nature photonics

Article https://doi.org/10.1038/s41566-023-01213-0

# Boson bunching is not maximized by indistinguishable particles

In the format provided by the authors and unedited

**Resilience to perturbations** — It is natural to ask whether small perturbations of the interferometer $U$ or partial distinguishability matrix $S$ would immediately negate the violation of the generalized bunching conjecture, or, equivalently of the Bapat-Sunder conjecture, in which case the discussion would be physically insignificant. However, it is easy to prove that this is not the case, as we now show.

We consider, for instance, a perturbation of the matrix $A$ around a violation point where $\mathrm{perm}\,(A \odot B) > \mathrm{perm}\,(A)$, the reasoning being similar if we consider a perturbation of the matrix $B$. We consider the perturbation $A \to (1-\epsilon)A + \epsilon\,\Delta$, with $\Delta_{i,i} = 1, \Delta_{i,j} = \mathcal{O}(1)$, and $\epsilon \ll 1$. Note that

$$\begin{aligned} &\mathrm{perm}\,((1-\epsilon)A + \epsilon\,\Delta) \\ &= \mathrm{perm}\left((1-\epsilon)\left(A + \frac{\epsilon}{(1-\epsilon)}\Delta\right)\right) \\ &= (1-\epsilon)^n\,\mathrm{perm}\,(A + \delta\,\Delta)\,, \end{aligned} \tag{1}$$

with $\delta = \epsilon/(1-\epsilon) = \mathcal{O}(\epsilon)$.

Next, we exploit a formula from Minc [1] for the permanent of the sum of two matrices, namely

$$\begin{aligned} &\mathrm{perm}\,(A + A') \\ &= \sum_{r=0}^{n} \sum_{\alpha,\beta \in Q_{r,n}} \mathrm{perm}\,(A[\alpha,\beta])\;\mathrm{perm}\,(A'(\alpha,\beta))\,, \end{aligned} \tag{2}$$

where $Q_{r,n}$ is the set of increasing sequences. More precisely, if we denote $\Gamma_{r,n}$ as the set of all $n^r$ sequences $\omega = (\omega_1, \ldots, \omega_r)$ of integers, with $1 \leq \omega_i \leq n$ and $i = 1, \ldots, n$, we define

$$Q_{r,n} = \{(\omega_1, \ldots, \omega_r) \in \Gamma_{r,n} \mid 1 \leq \omega_1 < \cdots < \omega_r \leq n\}\,. \tag{3}$$

Moreover, $A[\alpha, \beta]$ denotes the $r \times r$ matrix constructed by choosing the rows and columns of $A$ corresponding to the elements of sequences $\alpha$ and $\beta$. In contrast, $A(\alpha, \beta)$ is the $(n-r) \times (n-r)$ matrix where these rows and columns have been excluded. Using Eq. (2), we can expand the permanents of interest to the first order in $\delta$, namely

$$\begin{aligned} &\mathrm{perm}\,(A + \delta\Delta) = \mathrm{perm}\,(A) \\ &\quad + \delta \sum_{i,j=1}^{n} \Delta_{i,j}\,\mathrm{perm}\,(A(i,j)) + \mathcal{O}(\delta^2), \\ &\mathrm{perm}\,((A + \delta\,\Delta) \odot B) = \mathrm{perm}\,(A \odot B) \\ &\quad + \delta \sum_{i,j=1}^{n} \Delta_{i,j}\,B_{i,j}\,\mathrm{perm}\,((A \odot B)(i,j)) + \mathcal{O}(\delta^2), \end{aligned} \tag{4}$$

so that if $\mathrm{perm}\,(A \odot B) > \mathrm{perm}\,(A)$, then, by choosing small enough $\epsilon$, we have that

$$\mathrm{perm}\,(((1-\epsilon)A + \epsilon\,\Delta) \odot B) > \mathrm{perm}\,((1-\epsilon)A + \epsilon\,\Delta)\,. \tag{5}$$

This means that small enough perturbations to the matrix $A$ still lead to a violation of the Bapat-Sunder inequality. A similar reasoning can be used to argue about the robustness to perturbations of matrix $B$. For the particular counterexample of the generalized bunching conjecture considered in the main text, the robustness to perturbations can be seen in Fig. 4.

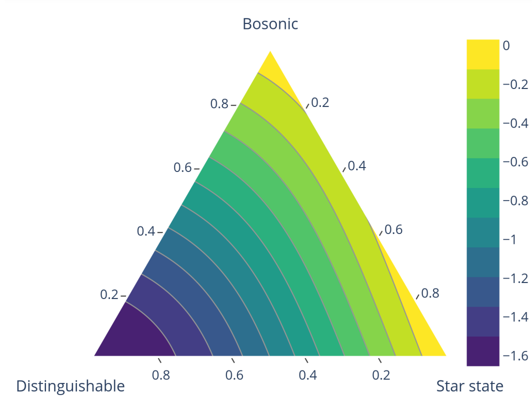


FIG. S1. Ternary plot of the logarithm (in base 10) of the bunching ratio $P_7(S(x,y))/P_7^{(\mathrm{bos})}$ for the optical scheme from Fig. 2, with a distinguishability matrix $S(x,y)$ belonging to the two-parameter family defined in Eq. (6). We see two regions where the ratio $\simeq 1$, one near the bosonic case (indistinguishable particles) and a separate one near the partially distinguishable case (star state).

In order to further visualize how the choice of distinguishability matrix can affect the bunching violation ratio, we consider the following two-parameter family of $S$ matrices of dimension 7

$$S(x,y) = (1-x-y)\,S^{(\star)} + x\,S^{(\mathrm{bos})} + y\,S^{(\mathrm{dist})}. \tag{6}$$

where $x, y \geq 0$ and $x + y \leq 1$. Here, $S^{(\star)}$ corresponds to the $S$ matrix of the partially distinguishable input state from Fig. 2, whereas $S^{(\mathrm{bos})} = \mathbb{E}$ and $S^{(\mathrm{dist})} = \mathbb{1}$ correspond to the fully indistinguishable and fully distinguishable cases, respectively. The bunching violation ratio for these different $S$ matrices is shown as a ternary plot in Fig. S1. As one gets closer to the case of distinguishable particles, the bunching decreases significantly, as expected. However, when we interpolate between the $S^{(\mathrm{bos})}$ and $S^{(\star)}$, the bunching probability behaves non-monotonically and the bunching violation ratio $P_7(S(x,y))/P_7^{(\mathrm{bos})}$ attains values larger than 1 in a small region around $S^{(\star)}$.

**Stability around the bosonic case** — We now turn to the question of whether the violation of the generalized bunching conjecture could possibly be observed in the neighborhood of the fully indistinguishable bosonic case. We demonstrate that first-order perturbations to the internal wave functions of the photons around the fully indistinguishable case leave multimode bunching probabilities invariant, which suggests that it is a local extremal point. We start with the internal wave functions of fully indistinguishable photons

$$|\phi_i\rangle = |\phi_0\rangle\,, \tag{7}$$

for all $i = 1, \ldots, n$, and consider a perturbation only to the first photon's internal wavefunction, namely

$$|\phi_1'\rangle = |\phi_0\rangle + |\delta\phi_1\rangle\,. \tag{8}$$

In order for this state to remain normalized, we need

$$\langle \phi_0 | \delta\phi_1 \rangle = ix, \tag{9}$$

with $x \in \mathbb{R}$ being a (small) perturbation parameter. Consider a perturbation around the bosonic case, *i.e.*, $S \to \mathbb{E} + \delta S$, with

$$\delta S = \begin{pmatrix} 0 & -ix & \cdots & -ix \\ ix & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ ix & 0 & \cdots & 0 \end{pmatrix}. \tag{10}$$

Note that $\delta S$ is not a Gram matrix. Then

$$\begin{aligned} \text{perm}\left(H \odot S^T\right) &= \text{perm}\left(H + H \odot \delta S^T\right) \\ &= \text{perm}\,(H) + \sum_{i,j=1}^{n} \delta S_{j,i}\, H_{i,j}\, \text{perm}\,(H(i,j)) + \mathcal{O}(x^2). \end{aligned} \tag{11}$$

To alleviate the notations, let us define the matrix $F$ such that

$$F_{i,j} = H_{i,j}\, \text{perm}\,(H(i,j)), \tag{12}$$

and note that $F$ can be proven to be Hermitian (since $H$ is an Hermitian matrix). Then, the first-order perturbation to the bunching probability is

$$\sum_{i,j=1}^{n} (\delta S^T \odot F)_{i,j} = -2x \sum_{j=2}^{n} \mathcal{I}(F_{1,j}) = 0 \tag{13}$$

where $\mathcal{I}$ stands for the imaginary part. The second equality holds because the Laplace expansion of the permanent of $H$ gives perm $(H) = \sum_{j=1}^{n} F_{1,j} = F_{1,1} + \sum_{j=2}^{n} F_{1,j}$, which is real (in fact $\geq 0$) since $H$ is Hermitian (positive semidefinite), and $F_{1,1}$ is real ($\geq 0$) since $F$ is Hermitian, hence $\sum_{j=2}^{n} F_{1,j}$ is real too. Thus, we have

$$\text{perm}\left(H \odot (\mathbb{E} + \delta S^T)\right) = \text{perm}\,(H) + \mathcal{O}(x^2). \tag{14}$$

A similar procedure can be applied when every photon's internal wave function is modified by a small quantity such that $\langle \phi_0 | \delta\phi_j \rangle = ix_j = \mathcal{O}(x)$ with $j = 1, \ldots, n$, giving the same result to first order. This means that a small physical perturbation of the internal wavefunctions of the photons around the bosonic case (fully indistinguishable particles) leads to a bunching probability that is unchanged to first order. We leave open the question of whether the bosonic case corresponds to a local maximum of the multimode bunching probability, as suggested by the scheme considered in the main text. This would be true only if second-order perturbations always give a negative contribution to this probability.

[1] Henryk Minc. *Permanents*. Cambridge University Press, 1984.