ragy, and G. Adesso, Quantification of Gaussian Quantum Steering, Phys. Rev. Lett. **114**, 060403 (2015).

[31] N. J. Cerf, A. Ipe, and X. Rottenberg, Cloning of Continuous Quantum Variables, Phys. Rev. Lett. **85**, 1754 (2000).

[32] N. N. Bogoljubov, On a new method in the theory of superconductivity, Il Nuovo Cimento (1955–1965) **7**, 794 (1958).

[33] K. Brádler and C. Adami, Black holes as bosonic Gaussian channels, Phys. Rev. D **92**, 025030 (2015).

[34] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevResearch.3.043065 for details on the calculation of the multiphoton transition probabilities and associated generating function for a beam splitter (and similarly for a two-mode squeezer), the classical benchmark of distinguishable photons, and the asymptotic calculation of the transition probabilities based on the generating function, which includes additional Refs. [52–56].

[35] H. Scutaru, Fidelity for displaced squeezed thermal states and the oscillator semigroup, J. Phys. A: Math. Theor. **31**, 3659 (1998).

[36] C. N. Gagatsos, O. Oreshkov, and N. J. Cerf, Majorization relations and entanglement generation in a beam splitter, Phys. Rev. A **87**, 042307 (2013).

[37] N. J. Cerf, The optical beam splitter under partial time reversal (2012), 9th Central European Workshop on Quantum Optics (CEWQO 2012), Sinaia, Romania.

[38] R. García-Patrón, C. Navarrete-Benlloch, S. Lloyd, J. H. Shapiro, and N. J. Cerf, Majorization Theory Approach to the Gaussian Channel Minimum Entropy Conjecture, Phys. Rev. Lett. **108**, 110505 (2012).

[39] S. Guha, B. I. Erkmen, and J. H. Shapiro, The entropy photon-number inequality and its consequences, arXiv:0710.5666.

[40] R. König and G. Smith, The entropy power inequality for quantum systems, IEEE Trans. Inf. Theory **60**, 1536 (2014).

[41] G. De Palma, A. Mari, and V. Giovannetti, A generalization of the entropy power inequality to bosonic quantum systems, Nat. Photonics **8**, 958 (2014).

[42] C. Dittel, G. Dufour, M. Walschaers, G. Weihs, A. Buchleitner, and R. Keil, Totally Destructive Many-Particle Interference, Phys. Rev. Lett. **120**, 240404 (2018).

[43] C. Dittel, G. Dufour, M. Walschaers, G. Weihs, A. Buchleitner, and R. Keil, Totally destructive interference for permutation-symmetric many-particle states, Phys. Rev. A **97**, 062116 (2018).

[44] H. Nakazato, S. Pascazio, M. Stobińska, and K. Yuasa, Photon distribution at the output of a beam splitter for imbalanced input states, Phys. Rev. A **93**, 023845 (2016).

[45] Y. Kim, K.-H. Hong, Y.-H. Kim, and J. Huh, Connection between Boson sampling with quantum and classical input states, Opt. Express **28**, 6929 (2020).

[46] C. Navarrete-Benlloch, R. García-Patrón, J. H. Shapiro, and N. J. Cerf, Enhancing quantum entanglement by photon addition and subtraction, Phys. Rev. A **86**, 012328 (2012).

[47] K. K. Sabapathy and A. Winter, Non-Gaussian operations on bosonic modes of light: Photon-added Gaussian channels, Phys. Rev. A **95**, 062309 (2017).

[48] S. M. Barnett, G. Ferenczi, C. R. Gilson, and F. C. Speirits, Statistics of photon-subtracted and photon-added states, Phys. Rev. A **98**, 013809 (2018).

[49] M. G. Jabbour and N. J. Cerf, Fock majorization in bosonic quantum channels with a passive environment, J. Phys. A: Math. Theor. **52**, 105302 (2019).

[50] M. G. Jabbour, Bosonic systems in quantum information theory: Gaussian-dilatable channels, passive states, and beyond, Ph.D. thesis, Université libre de Bruxelles, 2018.

[51] M. G. Jabbour and N. J. Cerf, Revisiting quantum interferences in multimode Gaussian circuits (unpublished).

[52] M. S. Kim, W. Son, V. Bužek, and P. L. Knight, Entanglement by a beam splitter: Nonclassicality as a prerequisite for entanglement, Phys. Rev. A **65**, 032323 (2002).

[53] J. S. Ivan, K. K. Sabapathy, and R. Simon, Operator-sum representation for bosonic Gaussian channels, Phys. Rev. A **84**, 042311 (2011).

[54] P. Flajolet and R. Sedgewick, *Analytic Combinatorics* (Cambridge University Press, Cambridge, 2009).

[55] G. H. Hardy and J. E. Littlewood, Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive, Proc. London Math. Soc. **s2-13**, 174 (1914).

[56] J. Karamata, Über die Hardy-Littlewoodschen umkehrungen des abelschen stetigkeitssatzes, Math. Z. **32**, 319 (1930).