Supplemental Material for
“Multiparticle quantum interference in Bogoliubov bosonic transformations”

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The present material provides supplementary information to the main document [1]. It includes calculations which are standard in both quantum optics and the theory of generating functions, which we chose not to cover in the main document. Section I provides the calculation of the explicit expression of the multiphoton transition probabilities in a beam splitter, which is used as a benchmark to exhibit the interest of our developed framework. Section II covers the calculation of the generating functions of these probabilities as well as of the corresponding probabilities for a two-mode squeezer (whose explicit form we chose not to include as its complexity makes it of little interest). The recurrence equations on the multiphoton transition probabilities that can be deduced from these generating functions (see [1]) are then discussed in the case of a beam splitter with a rational transmittance or two-mode squeezer with a rational gain. The classical transition probabilities for distinguishable photons impinging on a beam splitter, which serves as a reference in the analysis of our main results, is summarized in Section III. Finally, Section IV explores the asymptotic behavior of the transition probabilities, illustrating the power of generating functions.

I. MULTIPHOTON TRANSITION PROBABILITIES IN A BEAM SPLITTER

Consider a beam splitter (BS) of transmittance $\eta \in [0,1]$ characterized by the unitary U_η^{BS} of the form

$$U_\eta^{\mathrm{BS}} = \exp\left[\theta\left(\hat{a}^\dagger \hat{b} - \hat{a}\hat{b}^\dagger\right)\right], \quad \eta = \cos^2\theta. \tag{1}$$

An expression for the transition amplitudes $b_n^{(i,k)} \coloneqq \langle n,m| U_\eta^{\mathrm{BS}} |i,k\rangle$ (for $i,k,n \in \mathbb{N}_0$ and $m = i+k-n$) can be computed by first deriving an expression for the following state:

$$\left|\psi^{(i,k)}\right\rangle = U_\eta^{\mathrm{BS}}|i,k\rangle = \frac{1}{\sqrt{k!}} U_\eta^{\mathrm{BS}} (\hat{b}^\dagger)^k U_\eta^{\mathrm{BS}\dagger} U_\eta^{\mathrm{BS}} |i,0\rangle = \frac{1}{\sqrt{k!}}\left(U_\eta^{\mathrm{BS}} \hat{b}^\dagger U_\eta^{\mathrm{BS}\dagger}\right)^k \left|\psi^{(i,0)}\right\rangle. \tag{2}$$

Exploiting the action of the BS in phase space, we get

$$\left|\psi^{(i,k)}\right\rangle = \frac{1}{\sqrt{k!}}\left(\sqrt{1-\eta}\hat{a}^\dagger + \sqrt{\eta}\hat{b}^\dagger\right)^k \left|\psi^{(i,0)}\right\rangle = \frac{1}{\sqrt{k!}} \sum_{m=0}^{k} \binom{k}{m} \left(\sqrt{1-\eta}\hat{a}^\dagger\right)^m \left(\sqrt{\eta}\hat{b}^\dagger\right)^{k-m} \left|\psi^{(i,0)}\right\rangle. \tag{3}$$

Similarly, we have

$$\left|\psi^{(i,0)}\right\rangle = \frac{1}{\sqrt{i!}}\left(\sqrt{\eta}\hat{a}^\dagger - \sqrt{1-\eta}\hat{b}^\dagger\right)^i |0,0\rangle = \sum_{n=0}^{i} (-1)^{i-n} \sqrt{\binom{i}{n} \eta^n (1-\eta)^{i-n}}\, |n, i-n\rangle. \tag{4}$$

Combining Eqs. (3) and (4), we obtain

$$\left|\psi^{(i,k)}\right\rangle = \sum_{n,m=0}^{i,k} (-1)^{i-n} \sqrt{\Gamma_{n,m}^{(i,k)} \eta^{n+k-m} (1-\eta)^{i-n+m}}\, |n+m, i-n+k-m\rangle, \tag{5}$$

where we defined

$$\Gamma_{n,m}^{(i,k)} = \binom{i}{n}\binom{k}{m}\binom{n+m}{n}\binom{i-n+k-m}{i-n}. \tag{6}$$

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Form this, we obtain

$$b_n^{(i,k)} = \sum_{m=\max(0,n-k)}^{\min(i,n)} (-1)^{i-m} \sqrt{\Gamma_{m,n-m}^{(i,k)} \eta^{2m+k-n} (1-\eta)^{i-2m+n}}, \tag{7}$$

where

$$\Gamma_{n,m}^{(i,k)} := \binom{i}{n} \binom{k}{m} \binom{n+m}{n} \binom{i-n+k-m}{i-n}. \tag{8}$$

Thus, the transition probabilities $B_n^{(i,k)} := \left| b_n^{(i,k)} \right|^2$ in the BS U_η^{BS} are given by

$$B_n^{(i,k)} = \sum_{m,j=\max(0,n-k)}^{\min(i,n)} (-1)^{m+j} \sqrt{\Gamma_{m,n-m}^{(i,k)} \Gamma_{j,n-j}^{(i,k)}} \eta^{k-n+m+j} (1-\eta)^{i+n-m-j}. \tag{9}$$

Now, it can easily be shown that

$$\Gamma_{m,n-m}^{(i,k)} \Gamma_{j,n-j}^{(i,k)} = \gamma_{n,m,j}^{(i,k)} \gamma_{n,j,m}^{(i,k)}, \tag{10}$$

where we defined

$$\gamma_{n,m,j}^{(i,k)} := \binom{i}{m} \binom{k}{n-m} \binom{n}{j} \binom{i+k-n}{i-j}. \tag{11}$$

Since $\gamma_{n,m,j}^{(i,k)} = \gamma_{n,j,m}^{(i,k)}$, we end up with $\Gamma_{m,n-m}^{(i,k)} \Gamma_{j,n-j}^{(i,k)} = (\gamma_{n,m,j}^{(i,k)})^2$, so that

$$B_n^{(i,k)} = \sum_{m,j=\max(0,n-k)}^{\min(i,n)} (-1)^{m+j} \gamma_{n,m,j}^{(i,k)} \eta^{k-n+m+j} (1-\eta)^{i+n-m-j}. \tag{12}$$

This expression can of course be evaluated, but it is rather cumbersome and hence not very useful for the analytical investigation of multiphoton interferometry in a beam splitter. For instance, it is written as a double summation over terms with alternating signs, so that the positivity of the transition probability is not obvious from the expression. Similar derivations have for instance been given in [2–4].

II. GENERATING FUNCTIONS OF THE MULTIPHOTON TRANSITION PROBABILITIES

A. Case of a beam splitter

The generating function (GF) of the transition probability $| \langle n,m | U_\eta^{\mathrm{BS}} | i,k \rangle |^2$ in a BS is given by a function $f_\eta^{\mathrm{BS}} : [0,1)^2 \times [0,1]^2 \to [0,\infty)$ defined as

$$f_\eta^{\mathrm{BS}}(x,y,z,w) := \sum_{i,k,n,m} | \langle n,m | U_\eta^{\mathrm{BS}} | i,k \rangle |^2 x^i y^k z^n w^m, \tag{13}$$

where, when we omit limits in summations, it means that the summation is carried out over all natural numbers in $\mathbb{N}_0$ (including 0). The trick is to realize that it can rewritten as

$$\begin{aligned} f_\eta^{\mathrm{BS}}(x,y,z,w) &= \mathrm{Tr} \left[U_\eta^{\mathrm{BS}} \left(\sum_i x^i |i\rangle \langle i| \otimes \sum_k y^k |k\rangle \langle k| \right) U_\eta^{\mathrm{BS}\dagger} \left(\sum_n z^n |n\rangle \langle n| \otimes \sum_m w^m |m\rangle \langle m| \right) \right] \\ &= \frac{1}{(1-x)(1-y)(1-z)(1-w)} \mathrm{Tr} \left[U_\eta^{\mathrm{BS}} (\tau_x \otimes \tau_y) U_\eta^{\mathrm{BS}\dagger} (\tau_z \otimes \tau_w) \right] \end{aligned}$$

where τ_t is a Gaussian thermal state of parameter t [5], i.e,

$$\tau_t := (1-t) \sum_m t^m |m\rangle \langle m| . \tag{14}$$

Now, the object $\rho_1 \coloneqq U_\eta^{\rm BS}\left(\tau_x \otimes \tau_y\right) U_\eta^{\rm BS\dagger}$ actually represents the effect of a beam-splitter unitary on the tensor product of two Gaussian thermal states, making it a two-mode Gaussian state. The object $\rho_2 \coloneqq \tau_z \otimes \tau_w$ is obviously a two-mode Gaussian state as well. This means that $f_\eta^{\rm BS}(x,y,z,w)$ is proportional to the overlap $\operatorname{Tr}[\rho_1\rho_2]$ between the two Gaussian states ρ_1 and ρ_2,

$$f_\eta^{\rm BS}(x,y,z,w) = \frac{1}{(1-x)(1-y)(1-z)(1-w)} \operatorname{Tr}[\rho_1\rho_2]\,. \tag{15}$$

The above quantity can therefore be computed easily using standard tools of Gaussian quantum optics, *i.e.*, the symplectic formalism applied to the phase-space representation of bosonic quantum systems [5]. Since the first moments of each of the two Gaussian states ρ_1 and ρ_2 is zero, their overlap can be computed using the formula [6]

$$\operatorname{Tr}[\rho_1\rho_2] = \left(\det\left[\frac{V_1+V_2}{2}\right]\right)^{-\frac{1}{2}} = \frac{4}{\sqrt{\det[V_1+V_2]}}, \tag{16}$$

where V_1 and V_2 are the respective covariance matrices of ρ_1 and ρ_2. Some easy matrix algebra involving covariance matrices and the symplectic matrix of the BS in phase space finally yields

$$f_\eta^{\rm BS}(x,y,z,w) = \frac{1}{1-\eta xz-(1-\eta)xw-\eta yw-(1-\eta)yz+xyzw}. \tag{17}$$

The conservation of energy in the BS can be easily verified using the GF given by the above equation. Define the function $\tilde{f}_\eta^{\rm BS} : [0,1)^2 \times [0,1]^3 \to [0,\infty)$ as

$$\tilde{f}_\eta^{\rm BS}(x,y,z,w,t) \coloneqq \sum_{i,k,n,m} |\langle n,m| U_\eta^{\rm BS} |i,k\rangle|^2 x^i y^k z^n w^m t^{i+k-n-m} = f_\eta^{\rm BS}(xt, yt, \frac{z}{t}, \frac{w}{t}). \tag{18}$$

From Eq. (17), we have

$$\tilde{f}_\eta^{\rm BS}(x,y,z,w,t) = f_\eta^{\rm BS}(x,y,z,w), \quad \forall t. \tag{19}$$

This actually means that $\tilde{f}_\eta^{\rm BS}$ as defined in Eq. (18) does not depend on variable t, so that the only non-zero elements in the sums of the right-hand side of Eq. (18) verify $i+k-n-m=0$. Consequently,

$$\langle n,m| U_\eta^{\rm BS} |i,k\rangle = 0 \quad \text{if} \quad i+k \neq n+m. \tag{20}$$

B. Symmetric inputs to the beam splitter

We now consider the case in which the same Fock states impinge on both inputs of the BS and compute the GF of the corresponding transition probabilities, which will be useful when investigating the asymptotic behavior of $B_n^{(i,i)}$ (see Section IV). The sequence $B_n^{(i,k)}$ depends on 3 indices only, index m in $|\langle n,m| U_\eta^{\rm BS} |i,k\rangle|^2$ being redundant as a consequence of energy conservation in a BS. The GF of $B_n^{(i,k)}$ is then simply given by

$$\begin{aligned}\mathcal{T}_{i,k,n}\left[B_n^{(i,k)}\right](x,y,z) &\coloneqq \sum_{i,k,n} B_n^{(i,k)} x^i y^k z^n \\ &= \sum_{i,k,n}\left(\sum_m |\langle n,m| U_\eta^{\rm BS} |i,k\rangle|^2\right) x^i y^k z^n \\ &= f_\eta^{\rm BS}(x,y,z,1).\end{aligned}$$

In order to derive the GF of the diagonal elements $B_n^{(i,i)}$, we force the relation $k=i$ in the GF of $B_n^{(i,k)}$ by only considering the elements which satisfy it. Using the notation $c_n = [z^n]g(z)$ to mean that we select the coefficient of the z^n term in $g(z) \coloneqq \sum_{n=0}^{\infty} c_n z^n$, we write

$$\mathcal{T}_{i,n}\left[B_n^{(i,i)}\right](x,z) = [y^0] \sum_{i,k,n} B_n^{(i,k)} x^i y^{k-i} z^n = [y^0] f_\eta^{\rm BS}\left(\frac{x}{y}, y, z, 1\right). \tag{21}$$

By Cauchy's integral formula for any function $g(z)$, one has

$$g(a)=\frac{1}{2\pi i}\oint \mathrm{d}z\frac{g(z)}{z-a}. \tag{22}$$

Applying this to our case, we get that, for some circle γ_x around $y=0$,

$$\mathcal{T}_{i,n}\left[B_n^{(i,i)}\right](x,z)=\frac{1}{2\pi i}\int_{\gamma_x}\mathrm{d}y\frac{f_\eta^{\mathrm{BS}}\left(x/y,y,z,1\right)}{y}. \tag{23}$$

Now, using the Residue Theorem, the above equation amounts to

$$\mathcal{T}_{i,n}\left[B_n^{(i,i)}\right](x,z)=\sum_l \mathrm{Res}\left[\frac{f_\eta^{\mathrm{BS}}\left(x/y,y,z,1\right)}{y};y=s_l(x,z,\eta)\right], \tag{24}$$

where the s_l represent the singularities of $f_\eta^{\mathrm{BS}}\left(x/y,y,z,1\right)/y$ satisfying $\lim_{x\to 0}s_l(x,z,\eta)=0$. Some standard calculations yield

$$s_{1,2}(x,z,\eta)=\frac{1+xz\pm\sqrt{(1+xz)^2-4(\eta+(1-\eta)z)(x(1-\eta)+\eta xz))}}{2(\eta+z(1-\eta))}, \tag{25}$$

with the subscript $1(2)$ corresponding to $+(-)$ in the $\pm$ in the above equation. If we take their limits for x approaching zero, we obtain

$$\lim_{x\to 0}s_1(x,z,\eta)=0 \quad\text{and}\quad \lim_{x\to 0}s_2(x,z,\eta)=\frac{1}{\eta+z(1-\eta)}. \tag{26}$$

The residue of the function we are interested in reduces to

$$\mathrm{Res}\left[\frac{f_\eta^{\mathrm{BS}}\left(x/y,y,z,1\right)}{y};y=s_1(x,z,\eta)\right]=\frac{1}{\sqrt{(1+xz)^2-4(\eta+(1-\eta)z)(x(1-\eta)+\eta xz))}}, \tag{27}$$

so that

$$\mathcal{T}_{i,n}\left[B_n^{(i,i)}\right](x,z)=\frac{1}{\sqrt{(1+xz)^2-4(\eta+(1-\eta)z)(x(1-\eta)+\eta xz))}}. \tag{28}$$

If we particularize this to a balanced BS ($\eta=1/2$), we obtain the simple expression

$$\mathcal{T}_{i,n}\left[B_n^{(i,i)}\Big|_{\eta=1/2}\right](x,z)=\frac{1}{\sqrt{(1-x)(1-z^2x)}}, \tag{29}$$

which is the GF in i,n of the diagonal sequence $B_n^{(i,i)}$ for $\eta=1/2$. This will be useful for analyzing the asymptotic behavior of the transition probabilities in Section IV.

C. Case of a two-mode squeezer

The GF of the transition probability $|\langle n,m|\,U_r^{\mathrm{TMS}}\,|i,k\rangle|^2$ in a two-mode squeezer (TMS) is given by a function $f_\lambda^{\mathrm{TMS}}:[0,1)^2\times[0,1]^2\to[0,\infty)$ defined as

$$f_\lambda^{\mathrm{TMS}}(x,y,z,w):=\sum_{i,k,n,m}|\langle n,m|\,U_r^{\mathrm{TMS}}\,|i,k\rangle|^2x^iy^kz^nw^m, \tag{30}$$

with $\lambda=\tanh^2(r)$. One could compute it from scratch similarly as in the previous section. A much more elegant option is to use a fundamental relation that links the BS and the TMS, which was proven in [7]. There, it was shown that the TMS may be viewed as a BS undergoing "partial time reversal" [8], $|\langle n,m|\,U_\lambda^{\mathrm{TMS}}\,|i,k\rangle|^2=(1-\lambda)\,|\langle n,k|\,U_{1-\lambda}^{\mathrm{BS}}\,|i,m\rangle|^2$. As explained in [1], it implies that the GFs are connected by the relation

$$f_\lambda^{\mathrm{TMS}}(x,y,z,w)=(1-\lambda)\,f_{1-\lambda}^{\mathrm{BS}}(x,w,z,y), \tag{31}$$

which then yields

$$f_\lambda^{\mathrm{TMS}}(x,y,z,w)=\frac{1-\lambda}{1-\lambda(xy+zw)-(1-\lambda)(xz+yw)+xyzw}. \tag{32}$$

D. Rational value of the transmittance or gain

We now discuss the interferometric suppression effect that exists for a BS of rational transmittance $\eta < 1$ or for a TMS of rational gain $g > 1$. It can be checked that

$$B_1^{(i,k)} = 0 \qquad \text{if } \eta = \frac{k}{i+k}, \qquad \text{where } i, k \in \mathbb{N}_0, i+k > 0, \tag{33}$$

which gives rise a to full interferometric suppression for any rational $\eta < 1$, extending the HOM effect for $i = k = 1$ and $\eta = 1/2$. Indeed, using the mode transformation characterizing the BS [1], a closed expression for $B_1^{(i,k)}$ can easily be written, which entails the sum of the amplitudes where either a single photon from mode a is transmitted (all k photons on mode b being transmitted) or a single photon from mode b is reflected (all i photons on mode a being reflected) weighted with the appropriate combinatorial factors, namely

$$B_1^{(i,k)} = \frac{i+k-1}{i\,k} \, (i\,\eta - k\,\bar{\eta})^2 \, B_0^{(i-1,k-1)} \,, \tag{34}$$

where we have used the trivial expression

$$B_0^{(i,k)} = \binom{i+k}{i} \bar{\eta}^i \, \eta^k \,. \tag{35}$$

Note that given the symmetry between the two modes as well as the input-output symmetry, we have three associated interferometric suppressions (having in common a single photon in either one of the input or output mode):

$$\begin{aligned} B_{i+k-1}^{(i,k)} &= 0 && \text{if } \eta = \frac{i}{i+k}, i+k > 0, \\ B_n^{(1,k+n-1)} &= 0 && \text{if } \eta = \frac{k}{k+n}, k+n > 0, \\ B_n^{(i+n-1,1)} &= 0 && \text{if } \eta = \frac{n}{n+i}, i+n > 0. \end{aligned} \tag{36}$$

It is instructive to examine the interference effect (33) by using the recurrence equation for a BS derived in [1], yielding

$$B_1^{(i,k)} = \eta\, B_0^{(i-1,k)} + \eta\, B_1^{(i,k-1)} + \bar{\eta}\, B_1^{(i-1,k)} + \bar{\eta}\, B_0^{(i,k-1)} - B_0^{(i-1,k-1)}. \tag{37}$$

Note first that applying the recurrence equation to $B_0^{(i,k)}$ instead of $B_1^{(i,k)}$ gives

$$B_0^{(i,k)} = \eta\, B_0^{(i,k-1)} + \bar{\eta}\, B_0^{(i-1,k)} \,, \tag{38}$$

which, using Eq. (35), simply reduces to Pascal's formula for binomial coefficients

$$\binom{i+k}{i} = \binom{i+k-1}{i} + \binom{i+k-1}{i-1}. \tag{39}$$

Thus, as expected, the recurrence equation (38) is classical and can be recovered using simple combinatorial analysis. By comparison, Eq. (37) gives a more interesting recurrence. Using the above closed expression for $B_0^{(i,k)}$, it can be rewritten as

$$B_1^{(i,k)} = \eta\, B_1^{(i,k-1)} + \bar{\eta}\, B_1^{(i-1,k)} - \kappa\, B_0^{(i-1,k-1)}, \qquad \text{with } \kappa = 1 - (i+k-1)\left(\frac{\eta^2}{k} + \frac{\bar{\eta}^2}{i}\right). \tag{40}$$

The closed expression (34) is of course a solution of Eq. (40), but we see that the recurrence here is more interesting as it exhibits again a negative probability term. If we choose $\eta = k/(i+k)$ (with $i+k > 0$), then $\kappa = 1/(i+k)$, so the interferometric suppression effect $B_1^{(i,k)} = 0$ translates Eq. (40) into

$$k\, B_1^{(i,k-1)} + i\, B_1^{(i-1,k)} - B_0^{(i-1,k-1)} = 0\,. \tag{41}$$

The first and second terms can be interpreted classically : starting from the case where the output mode a already contains a single photon, the first term accounts for the kth photon in mode b being transmitted (so it does not give an extra photon on output mode a) while the second term accounts for the ith photon in mode a being reflected (so it does not give an extra photon on output mode a). Remarkably, the third (negative) term has no classical meaning and results in the full cancellation of probability $B_1^{(i,k)}$.

The same analysis can be applied to a quantum optical amplifier. As shown in [1], we have

$$A_n^{(1,k)} = 0 \qquad \text{if } \lambda = \frac{n}{n+k} \text{ or } g = 1 + \frac{n}{k}, \qquad \text{where } n,k \in \mathbb{N}_0, k > 0, \tag{42}$$

which gives rise to full a interferometric suppression for any rational $\lambda < 1$ (or rational gain $g = 1/(1-\lambda) > 1$). When $k = n = 1$ and $\lambda = 1/2$, we confirm the existence of an interferometric suppression effect in a parametric amplifier of gain $g = 2$ [7]. Similarly as for a BS, we also have three associated interferometric suppressions in a TMS, namely

$$\begin{aligned} A_{i+n-1}^{(i,1)} &= 0 \qquad && \text{if } \lambda = \frac{n}{i+n} \text{ or } g = 1 + \frac{n}{i}, i > 0, \\ A_1^{(i,i+k-1)} &= 0 && \text{if } \lambda = \frac{i}{i+k} \text{ or } g = 1 + \frac{k}{i}, i > 0, \\ A_n^{(k+n-1,k)} &= 0 && \text{if } \lambda = \frac{k}{k+n} \text{ or } g = 1 + \frac{k}{n}, n > 0. \end{aligned} \tag{43}$$

Let us examine the interference Eq. (42) by using the recurrence equation for a TMS,

$$A_n^{(1,k)} = \lambda\, A_n^{(0,k-1)} + \lambda\, A_{n-1}^{(1,k)} + \bar{\lambda}\, A_{n-1}^{(0,k)} + \bar{\lambda}\, A_n^{(1,k-1)} - A_{n-1}^{(0,k-1)}. \tag{44}$$

Using the closed expression

$$A_n^{(0,k)} = (1-\lambda)\, B_0^{(n,k)} = \binom{n+k}{k} \lambda^n \bar{\lambda}^{k+1}, \tag{45}$$

we may reexpress it as

$$A_n^{(1,k)} = \lambda\, A_{n-1}^{(1,k)} + \bar{\lambda}\, A_n^{(1,k-1)} - \kappa' A_{n-1}^{(0,k-1)}, \qquad \text{with } \kappa' = 1 - (n+k-1)\left(\frac{\lambda^2}{n} + \frac{\bar{\lambda}^2}{k}\right). \tag{46}$$

Similarly as for a BS, if we choose $\lambda = n/(n+k)$ (or $g = 1 + n/k$, with $k > 0$), the interferometric suppression $A_n^{(1,k)} = 0$ implies

$$n\, A_{n-1}^{(1,k)} + k\, A_n^{(1,k-1)} - A_{n-1}^{(0,k-1)} = 0 \tag{47}$$

Here, taking as a reference the situation where the input mode a contains a single photon, the first term accounts for the stimulated emission of a photon pair at the output (with probability $\propto \lambda$) while the second term accounts for the single photon in mode b being transmitted (with probability $\propto \bar{\lambda}$). Again, the third (negative) term has no classical interpretation and is responsible for the cancellation $A_n^{(1,k)} = 0$.

III. MULTIPHOTON TRANSITION PROBABILITIES IN A BEAM SPLITTER WITH DISTINGUISHABLE PHOTONS

Consider a situation in which the photons impinging on the two input modes a and b of the BS are distinguishable. The incident photons may for instance have different polarizations. We now count the photons exiting the BS in mode a', without making a distinction between different photons. The fact that they are distinguishable will however affect the distributions of photons in the output modes. The probability $\mathbb{P}(n \text{ in } a'|i \text{ in } a)$ that we detect n photons in output mode a' when we sent i photons in mode a is given by a simple binomial of parameter η, *i.e.*,

$$p_a(n|i) \coloneqq \mathbb{P}(n \text{ in } a'|i \text{ in } a) = \binom{i}{n} \eta^n (1-\eta)^{i-n}. \tag{48}$$

Similarly, the probability $\mathbb{P}(n \text{ in } a'|k \text{ in } b)$ that we detect n photons in mode a' when we sent k photons in mode b is given by

$$p_b(n|k) \coloneqq \mathbb{P}(n \text{ in } a'|k \text{ in } b) = \binom{k}{n} (1-\eta)^n \eta^{k-n}. \tag{49}$$

Using this, the probability $p(n|i,k) \coloneqq \mathbb{P}(n \text{ in } a'|i \text{ in } a \text{ and } k \text{ in } b)$ that we detect n photons in mode a' when we sent i photons in mode a and k photons in mode b can be calculated using a convolution

$$p(n|i,k) = \sum_{n'=0}^{n} p_a(n'|i) p_b(n-n'|k). \tag{50}$$

The 3-variate GF of the sequence $p(n|i,k)$ is given by a function $g_\eta^{\text{cl}} : [0,1)^2 \times [0,1] \to [0,\infty)$ defined as

$$g_\eta^{\text{cl}}(x,y,z) \coloneqq \mathcal{T}_{i,k,n}\left[p(n|i,k)\right](x,y,z) = \sum_{i,k,n} p(n|i,k) x^i y^k z^n. \tag{51}$$

Since the sequence $p(n|i,k)$ is given by a convolution over index n of the sequences $p_a(n|i)$ and $p_b(n|k)$, the GF $g_\eta^{\text{cl}}(x,y,z)$ is simply given by the product of their two respective GFs $g_\eta^a : [0,1) \times [0,1]$ and $g_\eta^b : [0,1) \times [0,1]$, which can simply be computed as

$$g_\eta^a(x,z) = \frac{1}{1-\eta xz-(1-\eta)x}, \qquad g_\eta^b(y,z) = \frac{1}{1-\eta y-(1-\eta)yz}. \tag{52}$$

This means that g_η^{cl} satisfies the relation

$$\left[1-\eta xz-(1-\eta)x\right] g_\eta^{\text{cl}}(x,y,z) = g_\eta^b(y,z) = \sum_{i=0}^{\infty} g_\eta^b(y,z)\delta_{i0}x^i, \tag{53}$$

where $\delta_{..}$ denotes a Kronecker delta. Using the shifting property of the GFs, the counterpart of Eq. (53) for sequences is

$$p(n|i,k) - \eta\, p(n-1|i-1,k) - \bar{\eta}\, p(n|i-1,k) = \delta_{i0}\, p_a(n|k), \tag{54}$$

where $\bar{\eta} = 1-\eta$ and $p(n|i,k) = 0$ if either of the indices n, i, k is negative. A similar reasoning yields

$$p(n|i,k) - \bar{\eta}\, p(n-1|i,k-1) - \eta\, p(n|i,k-1) = \delta_{k0}\, p_b(n|i). \tag{55}$$

Obviously, by summing the two relations one can always write the weaker relation

$$\begin{aligned} p(n|i,k) = {} & \nu\left[\eta\, p(n-1|i-1,k) + \bar{\eta}\, p(n|i-1,k) + \delta_{i0}\, p_b(n|k)\right] \\ & + (1-\nu)\left[\bar{\eta}\, p(n-1|i,k-1) + \eta\, p(n|i,k-1) + \delta_{k0}\, p_a(n|i)\right] \end{aligned} \tag{56}$$

for any $\nu \in [0,1]$, which amounts to

$$p(n|i,k) = \begin{cases} 1, & \text{if } i=0 \text{ and } k=0, \\ \nu\, p_b(n|k) + (1-\nu)\left[\bar{\eta}\, p(n-1|i,k-1) + \eta\, p(n|i,k-1)\right], & \text{if } i=0 \text{ and } k\neq 0, \\ \nu\left[\eta\, p(n-1|i-1,k) + \bar{\eta}\, p(n|i-1,k)\right] + (1-\nu)\, p_a(n|i), & \text{if } i\neq 0 \text{ and } k=0, \\ \nu\left[\eta\, p(n-1|i-1,k) + \bar{\eta}\, p(n|i-1,k)\right] & \\ \quad +(1-\nu)\left[\bar{\eta}\, p(n-1|i,k-1) + \eta\, p(n|i,k-1)\right], & \text{else.} \end{cases} \tag{57}$$

The last relation can be written more simply as

$$p(n|i,k) = \begin{cases} 1, & \text{if } i=0 \text{ and } k=0, \\ p_b(n|k), & \text{if } i=0 \text{ and } k\neq 0, \\ p_a(n|i), & \text{if } i\neq 0 \text{ and } k=0, \\ \nu\left[\eta\, p(n-1|i-1,k) + \bar{\eta}\, p(n|i-1,k)\right] & \\ \quad +(1-\nu)\left[\bar{\eta}\, p(n-1|i,k-1) + \eta\, p(n|i,k-1)\right], & \text{else.} \end{cases} \tag{58}$$

IV. ASYMPTOTICS OF THE TRANSITION PROBABILITIES

The asymptotic behavior of a sequence $\{c_n\}$ for a growing index can be studied by analyzing the asymptotic behavior of the corresponding GF $g(z)$ around its singularities. This is encompassed in the Tauberian theorems [9], the most famous of which being due to Hardy, Littlewood [10] and Karamata [11].

Theorem 1 (The HLK Tauberian theorem) *Let $g(z)$ be a power series with radius of convergence equal to 1, satisfying*

$$g(z) \sim \frac{1}{(1-z)^{\alpha}} \Lambda\left(\frac{1}{1-z}\right), \quad z \to 1, \tag{59}$$

for some $\alpha \geq 0$ with Λ a slowly varying function. Assume that the coefficients $c_n = [z^n]g(z)$ are all non-negative. Then

$$\sum_{k=0}^{n} c_k \sim \frac{n^{\alpha}}{\Gamma(\alpha+1)} \Lambda(n), \quad n \to \infty. \tag{60}$$

A function Λ is said to be slowly varying at infinity if and only if, for any $\beta > 0$, one has

$$\frac{\Lambda(\beta x)}{\Lambda(x)} \to 1 \quad \text{as} \quad x \to +\infty. \tag{61}$$

Our aim is now to use Tauberian theorems in order to study the asymptotic behavior of $B_n^{(i,i)}$ for $\eta = 1/2$. Theorem 1 can be generalized, and in case of multiple singularities, each one can be analyzed separately, and the different contributions can be combined in the end [9]. In our case, this must be done in two steps, since our sequence has two indices i and n. We begin by analyzing the behavior of

$$[z^n]\mathcal{T}_{i,n}\left[B_n^{(i,i)}\Big|_{\eta=1/2}\right](x,z) = \mathcal{T}_i\left[B_n^{(i,i)}\Big|_{\eta=1/2}\right](x)\,, \tag{62}$$

the GF in i, by studying the behavior of

$$\mathcal{T}_{i,n}\left[B_n^{(i,i)}\Big|_{\eta=1/2}\right](x,z)\,, \tag{63}$$

the GF in i and n. We then investigate the resulting

$$\mathcal{T}_i\left[B_n^{(i,i)}\Big|_{\eta=1/2}\right](x) \tag{64}$$

in order to conclude about

$$B_n^{(i,i)}\Big|_{\eta=1/2}. \tag{65}$$

Behavior of $\mathcal{T}_i\left[B_n^{(i,i)}\Big|_{\eta=1/2}\right](x)$. The function given in Eq. (29) has two singularities $z_1(x) := 1/\sqrt{x}$ and $z_2(x) := -1/\sqrt{x}$. First,

$$\mathcal{T}_{i,n}\left[B_n^{(i,i)}\Big|_{\eta=1/2}\right](x,z) \sim \frac{1}{\sqrt{2(1-x)(1-\sqrt{x}z)}}, \quad \text{when } z \to z_1(x). \tag{66}$$

Define the sequence $\beta_{i,n}^{(1)}$ such that

$$\sum_{i,n} \beta_{i,n}^{(1)} x^i z^n = \frac{1}{\sqrt{2(1-x)(1-\sqrt{x}z)}}. \tag{67}$$

In other words,

$$\mathcal{T}_{i,n}\left[B_n^{(i,i)}\Big|_{\eta=1/2}\right](x,z)\sim\sum_{i,n}\beta_{i,n}^{(1)}x^i z^n,\quad \text{when } z\to z_1(x). \tag{68}$$

Equation (67) is the same as (x is positive)

$$\sum_{i,n}\beta_{i,n}^{(1)}x^i\left(\frac{z}{\sqrt{x}}\right)^n=\frac{1}{\sqrt{2(1-x)(1-z)}}, \tag{69}$$

or,

$$\sum_{i,n}\beta_{i,n}^{(1)}x^{i-\frac{n}{2}}z^n=\frac{1}{\sqrt{2(1-x)(1-z)}}. \tag{70}$$

Now, for n increasing, according to the Tauberian theorems,

$$[z^n]\frac{1}{\sqrt{2(1-x)(1-z)}}\sim\frac{1}{\sqrt{2(1-x)\pi n}}, \tag{71}$$

so that

$$[z^n]\sum_{i,n}\beta_{i,n}^{(1)}x^{i-\frac{n}{2}}z^n\sim\frac{1}{\sqrt{2(1-x)\pi n}}, \tag{72}$$

$$[z^n]\sum_{i,n}\beta_{i,n}^{(1)}x^i z^n\sim\frac{x^{\frac{n}{2}}}{\sqrt{2(1-x)\pi n}}. \tag{73}$$

Using Definition (67), we end up with

$$[z^n]\frac{1}{\sqrt{2(1-x)(1-\sqrt{x}z)}}\sim\frac{x^{\frac{n}{2}}}{\sqrt{2(1-x)\pi n}}. \tag{74}$$

Secondly,

$$\mathcal{T}_{i,n}\left[B_n^{(i,i)}\Big|_{\eta=1/2}\right](x,z)\sim\frac{1}{\sqrt{2(1-x)(1+\sqrt{x}z)}},\quad \text{when } z\to z_2(x). \tag{75}$$

We can do the same analysis, and obtain

$$[z^n]\frac{1}{\sqrt{2(1-x)(1+\sqrt{x}z)}}\sim\frac{(-1)^n x^{\frac{n}{2}}}{\sqrt{2(1-x)\pi n}}. \tag{76}$$

As we explained earlier, in the case of two singularities (having the same absolute value), the two asymptotic contributions can be added up [9], so that

$$[z^n]\mathcal{T}_{i,n}\left[B_n^{(i,i)}\Big|_{\eta=1/2}\right](x,z)\sim\frac{x^{\frac{n}{2}}}{\sqrt{2(1-x)\pi n}}+\frac{(-1)^n x^{\frac{n}{2}}}{\sqrt{2(1-x)\pi n}}, \tag{77}$$

or,

$$\mathcal{T}_{i}\left[B_n^{(i,i)}\Big|_{\eta=1/2}\right](x)\sim\frac{1+(-1)^n}{\sqrt{2\pi n}}\frac{x^{\frac{n}{2}}}{\sqrt{1-x}}. \tag{78}$$

The zero contribution for odd n is consistent with the fact that the total input photon number $2i$ is even.

Behavior of $B_n^{(i,i)}\Big|_{\eta=1/2}$. The function on the right-hand side of Eq. (78) has only one singularity, $x_0 \coloneqq 1$. Since the dominant factor is $1/\sqrt{1-x}$ (compared to $x^{\frac{n}{2}}$) when $x \to x_0$, we can focus on it. We have [9]

$$[x^i]\frac{1}{\sqrt{1-x}} \sim \frac{1}{\sqrt{\pi i}}, \tag{79}$$

meaning that

$$[x^i]\left(x^{-\frac{n}{2}}\mathcal{T}_i\left[B_n^{(i,i)}\Big|_{\eta=1/2}\right](x)\right) \sim \frac{1+(-1)^n}{\sqrt{2\pi n}}\frac{1}{\sqrt{\pi i}}. \tag{80}$$

Now,

$$\begin{aligned} x^{-\frac{n}{2}}\mathcal{T}_i\left[B_n^{(i,i)}\Big|_{\eta=1/2}\right](x) &= x^{-\frac{n}{2}}\sum_{i=0}^{\infty} B_n^{(i,i)}\Big|_{\eta=1/2} x^i \\ &= \sum_{i=0}^{\infty} B_n^{(i,i)}\Big|_{\eta=1/2} x^{i-\frac{n}{2}} \\ &= \sum_{j=-n/2}^{\infty} B_n^{(j+\frac{n}{2},j+\frac{n}{2})}\Big|_{\eta=1/2} x^j, \end{aligned}$$

and $B_n^{(i,i)} = 0$ if $n > 2i$, so that

$$x^{-\frac{n}{2}}\mathcal{T}_i\left[B_n^{(i,i)}\Big|_{\eta=1/2}\right](x) = \sum_{j=0}^{\infty} B_n^{(j+\frac{n}{2},j+\frac{n}{2})}\Big|_{\eta=1/2} x^j, \tag{81}$$

and

$$[x^i]\left(x^{-\frac{n}{2}}\mathcal{T}_i\left[B_n^{(i,i)}\Big|_{\eta=1/2}\right](x)\right) = B_n^{(i+\frac{n}{2},i+\frac{n}{2})}\Big|_{\eta=1/2}. \tag{82}$$

As a consequence of Equation (80),

$$B_n^{(i+\frac{n}{2},i+\frac{n}{2})}\Big|_{\eta=1/2} \sim \frac{1+(-1)^n}{\sqrt{2\pi n}}\frac{1}{\sqrt{\pi i}}, \tag{83}$$

or,

$$B_n^{(i,i)}\Big|_{\eta=1/2} \sim \frac{1+(-1)^n}{\sqrt{2\pi n}}\frac{1}{\sqrt{\pi\left(i-\frac{n}{2}\right)}}. \tag{84}$$

After some simplification, we obtain

$$B_n^{(i,i)}\Big|_{\eta=1/2} \sim \frac{1+(-1)^n}{\pi\sqrt{n\,(2i-n)}}. \tag{85}$$

which exactly coincides with the result of the analysis performed in [12]. The output terms around $n \sim i$ are maximally suppressed, which is reminiscent of the HOM effect. Interestingly, we can again exploit partial time reversal and extend this analysis to a TMS with $\lambda = 1/2$, giving

$$A_k^{(i,k)} \sim \frac{1+(-1)^i}{2\pi\sqrt{i(2k-i)}}, \quad k, i \to \infty. \tag{86}$$

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